FUZZY SUBGROUPS OF THE DIRECT PRODUCT OF A GENERALIZED QUATERNION GROUP AND A CYCLIC GROUP OF ANY ODD ORDER

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ABSTRACT. Bentea and Tărnăuceanu (An. Ştiinţ. Univ. Al. I. Cuza Iaş, Ser. Nouă, Mat., **54(1)** (2008), 209-220) proposed the following problem: Find an explicit formula for the number of fuzzy subgroups of a finite hamiltonian group of type $Q_8 \times \mathbb{Z}_n$ where Q_8 is the quaternion group of order 8 and n is an arbitrary odd integer. In this paper we consider more general group: the direct product of a generalized quaternion group of any even order and a cyclic group of any odd order. For this group we give an explicit formula for the number of fuzzy subgroups.

1. **Introduction**

Let G be a group with a multiplicative binary operation and identity e, and let $\mu : G \to [0,1]$ be a fuzzy subset of G. Then μ is said to be a *fuzzy subgroup* of G if (1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, and (2) $\mu(x^{-1}) \geq \mu(x)$ for all $x, y \in G$. The set $\{\mu(x) \mid x \in G\}$ is called the *image* of μ and is denoted by $\mu(G)$. For each $a \in \mu(G)$, the set $\mu_a := \{x \in G \mid \mu(x) \geq a\}$ is called a *level subset* of μ. It follows that μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G (see [3]).

For given two fuzzy subgroups μ and ν in G, μ and ν are *equivalent*, written as $\mu \sim \nu$, if $\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y)$ for all $x, y \in G$. It follows that $\mu \sim \nu$ if and only if μ and ν have the same set of level subgroups (see [13]). Hence there exists a one-to-one correspondence between the collection of the equivalence classes of fuzzy subgroups of G and the collection of chains of subgroups of G which end in G. This notion of equivalence relation was used in [2, 11, 13] in order to enumerate fuzzy subgroups of certain families of finite groups. We call a chain of subgroups which ends in G *rooted* (or more exactly G*-rooted*). Otherwise we call it *unrooted*. There is another equivalence relation on the set of fuzzy subgroups used by Murali and Makamba [4, 5, 6, 7] in order to enumerate fuzzy subgroups of certain families of finite abelian groups. In this paper we follow the notion of the equivalence relation used in [2, 11, 13].

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It is still an open problem to determine the numbers of fuzzy subgroups of arbitrary finite abelian groups. As a partial result about the numbers of fuzzy subgroups of abelian groups Tărnăuceanu and Bentea [11] gave an explicit formula for the number of fuzzy subgroups of a finite cyclic group of any order by finding its generating function of one variable. The author [8] did a similar work by finding its generating function of multi-variables. As this problem to non-abelian groups Bentea and Tărnăuceanu [2] gave an explicit formula for the number of fuzzy subgroups of a hamiltonian group of type $Q_8 \times \mathbb{Z}_{p^n}$ where Q_8 is the quaternion group of order 8 and p is an odd prime number, and then propose the following problem: *Find an explicit formula for the number of fuzzy subgroups of a finite hamiltonian group of type $Q_8 \times \mathbb{Z}_n$ where n is an arbitrary odd integer.* In this paper we consider more general group: the direct product of a generalized quaternion group of any even order and a cyclic group of any odd order. For this group we give an explicit formula for the number of fuzzy subgroups.

This paper is organized as follows. In section 2 we present some definitions and results. In section 3 we find a generating function of multi-variables for the number of fuzzy subgroups of the direct product of a generalized quaternion group and a cyclic group of any odd order, and then give an explicit formula for that number.

2. Preliminaries

Given a group G let $\mathcal{C}(G), \mathcal{U}(G)$ and $\mathcal{R}(G)$ be the collection of chains of subgroups of G, of unrooted chains of subgroups of G and of rooted chains of subgroups of G, respectively. Let $C(G) := |\mathcal{C}(G)|$, $U(G) := |\mathcal{U}(G)|$ and $R(G) := |\mathcal{R}(G)|$.

The following simple observation is useful for enumerating chains of subgroups in the lattice of subgroups of a given finite group.

Proposition 2.1. *Let G be a finite group. Then $R(G) = U(G) + 1$ and $C(G) = R(G) + U(G) = 2R(G) - 1$.*

For a fixed positive integer k we define a function λ as follows.

$$\lambda(x_k) := 1 - 2x_k,$$
$$\lambda(x_k, x_{k-1}, \ldots, x_j) := \lambda(x_k, x_{k-1}, \ldots, x_{j+1}) - (1 + \lambda(x_k, x_{k-1}, \ldots, x_{j+1}))x_j$$

for any $j = k-1, k-2, \ldots, 1$.

