LINEAR OBJECTIVE FUNCTION OPTIMIZATION WITH THE MAX-PRODUCT FUZZY RELATION INEQUALITY CONSTRAINTS

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Abstract. In this paper, an optimization problem with a linear objective function subject to a consistent finite system of fuzzy relation inequalities using the max-product composition is studied. Since its feasible domain is non-convex, traditional linear programming methods cannot be applied to solve it. We study this problem and capture some special characteristics of its feasible domain and optimal solutions. Some procedures are proposed to reduce and decompose the original problem into several sub-problems with smaller dimensions. Combining the procedures, a new algorithm is proposed to solve the original problem. An example is also provided to show the efficiency of the algorithm.

1. Introduction

The notion of Fuzzy Relation Equations (FRE) based on the max-min composition was first investigated by Sanchez (1976). He studied conditions and theoretical methods to resolve fuzzy relations on fuzzy sets defined as mappings from sets into complete Brouwerian lattices. Some theorems about the existence and determination of solutions of certain basic fuzzy relation equations were presented in his work. However, the obtained solution in the work is only the derived greatest element in the max-min fuzzy relation equations.

Sanchez's work has shed some light on this important subject. Then many researchers have been trying to explore the problem and develop solution procedures (for instance, see references, [1, 3, 4, 6, 8, 9, 11, 13, 14, 17, 18, 19, 20, 21, 23, 26]). Fuzzy relation equations are a special case of Fuzzy Relation Inequalities (FRI). Some researchers extended the study of fuzzy relation equations and problems related to them into fuzzy relation inequalities (for instance, see references, [10, 12, 15, 27, 28]). Their applications can be seen in many areas, for instance, fuzzy control, fuzzy decision-making, fuzzy symptom diagnosis and especially fuzzy medical diagnosis (see references, [2, 5, 16, 22]). In some applications, we require to consider FRE and FRI as constraints of an optimization problem. At first, Fang and Li [8] were confronted such a model in the textile industry. They obtained a linear objective function minimization model with the FRE constraints and proposed a branch-and-bound algorithm to solve it. Recently, Zhang et al. [28] extended

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the model to a linear objective function optimization model with the max-min FRI constraints. The max-min composition is commonly used when a system requires conservative solutions in the sense that the goodness of one value cannot compensate the badness of another value. In reality, there are situations that allow compensatability among the values of a solution vector. In this case, the min operator is not the best choice for the intersection of fuzzy sets. Instead, the max-product composition is preferred since it can yield better, or at least equivalent, results (see references, [7, 25, 29]). Therefore, it is motivated to study the linear objective function optimization problem with the max-product FRI constraints.

Zhang et al. [28] designed an algorithm to solve the problem under a determinate condition. Guo and Xia [10] presented an approach to solve the problem based on a necessary optimality condition. The obtained results by Zhang et al. [28] and Guo and Xia [10] are true for the problem with the max-min composition. But they cannot be correct for the problem with the max-product composition, in a general case. On the other hand, since the feasible domain has a unique maximal point and finitely many minimal points, we need to verify its every minimal point to obtain an optimal solution. For large scale problems, too many minimal and quasi-minimal points must be verified to find an optimal solution. Hence, we are motivated to design a new algorithm for its resolution. First of all, some procedures are proposed to simplify the original problem and some optimal solution components of the original problem are directly obtained without solving the original problem. Determining the components, the size of the original problem is reduced. Some procedures are also proposed to decompose the reduced problem into several sub-problems with smaller dimensions. Then an algorithm is proposed to solve the sub-problems. Combining the procedures and the algorithm, a new algorithm is proposed to solve the original problem.

2. Linear Programming Problem with FRI Constraints

First of all, we formulate the linear programming problem with the FRI constraints in the following subsection.

2.1. **Formulation of the Problem.** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ and $m \times l$ fuzzy matrices, respectively. Also, let $d^1 = (d_1^1, ..., d_n^1)^t \in [0, 1]^n$ and $d^2 = (d_1^2, ..., d_l^2)^t \in [0, 1]^l$ be two fuzzy vectors. A system of fuzzy relation inequalities is to find vectors $x = (x_1, ..., x_m) \in [0, 1]^m$ such that

$$xoA \geq d^1, \tag{1}$$

$$xoB \leq d^2, \tag{2}$$

where the notation "o" denotes the max-product composition. In other words, the feasible solution set of the system (1)-(2) is a set of vectors $x \in [0, 1]^m$ such that

$$\max_{i \in \underline{m}} \{x_i.a_{ij}\} \geq d_j^1, \ for \ j \in \underline{n} \tag{3}$$

$$\max_{i \in \underline{m}} \{x_i.b_{ij}\} \leq d_j^2, \ for \ j \in \underline{l} \tag{4}$$

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where $\underline{k}, \forall k \in N$, is defined as $\underline{k} = \{1, ..., k\}$. We are now ready to formulate the Linear Programming Problem with FRI Constraints (LPPFRIC). Let $c = (c_1, ..., c_m) \in R^m$ be an m-dimensional vector where the component c_i denotes the weight (or cost) corresponding to the variable x_i, for each $i \in \underline{m}$. LPPFRIC is formulated as follows:

$$Z = Min\ c.x^t, \tag{5}$$

$$s.t.\ xoA \geq d^1, \tag{6}$$

$$xoB \leq d^2, \tag{7}$$

$$x \in [0,1]^m. \tag{8}$$

We will now study the structure of its feasible solution set in the next subsection.

2.2. **The Structure of Feasible Domain of LPPFRIC.** In this subsection, the structure of the feasible domain of LPPFRIC will be investigated. At first, Czogala and Pedrycz [5] gave the structure of the solution set of FRE with the max-min composition. Then Wang et al. [27] presented an algorithm to solve a system of FRI with the max-min composition. It now becomes well-known that the complete solution set of a consistent finite system of sup-T equations (or inequalities) can be determined by a maximum solution and a finite number of minimal solutions [18]. It is necessary to remind that in a general case, over a distributive lattice with the max-min composition, the number of minimal solutions is not finite [14]. In the following, the structure of the feasible domain of LPPFRIC is investigated. To do this, we try to present methods for determination of the maximum solution and minimal solutions. Doing the work, the feasible domain is completely determined. First of all, we express the following lemma. The lemma helps us to determine the feasible solution set of the system (1)-(2).

