UNIVERSAL TRIPLE I METHOD FOR FUZZY REASONING AND FUZZY CONTROLLER

Y. M. TANG AND F. J. REN

Abstract. As a generalization of the triple I method, the universal triple I method is investigated from the viewpoints of both fuzzy reasoning and fuzzy controller. The universal triple I principle is put forward, which improves the previous triple I principle. Then, unified form of universal triple I method is established based on the (0,1)-implication or R-implication. Moreover, the reversibility property of universal triple I method is analyzed from expansion, reduction and other type operators, which demonstrate that its reversibility property seems fine, especially for the case employing the (0,1)-implication. Lastly, we analyze the response ability of fuzzy controllers based on universal triple I method, then the practicability of triple I method is improved.

1. **Introduction**

Fuzzy reasoning plays a significant role in fuzzy control, artificial intelligence, affective computing, image processing and complex system (see [1, 7, 17, 18, 26, 28]). It is well-known that the most basic problem of fuzzy reasoning is fuzzy modus ponens (FMP) as follows:

FMP: For a given rule "If x is A then y is B " and input "x is A^* ",
to compute B^* (output), (1)

where $A, A^* \in F(X), B, B^* \in F(Y)$ $(F(X), F(Y)$ respectively denote the set of all fuzzy subsets on X and $Y)$. As for the FMP problem (1), the broadly used method in fuzzy control is the famous CRI (Compositional Rule of Inference) method proposed by Zadeh (see [4, 8, 20, 37]). The CRI solution is as follows:

$$B^*(y) = \sup_{x \in X}\{A^*(x) \wedge (A(x) \to B(y))\} \ (y \in Y) \tag{2}$$

where $\to$ is an implication. In 1999, Wang pointed out that the CRI method had some blemishes (see [33]). To improve the CRI method Wang put forward the triple

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I method (see [33]), whose idea is to seek the optimal $B^* \in F(Y)$ such that

$$(A(x) \to B(y)) \to (A^*(x) \to B^*(y)) \tag{3}$$

takes the maximum for any $x \in X, y \in Y$. Following that, lots of scholars carried through a series of researches related to the triple I method (see [5, 14, 34, 38]), demonstrating that the triple I method possesses many advantages such as its logic basis, excellent reversibility property, and the property of pointwise optimization (see [11, 24, 27, 36]), which is excellent from the viewpoint of logic.

It is pointed out in [10,12] that many usual fuzzy controllers (including the one based on the CRI method) can be regarded as an interpolation method. Following that, the interpolation mechanism (or more general terminology called " response ability ") of fuzzy controllers based on the triple I method or CRI method, have attracted rapidly growing interests (see [6, 14, 15, 25, 29]). Li et al. discussed the response ability of 23 fuzzy controllers based on the CRI method, where got the fact that 12 fuzzy controllers can be used (see [14]). In [6], Hou et al. analyzed the response ability of 51 fuzzy controllers based on the triple I method, and achieved only 2 usable fuzzy controllers. Li et al. discussed the response ability of 4 fuzzy controllers based on the triple I method, and obtained 2 usable fuzzy controllers (see [15]). We drew the conclusion that 2 fuzzy controllers are practicable in 11 fuzzy controllers via the triple I method, while 4 fuzzy controllers are usable in 11 ones via the CRI method, which are constructed by the same 11 implications (see [29]). As a result, there are very few usable fuzzy controllers based on the triple I method, implying that the response ability and practicability of the triple I method are imperfect (from the viewpoint of fuzzy controllers).

To solve this problem, an important way is to improve the triple I method. Li pointed out in [11] the fact that the CRI method is a special case of the triple I method only if three implications in (3) are different. In detail, the CRI method can be regarded as the triple I method where (3) is changed into

$$(A(x) \to B(y)) \to_2 (A^*(x) \to_2 B^*(y)),$$

where $\to_2$ takes the Mamdani operator I_M.

Enlightened by this idea, we can let the latter two implications be the same and the first one unlimited, that is, generalize (3) to:

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)), \tag{4}$$

where $\to_1$ and $\to_2$ (respectively called the first implication and second implication in the sequel) can take different implications, and the triple I method derived from (4) is called the differently implicational universal triple I method of (1, 2, 2) type (universal triple I method for short). In [30, 31], we have already proposed and discussed the universal triple I method with some preliminary results.

In the theory of fuzzy reasoning, there are no acknowledged standards to judge whether a fuzzy reasoning method is excellent; but the reversibility property of fuzzy reasoning method is a basic demand (embodying the compatibility), which is recognized by a lot of scholars (see [5, 23, 24, 27]). The reversibility property of universal tripe I method is not researched. Therefore, one aim of this paper is to analyze the reversibility property of universal tripe I method. Following that, we shall inspect the universal tripe I method from the viewpoint of fuzzy controllers, i.e., the response ability of fuzzy controllers based on universal triple I methods.

The rest of this paper is organized as follows. Section 2 is the preliminaries. In sections 3 and 4, the universal triple I method is investigated from the viewpoint of fuzzy reasoning. The solutions of universal triple I method are discussed, then based on them, the reversibility property of universal triple I method is analyzed. In section 5, the universal triple I method is investigated from the viewpoint of fuzzy controllers. The response ability of single-input single-output (SISO) fuzzy controllers and double-input single-output (DISO) fuzzy controllers based on the universal triple I method are respectively researched. Section 6 provides several discussions of the universal triple I method. Section 7 concludes this paper.

2. **Preliminaries**

Definition 2.1. An implication on $[0,1]$ is a function $I : [0,1]^2 \to [0,1]$ satisfying
(C1) $I(0,0) = I(0,1) = I(1,1) = 1$, $I(1,0) = 0$.
$I(a,b)$ is also written as $a \to b$ $(a,b \in [0,1])$.

Definition 2.2. A function $T : [0,1]^2 \to [0,1]$ is called a t-norm if T is associative commutative, increasing and satisfies $T(1,a) = a$ $(a \in [0,1])$.

Definition 2.3. A function $S : [0,1]^2 \to [0,1]$ is called a t-conorm if S is associative commutative, increasing and satisfies $S(0,a) = a$ $(a \in [0,1])$.

Definition 2.4. A decreasing function $N : [0,1] \to [0,1]$ is called a strong negation, if $N(0) = 1$, $N(1) = 0$ and $N(N(a)) = a$ $(a \in [0,1])$.
Definition 2.5. [18] An implication $\to$ is said to be an R-implication if there exist a left-continuous t-norm T such that (where $\vee$ denotes supremum)

$$a \to b = \vee\{x \in [0,1] |\ T(a,x) \le b\}, \quad a,b \in [0,1]. \tag{5}$$

Definition 2.6. Let T, $\to$ be two $[0,1]^2 \to [0,1]$ functions, $(T, \to)$ is called a residual pair or, T and $\to$ are residual to each other, if the residual condition holds, i.e., (iff denotes "if and only if")

$$T(a,b) \le c \text{ iff } b \le a \to c, \quad a,b,c \in [0,1]. \tag{6}$$

Proposition 2.7. [3] *Let T be a t-norm, then the following are equivalent: (i) T is left-continuous; (ii) T and $\to$ form a residual pair, where $\to$ is from (5).*

Definition 2.8. An implication $\to$ is said to be an S-implication if there exist a t-conorm S and a strong negation N such that

$$a \to b = S(N(a), b), \ \ a,b \in [0,1].$$

Definition 2.9. An implication $\to$ is called a QL-implication if there exist a t-norm T, a t-conorm S and a strong negation N such that

$$a \to b = S(N(a), T(a,b)), \ \ a,b \in [0,1].$$

Definition 2.10. [32] Let $\to$ be a function $I : [0,1]^2 \to [0,1]$.

(i) $\to$ is called an expansion type operator, if $a \to b \geq b$ holds for all $a \in [0,1], b \in [0,1]$.

(ii) $\to$ is called a reduction type operator, if $a \to b \leq b$ holds for all $a \in [0,1], b \in [0,1]$.

(iii) $\to$ is called an other type operator if it is neither expansion type nor reduction type.

Proposition 2.11. [32] *If $\to$ is an R-implication or S-implication, then $\to$ is an expansion type operator.*

It follows from Theorem 3.1 of [30] that Proposition 2.12 can be proved.

Proposition 2.12. *Let $\to$ be an implication satisfying*
(C2) $a \to b$ is non-decreasing w.r.t. b $(a, b \in [0,1])$,
(C3) $a \to b$ is right-continuous w.r.t. b $(a \in [0,1],\ b \in [0,1))$,
(C4) $a \leq b$ iff $a \to b = 1$ $(a, b \in [0,1])$,
and define $T : [0,1]^2 \to [0,1]$ as follows:

$$T(a,b) = \wedge\{x \in [0,1] \mid b \leq a \to x\}, \quad a, b \in [0,1],$$

then $(T, \to)$ is a residual pair, and (5) holds, where $\wedge$ denotes infimum.

Definition 2.13. Let $\to$ be an implication satisfying (C2), (C3) and (C4), and its residual function T satisfy $T(1,b) = b,\ T(0,b) = 0$ $(b \in [0,1])$, then $\to$ is called a (0,1)-implication.

It is easy to prove Lemma 2.14 (from residual condition (6)).

Lemma 2.14. *Let $\to$ be an implication satisfying (C2), (C3) and (C4), and T the function residual to $\to$, then $(a, b, c \in [0,1])$ (i) if $b \leq c$, then $T(a,b) \leq T(a,c)$; (ii) $T(a,b) \leq a$; (iii) $T(0,b) = 0$; (iv) $T(a, a \to b) \leq b$.*

It follows from Lemma 2.14 that $T(0,b) = 0$ can be deleted in Definition 2.13.

Proposition 2.15. *If $\to$ is an R-implication, then $\to$ is a (0,1)-implication.*

Proof. There uniquely exists T which is the residual function w.r.t. $\to$, and T is a left-continuous t-norm, thus we have $T(1,b) = b$, $T(0,b) = 0$ $(b \in [0,1])$. It follows from Proposition 4.3 in [30] and Proposition 1 in [36] that $\to$ satisfies (C2), (C3) and (C4). Thus it is easy to get that $\to$ is a (0,1)-implication. □

Definition 2.16. [34] Let T, $\to$ be two $[0,1]^2 \to [0,1]$ functions. $([0,1], T, \to)$ is called a residuated lattice, if the following conditions hold:

(i) $([0,1], T)$ is a commutative semigroup with unit 1.