Proposition 2.2. [8] *Let $\mathbb{Z}_n$ be the cyclic group of order*

$$n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

where $p_1, \ldots, p_k$ are distinct prime numbers and $\beta_1, \ldots, \beta_k$ are positive integers. Then the number $R(\mathbb{Z}_n)$ of fuzzy subgroups of $\mathbb{Z}_n$ is the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of

$$\chi_{\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) = \frac{1}{\lambda(x_k, \ldots, x_1)}.$$

Let $\mathbb{Z}$ be the set of all integer numbers. Given distinct positive integers $i_1, \ldots, i_t$ we define a function

$$\pi_{i_1 \cdots i_t} : \mathbb{Z}^k \to \mathbb{Z}^k, (x_1, \ldots, x_k) \mapsto (y_1, \ldots, y_k)$$

where

$$y_\ell = \begin{cases} x_\ell & \text{if } \ell \neq i_j \text{ for all } j = 1, \dots, t \\ x_\ell - 1 & \ell = i_j \text{ for some } j \text{ such that } j = 1, \dots, t. \end{cases}$$

Given a positive integer α, the generalized quaternion group $Q_{2^{\alpha+2}}$ of order $2^{\alpha+2}$ has the following presentation:

$$Q_{2^{\alpha+2}} = \left\langle a, b \mid a^{2^{\alpha+1}} = e, b^2 = a^\alpha, bab^{-1} = a^{-1} \right\rangle.$$

Most of our notations are standard and for undefined group theoretical terminologies we refer the reader to [9, 10]. For a general theory of solving a recurrence relation using a generating function we refer the reader to [1, 12].

3. The Number of Chains of Subgroups of $Q_{2^{\alpha+2}} \times \mathbb{Z}_n$

Throughout the section we assume that

$$n := p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$$

where $p_1, \dots, p_k$ are distinct odd prime numbers and $\beta_1, \dots, \beta_k$ are non-negative integers.

Let $\mathbb{Z}_n := \langle c \mid c^n = e \rangle$ be the cyclic group of order n. For any prime factors $p_{i_1}, p_{i_2}, \dots, p_{i_t}$ of n let

$$\mathbb{Z}_{n/p_{i_1} \cdots p_{i_t}} := \langle c^{p_{i_1} \cdots p_{i_t}} \rangle$$

be the subgroup of $\mathbb{Z}_n$ of order $n/p_{i_1} \cdots p_{i_t}$.

Let

$$Q_{2^{\alpha+2}} \times \mathbb{Z}_n := \left\langle a, b, c \mid a^{2^{\alpha+1}} = e, b^2 = a^\alpha, bab^{-1} = a^{-1}, c^n = e \right\rangle$$

be the direct product of the generalized quaternion group $Q_{2^{\alpha+2}}$ of order $2^{\alpha+2}$ and the cyclic group $\mathbb{Z}_n$ of order n.

Lemma 3.1. *Assume that $\beta_1, \dots, \beta_k$ are positive integers. The group $Q_{2^{\alpha+2}} \times \mathbb{Z}_n$ has three index 2 subgroups $\langle a, c \rangle \cong \mathbb{Z}_{2^{\alpha+1}} \times \mathbb{Z}_n$, $\langle a^2, b, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$ and $\langle a^2, ab, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$, and one index p_i subgroup $\langle a, b, c^{p_i} \rangle \cong Q_{2^{\alpha+2}} \times \mathbb{Z}_{n/p_i}$ where $i = 1, 2, \dots, k$.*

Proof. Let $G := Q_{2^{\alpha+2}} \times \mathbb{Z}_n$. Clearly the subgroup $H := \langle a, c \rangle \cong \mathbb{Z}_{2^{\alpha+1}} \times \mathbb{Z}_n$ is index 2 subgroup of G. Let K be an index 2 subgroup of G with $K \neq H$. Then $G = HK$ and

$$2^{\alpha+2} n = |HK| = \frac{|H||K|}{|H \cap K|} = \frac{2^{\alpha+1} n 2^{\alpha+1} n}{|H \cap K|}.$$

Thus $|H \cap K| = 2^\alpha n$ and so $H \cap K = \langle a^2, c \rangle$. Therefore one can see that $H = \langle a^2, b, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$ or $H = \langle a^2, ab, c \rangle \cong Q_{2^{\alpha+1}} \times \mathbb{Z}_n$.

Let M be an index p_i subgroup of G where $i = 1, 2, \dots, k$. Then $|M| = 2^{\alpha+2} n / p_i$. Since

$$|G| = |MQ_{2^{\alpha+2}}| = \frac{|M||Q_{2^{\alpha+2}}|}{|M \cap Q_{2^{\alpha+2}}|},$$

we have $\langle a, b \rangle \leq M$. Now it is easy to see that $M = \langle a, b, c^{p_i} \rangle \cong Q_{2^{\alpha+2}} \times \mathbb{Z}_{n/p_i}$. □

By Lemma 3.1

$$\mathcal{U}(Q_{2^{\alpha+2}}\times\mathbb{Z}_n)=\mathcal{C}(\langle a,c\rangle\cong\mathbb{Z}_{2^{\alpha+1}}\times\mathbb{Z}_n)\bigcup\mathcal{C}(\langle a^2,b,c\rangle\cong Q_{2^{\alpha+1}}\times\mathbb{Z}_n)$$
$$\bigcup\mathcal{C}(\langle a^2,ab,c\rangle\cong Q_{2^{\alpha+1}}\times\mathbb{Z}_n)\bigcup_{i=1}^{k}\mathcal{C}(\langle a,b,c^{p_i}\rangle\cong Q_{2^{\alpha+2}}\times\mathbb{Z}_{n/p_i}).$$

Using the inclusion-exclusion principle one can see that

$$\begin{aligned}U(Q_{2^{\alpha+2}}\times\mathbb{Z}_n)&=2C(Q_{2^{\alpha+1}}\times\mathbb{Z}_n)+C(\mathbb{Z}_{2^{\alpha+1}}\times\mathbb{Z}_n)-2C(\mathbb{Z}_{2^\alpha}\times\mathbb{Z}_n)\\&+\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t+1}C(Q_{2^{\alpha+2}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}})\\&+2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t}C(Q_{2^{\alpha+1}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}})\\&+\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t}C(\mathbb{Z}_{2^{\alpha+1}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}})\\&+2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t+1}C(\mathbb{Z}_{2^{\alpha}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}}).\end{aligned}$$