Lemma 2.1. *Let $X_1 = \{x \in [0,1]^m | xoA \geq d^1, xoB \leq d^2\}$ and $X_2 = \{x \in [0,1]^m | xoB \leq d^2\}$. Then the maximum solution of two sets X_1 and X_2 is the same provided X_1 is non-empty.*

Proof. Assume that $\hat{x}^1$ and $\hat{x}^2$ are maximum solutions of sets X_1 and X_2, respectively. By contradiction, assume that $\hat{x}^1 \neq \hat{x}^2$. Since $X_1 \subseteq X_2$, we have $\hat{x}^1 \in X_2$. On the other hand, $\hat{x}^2$ is the maximum solution of X_2. Hence, it is concluded that $\hat{x}^2 \geq \hat{x}^1$, $\hat{x}^2 \neq \hat{x}^1$, and $\hat{x}^2 \notin X_1$. Since $\hat{x}^2 \notin X_1$, there exists $j \in \underline{n}$, such that $\hat{x}^2 oa_j < d_j^1 \leq \hat{x}^1 oa_j$. Also, since $\hat{x}^2 \geq \hat{x}^1$, we have $\hat{x}_i^2 \geq \hat{x}_i^1$, for each $i \in \underline{m}$. On the other hand, we know that $0 \leq a_{ij} \leq 1$, for each $i \in \underline{m}$. Hence, it is concluded that for each $i \in \underline{m}$, $\hat{x}_i^2.a_{ij} \geq \hat{x}_i^1.a_{ij}$. Therefore, we have $\max_{i \in \underline{m}}\{\hat{x}_i^2.a_{ij}\} \geq \max_{i \in \underline{m}}\{\hat{x}_i^1.a_{ij}\}$. According to the definition of operation "o", we can write $\hat{x}^2 oa_j \geq \hat{x}^1 oa_j$. This contradicts $\hat{x}^2 oa_j < \hat{x}^1 oa_j$. Therefore, $\hat{x}^1 = \hat{x}^2$. □

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In the following, the structure of the feasible domain of the problem (5)-(8), i.e., the solution set of FRI (1)-(2), will be investigated. Czogala and Predrycz [5] first gave the structure of the solution set of FRE. The algorithm for solving FRI was given by Wang et al. [27]. In a general case, the complete solution set of a consistent finite system of FRI is determined by a unique maximal solution and finitely many minimal solutions. Notice that the characteristic of the obtained solution sets with the max-min operator and the max-product operator are similar, i.e., when the solution set is not empty, it can be completely determined by a unique maximum solution and a finite number of minimal solutions (see references, [10, 27]). In the following, the structure of the feasible domain of the problem (5)-(8), i.e., the solution set of FRI (1)-(2), will be investigated.

Lemma 2.2. *Let $X[A,B] = \{x \in [0,1]^m | xoA \geq d^1, xoB \leq d^2\}$. If $x \in X[A,B]$, then for each $j \in \underline{n}$ and $t \in \underline{l}$, we have*
I) $\exists i_0 \in I_j^A$ s.t. $x_{i_0}.a_{i_0 j} \geq d_j^1$,
II) $\forall i \in I_t^B$ s.t. $x_i \leq \frac{d_t^2}{b_{it}}$,
where $I_j^A = \{i \in \underline{m} | a_{ij} \geq d_j^1\}$ and $I_t^B = \{i \in \underline{l} | b_{it} \geq d_t^2\}$.

Proof. Since $xoA \geq d^1$ and $xoB \leq d^2$, then for any $j \in \underline{n}$ and $t \in \underline{l}$, we have
a) $\exists i_0 \in \underline{m}$ s.t. $x_{i_0}.a_{i_0 j} \geq d_j^1$,
b) $\forall i \in \underline{m}$ s.t. $x_i.b_{it} \leq d_t^2$.
For any $x \in X[A,B]$, the condition (b) is always true for any $i \in \{i \in \underline{m} | b_{it} < d_t^2\}$ and the condition (a) is not true for any $i \in \{i \in \underline{m} | a_{ij} < d_j^1\}$. Hence, we can easily simplify the conditions (a) and (b) as the conditions (I) and (II). □

A direct result of Lemma 2.2 is as follows.

Corollary 2.3. *If the solution set of the system (1)-(2) is not empty, then its maximum solution, i.e., $\hat{x} = (\hat{x}_1, ..., \hat{x}_m)$, is as follows:*

$$\hat{x}_i = \bigwedge_{j=1}^{l} \{\frac{d_j^2}{b_{ij}} | b_{ij} \geq d_j^2\} \text{ where } i = 1, ..., m,$$

where the minimum over empty set is 1.

Compared to the maximum solution, the minimal solutions of the system (1)-(2) are more difficult to be obtained. Some concepts and theorems related to the minimal solutions are given to find the minimal solutions. An algorithm is then proposed to find the minimal solutions.

Definition 2.4. Suppose that $\hat{x}$ is the maximum solution of the set $X[B]$, where $X[B] = \{x \in [0,1]^m | xoB \leq d^2\}$. We define two sets $G(j)$ and G as follows.

$$G(j) = \{i \in I_j^A | \hat{x}_i.a_{ij} \geq d_j^1\} \text{ and } G = \prod_{j=1}^{n} G(j).$$

If there exists $j \in \{1, ..., n\}$ such that $G(j) = \emptyset$, then $G = \emptyset$ and the solution set of system (1)-(2) is empty.

Definition 2.5. Let $e = (e(1), e(2), ..., e(n))$ such that $e(j) \in G(j)$, for all $j \in \underline{n}$. Now, we define vector $e_x = (e_{x_1}, e_{x_2}, ..., e_{x_m})$ as follows:

$$e_{x_i} = \bigvee_{j=1}^{n} \{\frac{d_j^1}{a_{ij}} | e(j) = i\},$$

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where $\bigvee \emptyset = 0$. If $a_{ij} = d_j^1 = 0$ then the value $\frac{d_j^1}{a_{ij}}$ is defined zero.

Theorem 2.6. *The solution set of the system (1)-(2) is not empty if and only if* $G(j) \neq \emptyset$, $\forall j \in \underline{n}$.