(ii) $(T, \to)$ is a residual pair in which T is non-decreasing w.r.t. every variable, and $\to$ is an implication where $a \to b$ is non-increasing w.r.t. a and non-decreasing w.r.t. b $(a, b \in [0,1])$;

Lemma 2.17. [34] *If* $([0,1], T, \to)$ *is a residuated lattice, then*

(i) $\wedge\{a \to x_i | i \in I\} = a \to \wedge\{x_i | i \in I\}$ *holds for any* $a, x_i \in [0,1]$ $(I \neq \varnothing)$;

(ii) $\to$ *satisfies (C1) and (C4).*

Lemma 2.18. [30] *If* $\to$ *satisfies* $\wedge\{a \to x_i | i \in I\} = a \to \wedge\{x_i | i \in I\}$ *where* $a, x_i \in [0,1]$, $I \neq \varnothing$, *then* $\to$ *satisfies (C2) and (C3).*

Proposition 2.19. *If* $([0,1], T, \to)$ *is a residuated lattice, then* $\to$ *is a (0,1)-implication.*

Proof. Since $([0,1], T, \to)$ is a residuated lattice, $(T, \to)$ is a residual pair, and it follows from Lemma 2.17 that the implication $\to$ satisfies (C4), and $\wedge\{a \to x_i | i \in I\} = a \to \wedge\{x_i | i \in I\}$ holds for any $a, x_i \in [0,1]$ $(I \neq \varnothing)$. By Lemma 2.18, we get that $\to$ satisfies (C2) and (C3). Meanwhile, since $([0,1], T)$ is a commutative semigroup with unit 1, we have $T(a,b) = T(b,a)$, $T(1,b) = b$ $(a, b \in [0,1])$. Then considering $(T, \to)$ is a residual pair, we achieve that $(T, \to)$ is a residual pair, thus $\to$ is a (0,1)-implication. □

Here we mainly consider 9 familiar operators. They are Lukasiewicz implication I_L, Fodor implication I_{FD} (see [2], which is also called I_0 implication, see [22, 33]), Gödel implication I_G, Goguen implication I_{Go}, revised Reichenbach implication I_{RR} (see [14, 21]), Zadeh implication I_Z, Kleene-Dienes implication I_{KD}, Mamdani operator I_M and Larsen operator I_{La} as the following.

$$I_L(a,b) = \begin{cases} 1, & a \le b \\ 1-a+b, & a > b \end{cases}, \qquad I_{FD}(a,b) = \begin{cases} 1, & a \le b \\ (1-a) \vee b, & a > b \end{cases},$$

$$I_G(a,b) = \begin{cases} 1, & a \le b \\ b, & a > b \end{cases}, \qquad I_{Go}(a,b) = \begin{cases} 1, & a \le b \\ b/a, & a > b \end{cases},$$

$$I_{RR}(a,b) = \begin{cases} 1, & a \le b \\ 1-a+ab, & a > b \end{cases}, \qquad I_Z(a,b) = (1-a) \vee (a \wedge b),$$

$$I_{KD}(a,b) = (1-a) \vee b, \qquad I_M(a,b) = a \wedge b,$$

$$I_{La}(a,b) = a \times b.$$

Here $I_L, I_{FD}, I_G, I_{Go}, I_{RR}$ satisfies (C1), (C2), (C3) and (C4); I_Z, I_{KD} satisfies (C1), (C2) and (C3); I_M, I_{La} satisfies (C2), (C3).

It is noted that I_M, I_{La} do not satisfy (C1), but they are also recognized by some authors (see e.g. [6, 11, 14, 32]). Moreover it is pointed out in [19] that I_M, I_{La} are referred to collectively as engineering implications. Therefore, for convenience, I_M, I_{La} can also be regarded as special implications.

It is easy to prove Propositions 2.20 and 2.21.

Proposition 2.20. *The operations residual to* $I_L, I_{FD}, I_G, I_{Go}, I_{RR}$ *are respectively:*

$$T_L(a,b) = \begin{cases} a+b-1, & a+b > 1 \\ 0, & a+b \le 1 \end{cases}, \quad T_{FD}(a,b) = \begin{cases} a \wedge b, & a+b > 1 \\ 0, & a+b \le 1 \end{cases},$$

$$T_G(a,b) = a \wedge b, \qquad T_{Go}(a,b) = a \times b,$$

$$T_{RR}(a,b) = \begin{cases} [(a+b-1)/a] \wedge a, & a+b > 1 \\ 0, & a+b \le 1 \end{cases}.$$

Proposition 2.21. *In these implications, we have the following results:*

(i) $I_L, I_{FD}, I_G, I_{Go}, I_{RR}$ *are (0,1)-implications.*

(ii) I_L, I_{FD}, I_G, I_{Go} *are R-implications;* I_L, I_{FD}, I_{KD} *are S-implications;* I_L, I_{FD}, I_{KD}, I_Z *are QL-implications.*

(iii) $I_L, I_{FD}, I_G, I_{Go}, I_{RR}, I_{KD}$ *are expansion type operators;* I_M, I_{La} *are reduction type operators; and* I_Z *is an other type operator.*

Definition 2.22. Let Z be any nonempty set and $F(Z)$ the set of all fuzzy subsets on Z, define partial order relation $\leq_F$ on $F(Z)$ (according to pointwise order) as: $A \leq_F B$ iff $A(z_0) \leq B(z_0)$ for any $z_0 \in Z$, where $A, B \in F(Z)$.

Lemma 2.23. [35] $< F(Z), \leq_F >$ *is a complete lattice.*

3. Solutions of Universal Triple I Method

For the FMP problem (1), from the point of view of universal triple I method, we can obtain the following principle:

Universal triple I principle: The conclusion B^* (in $< F(Y), \leq_F >$) of the FMP problem (1) is the smallest fuzzy set which makes (4) get its maximum for any $x \in X, y \in Y$.

Such principle improves the previous triple I principle for FMP in [33] or [36], since (4) is a generalization of (3) and the former can provide bigger choosing space.

Definition 3.1. Suppose that $A, A^* \in F(X)$, $B \in F(Y)$, if B^* (in $< F(Y), \leq_F >$) makes (4) get its maximum for any $x \in X, y \in Y$, then B^* is called a universal triple I solution.

Definition 3.2. Suppose that $A, A^* \in F(X)$, $B \in F(Y)$, and that nonempty set $\mathbb{E}$ is the set of all universal triple I solutions, and finally that D^* (in $< F(Y), \leq_F >$) is the infimum of $\mathbb{E}$. Then D^* is called an Inf-quasi-solution. And, if D^* is the minimum of $\mathbb{E}$, then D^* is also called a Min-solution.

From Lemma 2.23, $< F(Y), \leq_F >$ is a complete lattice. Thus the Inf-quasi-solution (i.e., the infimum of $\mathbb{E}$) uniquely exists since the non-empty set $\mathbb{E} \subset F(Y)$.

Proposition 3.3. *Suppose that* $\rightarrow_2$ *is an implication satisfying (C2), and that* D_1 *is a universal triple I solution, and finally that* $D_1 \leq_F D_2$ *(in which* $D_1, D_2 \in < F(Y), \leq_F >$*). Then* D_2 *is a universal triple I solution.*

Proof. Since D_1 is a universal triple I solution, it follows that $(A(x) \rightarrow_1 B(y)) \rightarrow_2 (A^*(x) \rightarrow_2 D_1(y))$ takes its maximum for any $x \in X, y \in Y$. Because $D_1 \leq_F D_2$ and $\rightarrow_2$ satisfies (C2), we get that $A^*(x) \rightarrow_2 D_1(y) \leq A^*(x) \rightarrow_2 D_2(y)$ and

$$(A(x) \rightarrow_1 B(y)) \rightarrow_2 (A^*(x) \rightarrow_2 D_1(y)) \leq (A(x) \rightarrow_1 B(y)) \rightarrow_2 (A^*(x) \rightarrow_2 D_2(y))$$

hold for any $x \in X, y \in Y$. Therefore D_2 is also a universal triple I solution. □

Remark 3.4. Suppose that $\rightarrow_2$ satisfies (C2). For (4), once there exists a universal triple I solution B^*, then every fuzzy set D which is larger than B^* ($D \in F(Y)$), will be a solution (it is easy to know from Proposition 3.3). This means that there are many solutions, including $B^*(y) \equiv 1$ $(y \in Y)$. This last is a special solution, for which (4) always takes its maximum no matter what major premise $A \to_1 B$ and minor premise A^* are adopted. Therefore, when the optimal universal triple I solution exists, it should be the smallest one; in other words, it should be the infimum of all solutions (i.e. the infimum of $\mathbb{E}$).

Theorem 3.5. *Suppose that $\to_2$ is a (0,1)-implication, and that T the function residual to $\to_2$, then the Min-solution can be computed as follows:*

$$B^*(y) = \sup_{x \in X}\{T(A^*(x), (A(x) \to_1 B(y)))\},\ y \in Y. \tag{7}$$

Proof. To begin with, we shall show that B^* (given by (7)) is a universal triple I solution, that is, the following formula holds for any $x \in X, y \in Y$:

$$(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 B^*(y)) = 1. \tag{8}$$

In fact, it follows from (7) that $T(A^*(x), (A(x) \to_1 B(y))) \leq B^*(y)$, $x \in X, y \in Y$. Since $(T, \to_2)$ is a residual pair, we get that $A(x) \to_1 B(y) \leq A^*(x) \to_2 B^*(y)$ holds for any $x \in X, y \in Y$. Thus (8) holds (considering $\to_2$ satisfies (C4)).

Furthermore we shall prove that B^* is the minimum of all universal triple I solutions. Suppose that D is any universal triple I solution, thus $(A(x) \to_1 B(y)) \to_2 (A^*(x) \to_2 D(y)) = 1$ holds for any $x \in X, y \in Y$. So $A(x) \to_1 B(y) \leq A^*(x) \to_2 D(y)$ holds for any $x \in X, y \in Y$ (noting that $\to_2$ satisfies (C4)). Then, from the fact that $(T, \to_2)$ is a residual pair, we obtain $T(A^*(x), (A(x) \to_1 B(y))) \leq D(y)$ holds for any $x \in X, y \in Y$. Therefore, $D(y)$ is an upper bound of $\{T(A^*(x), (A(x) \to_1 B(y)))|\, x \in X\}$, $y \in Y$. Thus it follows from (7) that $B^* \leq_F D$. Thus, we obtain that B^* is the minimum of all universal triple I solutions.

