SOME (FUZZY) TOPOLOGIES ON GENERAL FUZZY AUTOMATA

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Abstract. In this paper, by presenting some notions and theorems, we obtain different types of fuzzy topologies. In fact, we obtain some Lowen-type and Chang-type fuzzy topologies on general fuzzy automata. To this end, first we define a Kuratowski fuzzy interior operator which induces a Lowen-type fuzzy topology on the set of states of a max- min general fuzzy automaton. Also by proving some theorems, we can define two fuzzy closure (two fuzzy interior) operators on the certain sets related to a general fuzzy automaton and then according to these notions we give some theorems and obtain some different Chang-type fuzzy topologies.

1. Introduction and preliminaries

The theory of fuzzy sets was introduced by Zadeh [22]. Wee [20] introduced the idea of fuzzy automata. Automata have a long history both in theory and application [1, 2]. Automata are the prime example of general computational systems over discrete spaces [8]. Among the conventional spectrum of automata (i.e. deterministic finite-state automata, non-deterministic finite-state automata, probabilistic automata and fuzzy finite-state automata), deterministic finite-state automata have been the most applied automata to different areas [3, 14, 15]. See [18] for more applications. Fuzzy automata not only provide a systematic approach to handle uncertainty in such systems, but also are able to handle continuous spaces [19]. In general, fuzzy automata provide an attractive systematic way for generalizing discrete applications [5]. Moreover, fuzzy automata are able to create capabilities which are hardly achievable by other tools [21].

Let X be a set. A word on X is the product of a finite sequence of elements in X, Λ is empty word and X^* is the set of all words on X. In fact, X^* is the free monoid on X. For a nonempty set X, $\tilde{\mathrm{P}}(\mathrm{X})$ denots the set of all fuzzy subsets on X and $P(X)$ denots the set of all subsets on X.

A deterministic finite-state automaton is a five-tuple denoted as $A = (Q, X, f, T, s)$, where Q is a finite set of states, X is a finite set of input symbols, the function f from $Q \times X$ into Q is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.

A word $x = x_1 x_2 \dots x_n \in X^*$ is said to be accepted by A if there exist states $q_0, q_1, \dots, q_n$ satisfying

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(1) $q_0 = s$
(2) $f(q_{i-1}, x_i) = q_i$ for $i = 1, 2, \ldots, n$,
(3) $q_n \in T$.

The empty word is accepted by A if and only if $s \in T$.
A nondeterministic finite-state automaton is a five-tuple denoted as $A = (Q, X, f, T, s)$, where Q is a finite set of states, X is a finite set of input symbols, the function f from $Q \times X$ into $P(Q)$ is the state transition, T is a subset of Q of accepting states and $s \in Q$ is the initial state.
A fuzzy finite-state automaton (FFA) is a six-tuple denoted as $\tilde{\mathrm{F}} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the start state of $\tilde{\mathrm{F}}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \longrightarrow [0, 1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval [0,1] and $\omega : Q \longrightarrow Z$ is the output function. In an FFA, as can be seen, associated with each fuzzy transition, there is a membership value in [0,1]. We call this membership value the weight of the transition. The transition from state q_i (current state) to state q_j(next state) upon input a_k is denoted as $\delta(q_i, a_k, q_j)$. We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition, otherwise, it specifies the transition itself. Also, the set of all transitions of $\tilde{\mathrm{F}}$ is denoted as Δ.
The above definition is generally accepted as a formal definition for FFA [13, 16, 17]. There is an important problem which should be clarified in the definition of FFA. It is the assignment of membership values to the next states. There are two issues within state membership assignment. The first one is how to assign a membership value to a next state upon the completion of a transition. Secondly, how should we deal with the cases where a state is forced to take several membership values simultaneously via overlapping transition?
In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and gave the notion of general fuzzy automata [7]. Then we followed it in [23] and now we follow [7,23] and give some new notions and results as mentioned in the abstract.

Definition 1.1. [7] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted as $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where
(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \subset \tilde{P}(Q)$,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \longrightarrow Z$ is the output function,
(vi) $\tilde{\delta} : (Q \times [0, 1]) \times \Sigma \times Q \longrightarrow [0, 1]$ is the augmented transition function,
(vii) $F_1 : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is called membership assignment function.
Function $F_1(\mu, \delta)$ as is seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition.

In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)),$$

which means that the membership value (mv) of the state q_j at time $t+1$ is computed by function F_1 by using both the membership value of q_i at time t and the weight of the transition.
There are many options which can be used for the function $F_1(\mu, \delta)$, for example $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ or $(\mu + \delta)/2$.

(viii) $F_2 : [0,1]^* \longrightarrow [0,1]$ is called multi-membership resolution function.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.
Let $Q_{act}(t_i)$ be the set of all active states at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$, $Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}),\ \ \exists a \in \Sigma,\ \delta(q', a, q) \in \Delta\},\ \ \forall i \geq 1.$
Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$. Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.
The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j will lead to the multi-membership resolution algorithm.

Algorithm 1.2. [7] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to that:

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i,

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they will be processed by another function F_2, called the multi-membership resolution function.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = \overset{n}{\underset{i=1}{F_2}}[v_i] = \overset{n}{\underset{i=1}{F_2}}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

Where

- n : is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$: is the weight of the transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$: is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$: is the final membership value of q_j at time $t+1$.

Example 1.3. [7] Consider the GFA in Fig.1 with several transition overlaps. It is specified as: $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where $Q = \{q_0, q_1, q_2, q_3, q_4\}$ is the set of states, $\Sigma = \{a, b\}$ is the set of input symbols, $Z = \emptyset$ and ω is not applicable, $\tilde{R} = Q_{act}(t_0) = \{(q_0, \mu^{t_0}(q_0))\} = \{(q_0, 1)\}$, $v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))$ and