Proof. (Sufficiency) Suppose that $G(j) \neq \emptyset$, for any $j \in \underline{n}$, . For any i satisfying $e_{x_i} \neq 0$, there exists j_i such that $e_{x_i} = \frac{d_{j_i}^1}{a_{ij_i}}$ and $e(j_i) = i$. This implies that $e_{x_i} = \frac{d_{j_i}^1}{a_{ij_i}} \leq \hat{x}_i$, $\forall i \in \underline{m}$. Hence, the vector e_x solves the system $xoB \leq d^2$. For any $j \in \underline{n}$, one has $e_{x_{i_j}} = \bigvee_{k=1}^n \{\frac{d_k^1}{a_{i_j k}} | e(k) = i_j\} \geq \frac{d_j^1}{a_{i_j j}}$ and hence $\bigvee_{i=1}^m (e_{x_i}.a_{ij}) \geq d_j^1$. It follows that $e_x oA \geq d^1$. Consequently, the vector e_x is a solution of the system (1)-(2).
(Necessity) Suppose that the solution set of the system (1)-(2) is not empty. If there exists j such that $G(j) = \emptyset$, then we have $\hat{x}_i.a_{ij} < d_j^1$, for each $i \in \underline{m}$. Hence, we conclude that for each $x^\circ \in X[B]$, and $i \in \underline{m}$, $x_i^\circ.a_{ij} < d_i^1$. Therefore, we have $x^\circ oA < d^1$. This implies that the solution set of the system (1)-(2) is empty. This contradicts the assumption that the solution set of the system (1)-(2) is not empty. □

Theorem 2.7. *Suppose that the solution set of the system (1)-(2) is not empty. Let* $e \in G$ *and* $\hat{x}$ *be the maximum solution of the system. Then its solution set is as follows:* $S = \bigcup_{e \in G} \{x | e_x \leq x \leq \hat{x}\}$.

Proof. Assume that x is an arbitrary solution of the system (1)-(2). Then x satisfies the inequality $xoB \leq d^2$ and hence $x \leq \hat{x}$. On the other hand, x also satisfies the inequality $xoA \geq d^1$. Hence, we can write $\bigvee_{i=1}^m (x_i.a_{ij}) \geq d_j^1$, for any $j \in \underline{n}$. It results that there exists an $i_j \in \underline{m}$ such that $x_{i_j}.a_{i_j j} \geq d_j^1$. We now define the sets $G'(j)$, for each $j \in \underline{n}$, and G', respectively, as: $G'(j) = \{i \in I_j^A | x_i.a_{ij} \geq d_j^1\}$ and $G' = \prod_{j=1}^n G'(j)$. Since $x_i \leq \hat{x}_i$, for each $i \in \underline{m}$, we have $G'(j) \subseteq G(j)$. Hence, $G' \subseteq G$. Now, let $e' = (e'(1), e'(2), ..., e'(n))$ such that $e'(j) \in G'(j)$, for each $j \in \underline{n}$. We also have $x_i \geq \bigvee_{j=1}^n \{\frac{d_j^1}{a_{ij}} | e'(j) = i\} = e'_{x_i}$. The first inequality is concluded the feasibility x and the second equality is another description of the vector e_x in the set G'. We display it by e'_x. Therefore, for x satisfying the system (1)-(2), there exists $e \in G$ such that $e_x \leq x$. To complete the proof, we show that for each $e \in G$, the vector e_x is a feasible solution of the system (1)-(2). We now suppose that $e \in G$. Since the feasible solution set of the system (1)-(2) is not empty, according to Theorem 2.6, for each $j \in \underline{n}$, there exists $i \in \underline{m}$ such that $e(j) = i$. Also, $e_{x_i} = \bigvee_{k=1}^n \{\frac{d_k^1}{a_{ik}} | e(k) = i\} \geq \frac{d_j^1}{a_{ij}}$. Thus, $\bigvee_{i=1}^m (e_{x_i}.a_{ij}) \geq d_j^1$. Hence, $e_x oA \geq d^1$. On the other hand, for any $i \in \underline{m}$, if $e_{x_i} \neq 0$ then there exists j such that $e(j) = i$ and $e_{x_i} = \frac{d_j^1}{a_{ij}}$. Also, we have $\hat{x}_i.a_{ij} \geq d_j^1$, i.e., $\hat{x}_i \geq \frac{d_j^1}{a_{ij}}$. Thus, we conclude that $e_{x_i} \leq \hat{x}_i$. Hence, $e_x \leq \hat{x}$. Therefore, $e_x oB \leq d^2$ and e_x is a feasible solution of the system (1)-(2). □

Theorem 2.7 results that for any $e \in G$, the vector e_x is a feasible solution of the system (1)-(2). We call e_x a quasi-minimal solution of the system (1)-(2). Theorem

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2.7 also shows that $\underline{X}[A,B] \subseteq \{e_x | e \in G\}$, where $\underline{X}[A,B]$ denotes the set of all the minimal solutions of the system (1)-(2). We display the set $\{e_x | e \in G\}$ by E. With regard to Theorem 2.7, two important results are obtained as follows.

Corollary 2.8. $\underline{X}[A,B] \subseteq E$.

Corollary 2.9. *$\underline{X}[A,B] = E_0$, where E_0 is the set of all the minimal elements of set E.*

We now illustrate the process of resolution of the problem (5)-(8), in the next subsection.

2.3. **The Process of Resolution of LPPFRIC.** In this subsection, we introduce two sub-problems. It is then proved that solving the problem (5)-(8) is equivalent to solving the two sub-problems. The sub-problems are formulated as follows.

$$Min \sum_{i=1}^{m} c_i^+ . x_i, \tag{9}$$

$$s.t.\ xoA \geq d^1, \tag{10}$$

$$xoB \leq d^2, \tag{11}$$

$$x_i \in [0,1], i \in \underline{m}, \tag{12}$$

and

$$Min \sum_{i=1}^{m} c_i^- . x_i, \tag{13}$$

$$s.t.\ xoB \leq d^2, \tag{14}$$

$$x_i \in [0,1], i \in \underline{m}, \tag{15}$$

where $c_i^+ = max\{c_i, 0\}$ and $c_i^- = min\{c_i, 0\}$, $i = 1, ..., m$.