Further, by Proposition 2.1 we have

$$\begin{aligned}R(Q_{2^{\alpha+2}}\times\mathbb{Z}_n)&=4R(Q_{2^{\alpha+1}}\times\mathbb{Z}_n)+2R(\mathbb{Z}_{2^{\alpha+1}}\times\mathbb{Z}_n)-4R(\mathbb{Z}_{2^\alpha}\times\mathbb{Z}_n)\\&+2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t+1}R(Q_{2^{\alpha+2}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}})\\&+4\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t}R(Q_{2^{\alpha+1}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}})\\&+2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t}R(\mathbb{Z}_{2^{\alpha+1}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}})\\&+4\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t+1}R(\mathbb{Z}_{2^{\alpha}}\times\mathbb{Z}_{n/p_{i_1}\cdots p_{i_t}}).\end{aligned}\qquad(1)$$

Let $a_{\alpha,\beta_1,\beta_2,\ldots,\beta_k} := R(Q_{2^{\alpha+2}}\times\mathbb{Z}_n)$ and $b_{\alpha,\beta_1,\beta_2,\ldots,\beta_k} := R(\mathbb{Z}_{2^{\alpha+1}}\times\mathbb{Z}_n)$. Then equation (1) becomes

$$\begin{aligned}a_{\alpha,\beta_1,\ldots,\beta_k}&=4a_{\alpha-1,\beta_1,\ldots,\beta_k}+2b_{\alpha,\beta_1,\ldots,\beta_k}-4b_{\alpha-1,\beta_1,\ldots,\beta_k}\\&+2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t+1}a_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}\\&+4\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t}a_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}\\&+2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t}b_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}\\&+4\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\1\le t\le k}}(-1)^{t+1}b_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}.\end{aligned}\qquad(2)$$

Let k be a positive integer. We define

$$\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,x_{k-1},\dots,x_j) := \sum_{\beta_j=0}^{\infty}\cdots\sum_{\beta_{k-1}=0}^{\infty}\sum_{\beta_k=0}^{\infty} a_{\alpha,\beta_1,\dots,\beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}}\cdots x_j^{\beta_j},$$

$$\phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,x_{k-1},\dots,x_j) := \sum_{\beta_j=0}^{\infty}\cdots\sum_{\beta_{k-1}=0}^{\infty}\sum_{\beta_k=0}^{\infty} b_{\alpha,\beta_1,\dots,\beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}}\cdots x_j^{\beta_j}$$

where $j = k, k-1, \dots, 1$ and

$$\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,x_{k-1},\dots,x_1,y) := \sum_{\alpha=0}^{\infty}\sum_{\beta_1=0}^{\infty}\cdots\sum_{\beta_{k-1}=0}^{\infty}\sum_{\beta_k=0}^{\infty} a_{\alpha,\beta_1,\dots,\beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}}\cdots x_1^{\beta_1} y^{\alpha},$$

$$\phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,x_{k-1},\dots,x_1,y) := \sum_{\alpha=0}^{\infty}\sum_{\beta_1=0}^{\infty}\cdots\sum_{\beta_{k-1}=0}^{\infty}\sum_{\beta_k=0}^{\infty} b_{\alpha,\beta_1,\dots,\beta_k} x_k^{\beta_k} x_{k-1}^{\beta_{k-1}}\cdots x_1^{\beta_1} y^{\alpha}.$$

From now on we explicitly find the function $\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,x_{k-1},\dots,x_1,y)$ through several lemmas.

Lemma 3.2. *Let k be a non-negative integer. If $k=0$, then*

$$\psi_{\alpha}(y) = \frac{4}{1-4y} \text{ and } a_{\alpha} = 2^{2\alpha+2}.$$

If $k=1$, then

$$\lambda(x_1)\psi_{\alpha,\beta_1}(x_1) = (1+\lambda(x_1))[2\psi_{\alpha-1,\beta_1}(x_1) + \phi_{\alpha,\beta_1}(x_1) - 2\phi_{\alpha-1,\beta_1}(x_1)]. \tag{3}$$

If $k \geq 2$, then

$$\begin{aligned}
\lambda(x_k,\dots,x_j)\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_j) &= (1+\lambda(x_k,\dots,x_j))\,[2\psi_{\alpha-1,\beta_1,\dots,\beta_k}(x_k,\dots,x_j) \\
&+ \phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_j) - 2\phi_{\alpha-1,\beta_1,\dots,\beta_k}(x_k,\dots,x_j) \\
&+ \sum_{\substack{1\leq i_1<\cdots<i_t\leq j-1,\\ 1\leq t\leq j-1}} (-1)^{t+1}\psi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j) \\
&+ 2\sum_{\substack{1\leq i_1<\cdots<i_t\leq j-1,\\ 1\leq t\leq j-1}} (-1)^{t}\psi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j) \\
&+ \sum_{\substack{1\leq i_1<\cdots<i_t\leq j-1,\\ 1\leq t\leq j-1}} (-1)^{t}\phi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j) \\
&+2\sum_{\substack{1\leq i_1<\cdots<i_t\leq j-1,\\ 1\leq t\leq j-1}} (-1)^{t+1}\phi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j)\Bigg]
\end{aligned} \tag{4}$$

for any $j = k, k-1, \ldots, 2$.