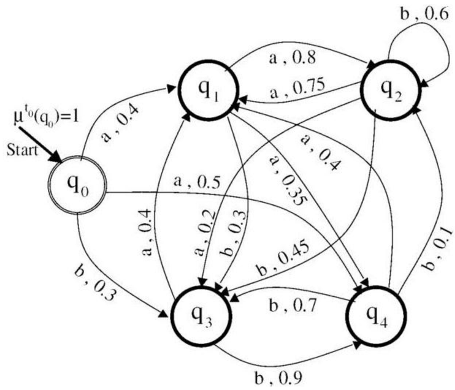

FIGURE 1. The GFA of Example 1.3

$$^1F_1(\mu,\delta)=\delta,\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{t+1}(q_m)=\bigwedge_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))),$$

$$^2F_1(\mu,\delta)=\min(\mu,\delta),\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{t+1}(q_m)=\bigwedge_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))),$$

$$^3F_1(\mu,\delta)=\min(\mu,\delta),\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{t+1}(q_m)=\bigvee_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))),$$

$$^4F_1(\mu,\delta)=\max(\mu,\delta),\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{t+1}(q_m)=\bigwedge_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))),$$

$$^5F_1(\mu,\delta)=\max(\mu,\delta),\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{t+1}(q_m)=\bigvee_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))),$$

$$^6F_1(\mu,\delta)=\min(\mu,\delta),\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{i+1}(q_m)=\sum_{i=1}^{n}F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))/n,$$

$$^7F_1(\mu,\delta)=\frac{\mu+\delta}{2},\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{t+1}(q_m)=\bigvee_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))),$$

Where n is the number of simultaneous transitions to the active state q_m at time $t+1$. If we choose

$$^1F_1(\mu,\delta)=\delta,\ \overset{n}{\underset{i=1}{F_2}}[v_i]=\mu^{t+1}(q_m)=\bigwedge_{i=1}^{n}(F_1(\mu^t(q_i),\delta(q_1,a_k,q_m))),$$

then we have :

$$\begin{aligned}
\mu^{t_0}(q_0) &= 1, \mu^{t_1}(q_1)=F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_1))=F_1(1,0.4)=0.4,\\
\mu^{t_1}(q_4) &= F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_4))=F_1(1,0.5)=0.5,\\
\mu^{t_2}(q_1) &= F_1(\mu^{t_1}(q_4),\delta(q_4,a,q_1))=F_1(0.5,0.4)=0.4,\\
\mu^{t_2}(q_2) &= F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_2))=F_1(0.4,0.8)=0.8,\\
\mu^{t_2}(q_4) &= F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_4))=F_1(0.4,0.35)=0.35,\\
\mu^{t_3}(q_2) &= F_1(\mu^{t_2}(q_4),\delta(q_4,b,q_2))\wedge F_1(\mu^{t_2}(q_2),\delta(q_2,b,q_2))\\
&= F_1(0.4,0.1)\wedge F_1(0.8,0.6)=0.1\wedge 0.6=0.1,\\
\mu^{t_3}(q_3) &= F_1(\mu^{t_2}(q_1),\delta(q_1,b,q_3))\wedge F_1(\mu^{t_2}(q_2),\delta(q_2,b,q_3))\\
&\quad \wedge F_1(\mu^{t_2}(q_4),\delta(q_4,b,q_3))\\
&= F_1(0.4,0.3)\wedge F_1(0.8,0.45)\wedge F_1(0.35,0.7)=0.3\wedge 0.45\wedge 0.7=0.3,
\end{aligned}$$

which there are two simultaneous transitions to the active state q_2 at time t_3 and there are three simultaneous transitions to the active state q_3 at time t_3. So we can draw the Table 1.
The operation of this fuzzy automaton upon input string a^2b^2 is shown in Table 1 for different membership assignment functions and multi-membership resolution strategies. In this table, we have considered different cases for combining functions F_1 and F_2.

time	t_0	t_1		t_2			t_3		t_4		
input	Λ	a		a			b		b		
$Q_{act}(t_i)$	q_0	q_1	q_4	q_1	q_2	q_4	q_2	q_3	q_2	q_3	q_4
mv^1	1.0	0.4	0.5	0.4	0.8	0.35	0.1	0.3	0.6	0.45	0.9
mv^2	1.0	0.4	0.5	0.4	0.4	0.35	0.1	0.3	0.1	0.1	0.3
mv^3	1.0	0.4	0.5	0.4	0.4	0.35	0.4	0.4	0.4	0.4	0.4
mv^4	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^5	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
mv^6	1.0	0.4	0.5	0.4	0.4	0.35	0.25	0.35	0.25	0.25	0.35
mv^7	1.0	0.7	0.75	0.575	0.75	0.525	0.763	0.613	0.682	0.607	0.756

Table 1. Active States and Their Membership Values (mv) at Different Times in Example 1.3

Definition 1.4. [23] Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton, which is defined in Definition 1.1. We defined max-min general fuzzy automata of the form:

$$\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$$

such that :

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \longrightarrow [0, 1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and let for every i, $i \geq 0$

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p \\ 0, & otherwise, \end{cases}$$

and for every i, $i \geq 1$

$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p),$$
$$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i u_{i+1}, p) = \bigvee_{q' \in Q_{act}(t_i)} (\tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, q')$$
$$\wedge \tilde{\delta}((q', \mu^{t_i}(q')), u_{i+1}, p)),$$

and recursively

$$\tilde{\delta}^*((q, \mu^{t_0}(q)), u_1 u_2 \dots u_n, p) = \vee\{\tilde{\delta}((q, \mu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}((p_1, \mu^{t_1}(p_1)), u_2, p_2)$$
$$\wedge \dots \wedge \tilde{\delta}((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), u_n, p) | p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \dots, p_{n-1}$$
$$\in Q_{act}(t_{n-1})\},$$

in which $u_i \in \Sigma$, $\forall 1 \leq i \leq n$ and assuming that the entered input at time t_i be u_i, $\forall 1 \leq i \leq n-1$.

Definition 1.5. [23] Let $\tilde{F}^*$ be a max-min general fuzzy automaton. The response function $r^{\tilde{F}^*} : \Sigma^* \times Q \to [0,1]$ of $\tilde{F}^*$, for any $x \in \Sigma^*$, $q \in Q$, is defined by

$$r^{\tilde{F}^*}(x,q) = \bigvee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, q).$$

Definition 1.6. [23] Let $q \in Q$ and $0 \leq c < 1$. Then q is called an accessible state of $\tilde{F}^*$ with threshold c if there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x,q) > c$.

Definition 1.7. [23] Let $A \subseteq Q$. Then $\tilde{F}^*$ is said to be connected with threshold c on A, if $A = \bar{Q}_c$, where $\bar{Q}_c$ is the set of all accessible states with threshold c.

Definition 1.8. [12] Let X be an arbitrary set. The function $\psi : \tilde{P}(X) \longrightarrow \tilde{P}(X)$ is called a Kuratowski fuzzy interior operator if for any two elements λ and γ of $\tilde{P}(X)$, we have

(i) $\psi(k) = k$, $\forall k$ constant,
(ii)$\psi(\lambda) \leq \lambda$,
(iii)$\psi(\lambda \bigwedge \gamma) = \psi(\lambda) \bigwedge \psi(\gamma)$,
(iv) $\psi(\psi(\lambda)) = \psi(\lambda)$.

