CATEGORIES ISOMORPHIC TO THE CATEGORY OF L-FUZZY CLOSURE SYSTEM SPACES

F. G. SHI AND B. PANG

Abstract. In this paper, new definitions of L-fuzzy closure operator, L-fuzzy interior operator, L-fuzzy remote neighborhood system, L-fuzzy neighborhood system and L-fuzzy quasi-coincident neighborhood system are proposed. It is proved that the category of L-fuzzy closure spaces, the category of L-fuzzy interior spaces, the category of L-fuzzy remote neighborhood spaces, the category of L-fuzzy quasi-coincident neighborhood spaces, the category of L-fuzzy neighborhood spaces are all isomorphic to the category L-**FCS** of L-fuzzy closure system spaces.

1. Introduction

Closure operators and closure systems are very useful tools in several areas of classical mathematics, involving the realm of topology, algebra, analysis, matroid theory, etc. In fuzzy set theory, different kinds of fuzzy closure operators and fuzzy closure systems are studied as extensions of closure operators and closure systems [1, 2, 3, 5, 6, 9, 10, 12, 15, 19, 21, 25, 26, 28, 30, 32].

In [4], Birkhoff introduced classical closure systems as a subset of the powerset 2^X. Later on, Biacino and Gerla [3] defined a kind of fuzzy closure system extending 2^X to I^X. Kim [16] proved that the lattice of fuzzy closure systems is isomorphic to the lattice of fuzzy closure operators for the respective notions defined in [3]. Bělohlávek [1] outlined a general theory of fuzzy closure operators and fuzzy closure systems (L_K-closure systems). He showed the existence of a one-to-one correspondence between his fuzzy closure operators and fuzzy closure systems.

However, the above-mentioned fuzzy closure systems are crisp families of fuzzy subsets on a universe set X. Following the idea of [13, 17, 27], Fang [8] proposed the concept of L-fuzzy closure system. Unluckily, L-fuzzy closure operators in [8] are not equivalent to L-fuzzy closure systems. Later on, Luo and Fang [20] introduced the concepts of fuzzifying closure system and fuzzifying closure operator as a generalization of Birkhoff's closure operator. Moreover, a one-to-one correspondence between the notions was established.

In [25], Shi proposed the concept of L-fuzzy closure operators based on Kuratowski's closure operators and showed an equivalence between L-fuzzy closure operators and their respective L-fuzzy topologies.

Received: December 2011; Revised: May 2012 and June 2012; Accepted: December 2012

Key words and phrases: L-fuzzy closure system, L-fuzzy closure operator, L-fuzzy interior operator, L-fuzzy remote neighborhood, L-fuzzy quasi-coincident neighborhood, L-fuzzy neighborhood, Isomorphism of categories.

In this paper, we shall give a new definition of L-fuzzy closure operators as extensions of Birkhoff's closure operators, in order to characterize L-fuzzy closure systems. Moreover, some other characterizations of L-fuzzy closure systems will be also presented.

The structure of this paper is as follows. In Section 2, we provide some preliminary concepts and results. In Section 3, characterizations of L-fuzzy closure operators and L-fuzzy interior operators in the sense of Shi are given. In Section 4, we shall introduce the new notion of L-fuzzy closure operators as extensions of Birkhoff's closure operators. We prove that the category L-**FCS** of L-fuzzy closure system spaces and the category L-**FC** of L-fuzzy closure spaces are isomorphic. In Section 5, L-fuzzy remote neighborhood system and L-fuzzy quasi-coincident neighborhood system of an L-fuzzy closure system space are presented and it is shown that the category L-**FRN** of L-fuzzy remote neighborhood spaces and the category L-**FQN** of L-fuzzy quasi-coincident neighborhood spaces are both isomorphic to L-**FCS**. In Section 6, the concepts of L-fuzzy interior operator and L-fuzzy neighborhood system of an L-fuzzy closure system space are introduced, and it is shown that the category L-**FI** of L-fuzzy interior spaces and the category L-**FN** of L-fuzzy neighborhood spaces are isomorphic to the category L-**FCS**.

2. **Preliminaries**

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ denotes a completely distributive De Morgan algebra. The smallest element and the largest element in L are denoted $\perp$ and $\top$, respectively. The set of non-zero coprimes in L is denoted $J(L)$. For $a,\ b \in L$, we say "a is wedge below b", in symbols $a \prec b$, if for every subset $D \subseteq L, \bigvee D \geqslant b$ implies $a \leqslant d$ for some $d \in D$. We denote $\beta(a) = \{b \in L \mid b \prec a\}$ and $\beta^*(a) = \beta(a) \cap J(L)$ for each $a \in L$. For $a,\ b \in L$, $a \prec^{op} b$ means that if for every subset $D \subseteq L, \bigwedge D \leqslant a$ implies $d \leqslant b$ for some $d \in D$. We denote $\alpha(a) = \{b \in L \mid a \prec^{op} b\}$. A complete lattice L is completely distributive if and only if $a = \bigvee \beta^*(a) = \bigvee \beta(a) = \bigwedge \alpha(a)$ for each $a \in L$ [29]. The wedge below relation in a completely distributive lattice has the interpolation property, i.e., if $a \prec b$, then there exists $c \in L$ such that $a \prec c \prec b$. Moreover, it is easy to see that $a \prec \bigwedge_{i \in I} b_i$ implies $a \prec b_i$ for every $i \in I$, whereas $a \prec \bigvee_{i \in I} b_i$ is equivalent to $a \prec b_i$ for some $i \in I$.

For a completely distributive De Morgan algebra L and a non-empty set X, L^X denotes the set of all L-fuzzy subsets on X. L^X is also a completely distributive De Morgan algebra, when it inherits the structure of the lattice L in a natural way, by defining $\bigvee,\ \bigwedge,\ \leqslant$ and $'$ pointwise. The set of non-zero coprimes in L^X is denoted $J(L^X)$. It is easy to see that $J(L^X)$ is precisely the set of all fuzzy points x_λ $(\lambda \in J(L))$. The smallest element and the largest element in L^X are denoted $\underline{\perp}$ and $\underline{\top}$, respectively. For every L-fuzzy subset $A \in L^X$, and every $a \in L$, we use the following notations:

$$A_{[a]} = \{x \in X \mid A(x) \geqslant a\}, \quad A^{(a)} = \{x \in X \mid A(x) \nleqslant a\}.$$

Let $f : X \to Y$ be a set mapping. Define $f^{\to} : L^X \to L^Y$ and $f^{\leftarrow} : L^Y \to L^X$ by $f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for every $A \in L^X$ and every $y \in Y$, $f^{\leftarrow}(B) = B \circ f$ for every $B \in L^Y$, respectively. Moreover, $(f^{\leftarrow}(B))' = f^{\leftarrow}(B')$. For more details, we refer to [22, 23, 24].

Definition 2.1. [8] A mapping $\varphi : L^X \to L$ is called an L-fuzzy closure system on X provided that it satisfies the following conditions:

(LFCS) $\varphi\left(\bigwedge\limits_{i\in I} A_i\right) \geqslant \bigwedge\limits_{i\in I} \varphi(A_i),\ \forall\{A_i \mid i \in I\} \subseteq L^X$.

A pair (X, φ) is called an L-fuzzy closure system space provided that φ is an L-fuzzy closure system on X.

A mapping $f : X \to Y$ between two L-fuzzy closure system spaces (X, φ_X) and (Y, φ_Y) is called continuous if $\varphi_X(f^{\leftarrow}(A)) \geqslant \varphi_Y(A)$ for every $A \in L^Y$. The category of L-fuzzy closure system spaces and continuous mappings is denoted L-**FCS**.

Remark 2.2. In [8], the definition of L-fuzzy closure system requires that φ satisfy (LFCS) and another axiom (LFCS$'$) $\varphi(\underline{\top}) = \top$. Howerer, it is usually assumed that $\bigwedge \emptyset = \top$ for the empty set $\emptyset$ in lattice theory. Therefore, (LFCS) implies (LFCS$'$).

Definition 2.3. [11] A mapping $\tau : L^X \to L$ is called an L-fuzzy pretopology on X provided that it satisfies the following conditions:

(LFPT1) $\tau(\underline{\top}) = \top$;

(LFPT2) $\tau\left(\bigvee\limits_{i\in I} A_i\right) \geqslant \bigwedge\limits_{i\in I} \tau(A_i),\ \forall\{A_i \mid i \in I\} \subseteq L^X$.

A pair (X, τ) is called an L-fuzzy pretopological space provided that τ is an L-fuzzy pretopology on X.

A mapping $f : X \to Y$ between two L-fuzzy pretopological spaces (X, τ_X) and (Y, τ_Y) is called continuous if $\tau_X(f^{\leftarrow}(A)) \geqslant \tau_Y(A)$ for every $A \in L^Y$. The category of L-fuzzy pretopological spaces and continuous mappings is denoted L-**FPTOP**.

Theorem 2.4. *The category* L-**FPTOP** *is a full subcategory of the category* L-**FCS**.