Suppose that $\underline{x}^*$ and $\overline{x}^*$ are the optimal solutions of (9)-(12) and (13)-(15), respectively. Define $x^* = (x_1^*, ..., x_m^*)$ by the following relation:

$$x_i^* = \begin{cases} \underline{x}_i^*, & c_i \geq 0, \\ \overline{x}_i^*, & c_i < 0, \end{cases} \quad i = 1, ..., m. \tag{16}$$

In the following theorem, we show that x^* solves the problem (5)-(8).

Theorem 2.10. *The vector x^* defined by (16) is an optimal solution of the problem (5)-(8). Furthermore, if $c_i = 0$, then x_i^* can also be equal to one of elements of interval $[\underline{x}_i^*, \overline{x}_i^*]$.*

Proof. The proof is similar to the proof of Theorem 2.1. in [10]. □

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With regard to Theorem 2.10, we know that the problem (5)-(8) can be solved by the resolution of the sub-problems (9)-(12) and (13)-(15). Since the objective function of (13)-(15) is monotone decreasing, the optimal solution of the sub-problem (13)-(15) must be the maximum solution of the feasible domain, i.e., $X[B]$. Similarly, since the objective function of the sub-problem (9)-(12) is monotone increasing, one of the minimal solutions of the feasible domain is the optimal solution of the sub-problem (9)-(12). Therefore, to solve problem (5)-(8), it is necessary to obtain the maximum solution and minimal solutions of the system (1)-(2). Hence, we pay our attention to the maximum solution and the minimal solutions of the system (1)-(2). Its maximum solution is obtained by Corollary 2.3. Its minimal solutions can be computed by Theorem 2.7 and Corollary 2.9. In the next section, we focus on the reduction of search domain of the minimal solutions to find the optimal solution of the sub-problem (13)-(15).

3. Problem Reduction

In this section, we present some properties that help us in order to reduce the size of the original problem so that solving the problem is minimized. The key idea in these reduction procedures is as follows: Some of x_i's can be determined immediately without solving the problem, only by identifying the special characteristic of the problem. We capture some special cases of the problem (5)-(8) in the following lemmas. We can recognize the situations and eliminate them in our considerations. The original problem is reduced by the reduction procedures.

Lemma 3.1. *Suppose that the system (1)-(2) satisfies two following conditions:*
I) There exists $j_0 \in \underline{n}$ *such that* $|G(j_0)| = 1$*, i.e., for* $j_0 \in \underline{n}$*, there exists only one* $i_0 \in \underline{m}$ *such that* $\hat{x}_{i_0}.a_{i_0 j_0} \geq d^1_{j_0}$*.*
II) $\forall j \in \{j \in \underline{n} | \hat{x}_{i_0}.a_{i_0 j} \geq d^1_j\}$, $\frac{d^1_{j_0}}{a_{i_0 j_0}} \geq \frac{d^1_j}{a_{i_0 j}}$*.*
Then for each minimal solution $\check{x} = (\check{x}_1, ..., \check{x}_m)$*, its* i_0^{th} *component is as:* $\check{x}_{i_0} = \frac{d^1_{j_0}}{a_{i_0 j_0}}$*.*

Proof. According to Corollary 2.8, for each minimal solution $x \in \underline{X}[A, B]$, there exists $e \in G$ such that $e_x = x$, where $e(j_0) \in G(j_0) = \{i_0\}$. Thus, $e(j_0) = i_0$. With regard to the assumption (II) and Definition 2.5, we have $x_{i_0} = e_{x_{i_0}} = \frac{d^1_{j_0}}{a_{i_0 j_0}}$. □

Lemma 3.2. *Suppose that for each* $j \in \underline{n}$ *and some* i_0*,* $i_0 \in \underline{m}$*, we have* $i_0 \in G(j)$*, then one of minimal solutions of the system (1)-(2) is as* $x_{i_0} = \bigvee_{j=1}^{n} \{\frac{d^1_j}{a_{i_0 j}} | a_{i_0 j} > d^1_j\}$ *and* $x_i = 0$ *for each* $i \neq i_0$*.*

Proof. It is obvious. □

We now consider two following cases:
Case I: $c_i \leq 0$.
With attention to Theorem 2.10 and the relation (16), if $c_i < 0$, it is concluded that $x^*_i = \hat{x}_i$ and if $c_i = 0$, then $x^*_i \in [\check{x}_i, \hat{x}_i]$.

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Case II: If the mentioned conditions (I) and (II) in Lemma 3.1 are true in the system (1)-(2), we define the following sets.

$$\overline{J_1} = \{j \in \underline{n} | |G(j)| = 1 \; and \; i \in G(j) \; and \; \frac{d_j^1}{a_{ij}} = \bigvee_{k=1}^{n} \{\frac{d_k^1}{a_{ik}}\}\},$$
$$\overline{I} = \{i \in \underline{m} | i \in G(j) \; and \; j \in \overline{J}_1\},$$
$$\overline{J}_2 = \{j \in \underline{n} | \hat{x}_i . a_{ij} \geq d_j^1 \; and \; i \in \overline{I}\}.$$

We eliminate row i, $i \in \overline{I}$, and column j, $j \in \overline{J}_2$, from the fuzzy matrix A as well as the j^{th} element, $j \in \overline{J}_2$, from the updated vector d^1. Let A' and d'^1 be the reduced fuzzy matrix and fuzzy vector corresponding to A and d^1, respectively. Define $J = \underline{n} - \overline{J}_2$, and $I = \underline{m} - \overline{I}$. Update G. Let $x_i^* = \frac{d_j^1}{a_{ij}}$, for $i \in G(j)$, $i \in \overline{I}$, and $j \in \overline{J}_1$. The set J is an updated index set of constraints of an optimization problem which needs to be solved later by Algorithm 4.1 in the next section. The updated set G' is as $G' = \prod_{j \in J} G(j)$. The Algorithm 4.1 will be performed on the matrices A' and d'^1. If d'^1 is empty, then all constraints have been satisfied. We are now left with positive c_i's. Hence, we can assign the minimum value, i.e., zero, to all x_i's whose values have not been assigned yet. When d'^1 is not empty, we need to proceed further. Details will be discussed in the next section. In a special case, if $A = B$ and $d^1 = d^2 = b$, then the system (1)-(2) is converted to $xoA = b$. Also, there are other reductions from the problem. Since these reductions were considered by Loetamonphong and Fang [19], we don't repeat them here.