Definition 1.9. [12] A subset τ of $\tilde{P}(X)$ is called a Lowen -type fuzzy topology on X if

(i) $\forall k$ constant, $k \in \tau$,
(ii) If λ_1, $\lambda_2 \in \tau$, then $\lambda_1 \bigwedge \lambda_2 \in \tau$,
(iii) If $\lambda_i \in \tau$, $\forall i \in I$, then $\bigvee_{i \in I} \lambda_i \in \tau$.

Remark 1.10. [12] Let τ be a Lowen-type fuzzy topology on a set X. Then for any $\alpha \in [0,1)$, $\ell_\alpha(\tau) = \{\lambda^{-1}(\alpha, 1] : \lambda \in \tau\}$ is a topology on X, referred to as the α-level topology of τ.

Remark 1.11. [12] Let ψ be a Kuratowski fuzzy interior operator.Then ψ induces a Lowen-type fuzzy topology of the form $\tau(\psi) = \{\lambda : \psi(\lambda) = \lambda\}$.

Definition 1.12. [11] Let X be an arbitrary set. The function $\psi : P(X) \longrightarrow P(X)$ is called a closure operator on X, if for any two elements A and B of $P(X)$, we have

(i) $\psi(\emptyset) = \emptyset$,
(ii) $A \subseteq \psi(A)$,
(iii) $\psi(A \bigcup B) = \psi(A) \bigcup \psi(B)$,
(iv) $\psi(\psi(A)) = \psi(A)$.

Definition 1.13. [11] Let X be an arbitrary set. The function $\psi : \tilde{P}(X) \longrightarrow \tilde{P}(X)$ is called a fuzzy closure operator if for any two elements λ and γ of $\tilde{P}(X)$, we have

(i) $\psi(0) = 0$,
(ii) $\psi(\lambda) \geq \lambda$,
(iii) $\psi(\lambda \bigvee \gamma) = \psi(\lambda) \bigvee \psi(\gamma)$,
(iv) $\psi(\psi(\lambda)) = \psi(\lambda)$.

Also, a fuzzy closure operator ψ is called saturation if for any family $\{\lambda_i\}_{i \in I}$ of elements of $\tilde{P}(X)$, we have $\psi(\bigvee_{i \in I} \lambda_i) = \bigvee_{i \in I} \psi(\lambda_i)$.

Definition 1.14. [11] Let X be an arbitrary set. The function $\psi : \tilde{P}(X) \longrightarrow \tilde{P}(X)$ is called a fuzzy interior operator if for any two elements λ and γ of $\tilde{P}(X)$, we have

(i) $\psi(1) = 1$,
(ii) $\psi(\lambda) \leq \lambda$,
(iii) $\psi(\lambda \bigwedge \gamma) = \psi(\lambda) \bigwedge \psi(\gamma)$,
(iv) $\psi(\psi(\lambda)) = \psi(\lambda)$.

Definition 1.15. [4] A subset τ of $\tilde{P}(X)$ is called a Chang-type fuzzy topology on X if

(i) $0, 1 \in \tau$,
(ii) If λ_1, $\lambda_2 \in \tau$, then $\lambda_1 \bigwedge \lambda_2 \in \tau$,
(iii) If $\lambda_i \in \tau$, $\forall i \in I$, then $\bigvee_{i \in I} \lambda_i \in \tau$.

Remark 1.16. [12] Let ψ be a fuzzy closure operator. Then ψ induces a Chang-type fuzzy topology of the form $\tau(\psi) = \{\lambda^C : \psi(\lambda) = \lambda\}$, where $\lambda^C = 1 - \lambda$.
Let ψ be a fuzzy interior operator. Then ψ induces a Chang-type fuzzy topology of the form $\tau(\psi) = \{\lambda : \psi(\lambda) = \lambda\}$.

2. Some Lowen-type Fuzzy Topologies on General Fuzzy Automata

Theorem 2.1. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and for any $p \in Q$, consider $\bar{D}(\lambda)(p) = \bigvee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$. Define*

$$\bar{S}_\lambda : P(Q) \longrightarrow P(Q)$$
$$A \longrightarrow \bar{S}_\lambda(A)$$

where

$$\bar{S}_\lambda(q) = \{p \in Q : \bar{D}(\lambda)(p) = \bar{D}(\lambda)(q)\}, \bar{S}_\lambda(A) = \bigcup_{q \in A} \bar{S}_\lambda(q).$$

Then $\bar{S}_\lambda$ is a closure operator on Q.

Proof. (i) $\bar{S}_\lambda(\emptyset) = \emptyset$ is obvious.
(ii) Let $q \in A$. Since $q \in \bar{S}_\lambda(q)$ and $\bar{S}_\lambda(q) \subseteq \bar{S}_\lambda(A)$, we get that $A \subseteq \bar{S}_\lambda(A)$.
(iii) $\bar{S}_\lambda(A \bigcup B) = \bigcup_{q \in A \bigcup B} \bar{S}_\lambda(q) = (\bigcup_{q \in A} \bar{S}_\lambda(q)) \bigcup (\bigcup_{q \in B} \bar{S}_\lambda(q)) = \bar{S}_\lambda(A) \bigcup \bar{S}_\lambda(B)$.
(iv) By (ii), we have $\bar{S}_\lambda(A) \subseteq \bar{S}_\lambda(\bar{S}_\lambda(A))$. Conversely, let $q \in \bar{S}_\lambda(\bar{S}_\lambda(A))$. Then there exists $q' \in \bar{S}_\lambda(A)$ such that $q \in \bar{S}_\lambda(q')$. Thus $q' \in \bar{S}_\lambda(q'')$, for some $q'' \in A$. Consequently, we have

$$\bar{D}(\lambda)(q) = \bar{D}(\lambda)(q'), \bar{D}(\lambda)(q') = \bar{D}(\lambda)(q'').$$

So $\bar{D}(\lambda)(q) = \bar{D}(\lambda)(q'')$. Hence, $q \in \bar{S}_\lambda(q'') \subseteq \bar{S}_\lambda(A)$. Therefore $\bar{S}_\lambda(\bar{S}_\lambda(A)) \subseteq \bar{S}_\lambda(A)$. □

Theorem 2.2. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $\bar{S}_\lambda(A) = \bigcup_{q \in A} \bar{S}_\lambda(q)$. Then $\bar{T}(Q) = \{A^C : A \subseteq Q, \bar{S}_\lambda(A) = A\}$ is a topology on Q.*