Proof. We first assume that $k = 0$. Equation (2) with $k = 0$ gives us that

$$a_\alpha = 4a_{\alpha-1} + 2b_\alpha - 4b_{\alpha-1}. \tag{5}$$

Taking $\sum_{\alpha=1}^{\infty} y^\alpha$ to both sides of equation (5) we have

$$(1-4y)\psi_\alpha(y) = 2(1-2y)\phi_\alpha(y)$$

because $a_0 = R(Q_{2^2}) = R(\mathbb{Z}_{2^2}) = 2^2$, $b_0 = R(\mathbb{Z}_2) = 2$. Since

$$\phi_\alpha(y) = \sum_{\alpha=0}^{\infty} 2^{\alpha+1} y^\alpha = \frac{2}{1-2y},$$

we get that

$$\psi_\alpha(y) = \frac{4}{1-4y},$$

and hence $a_\alpha = 2^{2\alpha+2}$.

Second we assume that $k = 1$. Equation (2) with $k = 1$ gives us that

$$a_{\alpha,\beta_1,} = 4a_{\alpha-1,\beta_1} + 2b_{\alpha,\beta_1} - 4b_{\alpha-1,\beta_1} + 2a_{\alpha,\beta_1-1} - 4a_{\alpha-1,\beta_1-1} - 2b_{\alpha,\beta_1-1} + 4b_{\alpha-1,\beta_1-1}. \tag{6}$$

Taking $\sum_{\beta_1=1}^{\infty} x_1^{\beta_1}$ to both sides of equation (6) one can see that

$$(1-2x_1)\psi_{\alpha,\beta_1}(x_1) = 2(1-x_1)[2\psi_{\alpha-1,\beta_1}(x_1) + \phi_{\alpha,\beta_1}(x_1) - 2\phi_{\alpha-1,\beta_1}(x_1)].$$

because

$$a_{\alpha,0} - 4a_{\alpha-1,0} - 2b_{\alpha,0} + 4b_{\alpha-1,0} = 0$$

by equation (5). Thus equation (3) holds.

From now on we assume that $k \geq 2$. We prove equation (4) by double induction on k and j. If $k = 2$, then equation (2) with $k = 2$ gives us that

$$\begin{aligned} a_{\alpha,\beta_1,\beta_2} = {} & 4a_{\alpha-1,\beta_1,\beta_2} + 2b_{\alpha,\beta_1,\beta_2} - 4b_{\alpha-1,\beta_1,\beta_2} \\ & + 2a_{\alpha,\beta_1-1,\beta_2} + 2a_{\alpha,\beta_1,\beta_2-1} - 2a_{\alpha,\beta_1-1,\beta_2-1} - 4a_{\alpha-1,\beta_1-1,\beta_2} \\ & - 4a_{\alpha-1,\beta_1,\beta_2-1} + 4a_{\alpha-1,\beta_1-1,\beta_2-1} - 2b_{\alpha,\beta_1-1,\beta_2} - 2b_{\alpha,\beta_1,\beta_2-1} \\ & + 2b_{\alpha,\beta_1-1,\beta_2-1} + 4b_{\alpha-1,\beta_1-1,\beta_2} + 4b_{\alpha-1,\beta_1,\beta_2-1} - 4b_{\alpha-1,\beta_1-1,\beta_2-1}. \end{aligned} \tag{7}$$

Taking $\sum_{\beta_2=1}^{\infty} x_2^{\beta_2}$ of both sides of equation (2) one can see that

$$\begin{aligned} (1-2x_2)\psi_{\alpha,\beta_1,\beta_2}(x_2) = {} & 2(1-x_2)\,[2\psi_{\alpha-1,\beta_1,\beta_2}(x_2) + \phi_{\alpha,\beta_1,\beta_2}(x_2) \\ & - 2\phi_{\alpha-1,\beta_1,\beta_2}(x_2) + \psi_{\alpha,\beta_1-1,\beta_2}(x_2) - 2\psi_{\alpha-1,\beta_1-1,\beta_2}(x_2) \\ & -\phi_{\alpha,\beta_1-1,\beta_2}(x_2) + 2\phi_{\alpha-1,\beta_1-1,\beta_2}(x_2)] \end{aligned}$$

because

$$\begin{aligned} a_{\alpha,\beta_1,0} - [& 4a_{\alpha-1,\beta_1,0} + 2b_{\alpha,\beta_1,0} - 4b_{\alpha-1,\beta_1,0} + 2a_{\alpha,\beta_1-1,0} - 4a_{\alpha-1,\beta_1-1,0} \\ & -2b_{\alpha,\beta_1-1,0} + 4b_{\alpha-1,\beta_1-1,0}] = 0 \end{aligned}$$

by equation (2). Thus equation (4) holds for $k = 2$.