Therefore B^* expressed as (7) is the Min-solution (from Definition 3.2). □

We can get Proposition 3.6 from Theorem 3.5 and Proposition 2.15.

Proposition 3.6. *Suppose that $\to_2$ is an R-implication, and that T the function residual to $\to_2$, then the Min-solution can be computed as (7).*

Lemma 3.7. [30] *(i) If $\to_2$ is I_M, then the Min-solution is $B^*(y) = \sup_{x \in X}$ $\{A^*(x) \wedge (A(x) \to_1 B(y))\}$, $y \in Y$.*

(ii) If $\to_2$ is I_Z, then the Min-solution is $B^*(y) = \sup_{x \in E_y}\{A^*(x) \wedge (A(x) \to_1 B(y))\}$, $y \in Y$, where $E_y = \{x \in X|\ (1 - A^*(x)) \vee 0.5 < A(x) \to_1 B(y)\}$.

(iii) If $\to_2 \in \{I_{KD}, I_{La}\}$, then the Min-solution is $B^*(y) = 1$ if $y \in E$, $B^*(y) = 0$ if $y \in Y - E$, where $E = \{y \in Y|\ \sup_{x \in X}\{(A(x) \to_1 B(y)) \times A^*(x)\} > 0\}$.

Remark 3.8. When $\to_1 = \to_2$ in (4), the universal triple I method degenerates into the triple I method. From Lemma 3.7, it is easy to know that when $\to_2$ takes I_M, the universal triple I method degenerates into the CRI method.

4. Reversibility Property of Some Universal Triple I Methods

As pointed out in [36], the most fundamental deduction rule in logic is the classical modus ponens (meaning that if $A \to B$ and A are given, then B follows), thus it is natural to hope that the FMP conclusion B^* of (1) should be equal to

B if $A^* = A$, which is also called that such fuzzy reasoning method possesses the reversibility property (see Definition 4.1).

Definition 4.1. For a method (of fuzzy reasoning) to solve the FMP problem, if this method satisfies the classical modus ponens whenever condition P holds, then this method is said to be reversible under condition P, or P-reversible.

Here 3 conditions are considered as follows: (i) P_1 means that A is a normal fuzzy set (i.e., there exists $a \in X$ such that $A(a) = 1$). (ii) P_2 means that P_1 and $\{A(x)|x \in X\} \supset \{B(y)|y \in Y\}$ hold. (iii) P_3 means that P_1 and $B(y) \in (0,5,1] \cup \{0\}$ (for any $y \in Y$) hold.

When Wang proposed the triple I method, he pointed out the fact that the triple I method has the reversibility property while the CRI method has not (see [33]), which is an important evidence illustrating that the triple I method has stronger logicality. Following that, the reversibility property of triple I method becomes a hot research topic (see [24, 27, 34, 36]).

Similarly, we also need to analyze the reversibility property of universal triple I method, which is not discussed in [30]. Since the universal triple I method is decided by the first implication and second implication, we shall research aiming at some familiar implications (i.e., $\rightarrow_1, \rightarrow_2 \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_{KD}, I_M, I_{La}, I_Z\}$).

For convenience, for any fuzzy set $A(x)$, denote $A^{'}(x) \triangleq 1 - A(x)$.

4.1. **Reversibility Property for Expansion Type Operators.** We shall investigate the case that $\rightarrow_2$ takes an expansion type operator (in detail, I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_{KD}).

Lemma 4.2. *Let $\rightarrow$ be a (0,1)-implication, then $\rightarrow$ satisfies the following condition:*
(C5) $1 \rightarrow b \geq b$, ($b \in [0,1]$).

Theorem 4.3. *If $\rightarrow_2$ is a (0,1)-implication, and $\rightarrow_1$ satisfies (C5) and*
(C6) $a \rightarrow_1 b \leq a \rightarrow_2 b$ whenever $a > b$ ($a, b \in [0,1]$),
then the universal triple I method is P_1-reversible.

Proof. Since $\rightarrow_2$ is a (0,1)-implication, there uniquely exists the function T which is residual to $\rightarrow_2$ and $T(1,b) = b$ ($b \in [0,1]$). When $A^* = A$, it follows from Theorem 3.5 that the Min-solution is $B^*(y) = \sup_{x \in X}\{T(A(x),(A(x) \rightarrow_1 B(y)))\},\ y \in Y$.

Taking into account that P_1 holds, there exists $a \in X$ such that $A(a) = 1$. Since $\rightarrow_2$ is a (0,1)-implication and $\rightarrow_1$ satisfies (C5), we obtain

$$B^*(y) \geq T(A(a),(A(a) \rightarrow_1 B(y))) = T(1,(1 \rightarrow_1 B(y))) = 1 \rightarrow_1 B(y) \geq B(y),$$

i.e. $B^*(y) \geq B(y)$.

We shall show $B^*(y) \leq B(y)$. For any $x \in X$, it follows Lemma 2.14(iv) that $T(A(x),(A(x) \rightarrow_2 B(y))) \leq B(y)$. Since $\rightarrow_1$ satisfies (C6), we get $a \rightarrow_1 b \leq a \rightarrow_2 b$ if $a > b$, and $a \rightarrow_2 b = 1 \geq a \rightarrow_1 b$ if $a \leq b$ (noting that $\rightarrow_2$ satisfies (C4)). Thus $a \rightarrow_1 b \leq a \rightarrow_2 b$ holds for any $a, b \in [0,1]$. So, we get by Lemma 2.14(i) that $T(A(x),(A(x) \rightarrow_1 B(y))) \leq T(A(x),(A(x) \rightarrow_2 B(y))) \leq B(y)$. Then we have $B^*(y) \leq B(y)$.

Therefore $B^* = B$. □

From Proposition 2.15, we can further get Theorem 4.4 and Proposition 4.5.

Theorem 4.4. *If $\rightarrow_2$ is an R-implication, and $\rightarrow_1$ satisfies (C5) and (C6), then the universal triple I method is P_1-reversible.*

Proposition 4.5. *(i)If $\rightarrow_2$ takes I_{FD}, and $\rightarrow_1$ satisfies (C5) and*
(C7) $a \rightarrow_1 b \leq (1-a) \vee b$ whenever $a > b$,
then the universal triple I method is P_1-reversible, especially for the case of $\rightarrow_1 \in \{I_{FD}, I_G, I_{KD}, I_M, I_{La}, I_Z\}$.
(ii) If $\rightarrow_2$ takes I_L, and $\rightarrow_1$ satisfies (C5) and
(C8) $a \rightarrow_1 b \leq 1-a+b$ whenever $a > b$ $(a, b \in [0,1])$,
then the universal triple I method is P_1-reversible, especially for the case of $\rightarrow_1 \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_{KD}, I_M, I_{La}, I_Z\}$.
(iii)If $\rightarrow_2$ takes I_{Go}, and $\rightarrow_1$ satisfies (C5) and
(C9) $a \rightarrow_1 b \leq b/a$ whenever $a > b$,
then the universal triple I method is P_1-reversible, especially for the case of $\rightarrow_1 \in \{I_{Go}, I_G, I_M, I_{La}\}$.
(iv) If $\rightarrow_2$ takes I_G, and $\rightarrow_1$ satisfies (C5) and
(C10) $a \rightarrow_1 b \leq b$ whenever $a > b$ $(a, b \in [0,1])$,
then the universal triple I method is P_1-reversible, especially for the case of $\rightarrow_1 \in \{I_G, I_M, I_{La}\}$.
(v) If $\rightarrow_2$ takes I_{RR}, and $\rightarrow_1$ satisfies (C5) and
(C11) $a \rightarrow_1 b \leq 1-a+ab$ whenever $a > b$ $(a, b \in [0,1])$,
then the universal triple I method is P_1-reversible, especially for the case of $\rightarrow_1 \in \{I_{FD}, I_G, I_{RR}, I_{KD}, I_M, I_{La}, I_Z\}$.

Remark 4.6. If $\rightarrow_2$ takes I_{FD}, and $\rightarrow_1$ satisfies (C5) but does not satisfy (C7), then the universal triple I method can not ensure to be P_1-reversible. In fact, when $A(x) > B(y) > 0$, considering $\rightarrow_1$ does not satisfy (C7), there may exist $x_0 \in E_y$ such that $A(x_0) > B(y) > 0$ and $A(x_0) \rightarrow_1 B(y) > A'(x_0) \vee B(y)$, then $A(x_0) \wedge (A(x_0) \rightarrow_1 B(y)) > B(y)$, thus $B^*(y) > B(y)$. Especially, if $\rightarrow_2$ takes I_{FD}, $\rightarrow_1 \in \{I_L, I_{Go}, I_{RR}\}$, then we have the same result. When $\rightarrow_2 \in \{I_L, I_{Go}, I_{RR}, I_G\}$, we can get similar analysis.

If $\rightarrow_2$ takes I_{KD} , then the universal triple I method can not ensure to be $P-$reversible where $P \in \{P_1, P_2, P_3\}$. In fact, when $A^* = A$, it follows from Lemma 3.7 that we can get the Min-solution $B^*(y)$. When $0 < B(y) < 1$, then $B^*(y) \neq B(y)$ holds (noting that $B^*(y) \in \{0, 1\}$).

Corollary 4.7. *Let $\rightarrow_2$ be a (0,1)-implication, and $\rightarrow_1 = \rightarrow_2 \triangleq \rightarrow$, then the universal triple I method (i.e. the triple I method) is P_1-reversible.*

Proof. Since $\rightarrow_1 = \rightarrow_2$, we have that $\rightarrow_1$ satisfies (C6). Considering $\rightarrow_1$ is a (0,1)-implication, we get from Lemma 4.2 that $1 \rightarrow_1 b \geq b$ holds (i.e. $\rightarrow_1$ satisfies (C5)), thus it follows from Theorem 4.3 that the conclusion is correct. □

From Corollary 4.7, Propositions 2.15, 2.19 and 2.21, we can easily get Corollaries 4.8, 4.9 and 4.10.