Proof. (i) Since $\bar{S}_\lambda(\emptyset) = \emptyset$, then $Q = (\emptyset)^C \in \bar{T}(Q)$.
(ii) Since $\bar{S}_\lambda$ is a closure operator on Q, we have $Q \subseteq \bar{S}_\lambda(Q)$. On the other hand, since $\bar{S}_\lambda(Q) \in P(Q)$, we have $\bar{S}_\lambda(Q) \subseteq Q$. Thus, $\bar{S}_\lambda(Q) = Q$. Therefore we conclude that $\emptyset = (Q)^C \in \bar{T}(Q)$.
(iii) Let A_1^C and A_2^C belong to $\bar{T}(Q)$. Then $\bar{S}_\lambda(A_1) = A_1$ and $\bar{S}_\lambda(A_2) = A_2$. Thus, we have

$$\bar{S}_\lambda(A_1 \bigcup A_2) = \bar{S}_\lambda(A_1) \bigcup \bar{S}_\lambda(A_2) = A_1 \bigcup A_1.$$

That is, $A_1^C \bigcap A_2^C = (A_1 \bigcup A_2)^C \in \bar{T}(Q)$.
(iv) Let $A_i^C \in \bar{T}(Q)$, $\forall i \in I$. Then $\bar{S}_\lambda(A_i) = A_i$, $\forall i \in I$. Since $\bar{S}_\lambda$ is a closure operator on Q, we have $\bigcap_{i\in I} A_i \subseteq \bar{S}_\lambda(\bigcap_{i\in I} A_i)$. On the other hand, since $A_i \bigcup (\bigcap_{i\in I} A_i) = A_i$, we get that $\bar{S}_\lambda(A_i) \bigcup (\bar{S}_\lambda(\bigcap_{i\in I} A_i)) = \bar{S}_\lambda(A_i)$. Then $\bar{S}_\lambda(\bigcap_{i\in I} A_i) \subseteq \bar{S}_\lambda(A_i) = A_i$. Thus $\bar{S}_\lambda(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} A_i$. Hence, $\bar{S}_\lambda(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} A_i$. That is, $\bigcup_{i\in I} A_i^C = (\bigcap_{i\in I} A_i)^C \in \bar{T}(Q)$. □

Theorem 2.3. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton. Define*

$$C : \tilde{P}(Q) \longrightarrow \tilde{P}(Q)$$
$$\lambda \longrightarrow C(\lambda)$$

where

$$C(\lambda)(p) = \bigwedge\{\lambda(q) : q \in \bar{S}_\lambda(p)\}.$$

Then C is a Kuratowski fuzzy interior operator.

Proof. (i) $C(k) = k$, $\forall k$ constant.
(ii) Since $p \in \bar{S}_\lambda(p) = \{q \in Q : \bar{D}(\lambda)(q) = \bar{D}(\lambda)(p)\}$, we have

$$C(\lambda)(p) = \bigwedge\{\lambda(q) : q \in \bar{S}_\lambda(p)\} \leq \lambda(p).$$

(iii) $C(\lambda_1 \bigwedge \lambda_2)(p) = \bigwedge\{\lambda_1(q) \bigwedge \lambda_2(q) : q \in \bar{S}_\lambda(p)\} = (\bigwedge\{\lambda_1(q) : q \in \bar{S}_\lambda(p)\}) \bigwedge (\bigwedge\{\lambda_2(q) : q \in \bar{S}_\lambda(p)\}) = C(\lambda_1(p)) \bigwedge C(\lambda_2)(p)$.
(iv) By (ii), we have $C(C(\lambda))(p) \leq C(\lambda)(p)$. For the reverse inequality, we have $C(C(\lambda))(p) = \bigwedge\{C(\lambda)(q) : q \in \bar{S}_\lambda(p)\} = \bigwedge\{\bigwedge\{\lambda(r) : r \in \bar{S}_\lambda(q)\} : q \in \bar{S}_\lambda(p)\}$. Since for any $r \in \bar{S}_\lambda(q)$ and $q \in \bar{S}_\lambda(p)$, we have $\bar{D}(\lambda)(r) = \bar{D}(\lambda)(q)$ and $\bar{D}(\lambda)(q) = \bar{D}(\lambda)(p)$. Then $\bar{D}(\lambda)(r) = \bar{D}(\lambda)(p)$. Thus $r \in \bar{S}_\lambda(p)$. Therefore we have $C(C(\lambda))(p) \geq \bigwedge\{\lambda(r) : r \in \bar{S}_\lambda(p)\} = C(\lambda)(p)$. Consequently, $C(C(\lambda)) = C(\lambda)$. □

Now, by Remark 1.11, we conclude that :

Corollary 2.4. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automata and $\tau(C) = \{\lambda \in \tilde{P}(Q) : C(\lambda) = \lambda\}$. Then $\tau(C)$ is a Lowen-type fuzzy topology on Q.*

Theorem 2.5. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton. Then*

$$\ell_\alpha(\tau(C)) = \{A : A \subseteq Q, A^C \in \bar{T}(Q)\}, \forall \alpha \in [0, 1).$$

Proof. Let $\alpha \in [0, 1)$. By Remark 1.10, we have :
$\ell_\alpha(\tau(C)) = \{\lambda^{-1}(\alpha, 1] : \lambda \in \tau(C)\} = \{\lambda^{-1}(\alpha, 1] : \lambda \in \tilde{P}(Q) : C(\lambda) = \lambda\}$.
Let $A = \lambda^{-1}(\alpha, 1] \in \ell_\alpha(\tau(C))$. Then $C(\lambda) = \lambda$. Since $\bar{S}_\lambda$ is a closure operator on Q, thus $A \subseteq \bar{S}_\lambda(A)$. On the other hand, let $p \in \bar{S}_\lambda(A)$. Then $p \in \bar{S}_\lambda(q)$, for some $q \in A$. Thus $\bar{D}(\lambda)(p) = \bar{D}(\lambda)(q)$ and $\lambda(q) > \alpha$. So $q \in \bar{S}_\lambda(p)$. Now, we have $\lambda(p) = C(\lambda)(p) = \bigwedge\{\lambda(q) : q \in \bar{S}_\lambda(p)\} > \alpha$. Hence, $p \in A$. Therefore $\bar{S}_\lambda(A) \subseteq A$. So we get that $A^C \in \bar{T}(Q)$. Consequently, $\ell_\alpha(\tau(C)) \subseteq \{A : A \subseteq Q, A^C \in \bar{T}(Q)\}$. Conversely, let $A^C \in \bar{T}(Q)$. Then $\bar{S}_\lambda(A) = A$. To prove $A \in \ell_\alpha(\tau(C))$, it is enough to show that
(i) $A = 1_A^{-1}(\alpha, 1]$,
(ii) $C(1_A) = 1_A$.
(i) is true. To show part (ii), let $p \in Q$. If $C(1_A)(p) = 1$, then (ii) follows obviously. If $C(1_A)(p) < 1$, then there exists $q \in \bar{S}_\lambda(p)$ such that $1_A(q) < 1$. Thus $q \notin A = \bar{S}_\lambda(A)$. Since $q \in \bar{S}_\lambda(p)$, then $p \notin A$. Hence $1_A(p) = 0$. Therefore $C(1_A)(p) = \bigwedge\{1_A(q) : q \in \bar{S}_\lambda(p)\} \geq 1_A(p)$. On the other hand, since C is a Kuratowski fuzzy interior operator, we have $C(1_A)(p) \leq 1_A(p)$. Consequently, $C(1_A) = 1_A$. Thus $\ell_\alpha(\tau(C)) \supseteq \{A : A \subseteq Q, A^C \in \bar{T}(Q)\}$. □