Proof. Given an L-fuzzy pretopological space (X, τ), define a mapping $\varphi_\tau : L^X \to L$ by

$$\forall A \in L^X, \quad \varphi_\tau(A) = \tau(A').$$

Then (X, φ_τ) is an L-fuzzy closure system space. Since $'$ is an order-reversing involution on L, it can be easily checked that the category L-**FPTOP** is a full subcategory of the category L-**FCS**. $\square$

Definition 2.5. [25] An L-fuzzy closure operator on X is a mapping $Cl : L^X \to L^{J(L^X)}$ satisfying the following conditions:

(LFC1) $(Cl(A))(x_\lambda) = \bigwedge\limits_{\mu \prec \lambda} (Cl(A))(x_\mu)$ for every $x_\lambda \in J(L^X)$;

(LFC2) $(Cl(\underline{\bot}))(x_\lambda) = \bot$ for every $x_\lambda \in J(L^X)$;

(LFC3) $(Cl(A))(x_\lambda) = \top$ for every $x_\lambda \leqslant A$;

(LFC4) $Cl(A \vee B) = Cl(A) \vee Cl(B)$;

(LFC5) $\forall a \in L\backslash\{\bot\},\ (Cl(\bigvee (Cl(A))_{[a]}))_{[a]} \subseteq (Cl(A))_{[a]}$.

$(Cl(A))(x_\lambda)$ is called the degree to which x_λ belongs to the closure of A.

Definition 2.6. [25] An L-fuzzy interior operator on X is a mapping $Int : L^X \to L^{J(L^X)}$ satisfying the following conditions:

(LFI1) $(Int(A))(x_\lambda) = \bigwedge_{\mu \prec \lambda} (Int(A))(x_\mu)$ for every $x_\lambda \in J(L^X)$;

(LFI2) $(Int(\underline{\top}))(x_\lambda) = \top$ for every $x_\lambda \in J(L^X)$;

(LFI3) $(Int(A))(x_\lambda) = \bot$ for every $x_\lambda \not\leqslant A$;

(LFI4) $Int(A \wedge B) = Int(A) \wedge Int(B)$;

(LFI5) $\forall a \in L\backslash\{\top\},\ (Int(A))^{(a)} \subseteq (Int(\bigvee (Int(A))^{(a)}))^{(a)}$.

$(Int(A))(x_\lambda)$ is called the degree to which x_λ belongs to the interior of A.

Definition 2.7. [25] An L-fuzzy neighborhood system on X is a set $\mathcal{N} = \{\mathcal{N}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{N}_{x_\lambda} : L^X \to L$ satisfying the following conditions:

(LFN1) $\mathcal{N}_{x_\lambda}(\underline{\top}) = \top,\ \mathcal{N}_{x_\lambda}(\underline{\bot}) = \bot$;

(LFN2) $\mathcal{N}_{x_\lambda}(A) = \bot$ for every $x_\lambda \not\leqslant A$;

(LFN3) $\mathcal{N}_{x_\lambda}(A \wedge B) = \mathcal{N}_{x_\lambda}(A) \wedge \mathcal{N}_{x_\lambda}(B)$;

(LFN4) $\mathcal{N}_{x_\lambda}(A) = \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} \mathcal{N}_{y_\mu}(B)$.

$\mathcal{N}_{x_\lambda}(A)$ is called the degree to which A is a neighborhood of x_λ.

Definition 2.8. [7] An L-fuzzy quasi-coincident neighborhood system on X is a set $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{Q}_{x_\lambda} : L^X \to L$ satisfying the following conditions:

(LFQ1) $\mathcal{Q}_{x_\lambda}(\underline{\top}) = \top, \mathcal{Q}_{x_\lambda}(\underline{\bot}) = \bot$;

(LFQ2) $\mathcal{Q}_{x_\lambda}(A) \neq \bot \Rightarrow x_\lambda \not\leqslant A'$;

(LFQ3) $\mathcal{Q}_{x_\lambda}(A \wedge B) = \mathcal{Q}_{x_\lambda}(A) \wedge \mathcal{Q}_{x_\lambda}(B)$;

(LFQ4) $\mathcal{Q}_{x_\lambda}(A) = \bigvee_{x_\lambda \not\leqslant D \geqslant A'} \bigwedge_{y_\mu \not\leqslant D} \mathcal{Q}_{y_\mu}(D')$.

Definition 2.9. [31] An L-fuzzy remote neighborhood system on X is a set $\eta = \{\eta_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\eta_{x_\lambda} : L^X \to L$ satisfying the following conditions:

(LFR1) $\eta_{x_\lambda}(\underline{\bot}) = \top,\ \eta_{x_\lambda}(\underline{\top}) = \bot$;

(LFR2) $\eta_{x_\lambda}(A) \neq \bot \Rightarrow x_\lambda \not\leqslant A$;

(LFR3) $\eta_{x_\lambda}(A \vee B) = \eta_{x_\lambda}(A) \wedge \eta_{x_\lambda}(B)$;

(LFR4) $\eta_{x_\lambda}(A) = \bigvee_{x_\lambda \not\leqslant B \geqslant A} \bigwedge_{y_\mu \not\leqslant B} \eta_{y_\mu}(B)$.

3. Characterizations of L-fuzzy Closure Operators and L-fuzzy Interior Operators in the Sense of Shi

In [25], Shi introduced the notions of L-fuzzy closure operator and L-fuzzy interior operator, which are equivalent to L-fuzzy topologies. In this section, we shall give their characterizations.

Theorem 3.1. *If a mapping $Cl : L^X \to L^{J(L^X)}$ is order-preserving and satisfies* (LFC3), *then* (LFC1) *and* (LFC5) *together are equivalent to the following condition:*

(LFC) $(Cl(A))(x_\lambda) = \bigwedge_{x_\lambda \not\leqslant B \geqslant A} \bigvee_{y_\mu \not\leqslant B} (Cl(B))(y_\mu)$.

Proof. Sufficiency. (LFC1) $(Cl(A))(x_\lambda) \leqslant \bigwedge_{\mu \prec \lambda} (Cl(A))(x_\mu)$ is obvious. In order to prove $(Cl(A))(x_\lambda) \geqslant \bigwedge_{\mu \prec \lambda} (Cl(A))(x_\mu)$, we have to show $((Cl(A))(x_\lambda))' \leqslant \bigvee_{\mu \prec \lambda} ((Cl(A))(x_\mu))'$. Let $\alpha \in J(M)$ be such that

$$\alpha \prec ((Cl(A))(x_\lambda))' = \bigvee_{x_\lambda \nleqslant B \geqslant A} \bigwedge_{y_\mu \nleqslant B} ((Cl(B))(y_\mu))'.$$

Then there exists B_α such that $x_\lambda \nleqslant B_\alpha \geqslant A$ and $\forall y_\mu \nleqslant B_\alpha, \alpha \leqslant ((Cl(B_\alpha))(y_\mu))'$. By $\lambda = \bigvee \beta^*(\lambda)$, we know that there exists $\mu \in \beta^*(\lambda)$ such that $x_\mu \nleqslant B_\alpha$. Further, we have

$$\alpha \leqslant ((Cl(B_\alpha))(x_\mu))' \leqslant ((Cl(A))(x_\mu))'.$$

This shows that $\alpha \leqslant \bigvee_{\mu \prec \lambda} ((Cl(A))(x_\mu))'$. So, $((Cl(A))(x_\lambda))' \leqslant \bigvee_{\mu \prec \lambda} ((Cl(A))(x_\mu))'$.

(LFC5) Let $x_\lambda \notin (Cl(A))_{[a]}$. Then $a \nleqslant Cl(A)(x_\lambda)$. This implies that

$$\bigvee_{x_\lambda \nleqslant B \geqslant A} \bigwedge_{y_\mu \nleqslant B} ((Cl(B))(y_\mu))' \nleqslant a'.$$

Hence, there exists B_a such that $x_\lambda \nleqslant B_a \geqslant A$ and $\bigwedge_{y_\mu \nleqslant B_a} ((Cl(B_a))(y_\mu))' \nleqslant a'$. This shows that $y_\mu \notin (Cl(B_a))_{[a]}$ for every $y_\mu \nleqslant B_a$. Then we have

$$x_\lambda \nleqslant B_a \geqslant \bigvee (Cl(B_a))_{[a]} \geqslant \bigvee (Cl(A))_{[a]}.$$

Thus, we obtain that

$$Cl(\bigvee (Cl(A))_{[a]})(x_\lambda)' = \bigvee_{x_\lambda \nleqslant D \geqslant \bigvee (Cl(A))_{[a]}} \bigwedge_{y_\mu \nleqslant D} ((Cl(D))(y_\mu))' \nleqslant a'.$$

So, $x_\lambda \notin (Cl(\bigvee (Cl(A))_{[a]}))_{[a]}$. This proves that $(Cl(\bigvee (Cl(A))_{[a]}))_{[a]} \subseteq (Cl(A))_{[a]}$.

Necessity. It is obvious that $(Cl(A))(x_\lambda) \leqslant \bigwedge_{x_\lambda \nleqslant B \geqslant A} \bigvee_{y_\mu \nleqslant B} (Cl(B))(y_\mu)$. In order to show that $(Cl(A))(x_\lambda) \geqslant \bigwedge_{x_\lambda \nleqslant B \geqslant A} \bigvee_{y_\mu \nleqslant B} (Cl(B))(y_\mu)$, we only need to show that $((Cl(A))(x_\lambda))' \leqslant \bigvee_{x_\lambda \nleqslant B \geqslant A} \bigwedge_{y_\mu \nleqslant B} ((Cl(B))(y_\mu))'$.