4. An Algorithm for Finding an Optimal Solution

First of all, we see that the problem can sometimes be decomposed into several sub-problems which can be solved separately. This leads to smaller problems with smaller search domains.

4.1. **Decomposition of the Problem (9)-(12).** In this section, a set of constraints, say B, which can be satisfied by a certain set of variables, say X_B, is considered. If the decision-making for the selection of variables in the set X_B for satisfying a constraint in B does not depend on the decision-making on the rest of the problem, then we can solve this part of the problem, separately. Let k be the number of sub-problems, $1 \leq k \leq |J|$. We now partition the set J to k sets as $J^{(1)}, ..., J^{(k)}$ such that (1) $J = J^{(1)} \bigcup ... \bigcup J^{(k)}$, (2) $J^{(i)} \bigcap J^{(j)} = \emptyset$, for each $i, j = 1, ..., k$, and $i \neq j$, (3) $\bigcap_{j \in J^{(i)}} G(j) \neq \emptyset$, for each $i = 1, ..., k$, and (4) $\forall p \in J - J^{(i)}$, $\bigcap_{j \in J^{(i)} \bigcup \{p\}} G(j) = \emptyset$. Define

$$\Omega = \{G(j) | j \in J\}, \tag{17}$$

$$\Omega_l = \{G(j) | j \in J^{(l)}\}, \; l = 1, ..., k, \tag{18}$$

$$\Omega_l \bigcap \Omega_{l'} = \emptyset, \; l \neq l', \tag{19}$$

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$$\Omega = \Omega_1 \bigcup \Omega_2 \bigcup ... \bigcup \Omega_k, \quad (20)$$

$$G_l = \prod_{G(j)\in\Omega_l} G(j), \quad (21)$$

$$I^{(l)} = \{i | i \in G(j),\ G(j) \in \Omega_l\}, \quad (22)$$

$$J^{(l)} = \{j | G(j) \in \Omega_l\}. \quad (23)$$

In this method, Ω_l contains the sets $G(j)$'s which have some element(s) in common and we can decompose the original problem into k sub-problems. The sets $I^{(l)}$ and $J^{(l)}$ are corresponded to the sets of indices of variables and constraints, respectively, on which search is performed for sub-problem l by Algorithm 4.1 in Section 4.2.

4.2. **An Algorithm.** With regard to Case I and Theorem 2.10, it is sufficient to solve the sub-problem (9)-(12) with the cost coefficients $c_i \geq 0$. Corollary 2.8 implies that $\underline{X}[A, B] \subseteq \{e_x | e \in G\}$. Therefore, the resolution of problem (9)-(12) is equivalent to finding an $e^* \in G$ such that

$$\sum_{i=1}^{m} c_i^+ . e^*_{x_i} = min_{e\in G}\{\sum_{i=1}^{m} c_i^+ . e_{x_i}\}, \quad (24)$$

We now propose an algorithm to find an optimal solution of problem (5)-(8) with regard to Corollary 2.8, the special cases and the decomposition procedure.

Algorithm 4.1. (An Algorithm for Finding an Optimal Solution):

Step 1: Find the maximum solution of the system (1)-(2) by Corollary 2.3.

Step 2: Check the feasibility of the problem (5)-(8) by Theorem 2.6. On the other hand, if there exists $j \in \underline{n}$ such that $G(j) = \emptyset$, then the problem has no feasible solution and stop!

Step 3: Compute $G(j) = \{i \in I_j | \hat{x}_i . a_{ij} \geq d_i^1\}$, for each $j \in \underline{n}$, which represents the set of indices x_i's that can satisfy the j^{th} constraint of the fuzzy relation inequalities. Then compute G.

Step 4: Compute $\hat{I} = \{i \in \underline{m} | c_i \leq 0\}$. Assign an optimal value $x_i^* = \hat{x}_i$, for $i \in \hat{I}$.

Step 5: If the mentioned conditions (I) and (II) in Lemma 3.1 are true for the system (1)-(2), then define the following sets.

$\overline{J}_1 = \{j \in \underline{n} || G(j)| = 1\ and\ i \in G(j)\ and\ \frac{d_j^1}{a_{ij}} = \bigvee_{k=1}^{n}\{\frac{d_k^1}{a_{ik}}\}\}$,
$\overline{I} = \{i \in \underline{m} | i \in G(j)\ and\ j \in \overline{J}_1\}$,
$\overline{J}_2 = \{j \in \underline{n} | \hat{x}_i . a_{ij} \geq d_j^1\ and\ i \in \overline{I}\}$.

Eliminate row i, $i \in \overline{I}$, and column j, $j \in \overline{J}_2$, from the fuzzy matrix A as well as

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the j^{th} element, $j \in \overline{J}_2$, from the updated vector d^1. Let A' and d'^1 be the reduced fuzzy matrix and fuzzy vector corresponding to A and d^1, respectively. Update G. Define $J = \underline{n} - \overline{J}_2$ and $I = \underline{m} - \overline{I}$. Let $x_i^* = \frac{d_j^1}{a_{ij}}$, for $i \in G(j)$, $i \in \overline{I}$ and $j \in \overline{J}_1$. If d'^1 is empty, then all constraints are satisfied. We can assign the minimum value, i.e., zero, to all x_i's whose values have not been assigned yet and stop!

Step 6: Decompose the problem (if possible) by computing the relations (17)-(23).

Step 7: Define the sub-problems in Step 6. For each sub-problem l, compute the vectors e_x using the elements G_l. By comparing objective function values in e_x's, select the optimal solutions for each sub-problem l.

Step 8: Generate optimal solutions for each sub-problem. For each sub-problem l, define $e^l = (e(j))$, $j \in J^{(l)}$, with $e(j) = i$ if $e_{x_i} = \frac{d_j^1}{a_{ij}}$.

Step 9: Combining the obtained solutions from Steps (4), (5), (7), and (8) and assigning zero value to unassigned $e_{x_i}^*$, produce the optimal solution for the problem (5)-(8). Compute $\hat{I}' = \{i \in \underline{m} | c_i = 0\}$. All of the optimal values x_i^*, $i \in \hat{I}'$, are presented as: $[e_{x_i}^*, \hat{x}_i]$ and stop!