Corollary 4.8. *Let* $\rightarrow_2 \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}\}$, *and* $\rightarrow_1 = \rightarrow_2 \triangleq \rightarrow$, *then the universal triple I method (i.e. the triple I method) is* P_1-*reversible.*

Corollary 4.9. *Let* $\rightarrow_2$ *be an R-implication, and* $\rightarrow_1 = \rightarrow_2 \triangleq \rightarrow$, *then the universal triple I method (i.e. the triple I method) is* P_1-*reversible.*

Corollary 4.10. *Let* $([0,1], T, \rightarrow)$ *be a residuated lattice, and* $\rightarrow_1 = \rightarrow_2 \triangleq \rightarrow$, *then the universal triple I method (i.e. the triple I method) is* P_1-*reversible.*

Remark 4.11. In [33, 34], Wang discussed the reversibility property of triple I method (for the FMP problem), which was only aiming at the case of I_{FD}. And Theorem 4 in [33] and Theorem 4.4.12 in [34] pointed out the fact that the triple I method via I_{FD} is P_1-reversible. It is evident that this conclusion is the same as the related one of Corollary 4.8 in this paper.

Remark 4.12. In Theorem 5 of [36] and Theorem 6 of [24], Wang and Pei all gave the same result that the triple I method (for the FMP problem) is P_1-reversible if the implication takes an R-implication, which coincides with Corollary 4.9 in this paper. Moreover, it follows from Proposition 2.15 in this paper that Theorem 5 of [36] (or Theorem 6 in [24]) is a special case of Corollary 4.7 in this paper.

Remark 4.13. In Theorem 5 of [23], Pei got the result that if $([0,1], T, \rightarrow)$ is a residuated lattice, then the triple I method (for the FMP problem) employing $\rightarrow$ is P_1-reversible, which is the same as Corollary 4.10 in this paper. Moreover, it follows from Proposition 2.19 in this paper that Theorem 5 of [23] is a special case of Corollary 4.7 in this paper.

4.2. **Reversibility Property for Reduction Type Operator.** We shall investigate the case that $\rightarrow_2$ takes a reduction type operator (in detail, I_M, I_{La}).

Theorem 4.14. *If* $\rightarrow_2$ *takes* I_M, *and* $\rightarrow_1$ *satisfies (C5) and (C10), then the universal triple I method is* P_1-*reversible, especially for the case of* $\rightarrow_1 \in \{I_G, I_M, I_{La}\}$.

Proof. It follows from Lemma 3.7 that when $A^* = A$, the Min-solution $B^*(y) = \sup_{x\in X}\{A(x) \wedge (A(x) \rightarrow_1 B(y))\}$.

Since P_1 is satisfied, there exists $a \in X$ such that $A(a) = 1$, then we get $B^*(y) \geq 1 \wedge (1 \rightarrow_1 B(y)) \geq B(y)$ (noting that $\rightarrow_1$ satisfies (C5)), i.e. $B^*(y) \geq B(y)$. For any $x \in X$, if $A(x) \leq B(y)$, then $A(x) \wedge (A(x) \rightarrow_1 B(y)) \leq A(x) \leq B(y)$; if $A(x) > B(y)$, then $A(x) \wedge (A(x) \rightarrow_1 B(y)) \leq A(x) \rightarrow_1 B(y) \leq B(y)$ (noting that $\rightarrow_1$ satisfies (C10)), thus $B^*(y) \leq B(y)$.

Therefore, $B^* = B$. □

Remark 4.15. If $\rightarrow_2$ takes I_M, and $\rightarrow_1$ satisfies (C5) but does not satisfy (C10), then it is similar to Remark 4.6 that the universal triple I method can not ensure to be P_1-reversible. Especially, if $\rightarrow_1 \in \{I_{FD}, I_L, I_{Go}, I_{RR}, I_{KD}, I_Z\}$, we have the same conclusion.

If $\rightarrow_2$ takes I_{La}, then it is similar to the case of I_{KD} that the universal triple I method can not ensure to be $P-$reversible where $P \in \{P_1, P_2, P_3\}$.

4.3. **Reversibility Property for Other Type Operators.** Noting that I_Z is an other type operator (and also a typical QL-implication), we shall analyze the case that $\to_2$ takes I_Z.

Theorem 4.16. *If $\to_2$ takes I_Z, and $\to_1$ satisfies (C5), (C7) together with*
(C12) $a \to_1 a = 1$ whenever $0 < a \leq 0.5$ ($a \in [0,1]$),
then the universal triple I method is P_2-reversible.

Proof. When $A^* = A$, we get from Lemma 3.7 that the Min-solution is $B^*(y) = \sup_{x\in E_y}\{A(x) \wedge (A(x) \to_1 B(y))\}$ where $E_y = \{x \in X|\ A'(x) \vee 0.5 < A(x) \to_1 B(y)\}$.

(i) Suppose $B(y) = 0$. We shall observe the structure of E_y. If $A(x) > 0$, considering $\to_1$ satisfies (C7), we have $A'(x) < A(x) \to_1 B(y) \leq A'(x) \vee 0 = A'(x)$, which is a contradiction. If $A(x) = 0$, then $A'(x) \vee 0.5 = 1 < A(x) \to_1 B(y)$, which is also a contradiction. Thus we get $E_y = \varnothing$, therefore $B^*(y) = 0 = B(y)$.

(ii) Suppose $B(y) > 0$. At first, we shall show $B^*(y) \geq B(y)$. (a) Suppose $B(y) > 0.5$. Since P_2 holds, there exists $a \in X$ such that $A(a) = 1$, thus $A(a) \to_1 B(y) \geq B(y) > A'(a) \vee 0.5$ (noting that $\to_1$ satisfies (C5)). Thus $a \in E_y$, and then $B^*(y) \geq A(a) \wedge (A(a) \to_1 B(y)) \geq B(y)$, i.e. $B^*(y) \geq B(y)$. (b) Suppose $0 < B(y) \leq 0.5$. Since P_2 is satisfied, there exists $b \in X$ such that $A(b) = B(y)$, so $A(b) \to_1 B(y) = 1 > A'(b) \vee 0.5$ (noting that $\to_1$ satisfies (C12)). Thus $b \in E_y$, and then $B^*(y) \geq A(b) \wedge (A(b) \to_1 B(y)) = A(b) = B(y)$, i.e. $B^*(y) \geq B(y)$.

Furthermore, we shall prove $B^*(y) \leq B(y)$ when $B(y) > 0$. Inspecting the process above, it is easy to know E_y is not empty. For any $x \in E_y$, $A'(x) \vee 0.5 < A(x) \to_1 B(y)$ holds. If $A(x) \leq B(y)$, then $A(x) \wedge (A(x) \to_1 B(y)) \leq A(x) \leq B(y)$. If $A(x) > B(y)$, then $A'(x) \vee 0.5 < A(x) \to_1 B(y) \leq A'(x) \vee B(y)$ (noting that $\to_1$ satisfies (C7)), thus $A'(x) < B(y)$, and $A(x) \wedge (A(x) \to_1 B(y)) \leq A(x) \wedge [A'(x) \vee B(y)] = A(x) \wedge B(y) \leq B(y)$. To sum up, we have $B^*(y) \leq B(y)$.

Therefore $B^* = B$ whenever $B(y) > 0$. Summarizing above, $B^* = B$ holds. □

From Theorem 4.16, we can easily get Proposition 4.17.

Proposition 4.17. *Let $\to_2$ take I_Z, and $\to_1 \in \{I_{FD}, I_G\}$, then the universal triple I method is P_2-reversible.*

Remark 4.18. If $\to_2$ takes I_Z, and $\to_1$ satisfies (C5), (C12) but does not satisfy (C7), then it is similar to Remark 4.6 that $B^*(y) > B(y)$ may hold when $A(x) > B(y) > 0$, and thus the universal triple I method can not ensure to be P_2-reversible. Especially, if $I_1 \in \{I_L, I_{Go}, I_{RR}\}$, we have the same result.

Proposition 4.19. *If $\to_2$ takes I_Z, $\to_1 \in \{I_{KD}, I_M, I_{La}, I_Z\}$ and A is a normal fuzzy set. Then, when $A^* = A$, $B^*(y) = 0$ if $B(y) \leq 0.5$, $B^*(y) = B(y)$ if $B(y) > 0.5$.*

Proof. Suppose $A^* = A$. It is obvious that the Min-solution $B^*(y)$ is the same as Theorem 4.16. Note that I_{KD}, I_M, I_{La}, I_Z obviously satisfy (C5) and (C7) (but do not satisfy (C12)), then from the proving process of Theorem 4.16, it is easy to get $B^*(y) = 0$ if $B(y) = 0$, and $B^*(y) = B(y)$ if $B(y) > 0.5$. We shall show $B^*(y) = 0$

when $0 < B(y) \leq 0.5$. We only prove the case of $\to_1 = I_{KD}$ as an example. Suppose $\to_1 = I_{KD}$, then

$$E_y = \{x \in X|\ A^{'}(x) \vee 0.5 < A^{'}(x) \vee B(y)\} = \{x \in X|\ A^{'}(x) \vee 0.5 < B(y)\}.$$

When $0 < B(y) \leq 0.5$, we can easily get $E_y = \varnothing$, and then $B^*(y) = 0$. □

It follows from Proposition 4.19 that we can easily get Theorem 4.20.

Theorem 4.20. *Let $\to_2$ take I_Z, and $\to_1 \in \{I_{KD}, I_M, I_{La}, I_Z\}$, then the universal triple I method is P_3-reversible.*

Remark 4.21. If $\to_2$ takes I_Z, and $\to_1$ satisfies (C5), (C7) but does not satisfy (C12), then $B^*(y) \neq B(y)$ may hold when $0 < B(y) \leq 0.5$, and the universal triple I method can not ensure to be P_2-reversible. Especially, if $\to_1 \in \{I_{KD}, I_M, I_{La}, I_Z\}$, we have the same result (from Proposition 4.19 and Theorem 4.20); but under more strict condition (i.e. P_3), the universal triple I method possesses the reversibility property. In [27], Song investigated the triple I method based on I_Z and its reversibility property, and got Theorem 4 and Corollary 5 for the reversibility property of triple I method (for the FMP problem), which are respectively the same as the related conclusions of Proposition 4.19 and Theorem 4.20 in this paper.