Let $b \in L$ with $((Cl(A))(x_\lambda))' \nleqslant b$. Then there exists $a \in \alpha(b)$ such that $((Cl(A))(x_\lambda))' \nleqslant a$. This implies that $(Cl(A))(x_\lambda) \ngeqslant a'$. By (LFC5), we have

$$x_\lambda \notin (Cl(A))_{[a']} \supseteq (Cl(\bigvee (Cl(A))_{[a']}))_{[a']}.$$

Let $D = \bigvee (Cl(A))_{[a']}$. We will check that $x_\lambda \nleqslant D \geqslant A$.

In fact, if $x_\lambda \leqslant D$, then $x_\gamma \prec x_\lambda \leqslant \bigvee (Cl(A))_{[a']}$ for every $\gamma \prec \lambda$. Further, there exists $x_\nu \in (Cl(A))_{[a']}$ such that $x_\gamma \prec x_\nu$. By (LFC1), we know that $(Cl(A))(x_\gamma) \geqslant$

$(Cl(A))(x_\nu) \geqslant a'$. So, $(Cl(A))(x_\lambda) = \bigwedge_{\gamma \prec \lambda} (Cl(A))(x_\gamma) \geqslant a'$, that contradicts the aforesaid. By (LFC3) and the following fact

$$\forall z_\nu \leqslant A,\ (Cl(A))(z_\nu) = \top \geqslant a' \Rightarrow z_\nu \in (Cl(A))_{[a']} \Rightarrow z_\nu \leqslant D,$$

we can obtain that $A \leqslant D$. Therefore, $x_\lambda \nleqslant D \geqslant A$.

Since

$$(Cl(A))_{[a']} \supseteq (Cl(\bigvee (Cl(A))_{[a']}))_{[a']} = (Cl(D))_{[a']},$$

we have that $((Cl(D))(y_\mu))' \nleqslant a$ for every $y_\mu \nleqslant D$. Therefore, $\bigwedge_{y_\mu \nleqslant D} ((Cl(D))(y_\mu))' \nleqslant b$. This shows that

$$\bigvee_{x_\lambda \nleqslant D \geqslant A} \bigwedge_{y_\mu \nleqslant D} ((Cl(D))(y_\mu))' \nleqslant b.$$

From the arbitrariness of b, we obtain that

$$((Cl(A))(x_\lambda))' \leqslant \bigvee_{x_\lambda \nleqslant B \geqslant A} \bigwedge_{y_\mu \nleqslant B} ((Cl(B))(y_\mu))'.$$

□

Corollary 3.2. *L-fuzzy closure operators in the sense of Shi are precisely the mappings $Cl : L^X \to L^{J(L^X)}$ satisfying the following conditions:*

(LFC2) $(Cl(\underline{\perp}))(x_\lambda) = \perp$ *for every* $x_\lambda \in J(L^X)$;
(LFC3) $(Cl(A))(x_\lambda) = \top$ *for every* $x_\lambda \leqslant A$;
(LFC4) $Cl(A \vee B) = Cl(A) \vee Cl(B)$;
(LFC) $(Cl(A))(x_\lambda) = \bigwedge_{x_\lambda \nleqslant B \geqslant A} \bigvee_{y_\mu \nleqslant B} (Cl(B))(y_\mu)$.

Theorem 3.3. *If a mapping $Int : L^X \to L^{J(L^X)}$ is order-preserving and satisfies* (LFI3), *then* (LFI1) *and* (LFI5) *together are equivalent to the following condition:*
(LFI) $(Int(A))(x_\lambda) = \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (Int(B))(y_\mu)$.

Proof. Sufficiency. (LFI1) By (LFI), we have $(Int(A))(x_\lambda) \leqslant \bigwedge_{\mu \prec \lambda} (Int(A))(x_\mu)$, and

$$\bigwedge_{\mu \prec \lambda} (Int(A))(x_\mu) = \bigwedge_{\mu \prec \lambda} \bigvee_{x_\mu \leqslant B \leqslant A} \bigwedge_{y_\nu \prec B} (Int(B))(y_\nu).$$

Now, we check that $(Int(A))(x_\lambda) \geqslant \bigwedge_{\mu \prec \lambda} (Int(A))(x_\mu)$.

If $\alpha \prec \bigwedge_{\mu \prec \lambda} (Int(A))(x_\mu)$, then $\alpha \prec \bigvee_{x_\mu \leqslant B \leqslant A} \bigwedge_{y_\nu \prec B} (Int(B))(y_\nu)$ for every $\mu \prec \lambda$. Further, there exists $B_\mu \in L^X$ such that $x_\mu \leqslant B_\mu \leqslant A$ and $\alpha \leqslant \bigwedge_{y_\nu \prec B_\mu} (Int(B_\mu))(y_\nu)$. Let $W = \bigvee_{\mu \prec \lambda} B_\mu$. Then we have the following inequality,

$$x_\lambda = \bigvee_{\mu \prec \lambda} x_\mu \leqslant \bigvee_{\mu \prec \lambda} B_\mu = W \leqslant A.$$

This implies that

$$\begin{aligned}\alpha &\leqslant \bigwedge_{\mu\prec\lambda}\bigwedge_{y_\nu\prec B_\mu}(Int(B_\mu))(y_\nu) \leqslant \bigwedge_{\mu\prec\lambda}\bigwedge_{y_\nu\prec B_\mu}(Int(W))(y_\nu)\\ &= \bigwedge_{y_\nu\prec \bigvee\limits_{\mu\prec\lambda} B_\mu}(Int(W))(y_\nu) = \bigwedge_{y_\nu\prec W}(Int(W))(y_\nu)\\ &\leqslant \bigvee_{x_\lambda\leqslant V\leqslant A}\bigwedge_{y_\nu\prec V}(Int(V))(y_\nu) = (Int(A))(x_\lambda).\end{aligned}$$

By the arbitrariness of α, $(Int(A))(x_\lambda) \geqslant \bigwedge\limits_{\mu\prec\lambda}(Int(A))(x_\mu)$ holds.

(LFI5) For every $x_\lambda \in (Int(A))^{(a)}$, we have

$$(Int(A))(x_\lambda) = \bigvee_{x_\lambda\leqslant B\leqslant A}\bigwedge_{y_\mu\prec B}(Int(B))(y_\mu) \nleqslant a.$$

Hence, there exists $U \in L^X$ such that $x_\lambda \leqslant U \leqslant A$ and $\bigwedge\limits_{y_\mu\prec U}(Int(U))(y_\mu) \nleqslant a$. This implies that $y_\mu \in (Int(U))^{(a)}$ for every $y_\mu \prec U$. Since Int is order-preserving, we obtain that

$$x_\lambda \leqslant U = \bigvee\{y_\mu \mid y_\mu \prec U\} \leqslant \bigvee(Int(U))^{(a)} \leqslant \bigvee(Int(A))^{(a)}.$$

Therefore,

$$Int(\bigvee(Int(A))^{(a)})(x_\lambda) = \bigvee_{x_\lambda\leqslant C\leqslant\bigvee(Int(A))^{(a)}}\bigwedge_{y_\mu\prec C}(Int(C))(y_\mu) \nleqslant a.$$

As a result, $x_\lambda \in (Int(\bigvee(Int(A))^{(a)}))^{(a)}$. Therefore, we have that

$$(Int(A))^{(a)} \subseteq (Int(\bigvee(Int(A))^{(a)}))^{(a)}.$$

Necessity. Firstly, we check that $(Int(A))(x_\lambda) \leqslant \bigvee\limits_{x_\lambda\leqslant B\leqslant A}\bigwedge\limits_{y_\mu\prec B}(Int(B))(y_\mu)$.

Let $b \in L$ be such that $(Int(A))(x_\lambda) \nleqslant b$. Then there exists $a \in \alpha(b)$ such that $(Int(A))(x_\lambda) \nleqslant a$. By (LFI5), it follows that

$$x_\lambda \in (Int(A))^{(a)} \subseteq (Int(\bigvee(Int(A))^{(a)}))^{(a)}.$$

Let $V = \bigvee(Int(A))^{(a)}$. Then, by (LFI3), $x_\lambda \leqslant V \leqslant A$ holds obviously. Further, there exists $y_\nu \in (Int(A))^{(a)}$ such that $y_\mu \prec y_\nu$ for every $y_\mu \prec V$. So, we have that

$$y_\nu \in (Int(A))^{(a)} \subseteq (Int(\bigvee(Int(A))^{(a)}))^{(a)} = (Int(V))^{(a)}.$$

This implies that $(Int(V))(y_\nu) \nleqslant a$. By (LFI1), we know that $(Int(V))(y_\mu) \nleqslant a$. Therefore, $\bigwedge\limits_{y_\mu\prec V}(Int(V))(y_\mu) \nleqslant b$. As a consequence, we obtain that

$$\bigvee_{x_\lambda\leqslant B\leqslant A}\bigwedge_{y_\mu\prec B}(Int(B))(y_\mu) \nleqslant b.$$

From the arbitrariness of b, $(Int(A))(x_\lambda) \leqslant \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (Int(B))(y_\mu)$ holds.