We now prove the validity of Algorithm 4.1 in the following theorem.

Theorem 4.2. *If the feasible domain of problem (5)-(8) is non-empty, then Algorithm 4.1 correctly finds all the optimal solutions of problem (5)-(8).*

Proof. In step 1, the maximum solution of feasible domain of problem (5)-(8) is computed. According to Corollary 2.3, this solution is the maximum solution of the feasible domain. Then the necessary and sufficient conditions are checked for feasibility of the problem (5)-(8) according to Theorem 2.6. If the problem is feasible, the sets $G(j)$, for each $j \in \underline{n}$, and the set G are computed. In step 4, with regard to Theorem 2.10, the optimal value of variable x_i with negative cost coefficient, i.e., $c_i < 0$, is equal to the i^{th} component of the maximum solution, i.e., $\hat{x}_i$. After finding the optimal values of variables with negative cost coefficients, other variables have non-negative cost coefficients. In fact, we have found the optimal solution of sub-problem (13)-(15). We now need to find the optimal solution of problem (9)-(12). According to Theorem 2.10, one of the minimal solutions of the feasible domain is an optimal solution of problem (9)-(12). If the conditions of step 5 are true, then the conditions (I) and (II) in Lemma 3.1 are satisfied. According to Lemma 3.1, one of the components in all the minimal solutions of the feasible domain is the same. Therefore, we can determine the optimal value of the variable corresponding to this component by this lemma. Then we remove this variable and the rows and columns corresponding to this variable and simplify the problem. We consecutively apply the mentioned conditions in step 5 and simplify the problem as far as possible. If all the constraints of the problem are satisfied, we assign the minimum value to all the x_i's whose values have not been assigned yet. The obtained solutions for variables of sub-problem (9)-(12) are obviously the components of a minimal solution. If all the constraints of the problem are not

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satisfied, then we decompose the problem by computing relations (17)-(23) and form the sub-problems in step 6. If the problem is not decomposable, we will have a problem (or sub-problem) to solve. For each sub-problem, we compute vectors e_x using the elements G_l. With regard to Theorem 2.7, we can easily find the optimal solutions for each sub-problem l by producing all the quasi-minimal solutions e_x and comparing objective function values in e_x's. The obtained optimal solutions in steps 4, 5, 7, and 8 produce feasible optimal solutions. Moreover, if $\hat{I}' \neq \emptyset$, then there exists $i \in \underline{m}$ such that $c_i = 0$. We know that $c_i.x_i = 0$ for each value $x_i \in [e_{x_i}, \hat{x}_i]$. Hence, Algorithm 4.1 obtains all the optimal values of problem (5)-(8). □

Remark 4.3. When we produce all the quasi-minimal solutions e_x and compare objective function values in e_x's for sub-problems with non-negative cost coefficients as (9)-(12), one of the optimal solutions of sub-problems is one of the minimal solutions. Since for each feasible solution x there exists a minimal solution $\check{x}$ such that $\check{x} \leq x$, we have $c.\check{x} \leq c.x$ with $c \geq o$. Now, according to our procedure, $c.\check{x}^* = min\{c.\check{x} | \check{x} \in \underline{X}[A,B]\} \leq c.x$ for each feasible solution x. Since $\underline{X}[A,B]$ is a finite set and $\underline{X}[A,B] \subseteq E$, vector $\check{x}^*$ is one of the minimal solutions.

It is necessary to remind some points about Algorithm 4.1.

1- Optimal Solutions: One of the obtained optimal solutions by Algorithm 4.1 is a combination of the components of the maximum solution and one of the minimal solutions of the feasible domain of the problem (5)-(8), in a general case.

2- Alternative Optimal Solutions: The alternative optimal solutions of problem (5)-(8) (if exists) can easily be found with regard to step 7 and step 9. In step 7, by comparing objective function values in e_x's, a part of the alternative optimal solutions are obtained. In step 9, by computing $\hat{I}'$ (if $\hat{I}' \neq \emptyset$), we can obtain other part of the alternative optimal solutions.

4.3. **Examples.** In this subsection, the algorithm is illustrated by two examples. Moreover, we illustrate the method of computation of the alternative optimal solutions in Example 4.5.

Example 4.4. Consider the following optimization problem.

$$\begin{aligned} &min\ Z = -3x_1 + 5x_2 + x_3 + 2x_4 + 3x_5 + x_6 + 6x_7 + 4x_8, \\ &subject\ to\ xoA \geq d^1, \\ &\qquad\qquad xoB \leq d^2, \\ &\qquad\qquad 0 \leq x_i \leq 1,\ i = 1, ..., 8, \end{aligned}$$

where

$$A = \begin{bmatrix} 0.8 & 0.6 & 0.8 & 0.2 & 0.65 & 0.7 \\ 0.44 & 0.5 & 0.9 & 0.3 & 0.5 & 0.5 \\ 1 & 0.1 & 0.1 & 0.9 & 0.7 & 0.45 \\ 0.9 & 0.1 & 0.4 & 0.6 & 0.4 & 0.6 \\ 0.89 & 0.3 & 0.45 & 0.9 & 0.6 & 0.22 \\ 0.25 & 0.8 & 0.38 & 0.7 & 0.9 & 0.51 \\ 0.25 & 0.4 & 0.9 & 0.1 & 0.3 & 0.7 \\ 0.36 & 0.3 & 0.8 & 1 & 0.43 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.8 & 0.7 & 0.93 & 0.8 \\ 0.19 & 0.9 & 0.78 & 1 \\ 0.65 & 0.6 & 0.3 & 0.9 \\ 0.66 & 0.1 & 0.89 & 0.2 \\ 0.7 & 0.27 & 0.48 & 0.7 \\ 0.5 & 0.49 & 0.6 & 0.4 \\ 0.4 & 0.6 & 0.7 & 0.6 \\ 0.42 & 0.3 & 0.8 & 0.8 \end{bmatrix},$$

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$$d^1 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.15 \\ 0.26 \\ 0.14 \\ 0.15 \end{bmatrix}, and\ d^2 = \begin{bmatrix} 0.4 \\ 0.25 \\ 0.16 \\ 0.3 \end{bmatrix}.$$

Step 1: The maximum solution of the problem is as follows.