4.4. **Summarizations of Reversibility Property of Universal Triple I Methods.** We shall summarize the reversibility property of universal triple I methods. By Remark 3.8, when $\to_2 = I_M$, the universal triple I method degenerates into the CRI method, then we get the reversibility property of related CRI methods by Theorem 4.14 and Remark 4.15 (see Proposition 4.22). When $\to_1 = \to_2$, the universal triple I method degenerates into the triple I method. Inspecting the results mentioned above, we get the reversibility property of triple I methods (see Proposition 4.23).

Proposition 4.22. *Let $\to_2$ take I_M, $\to_1 \in \{I_G, I_M, I_{La}\}$, then the universal triple I method (i.e. the CRI method) is P_1-reversible.*

Proposition 4.23. *Take $\to_1 = \to_2 \triangleq \to$, then the universal triple I method (that is, the triple I method) is P_1-reversible if $\to \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_M\}$, and P_3-reversible if $\to \in \{I_Z\}$.*

Remark 4.24. In Theorem 3.1 of [5], Hou and Li pointed out the fact that the CRI method where $\to \in \{I_G, I_M, I_{La}\}$ is P_1-reversible, and the one where $\to \in \{I_{FD}, I_L, I_{Go}, I_{RR}, I_{KD}, I_Z\}$ is not, which are the same as the related conclusions of Proposition 4.22 and Remark 4.15 in this paper. In Theorem 3.2 of [5], Hou and Li drew the conclusions that the triple I method (for the FMP problem) where $\to \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_M\}$ is P_1-reversible, and the one where $\to \in \{I_{KD}, I_{La}, I_Z\}$ is not, which coincide with the related conclusions of Proposition 4.5, Remark 4.6 and Proposition 4.23 in this paper. From Propositions 4.22 and 4.23, the triple I method is superior to the CRI method from the viewpoint of reversibility property.

Table 1 summarizes the reversibility property of universal triple I method (except the case of $\to_2 \in \{I_{KD}, I_{La}\}$), where P_i represents that corresponding universal

$\rightarrow_1$ / $\rightarrow_2$	I_{FD}	I_L	I_{Go}	I_G	I_{RR}	I_{KD}	I_M	I_{La}	I_Z
I_{FD}	P_1			P_1		P_1	P_1	P_1	P_1
I_L	P_1	P_1	P_1	P_1	P_1	P_1	P_1	P_1	P_1
I_{Go}			P_1	P_1			P_1	P_1	
I_G				P_1			P_1	P_1	
I_{RR}	P_1			P_1	P_1	P_1	P_1	P_1	P_1
I_M				P_1			P_1	P_1	
I_Z	P_2			P_2		P_3	P_3	P_3	P_3

Table 1. The Reversibility Property of Some Universal Triple I Methods

triple I method (employing first implication $\rightarrow_1$ and second implication $\rightarrow_2$) is P_i-reversible. The universal triple I method possesses reversibility property under very strict condition if $\rightarrow_2$ takes I_Z (which is an other type operator); and it hardly is reversible if $\rightarrow_2 \in \{I_{KD}, I_{La}\}$. However, when $\rightarrow_2$ takes a (0,1)-implication (which belongs to an expansion type operator), the reversibility property seems excellent, which is embodied as that the universal triple I method is $P-$reversible for a lot of first implications. By the way, when $\rightarrow_1$ takes a reduction type operator (e.g., I_M, I_{La}), the reversibility property seems fine.

From another viewpoint, when $\rightarrow_2$ takes an R-implication (e.g., I_{FD}, I_L, I_{Go}, I_G), the reversibility property of universal triple I method seems fine. When $\rightarrow_2$ is an S-implication (e.g., I_{FD}, I_L, I_{KD}) or a QL-implication (e.g., I_{FD}, I_L, I_{KD}, I_Z), the reversibility property is uncertain, where the case of $\rightarrow_2 \in \{I_{KD}\}$ is unacceptable, and the case of $\rightarrow_2 \in \{I_Z\}$ is complicated, and finally the case of $\rightarrow_2 \in \{I_{FD}, I_L\}$ is excellent.

5. Response Ability of Fuzzy Controllers Based on Universal Triple I Methods

In this section, we shall investigate the universal triple I methods from the viewpoint of fuzzy controllers.

First of all, we shall review briefly Mamdanian fuzzy control algorithm (which approximates to an interpolation function), where takes the SISO fuzzy controller as an example, thus some necessary concepts and signs are introduced.

Let X and Y be the input and output universe, respectively. Denote $\mathbb{A} = \{A_i\}_{(1\leq i\leq n)}$, $\mathbb{B} = \{B_i\}_{(1\leq i\leq n)}$ where $A_i \in F(X)$, $B_i \in F(Y)$. $\mathbb{A}, \mathbb{B}$ are regarded as linguistic variables, thus the fuzzy reasoning rules can be expressed as follows:

$$\text{If } x \text{ is } A_i, \text{ then } y \text{ is } B_i,\ i = 1, \cdots, n, \tag{9}$$

where $x \in X, y \in Y$ are called base variables.

According to the Mamdanian fuzzy control algorithm, the inference relation of the i-th rule can be regarded as a fuzzy relation from X to Y ($i = 1, \cdots, n$), denoting by $A_i(x) \rightarrow_1 B_i(y)$ (where $\rightarrow_1$ is an implication). And such n rules can be connected by 'or' relation (i.e, taking 'max' operator for rules), thus the whole rule is

$$IR(x, y) \triangleq \vee_{i=1}^{n}(A_i(x) \rightarrow_1 B_i(y)).$$

Given $A^* \in F(X)$, the inference conclusion $B^* \in F(Y)$ can be obtained as $B^* \triangleq A^* \circ IR$, by the CRI method, in which

$$B^*(y) = \sup_{x \in X} \{A^*(x) \wedge IR(x,y)\},\ y \in Y. \tag{10}$$

For a fuzzy controller, since its input is a crisp quantity, it should be transformed into fuzzy set to utilize (10) by

$$A^*(x) = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases} \triangleq A^*_{x^*},$$

which is called a singleton fuzzification.

Furthermore, it is known that B^* should be turned into a crisp quantity by using some defuzzification methods, and the commonly used method is the so-called method of centroid:

$$y^* = \frac{\int_Y yB^*(y)dy}{\int_Y B^*(y)dy}. \tag{11}$$

Then, a natural problem arise: What is interpolation representation of such fuzzy controller if we replace the CRI method with the universal triple I method? This is the task of this section.

5.1. **SISO Fuzzy Controllers Based on Universal Triple I Methods.** Here we employ the universal triple I method instead of the CRI method. Thus, for fuzzy reasoning rules (9), it is easy to get that (4) should be turned into:

$$IR(x,y) \to_2 (A^*(x) \to_2 B^*(y)). \tag{12}$$

Therefore, there is an output $y^* = F(x^*)$ for each input x^*. Thus a SISO fuzzy controller based on universal triple I method is obtained.

Definition 5.1. Let Z be any nonempty set and $\mathbb{C} = \{C_i\}_{(1 \leq i \leq n)}$ a family of normal fuzzy sets on Z, where the peak-point of C_i is z_i (i.e. the unique point satisfying $C_i(z_i) = 1$ in Z). $\mathbb{C}$ is called a fuzzy partition of Z if $(\forall z \in Z)(\sum_{i=1}^n C_i(z) = 1)$ holds, and C_i is defined as a base element in $\mathbb{C}$. Thus $\mathbb{C}$ is also called a group of base elements of Z.

Remark 5.2. Definition 5.1 obviously implies $(\forall i,j)(i \neq j \Rightarrow z_i \neq z_j)$ and that $\mathbb{C}$ has Kronecker property (i.e. $C_i(z_j) = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$).

To investigate interpolation mechanism of fuzzy controller, suppose that $\mathbb{A}$ and $\mathbb{B}$ are respectively fuzzy partitions of X and Y (in which A_i, B_i are integrable functions). We assume that X and Y are all real number intervals, e.g. $X = [a,b]$ and $Y = [c,d]$ where $a < x_1 < x_2 < \cdots < x_n < b$, $c < y_1 < y_2 < \cdots < y_n < d$, in which x_i, y_i are respectively peak-points of A_i, B_i.

Let $h_1 = y_1 - c$, $h_i = y_i - y_{i-1}$ $(i = 2, 3, \cdots, n)$ and $h = \max_{1 \leq i \leq n}\{h_i\}$. Since $\mathbb{A}$ and $\mathbb{B}$ are all fuzzy partitions, they have Kronecker property: $A_i(x_j) = \delta_{ij} = B_i(y_j)$ $(i, j = 1, \cdots, n)$. By the definition of definite integral, we achieve for the method of centroid:

$$y^* = \frac{\int_Y yB^*(y)dy}{\int_Y B^*(y)dy} \approx \frac{\sum_{i=1}^n y_i B^*(y_i) h_i}{\sum_{i=1}^n B^*(y_i) h_i}. \tag{13}$$

Similar to Theorem 3.5, we can prove Proposition 5.3.

Proposition 5.3. *Suppose that $\to_2$ is a (0,1)-implication, and that T the function residual to $\to_2$, then the Min-solution derived from (12) can be expressed as follows:*

$$B^*(y) = \sup_{x\in X}\{T(A^*(x), IR(x,y))\},\ y \in Y.$$

Theorem 5.4. *Let $\to_2$ be a (0,1)-implication, then the Min-solution $B^*(y) = IR(x^*, y)$ for the SISO fuzzy controller based on universal triple I method.*

Proof. Let $\to_2$ be a (0,1)-implication. Then there uniquely exists T which is the residual function w.r.t. $\to_2$, and $T(1,b) = b$, $T(0,b) = 0$ $(b \in [0,1])$. It follows from Proposition 5.3 that the Min-solution $B^*(y) = \sup_{x\in X}\{T(A^*(x), IR(x,y))\}$, $y \in Y$. As for input x^* , we get $A^*_{x^*} = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases}$. Because $T(1,b) = b$, $T(0,b) = 0$ $(b \in [0,1])$, we obtain $T(A^*(x), IR(x,y)) = T(1, IR(x^*,y)) = IR(x^*,y)$ for the case of $x = x^*$, and $T(A^*(x), IR(x,y)) = T(0, IR(x,y)) = 0$ for the case of $x \in X - \{x^*\}$. Therefore $B^*(y) = \sup_{x\in X}\{T(A^*(x), IR(x,y))\} = IR(x^*,y)$. □

Theorem 5.4 gives the equivalent form of Min-solution in SISO fuzzy controller.