Secondly, we check that $(Int(A))(x_\lambda) \geqslant \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (Int(B))(y_\mu)$.

Let $b \in L$ be such that $\bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (Int(B))(y_\mu) \nleqslant b$. Then there exists $a \in \alpha(b)$ such that $\bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (Int(B))(y_\mu) \nleqslant a$. Further, there exists $V \in L^X$ such that $x_\lambda \leqslant V \leqslant A$ and $\bigwedge_{y_\mu \prec V} (Int(V))(y_\mu) \nleqslant a$. Hence, $(Int(V))(x_\nu) \nleqslant a$ for every $\nu \prec \lambda$. By (LFI1), we obtain that

$$(Int(A))(x_\lambda) = \bigwedge_{\nu \prec \lambda} (Int(A))(x_\nu) \geqslant \bigwedge_{\nu \prec \lambda} (Int(V))(x_\nu) \nleqslant b.$$

By the arbitrariness of b, $(Int(A))(x_\lambda) \geqslant \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (Int(B))(y_\mu)$ is proved. $\square$

Corollary 3.4. *L-fuzzy interior operators in the sense of Shi are precisely the mappings $Int : L^X \to L^{J(L^X)}$ satisfying the following conditions:*

(LFI2) $(Int(\top))(x_\lambda) = \top$ *for every* $x_\lambda \in J(L^X)$*;*
(LFI3) $(Int(A))(x_\lambda) = \bot$ *for every* $x_\lambda \nleqslant A$*;*
(LFI4) $Int(A \wedge B) = Int(A) \wedge Int(B)$*;*
(LFI) $(Int(A))(x_\lambda) = \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (Int(B))(y_\mu)$.

4. L-fuzzy Closure Systems Characterized by L-fuzzy Closure Operators

In this section, we will introduce a new definition of L-fuzzy closure operator, which is a generalization of Birkhoff's closure operator and L-fuzzy closure operator in the sense of Definition 2.5. Moreover, the relations between this kind of L-fuzzy closure operators and L-fuzzy closure systems are discussed.

Definition 4.1. An L-fuzzy closure operator on X is a mapping $\mathcal{C} : L^X \to L^{J(L^X)}$ satisfying the following conditions:

(C1) $(\mathcal{C}(A))(x_\lambda) = \top$ for every $x_\lambda \leqslant A$;
(C2) $A \leqslant B \Rightarrow \mathcal{C}(A) \leqslant \mathcal{C}(B)$;
(C3) $(\mathcal{C}(A))(x_\lambda) = \bigwedge_{x_\lambda \nleqslant B \geqslant A} \bigvee_{y_\mu \nleqslant B} (\mathcal{C}(B))(y_\mu)$.

A set X equipped with an L-fuzzy closure operator $\mathcal{C}$, denoted $(X, \mathcal{C})$, is called an L-fuzzy closure space.

A mapping $f : X \to Y$ between two L-fuzzy closure spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ is called continuous if $(\mathcal{C}_X(A))(x_\lambda) \leqslant (\mathcal{C}_Y(f^{\to}(A)))(f(x)_\lambda)$ for every $x_\lambda \in J(L^X)$ and every $A \in L^X$. The category of L-fuzzy closure spaces and continuous mappings is denoted L-**FC**.

The next theorem is obvious.

Theorem 4.2. *A mapping $f : X \to Y$ between two L-fuzzy closure spaces $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ is continuous if and only if* $\forall x_\lambda \in J(L^X),\ \forall B \in L^Y$,

$$(\mathcal{C}_X(f^{\leftarrow}(B)))(x_\lambda) \leqslant (\mathcal{C}_Y(B))(f(x)_\lambda).$$

With the help of an L-fuzzy closure system φ, we can obtain a mapping $\mathcal{C}_\varphi : L^X \to L^{J(L^X)}$, which is defined by

$$\forall x_\lambda \in J(L^X),\ \forall A \in L^X,\ \ (\mathcal{C}_\varphi(A))(x_\lambda) = \bigwedge_{x_\lambda \nleqslant B \geqslant A} \varphi(B)'.$$

Lemma 4.3. *If φ is an L-fuzzy closure system, then $\bigvee_{x_\lambda \nleqslant A} (\mathcal{C}_\varphi(A))(x_\lambda) = \varphi(A)'$ for all $A \in L^X$.*

Proof. In fact, it is enough to prove that $\bigvee_{x_\lambda \nleqslant A} (\mathcal{C}_\varphi(A))(x_\lambda) \geqslant \varphi(A)'$. By the definition of $\mathcal{C}_\varphi$, we have

$$\begin{aligned}
\bigvee_{x_\lambda \nleqslant A} (\mathcal{C}_\varphi(A))(x_\lambda) &= \bigvee_{x_\lambda \nleqslant A} \bigwedge_{x_\lambda \nleqslant B \geqslant A} \varphi(B)' = \Big(\bigwedge_{x_\lambda \nleqslant A} \bigvee_{x_\lambda \nleqslant B \geqslant A} \varphi(B) \Big)' \\
&= \left(\bigvee_{f \in \prod\limits_{x_\lambda \nleqslant A} \mathcal{B}_{x_\lambda}} \bigwedge_{x_\lambda \nleqslant A} \varphi(f(x_\lambda)) \right)' \\
&\geqslant \left(\bigvee_{f \in \prod\limits_{x_\lambda \nleqslant A} \mathcal{B}_{x_\lambda}} \varphi\Big(\bigwedge_{x_\lambda \nleqslant A} f(x_\lambda) \Big) \right)' = \varphi(A)',
\end{aligned}$$

where $\mathcal{B}_{x_\lambda} = \{B \mid x_\lambda \nleqslant B \geqslant A\}$. □

Theorem 4.4. *If φ is an L-fuzzy closure system, then $\mathcal{C}_\varphi$ is an L-fuzzy closure operator.*

Proof. (C1) and (C2) are trivial. By Lemma 4.3, (C3) follows from

$$\bigwedge_{x_\lambda \nleqslant B \geqslant A} \bigvee_{y_\mu \nleqslant B} (\mathcal{C}_\varphi(B))(y_\mu) = \bigwedge_{x_\lambda \nleqslant B \geqslant A} \varphi(B)' = (\mathcal{C}_\varphi(A))(x_\lambda).$$

□

Theorem 4.5. *If $f : (X, \varphi_X) \to (Y, \varphi_Y)$ is continuous with respect to L-fuzzy closure systems φ_X and φ_Y, then $f : (X, \mathcal{C}_{\varphi_X}) \to (Y, \mathcal{C}_{\varphi_Y})$ is continuous with respect to L-fuzzy closure operators $\mathcal{C}_{\varphi_X}$ and $\mathcal{C}_{\varphi_Y}$.*

Proof. Since $f : (X, \varphi_X) \to (Y, \varphi_Y)$ is continuous, it follows that

$$\forall B \in L^Y, \quad \varphi_X(f^{\leftarrow}(B)) \geqslant \varphi_Y(B).$$

This implies that $\forall x_\lambda \in J(L^X)$,

$$\begin{aligned}(\mathcal{C}_{\varphi_Y}(B))(f(x)_\lambda) &= \bigwedge_{f(x)_\lambda \not\leqslant A \geqslant B} \varphi_Y(A)' \\ &\geqslant \bigwedge_{x_\lambda \not\leqslant f^{\leftarrow}(A) \geqslant f^{\leftarrow}(B)} \varphi_X(f^{\leftarrow}(A))' \\ &\geqslant \bigwedge_{x_\lambda \not\leqslant C \geqslant f^{\leftarrow}(B)} \varphi_X(C)' = (\mathcal{C}_{\varphi_X}(f^{\leftarrow}(B)))(x_\lambda).\end{aligned}$$

Therefore, $f : (X, \mathcal{C}_{\varphi_X}) \to (Y, \mathcal{C}_{\varphi_Y})$ is continuous. □

On the one hand, an L-fuzzy closure operator can be induced by an L-fuzzy closure system. On the other hand, an L-fuzzy closure operator induces an L-fuzzy closure system as follows.

Theorem 4.6. *Let $(X, \mathcal{C})$ be an L-fuzzy closure space. Define $\varphi_{\mathcal{C}} : L^X \to L$ by*

$$\forall A \in L^X, \quad \varphi_{\mathcal{C}}(A) = \bigwedge_{x_\lambda \not\leqslant A} ((\mathcal{C}(A))(x_\lambda))'.$$

Then $\varphi_{\mathcal{C}}$ is an L-fuzzy closure system on X.

Proof. It is enough to show that $\varphi_{\mathcal{C}}$ satisfies (LFCS).