$\hat{x} = (0.172, 0.205, 0.333, 0.18, 0.333, 0.267, 0.228, 0.2)$.

Step 2: For each $j \in \underline{n}$, $G(j) \neq \emptyset$. Therefore, we know that the feasible solution set of the problem is not empty.

Step 3: Compute the sets $G(j)$, for $j \in \underline{n}$, as follows: $G(1) = \{1, 3, 4, 5\}$,$G(2) = \{6\}$,$G(3) = \{2, 7, 8\}$,$G(4) = \{3, 5\}$,$G(5) = \{3, 5, 6\}$,and $G(6) = \{7, 8\}$. Also, compute G as: $G = \prod_{j=1}^{6} G(j)$.

Step 4: Let $\hat{I} = \{1\}$ and $e^*_{x_1} = \hat{x}_1 = 0.172$.

Step 5: Let $\overline{J}_1 = \{2\}$, $\overline{I} = \{6\}$, and $\overline{J}_2 = \{2, 5\}$. Also, let $J = \{1, 3, 4, 6\}$ and $I = \{1, 2, 3, 4, 5, 7, 8\}$. Row i, $i \in \overline{I}$, and column j, $j \in \overline{J}_2$, from the fuzzy matrix A as well as the j^{th} element, $j \in \overline{J}_2$, from the updated vector d^1 is removed. Let A' and d' be the reduced fuzzy matrix and fuzzy vector corresponding to A and d^1, respectively. Let $e^*_{x_6} = \frac{d^1_2}{a_{62}} = 0.25$ and $G' = \prod_{j \in J} G(j)$.

Step 6: Let $\Omega = \{G(1), G(3), G(4), G(6)\}$, $\Omega_1 = \{G(3), G(6)\}$, and $\Omega_2 = \{G(1), G(4)\}$. Also, $\Omega_1 \bigcap \Omega_2 = \emptyset$, $\Omega = \Omega_1 \bigcup \Omega_2$, $G_1 = G(3) \times G(6)$, $G_2 = G(1) \times G(4)$, $I^{(1)} = \{2, 7, 8\}$, $J^{(1)} = \{3, 6\}$, $I^{(2)} = \{1, 3, 4, 5\}$, and $J^{(2)} = \{1, 4\}$.

Steps 7,8: Sub-problem 1 is as: $G_1 = G(3) \times G(6) = \{(2, 7), (2, 8), (7, 7), (7, 8), (8, 7), (8, 8)\}$ and $\{e^1_x | e^1 \in G_1\} = \{(0.167, 0.214), (0.167, 0.15), (0.214, 0.214), (0.167, 0.15), (0.187, 0.214), (0.187, 0.187)\}$. Since $(c_2, c_7, c_8) = (5, 6, 4)$, the optimal solution is as $e(3) = e(6) = 8$ and $e^1_{x_8} = e_{x_8} = 0.187$.

Sub-problem 2 is as: $G_2 = G(1) \times G(4) = \{(1, 3), (1, 5), (3, 3), (3, 5), (4, 3), (4, 5), (5, 3), (5, 5)\}$ and $\{e^2_x | e^2 \in G_2\} = \{(0.125, 0.289), (0.125, 0.289), (0.289, 0.289), (0.1, 0.289), (0.111, 0.289), (0.111, 0.289), (0.112, 0.289), (0.289, 0.289)\}$. Since $(c_3, c_4, c_5) = (1, 2, 3)$, the optimal solution is as $e(4) = 3$ and $e^2_{x_3} = e_{x_3} = 0.289$.

Step 9: From Steps (4), (5), (7), and (8), we conclude that the optimal vector of e^*_x is as follows: $e^*_x = (0.172, 0, 0.289, 0, 0, 0.25, 0, 0.187)$.

In this example, we obtain the optimal solution by verifying $|G_1| + |G_2| = 6 + 8 = 14$ quasi-minimal solutions. If we apply the search direct method for this example, we must verify all of the quasi-minimal solutions, i.e., $|G| = \prod_{j=1}^{6} |G(j)| = 4 \times 1 \times 3 \times 2 \times 3 \times 2 = 144$ quasi-minimal solutions.

We now illustrate the method of computation of the alternative optimal solutions by Algorithm 4.1 with the following example.

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Example 4.5. Consider the problem of Example 4.4 with the following objective function.

$Z = 0x_1 + 5x_2 + x_3 + x_4 + \frac{300}{289}x_5 + x_6 + 6x_7 + 4x_8$.

Steps 1-6: are the same to steps 1-6 of Example 4.4.

Steps 7,8: Sub-problem 1 is as:$G_1 = G(3) \times G(6) = \{(2,7),(2,8),(7,7),(7,8),(8,7),(8,8)\}$ and $\{e^1_x | e^1 \in G_1\} = \{(0.167, 0.214),(0.167, 0.15),(0.214, 0.214),(0.167, 0.15),(0.187, 0.214),(0.187, 0.187)\}$. Since $(c_2, c_7, c_8) = (5, 6, 4)$, the optimal solution is as $e(3) = e(6) = 8$ and $e^1_{x_8} = e_{x_8} = 0.187$.

Sub-problem 2 is as:$G_2 = G(1)\times G(4) = \{(1,3),(1,5),(3,3),(3,5),(4,3),(4,5),(5,3),(5,5)\}$ and $\{e^2_x | e^2 \in G_2\}$= $\{(0.125, 0.289),(0.125, 0.289),(0.289, 0.289),(0.1, 0.289),(0.111, 0.289),(0.111, 0.289),(0.112, 0.289),(0.289, 0.289)\}$.

Since $(c_3, c_4, c_5) = (1, 1, \frac{300}{289})$, the optimal solution is as: $e(1) = 4$, $e(4) = 3$ and $e^2_{x_4} = e_{x_4} = 0.111$, $e^2_{x_3} = e_{x_3} = 0.289$.

Step 9: From Steps (4), (5), (7), and (8), we conclude that the optimal vector of e^*_x is as follows: e^*_x=$(0.172, 0, 0.1, 0, 0.289, 0.25, 0, 0.187)$ and e^*_x=$(0.172, 0, 0.289, 0.111, 0, 0.25, 0, 0.187)$. Also, $\hat{I}' = \{1\}$. Since $e^*_{x_1} = \hat{x}_1 = 0.172$, we have two optimal solutions as follows: $\{(0.172, 0, 0.1, 0, 0.289, 0.25, 0, 0.187),(0.172, 0, 0.289, 0.111, 0, 0.25, 0, 0.187)\}$.