Definition 2.6. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $p \in Q, q \in Q_{act}(t_i), i \geq 0$ and $0 \leq c < 1$. Then p is called a successor of q with threshold c if there exists $x \in \Sigma^*$ such that $\tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) > c$.

Definition 2.7. Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $q \in Q_{act}(t_i), i \geq 0$ and $0 \leq c < 1$. Also let $S_c(q)$ denote the set of all successors of q with threshold c. If $T \subseteq Q$, then $S_c(T)$ the set of all successors of T with threshold c is defined by

$$S_c(T) = \bigcup\{S_c(q) : q \in T\}.$$

Theorem 2.8. *Let $\tilde{F}^*$ be a max-min general fuzzy automaton and $0 \leq c < 1$. Then*
(i) $q \in S_c(q), \forall q \in Q$.
(ii) If $r \in S_c(p)$, $p \in S_c(q)$, then $r \in S_c(q)$.

Proof. (i) Since for all $q \in Q$, there exists $i \geq 0$ such that $q \in Q_{act}(t_i)$ and $\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, q) = 1 > c$, then $q \in S_c(q)$.
(ii) Since $p \in S_c(q)$, then $q \in Q_{act}(t_i)$ and there exists $x \in \Sigma^*$ such that $\tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) > c$. Also, $r \in S_c(p)$ implies $p \in Q_{act}(t_j)$ and there exists $y \in \Sigma^*$ such that $\tilde{\delta}^*((p, \mu^{t_j}(p)), y, r) > c$. Thus, we have

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), xy, r) = \bigvee_{q' \in Q_{act}(t_j)}[\tilde{\delta}^*((q, \mu^{t_i}(q)), x, q') \bigwedge \tilde{\delta}^*((q', \mu^{t_j}(q')), y, r)] \geq \tilde{\delta}^*((q, \mu^{t_i}(q)), x, p) \bigwedge \tilde{\delta}^*((p, \mu^{t_j}(p)), y, r) > c.$$

So we get that $r \in S_c(q)$. □

Example 2.9. Let $\tilde{F}^*$ be a max-min general fuzzy automaton in Example 1.3. If we choose [5] $F_1(\mu, \delta) = \max(\mu, \delta)$, $\overset{n}{\underset{i=1}{F_2}}[v_i] = \mu^{t+1}(q_m) = \bigvee_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_1, a_k, q_m)))$, then we have :

$$\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_1)=\bigvee_{q'\in Q_{act}(t_1)}[\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a,q')$$
$$\bigwedge\tilde{\delta}^*((q',\mu^{t_1}(q')),a,q_1)]=[\tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_1)$$
$$\bigwedge\tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q_1)]\bigvee[\tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_4)$$
$$\bigwedge\tilde{\delta}((q_4,\mu^{t_1}(q_4)),a,q_1)]=[F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_1))$$
$$\bigwedge F_1(\mu^{t_1}(q_1),\delta(q_1,a,q_1))]\bigvee[F_1(\mu^{t_0}(q_0),\delta(q_0,a,q_4))$$
$$\bigwedge F_1(\mu^{t_1}(q_4),\delta(q_4,a,q_1))]$$
$$=[F_1(1,0.4)\bigwedge F_1(1,0)]\bigvee[F_1(1,0.5)\bigwedge F_1(1,0.4)]$$
$$=[1\bigwedge 1]\bigvee[1\bigwedge 1]=1\bigvee 1=1.$$

Thus $\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_1)=1$. Similarly, we have :
$\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_2)=1$,
$\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_3)=1$,
$\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),a^2,q_4)=1$.
Also, we have $\tilde{\delta}^*((q_0,\mu^{t_0}(q_0)),\Lambda,q_0)=1$. Thus $S_c(q_0)=Q$, $\forall c$, $0\le c<1$.

Theorem 2.10. *Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $0\le c<1$ and*

$$S_c:P(Q)\longrightarrow P(Q)$$
$$A\longrightarrow S_c(A).$$

Then S_c is a closure operator on Q.

Proof. (i) $S_c(\emptyset)=\emptyset$.
(ii) Let $q\in A\subseteq Q$. By Theorem 2.8, $q\in S_c(q)$. Then $q\in S_c(q)\subseteq S_c(A)$, Thus, $A\subseteq S_c(A)$.
(iii)$S_c(A\bigcup B)=\bigcup_{q\in A\bigcup B}S_c(q)=(\bigcup_{q\in A}S_c(q))\bigcup(\bigcup_{q\in B}S_c(q))=S_c(A)\bigcup S_c(B)$.
(iv) By (ii), we have $S_c(A)\subseteq S_c(S_c(A))$. Conversely, let $p\in S_c(S_c(A))$. Then there exists $q'\in S_c(A)$ such that $p\in S_c(q')$. Thus $q'\in S_c(q'')$, for some $q''\in A$. Consequently, by Theorem 2.8, $p\in S_c(q'')$. Hence, $p\in S_c(A)$. Therefore $S_c(S_c(A))\subseteq S_c(A)$. □

Theorem 2.11. *Let $\tilde{F}^*$ be a max-min general fuzzy automaton. Then $\tau=\{A^C: A\subseteq Q, S_c(A)=A\}$ is a topology on Q.*

Proof. The proof is similar to that of Theorem 2.2, by using suitable modification. □

Definition 2.12. Let $\tilde{F}^*$ be a max-min general fuzzy automaton and $0\le c<1$. Then we say that $\tilde{F}^*$ is good with threshold c, if $\forall q\in Q$, $\exists q'\in Q_{act}(t_0): q\in S_c(q')$.