$\forall \{A_i\}_{i\in I} \subseteq L^X$, we have that

$$\begin{aligned}\varphi_{\mathcal{C}}\left(\bigwedge_{i\in I} A_i\right) &= \bigwedge_{x_\lambda \not\leqslant \bigwedge_{i\in I} A_i} \left(\mathcal{C}\left(\bigwedge_{i\in I} A_i\right)(x_\lambda)\right)' \\ &= \bigwedge_{i\in I} \bigwedge_{x_\lambda \not\leqslant A_i} \left(\mathcal{C}\left(\bigwedge_{i\in I} A_i\right)(x_\lambda)\right)' \\ &\geqslant \bigwedge_{i\in I} \bigwedge_{x_\lambda \not\leqslant A_i} ((\mathcal{C}(A_i))(x_\lambda))' = \bigwedge_{i\in I} \varphi_{\mathcal{C}}(A_i).\end{aligned}$$

□

Theorem 4.7. *If $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is continuous with respect to L-fuzzy closure operators $\mathcal{C}_X$ and $\mathcal{C}_Y$, then $f : (X, \varphi_{\mathcal{C}_X}) \to (Y, \varphi_{\mathcal{C}_Y})$ is continuous with respect to L-fuzzy closure systems $\varphi_{\mathcal{C}_X}$ and $\varphi_{\mathcal{C}_Y}$.*

Proof. Since $f : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is continuous, it follows that

$$\forall x_\lambda \in J(L^X),\ \forall B \in L^Y, \quad (\mathcal{C}_X(f^{\leftarrow}(B)))(x_\lambda) \leqslant (\mathcal{C}_Y(B))(f(x)_\lambda).$$

This implies that

$$\begin{aligned}\varphi_{\mathcal{C}_X}(f^{\leftarrow}(B)) &= \bigwedge_{x_\lambda \not\leqslant f^{\leftarrow}(B)} ((\mathcal{C}_X(f^{\leftarrow}(B)))(x_\lambda))' \\ &\geqslant \bigwedge_{f(x)_\lambda \not\leqslant B} ((\mathcal{C}_Y(B))(f(x)_\lambda))' \\ &\geqslant \bigwedge_{y_\mu \not\leqslant B} ((\mathcal{C}_Y(B))(y_\mu))' = \varphi_{\mathcal{C}_Y}(B).\end{aligned}$$

Therefore, $f : (X, \varphi_{\mathcal{C}_X}) \to (Y, \varphi_{\mathcal{C}_Y})$ is continuous. □

The following theorem shows that there is a one-to-one correspondence between L-fuzzy closure systems and L-fuzzy closure operators.

Theorem 4.8. *If $\mathcal{C}$ is an L-fuzzy closure operator on X and φ is an L-fuzzy closure system on X, then $\mathcal{C}_{\varphi_{\mathcal{C}}} = \mathcal{C}$ and $\varphi_{\mathcal{C}_\varphi} = \varphi$.*

Proof. By (C3), the equality $\mathcal{C}_{\varphi_{\mathcal{C}}} = \mathcal{C}$ is shown by

$$(\mathcal{C}_{\varphi_{\mathcal{C}}}(A))(x_\lambda) = \bigwedge_{x_\lambda \nleqslant B \geqslant A} \bigvee_{y_\mu \nleqslant B} (\mathcal{C}(B))(y_\mu) = (\mathcal{C}(A))(x_\lambda).$$

By Lemma 4.3, $\varphi_{\mathcal{C}_\varphi} = \varphi$ follows from

$$\varphi_{\mathcal{C}_\varphi}(A) = \bigwedge_{x_\lambda \nleqslant A} ((\mathcal{C}_\varphi(A))(x_\lambda))' = \left(\bigvee_{x_\lambda \nleqslant A} (\mathcal{C}_\varphi(A))(x_\lambda)\right)' = \varphi(A).$$

□

The next result follows from Theorems 4.4, 4.5, 4.6, 4.7 and 4.8.

Theorem 4.9. *The category L-**FCS** is isomorphic to the category L-**FC**.*

5. L-fuzzy Remote Neighborhood Systems and L-fuzzy Quasi-coincident Neighborhood Systems

In this section, we will introduce new definitions of L-fuzzy remote neighborhood system and L-fuzzy quasi-coincident neighborhood system, in order to characterize L-fuzzy closure systems.

Definition 5.1. An L-fuzzy remote neighborhood system on X is a set $\eta = \{\eta_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\eta_{x_\lambda} : L^X \to L$ satisfying the following conditions:

(RN1) $\eta_{x_\lambda}(A) \neq \perp \Rightarrow x_\lambda \nleqslant A$;
(RN2) $A \leqslant B \Rightarrow \eta_{x_\lambda}(A) \geqslant \eta_{x_\lambda}(B)$;
(RN3) $\eta_{x_\lambda}(A) = \bigvee_{x_\lambda \nleqslant B \geqslant A} \bigwedge_{y_\mu \nleqslant B} \eta_{y_\mu}(B)$.

A set X equipped with an L-fuzzy remote neighborhood system $\eta = \{\eta_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, denoted (X, η), is called an L-fuzzy remote neighborhood space.

A mapping $f : X \to Y$ between two L-fuzzy remote neighborhood spaces (X, η_X) and (Y, η_Y) is called continuous if $\forall x_\lambda \in J(L^X),\ \forall B \in L^Y$,

$$(\eta_X)_{x_\lambda}(f^{\leftarrow}(B)) \geqslant (\eta_Y)_{f(x)_\lambda}(B).$$

The category of L-fuzzy remote neighborhood spaces and continuous mappings is denoted L-**FRN**.

Definition 5.2. An L-fuzzy quasi-coincident neighborhood system on X is a set $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{Q}_{x_\lambda} : L^X \to L$ satisfying the following conditions:

(QN1) $\mathcal{Q}_{x_\lambda}(A) \neq \perp \Rightarrow x_\lambda \nleqslant A'$;
(QN2) $A \leqslant B \Rightarrow \mathcal{Q}_{x_\lambda}(A) \leqslant \mathcal{Q}_{x_\lambda}(B)$;
(QN3) $\mathcal{Q}_{x_\lambda}(A) = \bigvee_{x_\lambda \nleqslant B \geqslant A'} \bigwedge_{y_\mu \nleqslant B} \mathcal{Q}_{y_\mu}(B')$.

A set X equipped with an L-fuzzy quasi-coincident neighborhood system $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, denoted $(X, \mathcal{Q})$, is called an L-fuzzy quasi-coincident neighborhood space.

A mapping $f : X \to Y$ between two L-fuzzy quasi-coincident neighborhood spaces $(X, \mathcal{Q}_X)$ and $(Y, \mathcal{Q}_Y)$ is called continuous if $\forall x_\lambda \in J(L^X),\ \forall B \in L^Y$,

$$(\mathcal{Q}_X)_{x_\lambda}(f^{\leftarrow}(B)) \geqslant (\mathcal{Q}_Y)_{f(x)_\lambda}(B).$$

The category of L-fuzzy quasi-coincident neighborhood spaces and continuous mappings is denoted L-**FQN**.

Theorem 5.3. *The category L-**FRN** is isomorphic to the category L-**FQN**.*

Proof. Given an L-fuzzy remote neighborhood space (X, η), define a set $\mathcal{Q}^\eta = \{\mathcal{Q}^\eta_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{Q}^\eta_{x_\lambda} : L^X \to L$ by

$$\forall A \in L^X, \quad \mathcal{Q}^\eta_{x_\lambda}(A) = \eta_{x_\lambda}(A').$$

Then $(X, \mathcal{Q}^\eta)$ is an L-fuzzy quasi-coincident neighborhood space. Similarly, given an L-fuzzy quasi-coincident neighborhood space $(X, \mathcal{Q})$, define a set $\eta^{\mathcal{Q}} = \{\eta^{\mathcal{Q}}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\eta^{\mathcal{Q}}_{x_\lambda} : L^X \to L$ by

$$\forall A \in L^X, \quad \eta^{\mathcal{Q}}_{x_\lambda}(A) = \mathcal{Q}_{x_\lambda}(A').$$

Then $(X, \eta^{\mathcal{Q}})$ is an L-fuzzy remote neighborhood space. Since $'$ is an order-reversing involution on L, it can be easily checked that the category L-**FRN** is isomorphic to the category L-**FQN**. □

Next, we will discuss the relations between L-fuzzy closure systems and L-fuzzy remote neighborhood systems.

Let $\varphi : L^X \to L$ be an L-fuzzy closure system on X. For every $x_\lambda \in J(L^X)$, define a mapping $\eta^\varphi_{x_\lambda} : L^X \to L$ by

$$\eta^\varphi_{x_\lambda}(A) = \bigvee_{x_\lambda \nleqslant B \geqslant A} \varphi(B).$$

Then we have the following lemma.