Assume now that equation (4) holds from 2 to $k-1$ and consider the case for k. Note that the last four terms of the right side of equation (2) can be divided into three terms respectively as follows.

$$\begin{aligned}2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}}(-1)^{t+1}a_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)} &= 2a_{\alpha,\beta_1,\ldots,\beta_{k-1},\beta_k-1}\\ &-2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t+1}a_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k-1)}\\ &+2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t+1}a_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)},\end{aligned}$$

$$\begin{aligned}4\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}}(-1)^{t}a_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)} &= -4a_{\alpha-1,\beta_1,\ldots,\beta_{k-1},\beta_k-1}\\ &-4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t}a_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k-1)}\\ &+4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t}a_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)},\end{aligned}$$

$$\begin{aligned}2\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}}(-1)^{t}b_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)} &= -2b_{\alpha,\beta_1,\ldots,\beta_{k-1},\beta_k-1}\\ &-2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t}b_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k-1)}\\ &+2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t}b_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}\end{aligned}$$

and

$$\begin{aligned}4\sum_{\substack{1\le i_1<\cdots<i_t\le k,\\ 1\le t\le k}}(-1)^{t+1}b_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)} &= 4b_{\alpha-1,\beta_1,\ldots,\beta_{k-1},\beta_k-1}\\ &-4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t+1}b_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},\beta_k-1)}\\ &+4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}}(-1)^{t+1}b_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}.\end{aligned}$$

Now taking $\sum_{\beta_k=1}^{\infty} x_k^{\beta_k}$ of both sides of equation (2) we get that

$$
\begin{aligned}
(1-2x_k)\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k) &= 2(1-x_k)\Big[2\psi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k) + \phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k) \\
&\quad - 2\phi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k) + \sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1}\psi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k) \\
&\quad + 2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t}\psi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k) \\
&\quad + \sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t}\phi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k) \\
&\quad + 2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1}\phi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k)\Big] \\
&\quad + a_{\alpha,\beta_1,\ldots,\beta_{k-1},0} - \Big[4a_{\alpha-1,\beta_1,\ldots,\beta_{k-1},0} + 2b_{\alpha,\beta_1,\ldots,\beta_{k-1},0} - 4b_{\alpha-1,\beta_1,\ldots,\beta_{k-1},0} \\
&\quad + 2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1}a_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} \\
&\quad + 4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t}a_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} \\
&\quad + 2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t}b_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} \\
&\quad + 4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1}b_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)}\Big].
\end{aligned}
$$

Note that by equation (2)

$$
\begin{aligned}
&a_{\alpha,\beta_1,\ldots,\beta_{k-1},0} - \Big[4a_{\alpha-1,\beta_1,\ldots,\beta_{k-1},0} + 2b_{\alpha,\beta_1,\ldots,\beta_{k-1},0} - 4b_{\alpha-1,\beta_1,\ldots,\beta_{k-1},0} \\
&\quad + 2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1}a_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} \\
&\quad + 4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t}a_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} \\
&\quad + 2\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t}b_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)} \\
&\quad + 4\sum_{\substack{1\le i_1<\cdots<i_t\le k-1,\\ 1\le t\le k-1}} (-1)^{t+1}b_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{k-1},0)}\Big] = 0.
\end{aligned}
$$

Thus equation (4) holds for $j = k$. Assume that equation (4) holds from k to j and consider the case for $j-1$. Note that the last four terms of the right side of equation (4) can be divided into three terms respectively as follows.

$$\sum_{\substack{1\le i_1<\cdots<i_t\le j-1,\\ 1\le t\le j-1}} (-1)^{t+1}\psi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$= \psi_{\alpha,\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k}(x_k,\ldots,x_j)$$
$$- \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1}\psi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$+ \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1}\psi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$

$$2\sum_{\substack{1\le i_1<\cdots<i_t\le j-1,\\ 1\le t\le j-1}} (-1)^{t}\psi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$= -2\psi_{\alpha-1,\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$-2 \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t}\psi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$+2 \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t}\psi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$

$$\sum_{\substack{1\le i_1<\cdots<i_t\le j-1,\\ 1\le t\le j-1}} (-1)^{t}\phi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$= -\phi_{\alpha,\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$- \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t}\phi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$+ \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t}\phi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$

$$2\sum_{\substack{1\le i_1<\cdots<i_t\le j-1,\\ 1\le t\le j-1}} (-1)^{t+1}\phi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$= 2\phi_{\alpha-1,\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$-2 \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1}\phi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_{j-2},\beta_{j-1}-1,\beta_j,\ldots,\beta_k)}(x_k,\ldots,x_j)$$
$$+2 \sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}} (-1)^{t+1}\phi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\ldots,\beta_k)}(x_k,\ldots,x_j)$$

Now taking $\sum_{\beta_{j-1}=1}^{\infty} x_{j-1}^{\beta_{j-1}}$ of both sides of equation (4) we have

$$\begin{aligned}
&[\lambda(x_k,\dots,x_j)-(1+\lambda(x_k,\dots,x_j))x_{j-1}]\,\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_j,x_{j-1})\\
&=[1+\lambda(x_k,\dots,x_j)-(1+\lambda(x_k,\dots,x_j))x_{j-1}]\,[2\psi_{\alpha-1,\beta_1,\dots,\beta_k}(x_k,\dots,x_j,x_{j-1})\\
&\quad+\phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_j,x_{j-1})-2\phi_{\alpha-1,\beta_1,\dots,\beta_k}(x_k,\dots,x_j,x_{j-1})\\
&\quad+\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t+1}\psi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j,x_{j-1})\\
&\quad+2\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t}\psi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j,x_{j-1})\\
&\quad+\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t}\phi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j,x_{j-1})\\
&\quad\left.+2\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t+1}\phi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_k)}(x_k,\dots,x_j,x_{j-1})\right]
\end{aligned}$$