Lemma 5.5. *In the SISO fuzzy controller based on universal triple I method,*

(i) if $\to_1$ is an expansion type operator, then $IR(x^*, y_j) > 0$ for any $x^* \in X$ $(j = 1, \cdots, n)$;

(ii) if $\to_1$ is a reduction type operator satisfying

(C13) $a \to 1 = a$ $(a \in [0,1])$,

then there exists y_j such that $IR(x^*, y_j) > 0$ for any $x^* \in X$ $(j \in \{1, \cdots, n\})$;

(iii) if $\to_1$ is an other type operator satisfying

(C14) $a \to 1 > 0$ $(a \in [0,1])$,

then $IR(x^*, y_j) > 0$ for any $x^* \in X$ $(j = 1, \cdots, n)$;

(iv) if $\to_1$ is an R-implication or S-implication, then $IR(x^*, y_j) > 0$ for any $x^* \in X$ $(j = 1, \cdots, n)$;

(v) if $\to_1$ is a QL-implication, then $IR(x^*, y_j) > 0$ for any $x^* \in X$ $(j = 1, \cdots, n)$.

Proof. In the SISO fuzzy controller based on universal triple I method, suppose any $x^* \in X$, then $IR(x^*, y) = \vee_{i=1}^n (A_i(x^*) \to_1 B_i(y))$.

(i) Suppose that $\to_1$ is an expansion type operator. Then $a \to_1 1 = 1$, and thus $IR(x^*, y_j) = \vee_{i=1}^n (A_i(x^*) \to_1 B_i(y_j)) = 1 > 0$ holds $(j = 1, \cdots, n)$ since $B_i(y_j) = \delta_{ij}$ $(i, j = 1, \cdots, n)$.

(ii) Suppose that $\to_1$ is a reduction type operator satisfying (C13). Then we have $a \to_1 0 = 0$ $(a \in [0,1])$. Suppose, on the contrary, that there exists $x^* \in X$ such that $IR(x^*, y_j) = 0$ for any y_j $(j = 1, \cdots, n)$. Since $B_i(y_j) = \delta_{ij}$ $(i, j = 1, \cdots, n)$ and $a \to_1 1 = a$, $a \to_1 0 = 0$, we get $0 = IR(x^*, y_j) = \vee_{i=1}^n (A_i(x^*) \to_1 B_i(y_j)) = A_j(x^*)$ $(j = 1, \cdots, n)$, and then $\sum_{j=1}^n A_j(x^*) = 0$. But, from the previous assumption, $\mathbb{A} = \{A_i\}_{(1\le i\le n)}$ is a fuzzy partition on X, then it follows from Definition 5.1 that $\sum_{j=1}^n A_j(x^*) = 1$, which is a contradiction. Therefore, there exists y_j such that $IR(x^*, y_j) > 0$ for any $x^* \in X$ $(j \in \{1, \cdots, n\})$.

(iii) Suppose that $\to_1$ is an other type operator satisfying (C14). Since $B_i(y_j) = \delta_{ij}$ $(i, j = 1, \cdots, n)$, we get $IR(x^*, y_j) = \vee_{i=1}^n (A_i(x^*) \to_1 B_i(y_j)) \geq A_j(x^*) \to_1 B_j(y_j) = A_j(x^*) \to_1 1 > 0$ holds $(j = 1, \cdots, n)$.

(iv) Since an R-implication or S-implication is also an expansion type operator, then similarly we get the conclusion.

(v) Suppose that $\to_1$ is a QL-implication, then it is easy to get $a \to_1 1 > 0$, thus we also get the conclusion. □

Theorem 5.6. *Let $\to_2$ be a (0,1)-implication. If $\to_1$ is a reduction type operator satisfying (C13), then there exists a group of base functions $\mathbb{A}^* = \{A_i^*\}_{(1 \leq i \leq n)}$ such that the SISO fuzzy controller based on universal triple I method is approximately a univariate piecewise interpolation function regarding A_i^* as its base functions, i.e. $F(x) = \sum_{i=1}^n A_i^*(x) y_i$, and $\mathbb{A}^*$ is a fuzzy partition of X. Moreover, if $\{y_i\}_{(1 \leq i \leq n)}$ is an equidistant partition, then $\mathbb{A}^*$ degenerates into $\mathbb{A}$, i.e. $F(x) = \sum_{i=1}^n A_i(x) y_i$.*

Proof. Since $\to_2$ is a (0,1)-implication, it follows from Theorem 5.4 that Min-solution $B^*(y) = IR(x^*, y) = \vee_{k=1}^n (A_k(x^*) \to_1 B_k(y))$. Since $\to_1$ is a reduction type operator satisfying (C13), then $a \to_1 1 = a$, $a \to_1 0 = 0$. Noting $B_k(y_i) = \delta_{ki}$ $(k, i = 1, \cdots, n)$, it follows from (13) that

$$\begin{aligned} y^* &\approx \frac{\sum_{i=1}^n y_i B^*(y_i) h_i}{\sum_{i=1}^n B^*(y_i) h_i} = \frac{\sum_{i=1}^n h_i [\vee_{k=1}^n (A_k(x^*) \to_1 B_k(y_i))] y_i}{\sum_{i=1}^n h_i [\vee_{k=1}^n (A_k(x^*) \to_1 B_k(y_i))]} \\ &= \frac{\sum_{i=1}^n h_i A_i(x^*) y_i}{\sum_{i=1}^n h_i A_i(x^*)}, \end{aligned} \tag{14}$$

where there exists y_i such that $B^*(y_i) = IR(x^*, y_i) > 0$ $(i \in \{1, \cdots, n\})$ by Lemma 5.5(ii), so $\sum_{i=1}^n B^*(y_i) h_i > 0$ and then (14) makes sense.

Denote $A_i^*(x^*) \triangleq h_i A_i(x^*) / (\sum_{i=1}^n h_i A_i(x^*))$, then $y^* \approx \sum_{i=1}^n A_i^*(x^*) y_i$. Let $\mathbb{A}^* \triangleq \{A_i^*\}_{(1 \leq i \leq n)}$, $F(x) \triangleq \sum_{i=1}^n A_i^*(x) y_i$. We get

$$F(x_i) = \sum_{k=1}^n A_k^*(x_i) y_k = \frac{\sum_{k=1}^n h_k A_k(x_i) y_k}{\sum_{k=1}^n h_k A_k(x_i)} = y_i \ (i = 1, \cdots, n),$$

noting that $A_k(x_i) = \delta_{ki}$ $(i, k = 1, \cdots, n)$, then $F(x)$ is a univariate piecewise interpolation function which regards A_i^* as its base functions.

Furthermore, $\sum_{i=1}^n A_i^*(x) = \sum_{i=1}^n [h_i A_i(x) / (\sum_{i=1}^n h_i A_i(x))] = 1$ holds for any $x \in X$, thus $\mathbb{A}^*$ is a fuzzy partition of X. At last, if $\{y_i\}_{(1 \leq i \leq n)}$ is an equidistant partition (i.e. $(\forall i)(h_i = h)$), then it is evident that $A_i^* = A_i$, $\mathbb{A}^* = \mathbb{A}$, and hence $F(x) = \sum_{i=1}^n A_i(x) y_i$. □

Note that I_M, I_{La} are reduction type operators satisfying (C13), which implies Corollary 5.7 (by virtue of Theorem 5.6).

Corollary 5.7. *Let $\to_2$ be a (0,1)-implication. If $\to_1 \in \{I_M, I_{La}\}$, then there exists a group of base functions $\mathbb{A}^* = \{A_i^*\}_{(1 \leq i \leq n)}$ such that the SISO fuzzy controller based on universal triple I method is approximately a univariate piecewise interpolation function regarding A_i^* as its base functions and $\mathbb{A}^*$ is a fuzzy partition of X. Moreover, if $\{y_i\}_{(1 \leq i \leq n)}$ is an equidistant partition, then $\mathbb{A}^*$ degenerates into $\mathbb{A}$.*

Since I_Z is an other type operator satisfying (C14), and it is also a typical QL-implication, we shall investigate the case that $\to_1$ takes I_Z.

Theorem 5.8. *Let $\to_2$ be a (0,1)-implication. If $\to_1$ is I_Z, then there exists a group of base functions $\mathbb{A}^* = \{A_i^*\}_{(1\leq i\leq n)}$ such that the SISO fuzzy controller based on universal triple I method is approximately a univariate piecewise fitted function regarding A_i^* as its base functions, i.e. $F(x) = \sum_{i=1}^n A_i^*(x)y_i$.*

Proof. It is similar to Theorem 5.6 that $B^*(y) = \vee_{k=1}^n (A_k(x^*) \to_1 B_k(y))$. Since $\to_1$ is I_Z and $B_k(y_i) = \delta_{ki}$ $(k, i = 1, \cdots, n)$, it follows from (13) that

$$\begin{aligned} y^* &\approx \frac{\sum_{i=1}^n y_i B^*(y_i)h_i}{\sum_{i=1}^n B^*(y_i)h_i} = \frac{\sum_{i=1}^n h_i[\vee_{k=1}^n((1-A_k(x^*)) \vee (A_k(x^*) \wedge B_k(y_i)))]y_i}{\sum_{i=1}^n h_i[\vee_{k=1}^n((1-A_k(x^*)) \vee (A_k(x^*) \wedge B_k(y_i)))]} \\ &= \frac{\sum_{i=1}^n h_i[A_i(x^*) \vee (\vee_{k=1}^n(1-A_k(x^*)))]y_i}{\sum_{i=1}^n h_i[A_i(x^*) \vee (\vee_{k=1}^n(1-A_k(x^*)))]}, \end{aligned} \tag{15}$$

where $B^*(y_i) = IR(x^*, y_i) > 0$ $(i = 1, \cdots, n)$ according to Lemma 5.5(iii)(v), so $\sum_{i=1}^n B^*(y_i)h_i > 0$ and then (15) makes sense.