Example 2.13. Let $\tilde{F}^*$ be the max-min general fuzzy automaton in Example 2. 9 and $0\le c<1$. Since $S_c(q_0)=Q$, then $\tilde{F}^*$ is good with threshold c.

Theorem 2.14. *Let $\tilde{F}^*$ be a max-min general fuzzy automaton and $0 \le c < 1$. Then $\tilde{F}^*$ is good with threshold c if and only if $\tilde{F}^*$ is connected with threshold c on Q.*

Proof. Let $\tilde{F}^*$ be good with threshold c and $q \in Q$. Then there exists $q' \in Q_{act}(t_0)$ such that $q \in S_c(q')$. Thus, there exists $x \in \Sigma^*$ such that $\tilde{\delta}^*((q', \mu^{t_0}(q')), x, q) > c$. So

$$r^{\tilde{F}^*}(x, q) = \bigvee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, q) > c.$$

Consequently, by Definitions 1.6 and 1.7, $\tilde{F}^*$ is connected with threshold c on Q. Conversely, let $\tilde{F}^*$ be connected with threshold c on Q and $q \in Q$. Then there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x, q) = \bigvee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, q) > c$.

Hence, there exists $q' \in Q_{act}(t_0)$ such that $\tilde{\delta}^*((q', \mu^{t_0}(q')), x, q) > c$. So $q \in S_c(q')$. Therefore $\tilde{F}^*$ is good with threshold c. □

Theorem 2.15. *Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $0 \le c < 1$ and suppose that*

$$pR_cq \Leftrightarrow p \in S_c(q), q \in S_c(p).$$

Then R_c is an equivalence relation on Q.

Proof. By Theorem 2.8 the proof is obvious. □

Theorem 2.16. *Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $0 \le c < 1$, $A \subseteq Q$, $B_c(q) = \{p \in Q : pR_cq\}$ and $B_c(A) = \bigcup_{q \in A} B_c(q)$. Define*

$$B_c : P(Q) \longrightarrow P(Q)$$
$$A \longrightarrow B_c(A).$$

Then B_c is a closure operator on Q.

Proof. (i) $B_c(\emptyset) = \emptyset$.

(ii) Let $q \in A$. Since qR_cq, then $q \in B_c(q) \subseteq B_c(A)$. Thus, $A \subseteq B_c(A)$.

(iii)$B_c(A \bigcup D) = \bigcup_{q \in A \bigcup D} B_c(q) = (\bigcup_{q \in A} B_c(q)) \bigcup (\bigcup_{q \in D} B_c(q)) = B_c(A) \bigcup B_c(D)$.

(iv) By (ii), we have $B_c(A) \subseteq B_c(B_c(A))$. Conversely, let $q \in B_c(B_c(A))$. Then there exists $q' \in B_c(A)$ such that $q \in B_c(q')$. Thus $q' \in B_c(q'')$, for some $q'' \in A$. Consequently, qR_cq' and $q'R_cq''$. By Theorem 2.15, qR_cq''. Thus $q \in B_c(q'') \subseteq B_c(A)$. Therefore $B_c(B_c(A)) \subseteq B_c(A)$. □

Theorem 2.17. *Let $\tilde{F}^*$ be a max-min general fuzzy automaton, $0 \le c < 1$ and $p, q \in Q$. Then $p \in B_c(q)$ if and only if $B_c(p) = B_c(q)$.*

Proof. Let $p \in B_c(q)$. If $q' \in B_c(q)$, then $q'R_cp$ and pR_cq. Thus, $q'R_cq$. Therefore we get that $q' \in B_c(q)$. So $B_c(p) \subseteq B_c(q)$. On the other hand, if $q' \in B_c(q)$, then $q'R_cq$ and pR_cq. Thus, $q'R_cp$. So we get that $q' \in B_c(q)$. Consequently, $B_c(q) \subseteq B_c(p)$. Conversely, let $B_c(p) = B_c(q)$. Since pR_cp, then $p \in B_c(p)$. Thus, $p \in B_c(q)$. □

3. Some Chang-type Fuzzy Topologies on General Fuzzy Automata

Theorem 3.1. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton and λ be a fuzzy subset on Q. Define*

$$\bar{D} : \tilde{P}(Q_{act}(t_0)) \longrightarrow \tilde{P}(Q_{act}(t_0))$$
$$\lambda \longrightarrow \bar{D}(\lambda)$$

where

$$\bar{D}(\lambda)(p) = \vee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$$

and

$$r^{\tilde{F}^*}(x, p) = \vee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, p).$$

Then $\bar{D}$ is a saturation fuzzy closure operator.

Proof. (i) $\bar{D}(0) = 0$ is obvious.

(ii) $\bar{D}(\lambda)(p) = \bigvee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} \geq \lambda(p) \wedge r^{\tilde{F}^*}(\Lambda, p) = \lambda(p) \wedge 1 = \lambda(p)$.

(iii)$\bar{D}(\bigvee_{i\in I} \lambda_i)(p) = \bigvee\{(\bigvee_{i\in I} \lambda_i(p)) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$

$$= \bigvee_{i\in I}(\bigvee\{\lambda_i(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}) = \bigvee_{i\in I} \bar{D}(\lambda_i)(p).$$

(iv) By (ii), we have $\bar{D}(\bar{D}(\lambda))(p) \geq \bar{D}(\lambda)(p)$. For the reverse inequality, let x, x' be fixed and $x \neq x'$. Then we have

$\lambda(p) \wedge r^{\tilde{F}^*}(x', p) \wedge r^{\tilde{F}^*}(x, p) \leq \lambda(p) \wedge r^{\tilde{F}^*}(x, p)$

$$\leq \bigvee\{\lambda(p) \wedge r^{\tilde{F}^*}(z, p) : z \in \Sigma^*\} = \bar{D}(\lambda)(p).$$

Then

$\bar{D}(\bar{D}(\lambda))(p) = \bigvee\{\bar{D}(\lambda)(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$

$$= \bigvee_{x\in\Sigma^*} \bigvee_{x'\in\Sigma^*} \{\lambda(p) \wedge r^{\tilde{F}^*}(x', p) \wedge r^{\tilde{F}^*}(x, p)\} \leq \bar{D}(\lambda)(p).$$

Thus, $\bar{D}(\bar{D}(\lambda)) = \bar{D}(\lambda)$.