because it holds by induction hypothesis that

$$\begin{aligned}
&\lambda(x_k,\dots,x_j)\psi_{\alpha,\beta_1,\dots,\beta_{j-2},0,\beta_j,\dots,\beta_k}(x_k,\dots,x_j)\\
&\quad-(1+\lambda(x_k,\dots,x_j))\,[2\psi_{\alpha-1,\beta_1,\dots,\beta_{j-2},0,\beta_j,\dots,\beta_k}(x_k,\dots,x_j)\\
&\quad+\phi_{\alpha,\beta_1,\dots,\beta_{j-2},0,\beta_j,\dots,\beta_k}(x_k,\dots,x_j)-2\phi_{\alpha-1,\beta_1,\dots,\beta_{j-2},0,\beta_j,\dots,\beta_k}(x_k,\dots,x_j)
\end{aligned}$$

$$\begin{aligned}
&+\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t+1}\psi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_{j-2},0,\beta_j,\dots,\beta_k)}(x_k,\dots,x_j)\\
&\quad+2\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t}\psi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_{j-2},0,\beta_j,\dots,\beta_k)}(x_k,\dots,x_j)\\
&\quad+\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t}\phi_{\alpha,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_{j-2},0,\beta_j,\dots,\beta_k)}(x_k,\dots,x_j)\\
&\quad\left.+2\sum_{\substack{1\le i_1<\cdots<i_t\le j-2,\\ 1\le t\le j-2}}(-1)^{t+1}\phi_{\alpha-1,\pi_{i_1\cdots i_t}(\beta_1,\dots,\beta_{j-2},0,\beta_j,\beta_k)}(x_k,\dots,x_j)\right]=0.
\end{aligned}$$

Thus equation (4) holds for $j-1$, and the lemma is proved. □

Equation (4) with $j = 2$ gives us that

$$\begin{aligned}\lambda(x_k, \ldots, x_2)\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_2) \\ = (1 + \lambda(x_k, \ldots, x_2))\,[2\psi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_2) \\ + \phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_2) - 2\phi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_2) \\ + \psi_{\alpha,\beta_1-1\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) - 2\psi_{\alpha-1,\beta_1-1,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) \\ -\phi_{\alpha,\beta_1-1,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) + 2\phi_{\alpha-1,\beta_1-1,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2)]\,. \quad (8)\end{aligned}$$

Taking $\sum_{\beta_1=1}^{\infty} x_1^{\beta_1}$ of both sides of equation (8) one can see that

$$\begin{aligned}\lambda(x_k, \ldots, x_1)\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) \\ = (1 + \lambda(x_k, \ldots, x_1))[2\psi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) \\ + \phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) - 2\phi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1)] \\ + \lambda(x_k, \ldots, x_2)\psi_{\alpha,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) \\ - (1 + \lambda(x_k, \ldots, x_2))[2\psi_{\alpha-1,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) \\ + \phi_{\alpha,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) - 2\phi_{\alpha-1,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2)]. \quad (9)\end{aligned}$$

Lemma 3.3. *If $k \geq 2$, then*

$$\begin{aligned}\lambda(x_k, \ldots, x_2)\psi_{\alpha,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) \\ = (1 + \lambda(x_k, \ldots, x_2))\,[2\psi_{\alpha-1,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) \\ +\phi_{\alpha,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2) - 2\phi_{\alpha-1,0,\beta_2,\ldots,\beta_k}(x_k, \ldots, x_2)]\,. \quad (10)\end{aligned}$$

Proof. If $k = 2$, then since $\psi_{\alpha,0,\beta_2}(x_2) = \psi_{\alpha,\beta_2}(x_2)$ and $\phi_{\alpha,0,\beta_2}(x_2) = \phi_{\alpha,\beta_2}(x_2)$, the equation

$$\lambda(x_2)\psi_{\alpha,0,\beta_2}(x_2) = (1 + \lambda(x_2))\,[2\psi_{\alpha-1,0,\beta_2}(x_2) \; +\phi_{\alpha,0,\beta_2}(x_2) - 2\phi_{\alpha-1,0,\beta_2}(x_2)]$$

holds by equation (3). Assume now that equation (10) holds for k. Then by equation (9) we get that

$$\begin{aligned}\lambda(x_k, \ldots, x_1)\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) \\ = (1 + \lambda(x_k, \ldots, x_1))\,[2\psi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) \\ +\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1) - 2\phi_{\alpha-1,\beta_1,\ldots,\beta_k}(x_k, \ldots, x_1)]\end{aligned}$$

which implies that

$$\begin{aligned}\lambda(x_{k+1}, \ldots, x_2)\psi_{\alpha,0,\beta_2,\ldots,\beta_{k+1}}(x_{k+1}, \ldots, x_2) = (1 + \lambda(x_{k+1}, \ldots, x_2))\times \\ \big[2\psi_{\alpha-1,0,\beta_2,\ldots,\beta_{k+1}}(x_{k+1}, \ldots, x_2) + \phi_{\alpha,0,\beta_2,\ldots,\beta_{k+1}}(x_{k+1}, \ldots, x_2) \\ -2\phi_{\alpha-1,0,\beta_2,\ldots,\beta_{k+1}}(x_{k+1}, \ldots, x_2)\big]\,.\end{aligned}$$