Denote

$$C_i(x^*) \triangleq A_i(x^*) \vee (\vee_{k=1}^n(1-A_k(x^*))),\ \ A_i^*(x^*) \triangleq h_iC_i(x^*)/(\sum_{i=1}^n h_iC_i(x^*)),$$

then $y^* \approx \sum_{i=1}^n A_i^*(x^*)y_i$. Let $\mathbb{A}^* \triangleq \{A_i^*\}_{(1\leq i\leq n)}$ and $F(x) \triangleq \sum_{i=1}^n A_i^*(x)y_i$. It is easy to verify that it can't make $F(x_i) = y_i$ hold for any i, thus $F(x)$ is a univariate piecewise fitted function which regards A_i^* as its base functions. □

Theorem 5.9. *Let $\to_2$ be a (0,1)-implication. If $\to_1$ is an expansion type operator, then the SISO fuzzy controller based on universal triple I method is approximately a step response function.*

Proof. It is similar to Theorem 5.6 that $B^*(y) = \vee_{k=1}^n (A_k(x^*) \to_1 B_k(y))$. Since $\to_1$ is an expansion type operator, then $a \to_1 1 = 1$. Noting $B_k(y_i) = \delta_{ki}$ $(k, i = 1, \cdots, n)$, it follows from (13) that

$$y^* \approx \frac{\sum_{i=1}^n h_i[\vee_{k=1}^n(A_k(x^*) \to_1 B_k(y_i))]y_i}{\sum_{i=1}^n h_i[\vee_{k=1}^n(A_k(x^*) \to_1 B_k(y_i))]} = \frac{\sum_{i=1}^n h_iy_i}{\sum_{i=1}^n h_i} \triangleq c_0, \tag{16}$$

where $B^*(y_i) = IR(x^*, y_i) > 0$ $(i = 1, \cdots, n)$ according to Lemma 5.5(i), thus $\sum_{i=1}^n B^*(y_i)h_i > 0$ and then (16) makes sense. □

We can prove Theorem 5.10 using Theorem 5.9 and Proposition 2.11.

Theorem 5.10. *Let $\to_2$ be a (0,1)-implication. If $\to_1$ is an R-implication or S-implication, then the SISO fuzzy controller based on universal triple I method is approximately a step response function.*

Note that $I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_{KD}$ are expansion type operators, we can get Corollary 5.11.

Corollary 5.11. *Let $\rightarrow_2$ be a (0,1)-implication. If $\rightarrow_1 \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_{KD}\}$, then the SISO fuzzy controller based on universal triple I method is approximately a step response function.*

Remark 5.12. When $\rightarrow_2 \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_M\}$, the related results (e.g., Theorems 5.4, 5.6, 5.8, 5.9 and 5.10) are also correct, since $I_{FD}, I_L, I_{Go}, I_G, I_{RR}$ are (0,1)-implications, and the case of $\rightarrow_2 = I_M$ is similar to the one of $\rightarrow_2 = I_G$ (which has the same expression of Min-solution).

5.2. **DISO Fuzzy Controllers Based on Universal Triple I Methods.** Let X, Y be the universes of input variables and Z the universe of output variable. Denote $\mathbb{A} = \{A_i\}_{(1\le i\le n)}$, $\mathbb{B} = \{B_i\}_{(1\le i\le n)}$ and $\mathbb{C} = \{C_i\}_{(1\le i\le n)}$, where $A_i \in F(X)$, $B_i \in F(Y)$, $C_i \in F(Z)$. $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are regarded as linguistic variables, thus the fuzzy reasoning rules can be expressed as follows:

$$\text{If } x \text{ is } A_i \text{ and } y \text{ is } B_i, \text{ then } z \text{ is } C_i,\ i = 1, \cdots, n, \tag{17}$$

where $x \in X, y \in Y, z \in Z$ are called base variables. Similarly, the inference relation of $i-$th inference rule can be changed into $(A_i(x) \wedge B_i(y)) \rightarrow_1 C_i(z)$, and we get the whole inference rule

$$IR(x, y, z) \triangleq \vee_{i=1}^{n}((A_i(x) \wedge B_i(y)) \rightarrow_1 C_i(z)).$$

For a DISO fuzzy controller, the input value is a crisp quantity $(x^*, y^*) \in X \times Y$. We treat (x^*, y^*) by singleton fuzzification, and get $A^*_{x^*} = \begin{cases} 1, & x = x^* \\ 0, & x \neq x^* \end{cases}$, $B^*_{y^*} = \begin{cases} 1, & y = y^* \\ 0, & y \neq y^* \end{cases}$. Then we achieve C^* by universal triple I method of fuzzy reasoning from the input $A^*_{x^*}$ and $B^*_{y^*}$, where (12) should be turned into:

$$IR(x, y, z) \rightarrow_2 ((A^*(x) \wedge B^*(y)) \rightarrow_2 C^*(z)). \tag{18}$$

Lastly, to defuzzify C^*, we adopt the method of centroid, that is,

$$z^* = \int_Z zC^*(z)dz \ / \int_Z C^*(z)dz.$$

Therefore, there is an output $z^* = G(x^*, y^*)$ for each input (x^*, y^*). Then a DISO fuzzy controller based on universal triple I method is achieved.

To analyze interpolation mechanism of DISO fuzzy controller, suppose that $\mathbb{A}, \mathbb{B}, \mathbb{C}$ are respectively the fuzzy partitions of X, Y and Z (where A_i, B_i, C_i are integrable functions). We assume that X, Y and Z are all real number intervals, e.g. $X = [a, b]$, $Y = [c, d]$ and $Z = [e, f]$ where $a < x_1 < x_2 < \cdots < x_n < b$, $c < y_1 < y_2 < \cdots < y_n < d$ and $e < z_1 < z_2 < \cdots < z_n < f$, in which x_i, y_i, z_i are respectively peak-points of A_i, B_i, C_i.

Let $h_1 = z_1 - e$, $h_i = z_i - z_{i-1}$ $(i = 2, 3, \cdots, n)$ and $h = \max_{1\le i\le n}\{h_i\}$. Since $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are all fuzzy partitions, they have Kronecker property: $A_i(x_j) = B_i(y_j) = C_i(z_j) = \delta_{ij}$ $(i, j = 1, \cdots, n)$. By the definition of definite integral, we obtain for the method of centroid:

$$z^* = \frac{\int_Z zC^*(z)dz}{\int_Z C^*(z)dz} \approx \frac{\sum_{i=1}^{n} z_i C^*(z_i)h_i}{\sum_{i=1}^{n} C^*(z_i)h_i}. \tag{19}$$

It is similar to Proposition 5.3 that Proposition 5.13 can be obtained.

Proposition 5.13. *Suppose that $\to_2$ is a (0,1)-implication, and that T the function residual to $\to_2$, then the Min-solution derived from (18) can be expressed as follows:*

$$C^*(z) = \sup_{(x,y)\in X\times Y}\{T((A^*(x)\wedge B^*(y)), IR(x,y,z))\},\ z\in Z.$$

Theorem 5.14. *Let $\to_2$ be a (0,1)-implication, then the Min-solution $C^*(z) = IR(x^*, y^*, z)$ for the DISO fuzzy controller based on universal triple I method.*

Proof. Let $\to_2$ be a (0,1)-implication. Then there uniquely exists T which is the residual function w.r.t. $\to_2$, and $T(1,b) = b$, $T(0,b) = 0$ ($b \in [0,1]$). It follows from Proposition 5.13 that the Min-solution is $C^*(z) = \sup_{(x,y)\in X\times Y}\{T((A^*(x)\wedge B^*(y)), IR(x,y,z))\}$, $z \in Z$. As for input (x^*, y^*), we get $A^*_{x^*}$ and $B^*_{y^*}$. Because $T(1,b) = b$, $T(0,b) = 0$ hold for any $b \in [0,1]$, we have: $T((A^*(x)\wedge B^*(y)), IR(x,y,z)) = T(1, IR(x^*,y^*,z)) = IR(x^*,y^*,z)$ for the case of $(x,y) = (x^*,y^*)$, and $T((A^*(x)\wedge B^*(y)), IR(x,y,z)) = T(0, IR(x,y,z)) = 0$ for the case of $(x,y) \in X\times Y - \{(x^*,y^*)\}$. Therefore $C^*(z) = IR(x^*,y^*,z)$. □

Similar to Lemma 5.5, we can prove Lemma 5.15.

Lemma 5.15. *In the DISO fuzzy controller based on universal triple I method,*

(i) if $\to_1$ is an expansion type operator, then $IR(x^*,y^*,z_j) > 0$ for any $(x^*,y^*) \in X\times Y$ ($j = 1,\cdots,n$);

(ii) if $\to_1$ is a reduction type operator satisfying (C13), then there exists z_j such that $IR(x^*,y^*,z_j) > 0$ for any $(x^*,y^*) \in X\times Y$ ($j \in \{1,\cdots,n\}$);

(iii) if $\to_1$ is an other type operator satisfying (C14), then $IR(x^*,y^*,z_j) > 0$ for any $(x^*,y^*) \in X\times Y$ ($j = 1,\cdots,n$);

(iv) if $\to_1$ is an R-implication, an S-implication or a QL-implication, then $IR(x^*,y^*,z_j) > 0$ for any $(x^*,y^*) \in X\times Y$ ($j = 1,\cdots,n$).

Theorem 5.16. *Let $\to_2$ be a (0,1)-implication. If $\to_1$ is a reduction type operator satisfying (C13), then there exists a group of base functions $\Phi = \{\varphi_i\}_{(1\le i\le n)}$ such that the DISO fuzzy controller based on universal triple I method is approximately a binary piecewise interpolation function taking φ_i as its base functions, i.e. $G(x,y) = \sum_{i=1}^n \varphi_i(x,y)z_i$.*

Proof. Note that $\to_2$ is a (0,1)-implication, it follows from Theorem 5.14 that Min-solution $C^*(z) = IR(x^*,y^*,z) = \vee_{k=1}^n((A_k(x^*)\wedge B_k(y^*)) \to_1 C_k(z))$.