5. Conclusions

In this paper, the solution set of a consistent finite system of fuzzy relation inequalities with max-product composition and an optimization problem with a linear objective function subject to such fuzzy relation inequalities were studied. Some components of the optimal solution(s) of the optimization problem were determined under some sufficient conditions. The optimization problem was also separated into two sub-problems, one with non-negative cost coefficients and the other with negative cost coefficients. The problem can be further reduced by removing the constraints that have been satisfied. The reduced problem may be further decomposed into several sub-problems with smaller dimensions which are then solved separately. Combining the decomposition and reduction procedures, an algorithm was proposed to solve the original problem. Since the size of original problem is reduced by the decomposition procedures, the algorithm is very efficient in the sense that we do not have to enumerate all the quasi-minimal solutions of the original problem.

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References

[1] G. I. Adamopoulos and C. P. Pappis, *Some results on the resolution of fuzzy relation equations*, Fuzzy Sets and Systems, **60** (1993), 83–88.

[2] K. P. Adlassnig, *Fuzzy set theory in medical diagnosis*, IEEE Trans. Systems Man Cybernet., **16** (1986), 260–265.

[3] M. M. Bourke and D. G. Fisher, *Solution algorithms for fuzzy relational equations with max-product composition*, Fuzzy Sets and Systems, **94** (1998), 61–69.

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[4] E. Czogala, J. Drewniak and W. Pedrycz, *Fuzzy relation equations on a finite set*, Fuzzy Sets and Systems, **7** (1982), 89–101.
[5] E. Czogala and W. Pedrycz, *On Identification in Fuzzy Systems and its applications in control problem*, Fuzzy Sets and Systems, **6** (1981), 73–83.
[6] A. Di Nola, *Relational equations in totally ordered lattices and their complete resolution*, J. Math. Anal. Appl., **107** (1985), 148–155.
[7] D. Dubois and H. Prade, *New results about properties and semantics of fuzzy set-theoretic operators*, In: P. P.Wang, S. K. Chang, eds., Fuzzy Sets, Plenum Press, New York, (1986), 59–75.
[8] S. C. Fang and G. Li, *Solving fuzzy relation equations with a linear objective function*, Fuzzy Sets and Systems, **103** (1999), 107–113.
[9] S. Z. Guo, P. Z. Wang, A. Di Nola and S. Sessa, *Further contributions to the study of finite fuzzy relation equations*, Fuzzy Sets and Systems, **26** (1988), 93–104.
[10] F. F. Guo and Z. Q. Xia, *An algorithm for solving optimization problems with one linear objective function and finitely many constraints of fuzzy relation inequalities*, Fuzzy optim. Decis. Making, **5** (2006), 33–47.
[11] S. M. Guu and Y. K. Wu, *Minimizing a linear objective function with fuzzy relation equation constraints*, Fuzzy Optim. Decis. Making, **12** (2002), 1568–4539.
[12] S. Z. Han, A. H. Song and T. Sekiguchi, *Fuzzy inequality relation system identification via sign matrix method*, Proceeding of 1995 IEEE International Conference 3, (1995), 1375–1382.
[13] M. Higashi and G. J. Klir, *Resolution of finite fuzzy relation equations*, Fuzzy Sets and Systems, **13** (1984), 65–82.
[14] M. Hosseinyazdi, *The optimization problem over a distributive lattice*, J. Glob. Optim., **41** (2008), 283–298.
[15] C. F. Hu, *Generalized variational inequalities with fuzzy relation*, J. Comput. Appl. Math., **146** (1998), 198–203.
[16] W. B. V. Kandasamy and F. Smarandache, *Fuzzy relational maps and neutrosophic relational maps*, hexis church rock, 2004.
[17] P. Li and S. C. Fang, *On the resolution and optimization of a system of fuzzy relational equations with sup-T composition*, Fuzzy Optim. Decis. Making, **7** (2008), 169–214.
[18] G. Li and S. C. Fang, *On the resolution of finite fuzzy relation equations*, OR Report No. 322, North Carolina State University, Raleigh, North Carolina, May 1996.
[19] J. Loetamonphong and S. C. Fang, *Optimization of fuzzy relation equations with max-product composition*, Fuzzy Sets and Systems, **118** (2001), 509–517.
[20] J. Loetamonphong, S. C. Fang and R. E. Young, *Multi-objective optimization problems with Fuzzy Relation Equation constraints*, Fuzzy Sets and Systems, **127** (2002), 141–164.
[21] W. Pedrycz, *Proceeding in relational structures: fuzzy relational equations*, Fuzzy Sets and Systems, **40** (1991), 77–106.
[22] K. Peeva and Y. Kyosev, *Fuzzy relational calculus: theory, applications and software*, World Scientific, New Jersey, 2004.
[23] M. Prevot, *Algorithm for the solution of fuzzy relations*, Fuzzy Sets and Systems, **5** (1981), 319–322.
[24] E. Sanchez, *Resolution of composite fuzzy relation equations*, Information and Control, **30** (1976), 38–48.
[25] U. Thole, H. J. Zimmermann and P. Zysno, *On the suitability of minimum and product operators for intersection of fuzzy sets*, Fuzzy Sets and Systems, **2** (1979), 167–180.
[26] P. Z. Wang, *How many lower solutions of finite fuzzy relation equations*, Fuzzy Mathematics (Chinese), **4** (1984), 67–73.
[27] P. Z. Wang, D. Z. Zhang, E. Sanchez and E. S. Lee, *Lattecized linear programming and fuzzy relation inequities*, J. Math. Anal. Appl., **159** (1991), 72–87.
[28] H. T. Zhang, H. M. Dong and R. H. Ren, *Programming problem with fuzzy relation inequality constraints*, J. Liaon. Nor. Univ., **3** (2003), 231–233.
[29] H. J. Zimmermann and P. Zysno, *Latent connectives in human decision-making*, Fuzzy Sets and Systems, **4** (1980), 37–51.

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