Lemma 5.4. *If φ is an L-fuzzy closure system on X, then $\varphi(A) = \bigwedge_{x_\lambda \nleqslant A} \eta^\varphi_{x_\lambda}(A)$.*

Proof. It is obvious that $\bigwedge_{x_\lambda \nleqslant A} \eta^\varphi_{x_\lambda}(A) = \bigwedge_{x_\lambda \nleqslant A} \bigvee_{x_\lambda \nleqslant B \geqslant A} \varphi(B) \geqslant \varphi(A)$. Now, we show that $\bigwedge_{x_\lambda \nleqslant A} \eta^\varphi_{x_\lambda}(A) \leqslant \varphi(A)$. By the completely distributive law, we can obtain the following.

$$\begin{aligned}
\bigwedge_{x_\lambda \nleqslant A} \eta^\varphi_{x_\lambda}(A) &= \bigwedge_{x_\lambda \nleqslant A} \bigvee_{x_\lambda \nleqslant B \geqslant A} \varphi(B) \\
&= \bigvee_{f \in \prod_{x_\lambda \nleqslant A} \mathcal{B}_{x_\lambda}} \bigwedge_{x_\lambda \nleqslant A} \varphi(f(x_\lambda)) \\
&\leqslant \bigvee_{f \in \prod_{x_\lambda \nleqslant A} \mathcal{B}_{x_\lambda}} \varphi\left(\bigwedge_{x_\lambda \nleqslant A} f(x_\lambda) \right) \quad = \quad \varphi(A),
\end{aligned}$$

where $\mathcal{B}_{x_\lambda} = \{B \mid x_\lambda \not\leqslant B \geqslant A\}$. $\square$

Theorem 5.5. *If φ is an L-fuzzy closure system on X, then $\eta^\varphi = \{\eta^\varphi_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ is an L-fuzzy remote neighborhood system, which is called the L-fuzzy remote neighborhood system induced by φ.*

Proof. We check that η^φ satisfies (RN1)–(RN3).

(RN1) and (RN2) are trivial.

(RN3) By Lemma 5.4, it follows that

$$\bigvee_{x_\lambda \not\leqslant B \geqslant A} \bigwedge_{y_\mu \not\leqslant B} \eta^\varphi_{y_\mu}(B) = \bigvee_{x_\lambda \not\leqslant B \geqslant A} \varphi(B) = \eta^\varphi_{x_\lambda}(A).$$

$\square$

Theorem 5.6. *If $f : (X, \varphi_X) \to (Y, \varphi_Y)$ is continuous with respect to L-fuzzy closure systems φ_X and φ_Y, then $f : (X, \eta^{\varphi_X}) \to (Y, \eta^{\varphi_Y})$ is continuous with respect to L-fuzzy remote neighborhood systems η^{φ_X} and η^{φ_Y}.*

Proof. Since $f : (X, \varphi_X) \to (Y, \varphi_Y)$ is continuous, it follows that

$$\forall B \in L^Y, \quad \varphi_X(f^\leftarrow(B)) \geqslant \varphi_Y(B).$$

This implies that $\forall x_\lambda \in J(L^X)$,

$$\begin{aligned}
\eta^{\varphi_Y}_{f(x)_\lambda}(B) &= \bigvee_{f(x)_\lambda \not\leqslant C \geqslant B} \varphi_Y(C) \\
&\leqslant \bigvee_{x_\lambda \not\leqslant f^\leftarrow(C) \geqslant f^\leftarrow(B)} \varphi_X(f^\leftarrow(C)) \\
&\leqslant \bigvee_{x_\lambda \not\leqslant A \geqslant f^\leftarrow(B)} \varphi_X(A) \quad = \quad \eta^{\varphi_X}_{x_\lambda}(f^\leftarrow(B)).
\end{aligned}$$

Therefore, $f : (X, \eta^{\varphi_X}) \to (Y, \eta^{\varphi_Y})$ is continuous. $\square$

Conversely, we can construct an L-fuzzy closure system from an L-fuzzy remote neighborhood system as follows.

Theorem 5.7. *Let (X, η) be an L-fuzzy remote neighborhood space. Define $\varphi^\eta : L^X \to L$ by*

$$\forall A \in L^X, \quad \varphi^\eta(A) = \bigwedge_{x_\lambda \not\leqslant A} \eta_{x_\lambda}(A).$$

Then φ^η is an L-fuzzy closure system on X, which is called the L-fuzzy closure system induced by η.

Proof. We check that φ^η satisfies (LFCS) as follows.

$\forall \{A_i\}_{i\in I} \subseteq L^X$, by the definition of φ^η, we have

$$\begin{aligned}
\varphi^\eta\left(\bigwedge_{i\in I} A_i\right) &= \bigwedge_{x_\lambda \not\leqslant \bigwedge_{i\in I} A_i} \eta_{x_\lambda}\left(\bigwedge_{i\in I} A_i\right) \\
&= \bigwedge_{i\in I} \bigwedge_{x_\lambda \not\leqslant A_i} \eta_{x_\lambda}\left(\bigwedge_{i\in I} A_i\right) \\
&\geqslant \bigwedge_{i\in I} \bigwedge_{x_\lambda \not\leqslant A_i} \eta_{x_\lambda}(A_i) \quad = \quad \bigwedge_{i\in I} \varphi^\eta(A_i).
\end{aligned}$$

Hence, φ^η is an L-fuzzy closure system. $\square$

Theorem 5.8. *If $f:(X,\eta_X)\to(Y,\eta_Y)$ is continuous with respect to L-fuzzy remote neighborhood systems η_X and η_Y, then $f:(X,\varphi^{\eta_X})\to(Y,\varphi^{\eta_Y})$ is continuous with respect to L-fuzzy closure systems φ^{η_X} and φ^{η_Y}.*

Proof. Since $f:(X,\eta_X)\to(Y,\eta_Y)$ is continuous, it follows that

$$\forall x_\lambda\in J(L^X),\ \forall B\in L^Y,\quad (\eta_X)_{x_\lambda}(f^\leftarrow(B))\geqslant(\eta_Y)_{f(x)_\lambda}(B).$$

This implies that

$$\begin{aligned}\varphi^{\eta_X}(f^\leftarrow(B)) &= \bigwedge_{x_\lambda\nleqslant f^\leftarrow(B)}(\eta_X)_{x_\lambda}(f^\leftarrow(B))\\ &\geqslant \bigwedge_{f(x)_\lambda\nleqslant B}(\eta_Y)_{f(x)_\lambda}(B)\\ &\geqslant \bigwedge_{y_\mu\nleqslant B}(\eta_Y)_{y_\mu}(B) \ = \ \varphi^{\eta_Y}(B).\end{aligned}$$

Therefore, $f:(X,\varphi^{\eta_X})\to(Y,\varphi^{\eta_Y})$ is continuous. $\square$

The following result shows that there exists a one-to-one correspondence between L-fuzzy closure systems and L-fuzzy remote neighborhood systems.

Theorem 5.9. *If φ is an L-fuzzy closure system on X and η is an L-fuzzy remote neighborhood system on X, then $\varphi^{\eta^\varphi}=\varphi$ and $\eta^{\varphi^\eta}=\eta$.*

Proof. By Lemma 5.4, $\varphi^{\eta^\varphi}=\varphi$ follows from

$$\varphi^{\eta^\varphi}(A)=\bigwedge_{x_\lambda\nleqslant A}\eta^\varphi_{x_\lambda}(A)=\varphi(A).$$

By (RN3), $\eta^{\varphi^\eta}=\eta$ is shown by

$$\eta^{\varphi^\eta}_{x_\lambda}(A)=\bigvee_{x_\lambda\nleqslant B\geqslant A}\varphi^\eta(B)=\bigvee_{x_\lambda\nleqslant B\geqslant A}\ \bigwedge_{y_\mu\nleqslant B}\eta_{y_\mu}(B)=\eta_{x_\lambda}(A).$$

$\square$

The next result follows from Theorems 5.5, 5.6, 5.7, 5.8 and 5.9.

Theorem 5.10. *The category L-**FCS** is isomorphic to the category L-**FRN**.*

6. L-fuzzy Interior Operators and L-fuzzy Neighborhood Systems

In this section, we shall introduce new definitions of L-fuzzy interior operator and L-fuzzy neighborhood system, in order to characterize L-fuzzy closure systems.

Definition 6.1. An L-fuzzy interior operator on X is a mapping $\mathcal{I}:L^X\to L^{J(L^X)}$ satisfying the following conditions:

(I1) $(\mathcal{I}(A))(x_\lambda)=\perp$ for every $x_\lambda\nleqslant A$;
(I2) $A\leqslant B\Rightarrow\mathcal{I}(A)\leqslant\mathcal{I}(B)$;
(I3) $(\mathcal{I}(A))(x_\lambda)=\bigvee_{x_\lambda\leqslant B\leqslant A}\ \bigwedge_{y_\mu\prec B}(\mathcal{I}(B))(y_\mu)$.

A set X equipped with an L-fuzzy interior operator $\mathcal{I}$, denoted $(X,\mathcal{I})$, is called an L-fuzzy interior space.

A mapping $f : X \to Y$ between two L-fuzzy interior spaces $(X,\mathcal{I}_X)$ and $(Y,\mathcal{I}_Y)$ is called continuous if $(\mathcal{I}_X(f^{\leftarrow}(B)))(x_\lambda) \geqslant (\mathcal{I}_Y(B))(f(x)_\lambda)$ for every $x_\lambda \in J(L^X)$ and every $B \in L^Y$. The category of L-fuzzy interior spaces and continuous mappings is denoted L-**FI**.

Definition 6.2. An L-fuzzy neighborhood system on X is a set $\mathcal{N} = \{\mathcal{N}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{N}_{x_\lambda} : L^X \to L$ satisfying the following conditions:

(LN1) $\mathcal{N}_{x_\lambda}(A) = \perp$ for every $x_\lambda \nleqslant A$;
(LN2) $A \leqslant B \Rightarrow \mathcal{N}_{x_\lambda}(A) \leqslant \mathcal{N}_{x_\lambda}(B)$;
(LN3) $\mathcal{N}_{x_\lambda}(A) = \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} N_{y_\mu}(B)$.