Since $\to_1$ is a reduction type operator satisfying (C13), then $a \to_1 1 = a$, $a \to_1 0 = 0$. Noting $C_k(z_i) = \delta_{ki}$ ($i,k = 1,\cdots,n$), it follows from (19) that

$$\begin{aligned} z^* &\approx \frac{\sum_{i=1}^n z_i C^*(z_i)h_i}{\sum_{i=1}^n C^*(z_i)h_i} = \frac{\sum_{i=1}^n z_i[\vee_{k=1}^n((A_k(x^*)\wedge B_k(y^*)) \to_1 C_k(z_i))]h_i}{\sum_{i=1}^n[\vee_{k=1}^n((A_k(x^*)\wedge B_k(y^*)) \to_1 C_k(z_i))]h_i} \\ &= \frac{\sum_{i=1}^n z_i(A_i(x^*)\wedge B_i(y^*))h_i}{\sum_{i=1}^n(A_i(x^*)\wedge B_i(y^*))h_i}, \end{aligned} \tag{20}$$

where there exists z_i such that $C^*(z_i) = IR(x^*,y^*,z_i) > 0$ ($i \in \{1,\cdots,n\}$) by Lemma 5.15(ii), so $\sum_{i=1}^n C^*(z_i)h_i > 0$ and then (20) makes sense.

Denote

$$C_i(x^*, y^*) \triangleq A_i(x^*) \wedge B_i(y^*),\ \varphi_i(x^*, y^*) \triangleq h_i C_i(x^*, y^*)/(\sum_{i=1}^{n} h_i C_i(x^*, y^*)),$$

then we get $z^* \approx \sum_{i=1}^{n} \varphi_i(x^*, y^*) z_i$. Let $\Phi \triangleq \{\varphi_i\}_{(1 \le i \le n)}$, $G(x, y) \triangleq \sum_{i=1}^{n} \varphi_i(x, y) z_i$. Considering $A_k(x_i) = B_k(y_i) = \delta_{ki}$ $(i, k = 1, \cdots, n)$, we get

$$G(x_i, y_i) = \frac{\sum_{k=1}^{n} z_k (A_k(x_i) \wedge B_k(y_i)) h_k}{\sum_{k=1}^{n} (A_k(x_i) \wedge B_k(y_i)) h_k} = \frac{z_i h_i}{h_i} = z_i,\ i = 1, \cdots, n,$$

then $G(x, y)$ is a binary piecewise interpolation function. □

Corollary 5.17. *Let $\rightarrow_2$ be a (0,1)-implication. If $\rightarrow_1 \in \{I_M, I_{La}\}$, then there exists a group of base functions $\Phi = \{\varphi_i\}_{(1 \le i \le n)}$ such that the DISO fuzzy controller based on universal triple I method is approximately a binary piecewise interpolation function taking φ_i as its base functions.*

Similarly Theorems 5.18, 5.19, 5.20 and Corollary 5.21 can be obtained.

Theorem 5.18. *Let $\rightarrow_2$ be a (0,1)-implication. If $\rightarrow_1 = I_Z$, then there exists a group of base functions $\Phi = \{\varphi_i\}_{(1 \le i \le n)}$ such that the DISO fuzzy controller based on universal triple I method is approximately a binary piecewise fitted function regarding φ_i as its base functions, i.e. $G(x, y) = \sum_{i=1}^{n} \varphi_i(x, y) z_i$.*

Theorem 5.19. *Let $\rightarrow_2$ be a (0,1)-implication. If $\rightarrow_1$ is an expansion type operator, then the DISO fuzzy controller based on universal triple I method is approximately a step response function.*

Theorem 5.20. *Let $\rightarrow_2$ be a (0,1)-implication. If $\rightarrow_1$ is an R-implication or S-implication, then the DISO fuzzy controller based on universal triple I method is approximately a step response function.*

Corollary 5.21. *Let $\rightarrow_2$ be a (0,1)-implication. If $\rightarrow_1 \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_{KD}\}$, then the DISO fuzzy controller based on universal triple I method is approximately a step response function.*

Remark 5.22. Similar to Remark 5.12, when $\rightarrow_2 \in \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_M\}$, the related results (e.g., Theorems 5.14, 5.16, 5.18, 5.19 and 5.20) are also right.

Remark 5.23. Since an R-implication is a (0,1)-implication, the related results in this section is also applicable to the case that $\rightarrow_2$ takes an R-implication.

Remark 5.24. When $\rightarrow_2$ takes I_M, the universal triple I method degenerates into the CRI method, thus we can easily get the response ability of SISO and DISO fuzzy controllers based on the CRI method (e.g., from Remarks 5.12, 5.22, Theorems 5.6, 5.8, 5.9, 5.16, 5.18 and 5.19). Obviously, there are more usable fuzzy controllers based on the universal triple I method than the ones based on the CRI method.

Remark 5.25. The response ability of fuzzy controllers based on universal triple I methods can be divided into 3 kinds as follows:

(i) If $\to_2$ is a (0,1)-implication and $\to_1$ is a reduction type operator satisfying (C13), then the fuzzy controller based on universal triple I method is approximately an interpolation function, thus it can be universal approximator and then usable in practice.

(ii) If $\to_2$ is a (0,1)-implication, and $\to_1$ takes I_Z (noting that I_Z is an other type operator, and also a typical QL-implication), then the fuzzy controller based on universal triple I method is approximately a fitted function, hence it may be usable.

(iii) If $\to_2$ is a (0,1)-implication, and $\to_1$ is an expansion type operator (including R-implication or S-implication), then the fuzzy controller based on universal triple I method is approximately a step response function, thus it only has step response ability, therefore it can hardly be used in practice.

6. Several Discussions of the Universal Triple I Method

To begin with, we shall discuss the important value of generalization from the triple I method to the universal triple I method. The reversibility property of universal triple I method is totally determined by the second and third implications in (3) (corresponding to $\to_2$ in (4)) if $\to_2 \in \{I_{KD}, I_{La}\}$; and is unitedly determined by $\to_1$ and $\to_2$, if $\to_2$ is a (0,1)-implication or $\to_2 \in \{I_M, I_Z\}$ (see e.g., Theorems 4.3, 4.4, 4.14 and 4.16). Therefore, it is reasonable to let the first implication be $\to_1$, and the second and third implications be $\to_2$ (i.e., generalize (3) to (4)).

Furthermore, when $\to_1$ and $\to_2$ are allowed to take different implications, more usable fuzzy controllers are achieved. For example, it follows from Remark 5.25 that we obtain 18 usable fuzzy controllers based on universal triple I method (in which $(\to_1, \to_2) \in \{I_M, I_{La}, I_Z\} \times \{I_{FD}, I_L, I_{Go}, I_G, I_{RR}, I_M\}$). In fact, if we get more (0,1)-implications (or reduction type operators satisfying (C13)), we can achieve more usable fuzzy controllers. However, we get from [6, 15] that there are only 2 usable fuzzy controllers based on the triple I method. Therefore, the practicability of the universal triple I method is superior to the triple I method.

To sum up, it is of significance to generalize the triple I method to the universal triple I method.

At last, we shall analyze the duty of first implication $\to_1$ and second implication $\to_2$, together with how to choose $\to_1$ and $\to_2$. It is easy to know that the form of universal triple I solution is basically determined only if $\to_2$ is chosen (i.e. $\to_2$ takes an implication), and hence $\to_2$ determines the reasoning mechanism to a large extent (see e.g. Theorem 3.5, Lemma 3.7 and Proposition 5.3). Meanwhile, $\to_1$ often exists as the form of $(A(x) \to_1 B(y))$ (or $IR(x, y)$ and so on) embodying the function of rule base (see e.g. Theorem 3.5, Propositions 5.3 and 5.13). Summarizing above, the second implication and first implication respectively embody the reasoning mechanism and function of rule base. What is more, the second implication has leading status in virtue of its effect on direction of inference.

Note that a universal triple I method is decided if $\to_1$ and $\to_2$ are chosen. Naturally, how to reasonably choose $\to_1$ and $\to_2$ becomes a key research topic in the universal triple I method. We will not discuss it here in detail, but give some instructional principles. From the results mentioned above, when $\to_2$ takes the

(0,1)-implication (or more strict R-implication), the universal triple I method has good property from the viewpoints of reversibility property and response ability, thus $\rightarrow_2$ can prefer to be the (0,1)-implication (or R-implication). Besides, it is natural to let the universal triple I method possess both the reversibility property and good response ability to the greatest extent for choosing $\rightarrow_1$ and $\rightarrow_2$.

7. Conclusions

The universal triple I method is investigated from the viewpoints of both fuzzy reasoning and fuzzy controllers. The main contributions are as follows.

(i) The universal triple I principle is brought forward, which improves the previous triple I principle. Then, unified form of universal triple I method is established (to allow different implications to be employed in the same manner), where $\rightarrow_2$ takes a (0,1)-implication or an R-implication. The CRI method and the triple I method can be regarded as special cases of the universal triple I method.

(ii) The reversibility property of universal triple I method is analyzed from expansion, reduction and other type operators. It is found that the universal triple I method can be reversible so long as we appropriately choose $\rightarrow_1$ and $\rightarrow_2$, and the reversibility property seems excellent if $\rightarrow_2$ takes a (0,1)-implication.

(iii) We investigate the response ability of fuzzy controllers based on universal triple I methods. When $\rightarrow_2$ takes a (0,1)-implication, and $\rightarrow_1$ takes I_Z or a reduction type operator satisfying (C13), the corresponding fuzzy controller can be practicable. There are more usable fuzzy controllers based on the universal triple I method than the ones based on the CRI method or the triple I method.

(iv) It is pointed out that, in universal triple I method, $\rightarrow_1$ and $\rightarrow_2$ respectively embody the function of rule base and the reasoning mechanism. Moreover, it is suggested that $\rightarrow_2$ should prefer to take the (0,1)-implication (or R-implication).

In the universal triple I method, how do we reasonably choose $\rightarrow_1$ and $\rightarrow_2$? There is only preparatory research involving it in this paper. Moreover, for the fuzzy controllers based on universal triple I methods, we analyze the case that the combination operator of inference rules takes 'max' operator, then how about the 'min' operator (where the whole inference rule is $\wedge_{i=1}^{n}(A_i(x) \rightarrow_1 B_i(y)))$? These problems will be investigated in the further research.

Furthermore, the properties related to universal triple I method (and corresponding fuzzy controllers), such as continuity, robustness as well as stability (see [9, 13, 16]), are also vital topics, which will be our research emphases in the future.

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Yiming Tang*, Information and Communication Engineering Postdoctoral Research Station, School of Computer and Information, Hefei University of Technology, Hefei 230009, China
E-mail address: tym608@163.com

Fuji Ren, Institute of Technology and Science, The University of Tokushima, Minami Josanjima, Tokushima, 770-8506, Japan
E-mail address: ren2fuji@gmail.com

*Corresponding author