Thus equation (10) holds for $k + 1$. □

By equations (3), (9) and Lemma 3.3 it holds that

$$\begin{aligned}\lambda(x_k,\dots,x_1)\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_1)\\ &= (1+\lambda(x_k,\dots,x_1))\left[2\psi_{\alpha-1,\beta_1,\dots,\beta_k}(x_k,\dots,x_1)\right.\\ &\left.+\phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_1)-2\phi_{\alpha-1,\beta_1,\dots,\beta_k}(x_k,\dots,x_1)\right] \quad (11)\end{aligned}$$

for any positive integer k. Taking $\sum_{\alpha=1}^{\infty} y^\alpha$ of both sides of equation (11) one can see that

$$\begin{aligned}&[\lambda(x_k,\dots,x_1)-2(1+\lambda(x_k,\dots,x_1))y]\,\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_1,y)\\ &= (1+\lambda(x_k,\dots,x_1))(1-2y)\phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_1,y)\\ &+\lambda(x_k,\dots,x_1)\psi_{0,\beta_1,\dots,\beta_k}(x_k,\dots,x_1)\\ &-(1+\lambda(x_k,\dots,x_1))\phi_{0,\beta_1,\dots,\beta_k}(x_k,\dots,x_1). \quad (12)\end{aligned}$$

Lemma 3.4. *For any positive integer k,*

$$\begin{aligned}&\lambda(x_k,\dots,x_1)\psi_{0,\beta_1,\dots,\beta_k}(x_k,\dots,x_1)\\ &= (1+\lambda(x_k,\dots,x_1))\phi_{0,\beta_1,\dots,\beta_k}(x_k,\dots,x_1). \quad (13)\end{aligned}$$

Proof. Note that $a_{0,\beta_1,\beta_2,\dots,\beta_k} = R(Q_{2^2}\times\mathbb{Z}_n) = R(\mathbb{Z}_{2^2}\times\mathbb{Z}_n)$ and $b_{0,\beta_1,\beta_2,\dots,\beta_k} := R(\mathbb{Z}_2\times\mathbb{Z}_n)$. Since n is a product of odd prime numbers, by Proposition 2.2 one can see that

$$\psi_{0,\beta_1,\dots,\beta_k}(x_k,\dots,x_1) = \frac{1}{\lambda(x_k,\dots,x_1)}\left[1+\frac{1}{\lambda(x_k,\dots,x_1)}\right]^2,$$
$$\phi_{0,\beta_1,\dots,\beta_k}(x_k,\dots,x_1) = \frac{1}{\lambda(x_k,\dots,x_1)}\left[1+\frac{1}{\lambda(x_k,\dots,x_1)}\right].$$

Now it is easy to see that equation (13) holds. □

By Lemma 3.4 equation (12) becomes

$$\begin{aligned}&\psi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_1,y)\\ &= \left[1+\frac{1}{\lambda(x_k,\dots,x_1)-2(1+\lambda(x_k,\dots,x_1))y}\right]\phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_1,y). \quad (14)\end{aligned}$$

Since $b_{\alpha,\beta_1,\beta_2,\dots,\beta_k} = R(\mathbb{Z}_{2^{\alpha+1}}\times\mathbb{Z}_n)$ by the definition and n is a product of odd prime numbers, by Proposition 2.2 one can see that

$$\phi_{\alpha,\beta_1,\dots,\beta_k}(x_k,\dots,x_1,y) = \frac{1}{\lambda(x_k,\dots,x_1)}\left[1+\frac{1}{\lambda(x_k,\dots,x_1)}\right]\frac{1}{1-\left[1+\frac{1}{\lambda(x_k,\dots,x_1)}\right]y}.$$

In summary we have proved the following.

Theorem 3.5. *Let*

$$n = p_1^{\beta_1}p_2^{\beta_2}\cdots p_k^{\beta_k}$$

where $p_1,\dots,p_k$ are distinct odd prime numbers and $\beta_1,\dots,\beta_k$ are positive integers. Let α be a non-negative integer. Let

$$G := Q_{2^{\alpha+2}}\times\mathbb{Z}_n = \left\langle a,b,c \mid a^{2^{\alpha+1}} = e, b^2 = a^\alpha, bab^{-1} = a^{-1}, c^n = e\right\rangle$$

be the direct product of the generalized quaternion group $Q_{2^{\alpha+2}}$ of order $2^{\alpha+2}$ and the cyclic group $\mathbb{Z}_n$ of order n. Then the number $R(G)$ of fuzzy subgroups of G is the coefficient of $y^\alpha x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of

$$\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y) = \left[1 + \frac{1}{\lambda(x_k,\ldots,x_1) - 2\left(1+\lambda(x_k,\ldots,x_1)\right)y}\right]\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y) \quad (15)$$

where

$$\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y) = \frac{1}{\lambda(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{1}{1-\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]y}.$$

The following is useful for specifically given integers n and α.

Corollary 3.6. *Let G and $\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y)$ be the group and the function defined in Theorem 3.5. Then*

$$R(G) = \left.\frac{\partial^{\beta_1+\beta_2+\cdots+\beta_k+\alpha}\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y)}{\beta_k!\cdots\beta_2!\beta_1!\alpha!(\partial x_k)^{\beta_k}\cdots(\partial x_2)^{\beta_2}(\partial x_1)^{\beta_1}(\partial y)^{\alpha}}\right|_{\substack{x_i=0,y=0,\\1\le i\le k}}.$$

We now want to find the coefficients of $y^\alpha x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ of

$$\psi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y) = \phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y) + \frac{1}{\lambda(x_k,\ldots,x_1)-2(1+\lambda(x_k,\ldots,x_1))y}\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y)$$

explicitly. Note that

$$\begin{aligned}\psi_{\alpha,\beta_1,\ldots,\beta_k}&(x_k,\ldots,x_1,y)\\ &= \frac{1}{\lambda(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{1}{1-\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]y}\\ &\quad -\frac{1}{\lambda^2(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{1}{1-\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]y}\\ &\quad\quad +\frac{1}{\lambda^2(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{2}{1-2\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]y}\end{aligned}$$