A set X equipped with an L-fuzzy neighborhood system $\mathcal{N} = \{\mathcal{N}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$, denoted $(X,\mathcal{N})$, is called an L-fuzzy neighborhood space.

A mapping $f : X \to Y$ between two L-fuzzy neighborhood spaces $(X,\mathcal{N}_X)$ and $(Y,\mathcal{N}_Y)$ is called continuous if $\forall x_\lambda \in J(L^X)$, $\forall B \in L^Y$,

$$(\mathcal{N}_X)_{x_\lambda}(f^{\leftarrow}(B)) \geqslant (\mathcal{N}_Y)_{f(x)_\lambda}(B).$$

The category of L-fuzzy neighborhood spaces and their continuous mappings is denoted L-**FN**.

Theorem 6.3. *The category L-**FI** is isomorphic to the category L-**FN**.*

Proof. Given an L-fuzzy interior space $(X,\mathcal{I})$, define a set $\mathcal{N}^{\mathcal{I}} = \{\mathcal{N}^{\mathcal{I}}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{N}^{\mathcal{I}}_{x_\lambda} : L^X \to L$ by

$$\forall A \in L^X, \quad \mathcal{N}^{\mathcal{I}}_{x_\lambda}(A) = (\mathcal{I}(A))(x_\lambda).$$

By Definition 6.1, $(X,\mathcal{N}^{\mathcal{I}})$ is an L-fuzzy neighborhood space. Similarly, given an L-fuzzy neighborhood space $(X,\mathcal{N})$, define a mapping $\mathcal{I}^{\mathcal{N}} : L^X \to L^{J(L^X)}$ by

$$\forall A \in L^X, \quad (\mathcal{I}^{\mathcal{N}}(A))(x_\lambda) = \mathcal{N}_{x_\lambda}(A).$$

By Definition 6.2, $(X,\mathcal{N}^{\mathcal{I}})$ is an L-fuzzy interior space. Then it can be easily checked that the category L-**FI** is isomorphic to the category L-**FN**. □

Now, we establish a relation between L-fuzzy interior operators and L-fuzzy closure systems.

Theorem 6.4. *For an L-fuzzy interior space $(X,\mathcal{I})$, define $\varphi^{\mathcal{I}} : L^X \to L$ by*

$$\varphi^{\mathcal{I}}(A) = \bigwedge_{x_\lambda \prec A'} (\mathcal{I}(A'))(x_\lambda).$$

Then $\varphi^{\mathcal{I}}$ is an L-fuzzy closure system on X.

Proof. We verify that $\varphi^{\mathcal{I}}$ satisfies (LFCS) as follows.

$\forall \{A_i\}_{i\in I} \subseteq L^X$, we have that

$$\begin{aligned}
\varphi^{\mathcal{I}}\left(\bigwedge_{i\in I} A_i\right) &= \bigwedge_{x_\lambda \prec \left(\bigwedge_{i\in I} A_i\right)'} \Big(\mathcal{I}\Big(\Big(\bigwedge_{i\in I} A_i\Big)'\Big)\Big)(x_\lambda) \\
&= \bigwedge_{x_\lambda \prec \bigvee_{i\in I} A_i'} \Big(\mathcal{I}\Big(\bigvee_{i\in I} A_i'\Big)\Big)(x_\lambda) \\
&= \bigwedge_{i\in I} \bigwedge_{x_\lambda \prec A_i'} \Big(\mathcal{I}\Big(\bigvee_{i\in I} A_i'\Big)\Big)(x_\lambda) \\
&\geqslant \bigwedge_{i\in I} \bigwedge_{x_\lambda \prec A_i'} (\mathcal{I}(A_i'))(x_\lambda) \;=\; \bigwedge_{i\in I} \varphi^{\mathcal{I}}(A_i).
\end{aligned}$$

Thus, $\varphi^{\mathcal{I}}$ is an L-fuzzy closure system on X. □

Theorem 6.5. *If $f : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ is continuous with respect to L-fuzzy interior operators $\mathcal{I}_X$ and $\mathcal{I}_Y$, then $f : (X, \varphi^{\mathcal{I}_X}) \to (Y, \varphi^{\mathcal{I}_Y})$ is continuous with respect to L-fuzzy closure systems $\varphi^{\mathcal{I}_X}$ and $\varphi^{\mathcal{I}_Y}$.*

Proof. Since $f : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ is continuous, it follows that

$$\forall x_\lambda \in J(L^X),\ \forall B \in L^Y, \quad (\mathcal{I}_X(f^{\leftarrow}(B)))(x_\lambda) \geqslant (\mathcal{I}_Y(B))(f(x)_\lambda).$$

This implies that

$$\begin{aligned}
\varphi^{\mathcal{I}_X}(f^{\leftarrow}(B)) &= \bigwedge_{x_\lambda \prec (f^{\leftarrow}(B))'} (\mathcal{I}_X((f^{\leftarrow}(B))'))(x_\lambda) \\
&= \bigwedge_{x_\lambda \prec f^{\leftarrow}(B')} (\mathcal{I}_X(f^{\leftarrow}(B')))(x_\lambda) \\
&\geqslant \bigwedge_{f(x)_\lambda \prec B'} (\mathcal{I}_Y(B'))(f(x)_\lambda) \\
&\geqslant \bigwedge_{y_\mu \prec B'} (\mathcal{I}_Y(B'))(y_\mu) \;=\; \varphi^{\mathcal{I}_Y}(B).
\end{aligned}$$

Therefore, $f : (X, \varphi^{\mathcal{I}_X}) \to (Y, \varphi^{\mathcal{I}_Y})$ is continuous. □

Given a mapping $\varphi : L^X \to L$, define $\mathcal{I}^\varphi : L^X \to L^{J(L^X)}$ by

$$\forall A \in L^X, \ \ x_\lambda \in J(L^X), \ \ (\mathcal{I}^\varphi(A))(x_\lambda) = \bigvee_{x_\lambda \leqslant B \leqslant A} \varphi(B').$$

Then the following lemma holds.

Lemma 6.6. *If φ is an L-fuzzy closure system on X, then $\bigwedge_{x_\lambda \prec A} (\mathcal{I}^\varphi(A))(x_\lambda) = \varphi(A')$ for every $A \in L^X$.*

Proof. It is enough to show that $\bigwedge_{x_\lambda \prec A} (\mathcal{I}^\varphi(A))(x_\lambda) \leqslant \varphi(A')$. By the definition of $\mathcal{I}^\varphi$ and the completely distributive law, we have

$$\begin{aligned}
\bigwedge_{x_\lambda \prec A} (\mathcal{I}^\varphi(A))(x_\lambda) &= \bigwedge_{x_\lambda \prec A} \bigvee_{x_\lambda \leqslant B \leqslant A} \varphi(B') \\
&= \bigvee_{f \in \prod\limits_{x_\lambda \prec A} \mathcal{B}_{x_\lambda}} \bigwedge_{x_\lambda \prec A} \varphi(f(x_\lambda)') \\
&\leqslant \bigvee_{f \in \prod\limits_{x_\lambda \prec A} \mathcal{B}_{x_\lambda}} \varphi\left(\bigwedge_{x_\lambda \prec A} f(x_\lambda)'\right) \\
&= \bigvee_{f \in \prod\limits_{x_\lambda \prec A} \mathcal{B}_{x_\lambda}} \varphi\left(\Big(\bigvee_{x_\lambda \prec A} f(x_\lambda)\Big)'\right) = \varphi(A'),
\end{aligned}$$

where $\mathcal{B}_{x_\lambda} = \{B \mid x_\lambda \leqslant B \leqslant A\}$. □

Theorem 6.7. *If φ is an L-fuzzy closure system on X, then $\mathcal{I}^\varphi$ is an L-fuzzy interior operator on X, which is called the L-fuzzy interior operator induced by φ.*

Proof. (I1)–(I2) are trivial. By Lemma 6.6, (I3) follows from

$$\bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (\mathcal{I}^\varphi(B))(y_\mu) = \bigvee_{x_\lambda \leqslant B \leqslant A} \varphi(B') = (\mathcal{I}^\varphi(A))(x_\lambda).$$

□

Theorem 6.8. *If $f : (X, \varphi_X) \to (Y, \varphi_Y)$ is continuous with respect to L-fuzzy closure systems φ_X and φ_Y, then $f : (X, \mathcal{I}^{\varphi_X}) \to (Y, \mathcal{I}^{\varphi_Y})$ is continuous with respect to L-fuzzy interior operators $\mathcal{I}^{\varphi_X}$ and $\mathcal{I}^{\varphi_Y}$.*

Proof. Since $f : (X, \varphi_X) \to (Y, \varphi_Y)$ is continuous, it follows that

$$\forall B \in L^Y, \quad \varphi_X(f^\leftarrow(B)) \geqslant \varphi_Y(B).$$

This implies that $\forall x_\lambda \in J(L^X)$,

$$\begin{aligned}
(\mathcal{I}^{\varphi_Y}(B))(f(x)_\lambda) &= \bigvee_{f(x)_\lambda \leqslant A \leqslant B} \varphi_Y(A') \\
&\leqslant \bigvee_{x_\lambda \leqslant f^\leftarrow(A) \leqslant f^\leftarrow(B)} \varphi_X(f^\leftarrow(A')) \\
&= \bigvee_{x_\lambda \leqslant f^\leftarrow(A) \leqslant f^\leftarrow(B)} \varphi_X((f^\leftarrow(A))') \\
&\leqslant \bigvee_{x_\lambda \leqslant C \leqslant f^\leftarrow(B)} \varphi_X(C') \quad = \quad (\mathcal{I}^{\varphi_X}(f^\leftarrow(B)))(x_\lambda).
\end{aligned}$$

Therefore, $f : (X, \mathcal{I}^{\varphi_X}) \to (Y, \mathcal{I}^{\varphi_Y})$ is continuous. □

The following result shows that there exists a one-to-one correspondence between L-fuzzy interior operators and L-fuzzy closure systems.