Now, by Remark 1.16, we conclude that: □

Corollary 3.2. *In Theorem 3.1, let $T_1 = \{\lambda \in \tilde{P}(Q_{act}(t_0)) : \bar{D}(\lambda) = \lambda\}$. Then $\tau(\bar{D}) = \{\lambda^C : \lambda \in T_1\}$ is a Chang-type fuzzy topology on $Q_{act}(t_0)$.*

Theorem 3.3. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $0 \leq c < 1$. Then $\bar{D}(\lambda) > c$ if and only if $\tilde{F}^*$ is connected with threshold c on $Q_{act}(t_0)$, and $\lambda(p) > c$, $\forall p \in Q_{act}(t_0)$.*

Proof. Let $\tilde{F}^*$ be connected with threshold c on $Q_{act}(t_0)$ and $\lambda(p) > c$, $\forall p \in Q_{act}(t_0)$. Then for any $p \in Q_{act}(t_0)$, there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x, p) > c$. Thus $\lambda(p) \bigwedge r^{\tilde{F}^*}(x, p) > c$ which implies that

$$\bar{D}(\lambda)(p) = \bigvee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} > c.$$

Conversely, let $\bar{D}(\lambda) > c$. Then we have

$$\bar{D}(\lambda)(p) = \bigvee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} > c.$$

Thus, for $p \in Q_{act}(t_0)$, there exists $x \in \Sigma^*$ such that $r^{\tilde{F}^*}(x, p) > c$ and $\lambda(p) > c$. So $\tilde{F}^*$ is connected with threshold c on $Q_{act}(t_0)$ and $\lambda(p) > c$, $\forall p \in Q_{act}(t_0)$. □

Definition 3.4. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $A \subseteq Q$. Then we say that λ is normal on A if

$$\lambda(p) \leq r^{\tilde{F}^*}(x, p), \forall p \in A, \forall x \in \Sigma^*.$$

Theorem 3.5. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q. If λ is normal on $Q_{act}(t_0)$, then $\lambda^C \in \tau(\bar{D})$.*

Proof. Let λ be normal on $Q_{act}(t_0)$. Then we have

$$\lambda(p) \leq r^{\tilde{F}^*}(x, p), \forall p \in Q_{act}(t_0), \forall x \in \Sigma^*$$
$$\Rightarrow \lambda(p) \bigwedge r^{\tilde{F}^*}(x, p) = \lambda(p), \forall x \in \Sigma^*$$
$$\Rightarrow \bar{D}(\lambda)(p) = \bigvee\{\lambda(p) \wedge r^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} = \lambda(p)$$
$$\Rightarrow \bar{D}(\lambda) = \lambda.$$

Therefore $\lambda^C \in \tau(\bar{D})$. □

Theorem 3.6. *Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $Q' = Q - Q_{act}(t_0)$. Suppose*

$$D : \tilde{P}(Q') \longrightarrow \tilde{P}(Q')$$
$$\lambda \longrightarrow D(\lambda)$$

where

$$D(\lambda)(p) = \bigwedge\{\lambda(p) \bigvee R^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$$

and

$$R^{\tilde{F}^*}(x, p) = \bigwedge_{q \in Q_{act}(t_0)} \tilde{\delta}^*((q, \mu^{t_0}(q)), x, p).$$

Then D is a fuzzy interior operator.

Proof. (i) $D(1) = 1$ is obvious.
(ii) $D(\lambda)(p) = \wedge\{\lambda(p) \vee R^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} \leq \lambda(p) \bigvee R^{\tilde{F}^*}(\Lambda, p) = \lambda(p) \bigvee 0 = \lambda(p)$.
(iii)$D(\lambda_1 \wedge \lambda_2)(p) = \wedge\{(\lambda_1(p) \wedge \lambda_2(p)) \vee R^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$

$$= (\wedge\{(\lambda_1(p) \vee R^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}) \wedge (\wedge\{(\lambda_2(p) \vee R^{\tilde{F}^*}(x, p) : x \in \Sigma^*\})$$
$$= D(\lambda_1)(p) \wedge D(\lambda_2)(p).$$

(iv) By (ii), we have $D(D(\lambda))(p) \leq D(\lambda)(p)$.
For the reverse inequality, let x, x' be fixed and $x \neq x'$. Then we have

$$\lambda(p) \bigvee R^{\tilde{F}^*}(x', p) \bigvee R^{\tilde{F}^*}(x, p) \geq \lambda(p) \bigvee R^{\tilde{F}^*}(x, p)$$
$$\geq \wedge\{\lambda(p) \vee R^{\tilde{F}^*}(z, p) : z \in \Sigma^*\} = D(\lambda)(p).$$

Then

$$D(D(\lambda))(p) = \wedge\{D(\lambda)(p) \vee R^{\tilde{F}^*}(x, p) : x \in \Sigma^*\}$$
$$= \wedge_{x \in \Sigma^*} \wedge_{x' \in \Sigma^*} \{\lambda(p) \vee R^{\tilde{F}^*}(x', p) \vee R^{\tilde{F}^*}(x, p)\} \geq D(\lambda)(p).$$

Thus, $D(D(\lambda)) = D(\lambda)$. □

Now, by Remark 1.16, we conclude that:

Corollary 3.7. *In Theorem 3.6, let* $\tau(D) = \{\lambda \in \tilde{P}(Q') : D(\lambda) = \lambda\}$*. Then* $\tau(D)$ *is a Chang-type fuzzy topology on* Q'*.*

Definition 3.8. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, λ be a fuzzy subset on Q and $A \subseteq Q$. Then we say that λ is subnormal on A if

$$\lambda(p) \geq R^{\tilde{F}^*}(x, p), \forall p \in A, \forall x \in \Sigma^*.$$

Theorem 3.9. *Let* $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* λ *be a fuzzy subset on* Q *and* $Q' = Q - Q_{act}(t_0)$*. If* λ *is subnormal on* Q'*, then* $\lambda \in \tau(D)$*.*