Since

$$\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y) = \frac{1}{\lambda(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{1}{1-\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]y},$$

the coefficient of y^α of $\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y)$ is

$$\frac{1}{\lambda(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]^{\alpha+1}.$$

Since

$$\frac{1}{\lambda(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]^{\alpha+1}=\sum_{i_0=0}^{\alpha+1}\binom{\alpha+1}{i_0}\left[\frac{1}{\lambda(x_k,\ldots,x_1)}\right]^{i_0+1}$$
$$=\sum_{i_0=0}^{\alpha+1}\binom{\alpha+1}{i_0}\left[\frac{1}{\lambda(x_k,\ldots,x_2)}\right]^{i_0+1}\left[\frac{1}{1-\left[1+\frac{1}{\lambda(x_k,\ldots,x_2)}\right]x_1}\right]^{i_0+1},$$

the coefficient of $y^\alpha x_1^{\beta_1}$ of $\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y)$ is

$$\sum_{i_0=0}^{\alpha+1}\binom{\alpha+1}{i_0}\binom{\beta_1+i_0}{\beta_1}\left[\frac{1}{\lambda(x_k,\ldots,x_2)}\right]^{i_0+1}\left[1+\frac{1}{\lambda(x_k,\ldots,x_2)}\right]^{\beta_1},$$

which is equal to

$$\sum_{i_0=0}^{\alpha+1}\sum_{i_1=0}^{\beta_1}\binom{\alpha+1}{i_0}\binom{\beta_1}{i_1}\binom{\beta_1+i_0}{\beta_1}\left[\frac{1}{\lambda(x_k,\ldots,x_2)}\right]^{i_0+i_1+1}.$$

By continuing this process, one can see that the coefficient of $y^\alpha x_1^{\beta_1}\cdots x_k^{\beta_k}$ of $\phi_{\alpha,\beta_1,\ldots,\beta_k}(x_k,\ldots,x_1,y)$ is

$$2^{\beta_k}\sum_{i_0=0}^{\alpha+1}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\binom{\alpha+1}{i_0}\binom{\beta_1+i_0}{\beta_1}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=0}^{r}i_m}{\beta_{r+1}}.$$

Similarly one can see that the coefficient of $y^\alpha x_1^{\beta_1}x_2^{\beta_2}\cdots x_{k-1}^{\beta_{k-1}}x_k^{\beta_k}$ of

$$\frac{1}{\lambda^2(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{1}{1-\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]y}$$

is

$$2^{\beta_k}\sum_{i_0=0}^{\alpha+1}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\binom{\alpha+1}{i_0}\binom{\beta_1+1+i_0}{\beta_1}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+1+\sum_{m=0}^{r}i_m}{\beta_{r+1}}$$

and the coefficient of $y^\alpha x_1^{\beta_1}x_2^{\beta_2}\cdots x_{k-1}^{\beta_{k-1}}x_k^{\beta_k}$ of

$$\frac{1}{\lambda^2(x_k,\ldots,x_1)}\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]\frac{2}{1-2\left[1+\frac{1}{\lambda(x_k,\ldots,x_1)}\right]y}$$

is

$$2^{\alpha+1+\beta_k}\sum_{i_0=0}^{\alpha+1}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\left[\binom{\alpha+1}{i_0}\times\binom{\beta_1+1+i_0}{\beta_1}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+1+\sum_{m=0}^{r}i_m}{\beta_{r+1}}\right].$$

Therefore we get the following.

Corollary 3.7. *Let G be the group defined in Theorem 3.5. Then the number $R(G)$ of fuzzy subgroups of G is*

$$R(G) = 2^{\beta_k}\sum_{i_0=0}^{\alpha+1}\sum_{i_1=0}^{\beta_1}\sum_{i_2=0}^{\beta_2}\cdots\sum_{i_{k-1}=0}^{\beta_{k-1}}\left[\binom{\alpha+1}{i_0}\binom{\beta_1+i_0}{\beta_1}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+\sum_{m=0}^{r}i_m}{\beta_{r+1}}\right.$$
$$\left.+(2^{\alpha+1}-1)\binom{\alpha+1}{i_0}\binom{\beta_1+1+i_0}{\beta_1}\prod_{r=1}^{k-1}\binom{\beta_r}{i_r}\binom{\beta_{r+1}+1+\sum_{m=0}^{r}i_m}{\beta_{r+1}}\right] \quad (16)$$

where if $k=0$, then $R(Q_{2^{\alpha+2}}) = 2^{2\alpha+2}$ and if $k=1$, then

$$R(Q_{2^{\alpha+2}}\times\mathbb{Z}_{p_1^{\beta_1}}) = 2^{\beta_1}\sum_{i_0=0}^{\alpha+1}\left[\binom{\alpha+1}{i_0}\binom{\beta_1+i_0}{\beta_1}+(2^{\alpha+1}-1)\binom{\alpha+1}{i_0}\binom{\beta_1+1+i_0}{\beta_1}\right].$$

Remark 3.8. If one adopts the equivalence relation on the set of fuzzy subgroups of a given finite group G used by Murali and Makamba [4, 5, 6, 7], then one can see that the number of fuzzy subgroups of G is equal to the number of chains of subgroups of G. Thus by Proposition 2.1 the result of this paper leads us to the number of fuzzy subgroups of $Q_{2^{\alpha+2}}\times\mathbb{Z}_n$ under the equivalence relation by Murali and Makamba.

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