Theorem 6.9. *If $\mathcal{I}$ is an L-fuzzy interior operator on X and φ is an L-fuzzy closure system on X, then $\mathcal{I}^{\varphi^{\mathcal{I}}} = \mathcal{I}$ and $\varphi^{\mathcal{I}^{\varphi}} = \varphi$.*

Proof. By (I3), $\mathcal{I}^{\varphi^{\mathcal{I}}} = \mathcal{I}$ is shown by

$$(\mathcal{I}^{\varphi^{\mathcal{I}}}(A))(x_\lambda) = \bigvee_{x_\lambda \leqslant B \leqslant A} \varphi^{\mathcal{I}}(B') = \bigvee_{x_\lambda \leqslant B \leqslant A} \bigwedge_{y_\mu \prec B} (\mathcal{I}(B))(y_\mu) = (\mathcal{I}(A))(x_\lambda).$$

By Lemma 6.6, $\varphi^{\mathcal{I}^{\varphi}} = \varphi$ follows from

$$\varphi^{\mathcal{I}^{\varphi}}(A) = \bigwedge_{x_\lambda \prec A'} (\mathcal{I}^{\varphi}(A'))(x_\lambda) = \varphi(A).$$

□

The next results follow from Theorems 4.9, 5.3, 5.10, 6.3, 6.4, 6.5, 6.7, 6.8 and 6.9.

Theorem 6.10. *The category L-**FCS** is isomorphic to the category L-**FI**.*

Corollary 6.11. *The categories L-**FCS**, L-**FC**, L-**FRN**, L-**FQN**, L-**FI**, L-**FN** are isomorphic.*

7. Conclusion

In this paper, new characterizations of L-fuzzy closure operator and L-fuzzy interior operator in the sense of Shi [25] are provided. Also, the definitions of L-fuzzy closure operator, L-fuzzy interior operator, L-fuzzy remote neighborhood system, L-fuzzy neighborhood system and L-fuzzy quasi-coincident neighborhood system are generalized, and it is shown that the respective categories are isomorphic to the category of L-fuzzy closure system spaces.

In fact, the notion of L-fuzzy closure system can be regarded as a generalization of L-fuzzy pretopology [11]. Thus, L-fuzzy pretopologies can also be characterized by means of the analogues of L-fuzzy closure operator, L-fuzzy interior operator, L-fuzzy remote neighborhood system, L-fuzzy neighborhood system and L-fuzzy quasi-coincident neighborhood system.

It is well known that the concept of (L, M)-fuzzy topology [18] is a generalization of L-fuzzy topology (for $M = L$). Motivated by this, we can generalize L-fuzzy closure system to (L, M)-fuzzy closure system. Analogously the other notions can also be generalized, and we can obtain the same results as those in this paper.

The lattice-valued approach in this paper is fixed-basis [14], we think that this approach can be generalized to the variable-basis setting [23], which can be the subject of our future research.

Acknowledgements. The authors would like to thank the anonymous reviewers for their valuable comments. This paper is supported by the National Natural Science Foundation of China (11371002)(10971242).

References

[1] R. Bělohlávek, *Fuzzy closure operators*, J. Math. Anal. Appl., **262** (2001), 473–489.
[2] L. Biacino and G. Gerla, *Closure systems and L-subalgebras*, Information Sciences, **33** (1984), 181–195.

[3] L. Biacino and G. Gerla, *An extension principle for closure operators*, J. Math. Anal. Appl., **198** (1996), 1–24.

[4] G. Birkhoff, *Lattice theory*, 3rd Edition, Amer. Math. Soc., Rhode Island, 1967.

[5] M. K. Chakraborty and J. Sen, *MV-algebras embedded in a CL-algebra*, Int. J. Approximate Reasoning, **18** (1998), 217–229.

[6] K. C. Chattopadhyay and S. K. Samanta, *Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness*, Fuzzy Sets and Systems, **54** (1993), 207–212.

[7] J. M. Fang, *Categories isomorphic to L-* **FTOP**, Fuzzy Sets and Systems, **157** (2006), 820–831.

[8] J. M. Fang and Y. L. Yue, *L-fuzzy closure systems*, Fuzzy Sets and Systems, **161** (2010), 1242–1252.

[9] G. Gerla, *An extension principle for fuzzy logics*, Math. Logic Quart., **40** (1994), 357–380.

[10] G. Gerla, *Graded consequence relations and fuzzy closure operators*, J. Appl. Non-Classical Logics, **6** (1996), 369–379.

[11] M. H. Ghanim, O. A. Tantawy and F. M. Selin, *Gradation of supra-openness*, Fuzzy Sets and Systems, **109** (2000), 245–250.

[12] R. Goetschel and W. Voxman, *Spanning properties for fuzzy matroids*, Fuzzy Sets and Systems, **51** (1992), 313–321.

[13] U. Höhle and S. E. Rodabaugh, et al., (eds), *Mathematics of fuzzy sets: logic, topology and measure theory*, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Boston, Dordrecht, London, **3** (1999).

[14] U. Höhle and A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, In [13], 123–173.

[15] Y. C. Kim, *Initial L-fuzzy closure spaces*, Fuzzy Sets and Systems, **133** (2003), 277–297.

[16] Y. C. Kim and J. M. Ko, *Fuzzy closure systems and fuzzy closure operators*, Commun. Korean Math. Soc., **19(1)** (2004), 35–51.

[17] T. Kubiak, *On fuzzy topologies*, Ph. D. Thesis, Adam Mickiewicz, Poznan, Poland, 1985.

[18] T. Kubiak and A. P. Šostak, *Foundations of the theory of (L, M)-fuzzy topological spaces*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory (U. Bodenhofer, B. De Bates, E. P. Klement, and S. Saminger-Platz, eds.), Johannes Kepler University, Linz, (2009), 70–73.

[19] S. G. Li, X. Xin and Y. L. Li, *Closure axioms for a class of fuzzy matroids and co-towers of matroids*, Fuzzy Sets and Systems, **158** (2007), 1246–1257.

[20] X. L. Luo and J. M. Fang, *Fuzzifying closure systems and closure operators*, Iranian Journal of Fuzzy Systems, **8(1)** (2011), 77–94.

[21] A. S. Mashhour and M. H. Ghanim, *Fuzzy closure spaces*, J. Math. Anal. Appl., **106** (1985), 154–170.

[22] S. E. Rodabaugh, *Powerset operator foundations for poslat fuzzy set theories and topologies*, in [13], 91–116.

[23] S. E. Rodabaugh, *Categorical foundations of variable-basis fuzzy topology*, in [13], 273–388.

[24] S. E. Rodabaugh, *Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics*, Int. J. Math. Math. Sci., **2007(3)** (2007), 1–71.

[25] F. G. Shi, *L-fuzzy interiors and L-fuzzy closures*, Fuzzy Sets and Systems, **160** (2009), 1218–1232.

[26] F. G. Shi, *Regularity and normality of (L, M)-fuzzy topological spaces*, Fuzzy Sets and Systems, **182** (2011), 37–52.

[27] A. P. Šostak, *On a fuzzy topological structure*, Rend. Circ. Mat. Palermo, (Supp. Ser. II), **11** (1985), 83–103 .

[28] R. Srivastava, A. K. Srivastava and A. Choubey, *Fuzzy closure spaces*, J. Fuzzy Math., **2** (1994), 525–534.

[29] G. J. Wang, *Theory of topological molecular lattices*, Fuzzy Sets and Systems, **47(3)** (1992), 351–376.

[30] L. Wang and F. G. Shi, *Charaterization of L-fuzzifying matroids by L-fuzzifying closure operators*, Iranian Journal of Fuzzy Systems, **7** (2010), 47–58.

[31] Y. L. Yue and J. M. Fang, *Categories isomorphic to the Kubiak-Šostak extension of TML*, Fuzzy Sets and Systems, **157** (2006), 832–842.

[32] W. Zhou, *Generalization of L-closure spaces*, Fuzzy Sets and Systems, **149** (2005), 415–432.

Fu-Gui Shi*, School of Mathematics, Beijing Institute of Technology, 5 South Zhongguancun Street, Haidian District, 100081 Beijing, P.R. China

E-mail address: fuguishi@bit.edu.cn

Bin Pang, School of Mathematics, Beijing Institute of Technology, 5 South Zhongguancun Street, Haidian District, 100081 Beijing, P.R. China

E-mail address: pangbin1205@163.com

*Corresponding author