Proof. Let λ be subnormal on Q'. Then we have

$$\lambda(p) \geq R^{\tilde{F}^*}(x, p), \forall p \in Q', \forall x \in \Sigma^*$$

$$\Rightarrow \lambda(p) \vee R^{\tilde{F}^*}(x, p) = \lambda(p), \forall x \in \Sigma^*$$

$$\Rightarrow D(\lambda)(p) = \wedge\{\lambda(p) \vee R^{\tilde{F}^*}(x, p) : x \in \Sigma^*\} = \lambda(p)$$

$$\Rightarrow D(\lambda) = \lambda.$$

Therefore $\lambda \in \tau(D)$. □

Theorem 3.10. *Let* $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* λ *be a fuzzy subset on* Σ^* *and* $\Sigma' = \{x \in \Sigma^* : r^{\tilde{F}^*}(x, p) = 1, \forall p \in Q_{act}(t_0)\}$*. Consider*

$$\bar{B} : \tilde{P}(\Sigma') \longrightarrow \tilde{P}(\Sigma')$$
$$\lambda \longrightarrow \bar{B}(\lambda)$$

where

$$\bar{B}(\lambda)(x) = \vee\{\lambda(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\}$$

and

$$r^{\tilde{F}^*}(x, p) = \vee_{q' \in Q_{act}(t_0)} \tilde{\delta}^*((q', \mu^{t_0}(q')), x, p).$$

Then $\bar{B}$ *is a saturation fuzzy closure operator.*

Proof. (i) $\bar{B}(0) = 0$ is obvious.
(ii) $\bar{B}(\lambda)(x) = \vee\{\lambda(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\} \geq \lambda(x) \wedge r^{\tilde{F}^*}(\Lambda, p_0) = \lambda(x) \wedge 1 = \lambda(x)$, where $p_0 \in Q_{act}(t_0)$.
(iii)$\bar{B}(\vee_{i \in I}\lambda_i)(x) = \vee\{(\vee_{i \in I}\lambda_i(x)) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\}$

$$= \vee_{i \in I}(\vee\{\lambda_i(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\}) = \vee_{i \in I}\bar{B}(\lambda_i)(x).$$

(iv) By (ii), we have $\bar{B}(\bar{B}(\lambda))(x) \geq \bar{B}(\lambda)(x)$. For the reverse inequality, let $p, q \in Q$ be fixed and $p \neq q$. Then we have

$$\lambda(x) \wedge r^{\tilde{F}^*}(x, q) \wedge r^{\tilde{F}^*}(x, p) \leq \lambda(x) \wedge r^{\tilde{F}^*}(x, p)$$

$$\leq \vee\{\lambda(x) \wedge r^{\tilde{F}^*}(x, r) : r \in Q\} = \bar{B}(\lambda)(x).$$

Then

$$\bar{B}(\bar{B}(\lambda))(x) = \vee\{\bar{B}(\lambda)(x) \wedge r^{\tilde{F}^*}(x, p) : p \in Q\}$$

$$= \vee_{p \in Q} \vee_{q \in Q} \{\lambda(x) \wedge r^{\tilde{F}^*}(x, q) \wedge r^{\tilde{F}^*}(x, p)\} \leq \bar{B}(\lambda)(x).$$

Thus, $\bar{B}(\bar{B}(\lambda)) = \bar{B}(\lambda)$.
Now, by Remark 1.16, we conclude that: □

Corollary 3.11. *In Theorem 3.10, let* $T_2 = \{\lambda \in \tilde{P}(\Sigma') : \bar{B}(\lambda) = \lambda\}$. *Then* $\tau(\bar{B}) = \{\lambda^C : \lambda \in T_2\}$ *is a Chang-type fuzzy topology on* Σ'.

Theorem 3.12. *Let* $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ *be a max-min general fuzzy automaton,* λ *be a fuzzy subset on* Σ^*, $Q' = Q - Q_{act}(t_0)$ *and* $\Sigma'' = \{x \in \Sigma^* : R^{\tilde{F}^*}(x, p) = 0, \forall p \in Q'\}$. *Define*

$$B : \tilde{P}(\Sigma'') \longrightarrow \tilde{P}(\Sigma'')$$
$$\lambda \longrightarrow B(\lambda)$$

where

$$B(\lambda)(x) = \wedge\{\lambda(x) \vee R^{\tilde{F}^*}(x, p) : p \in Q\}$$

and

$$R^{\tilde{F}^*}(x, p) = \wedge_{q \in Q_{act}(t_0)} \tilde{\delta}^*((q, \mu^{t_0}(q)), x, p).$$

Then B is a fuzzy interior operator.

Proof. (i) $B(1) = 1$ is obvious.

(ii) $B(\lambda)(x) = \wedge\{\lambda(x) \vee R^{\tilde{F}^*}(x, p) : p \in Q\} \leq \lambda(x) \bigvee R^{\tilde{F}^*}(x, p_0) = \lambda(x) \vee 0 = \lambda(x)$, where $p_0 \in Q'$.

(iii) $B(\lambda_1 \wedge \lambda_2)(x) = \wedge\{(\lambda_1(x) \wedge \lambda_2(x)) \vee R^{\tilde{F}^*}(x, p) : p \in Q\}$

$$= (\wedge\{(\lambda_1(x) \vee R^{\tilde{F}^*}(x, p) : p \in Q\}) \wedge (\wedge\{(\lambda_2(x) \vee R^{\tilde{F}^*}(x, p) : p \in Q\})$$
$$= B(\lambda_1)(x) \wedge B(\lambda_2)(x).$$

(iv) By (ii), we have $B(B(\lambda))(p) \leq B(\lambda)(p)$.

For the reverse inequality, let $p, q \in Q$ be fixed and $p \neq q$. Then we have

$$\lambda(x) \vee R^{\tilde{F}^*}(x, p) \vee R^{\tilde{F}^*}(x, q) \geq \lambda(x) \vee R^{\tilde{F}^*}(x, p)$$
$$\geq \wedge\{\lambda(x) \vee R^{\tilde{F}^*}(x, r) : r \in Q\} = B(\lambda)(x).$$

Then

$$B(B(\lambda))(x) = \wedge\{B(\lambda)(x) \vee R^{\tilde{F}^*}(x, p) : p \in Q\}$$
$$= \wedge_{p \in Q} \wedge_{q \in Q} \{\lambda(x) \vee R^{\tilde{F}^*}(x, p) \vee R^{\tilde{F}^*}(x, q)\} \geq B(\lambda)(x).$$

Thus, $B(B(\lambda)) = B(\lambda)$. □

Now, by Remark 1.16, we conclude that

Corollary 3.13. *In Theorem 3.12, let* $\tau(B) = \{\lambda \in \tilde{P}(\Sigma'') : B(\lambda) = \lambda\}$. *Then* $\tau(B)$ *is a Chang-type fuzzy topology on* Σ''.

4. Conclusions

As it is shown, in this manuscript by considering (in fact proving) some fuzzy interior (closure) operators, we have introduced some Lowen-type and Chang-type fuzzy topology structures on general fuzzy automata. So these results open a way to study deeply the properties of these structures in future, for example to discuss open sets, closed sets, Hausdorff spaces and compact spaces.

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