MEASURES OF FUZZY SEMICOMPACTNESS IN L-FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, the notion of fuzzy semicompactness degrees is introduced in L-fuzzy topological spaces by means of the implication operation of L. Characterizations of fuzzy semicompactness degrees in L-fuzzy topological spaces are obtained, and some properties of fuzzy semicompactness degrees are researched.

1. Introduction

In 1968, Chang [3] introduced fuzzy theory into topology. In Chang's fuzzy topology, open sets are fuzzy, but the topology comprising those open sets is a crisp subset of I-power set I^X . The notion of Chang's fuzzy topology was extended by Goguen to L-topology. Later many other authors gave more results and notions concerning fuzzy topology (see [1], [2], [5]- [9], [11]- [34]).

Especially, the notion of fuzzy compactness degrees and L-fuzzy semicompactness are introduced in L-fuzzy topological spaces in [14] and [29], respectively. Based on the idea of [14], a natural problem is: Can the degrees of fuzzy semicompactness be defined in an L-fuzzy topological space?

The aim of this paper is to present a new notion of fuzzy semicompactness degrees in L-fuzzy topological spaces by means of the implication operation of L. We will also characterize the fuzzy semicompactness degrees in L-fuzzy topological spaces, and research some properties of fuzzy semicompactness degrees.

2. Preliminaries

Throughout this paper, $(L, \bigvee, \bigwedge, \prime)$ is a complete DeMorgan frame (i.e., a complete lattice with order-reversing involution satisfying joint-infinite distributive law) [10, 21]. By $\perp$ and $\top$ we denote the smallest element and the largest element in L, respectively. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero coprime elements in L is denoted by $M(L)$. We say that a is wedge below b in L, denoted by $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$ [4]. $\beta(b)$ is the greatest minimal family of b, $\beta^*(b) = \beta(b) \cap M(L)$. $\alpha(b)$ is the greatest maximal family of b, $\alpha^*(b) = \alpha(b) \cap P(L)$.

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In a complete DeMorgan frame L, there exists a binary operation $\to$. Explicitly the implication is given by $a \to b = \bigvee\{c \in L : a \wedge c \leq b\}$.

It is easy to check the following properties of $\to$.

(1) $(a \to b) \geq c \Leftrightarrow a \wedge c \leq b$.
(2) $a \to b = \top \Leftrightarrow a \leq b$.
(3) $a \to \bigwedge_i b_i = \bigwedge_i (a \to b_i)$.
(4) $(\bigvee_i a_i) \to b = \bigwedge_i (a_i \to b)$.
(5) $a \leq b \Rightarrow a \to c \geq b \to c,\ \ c \to a \leq c \to b$.
(6) $(a \wedge b) \to c = a \to (b \to c)$.

We interpret $[a \leq b]$ as the degree to which $a \leq b$, then $[a \leq b] = a \to b$.

Let X be a nonempty set, L^X the set of all L-subsets on X, $\underline{\perp}$ the smallest element of L^X and $\underline{\top}$ the largest element of L^X. Then, L-fuzzy topology on a set X is a mapping $\mathcal{T} : L^X \longrightarrow L$ which satisfies the following conditions:

(1) $\mathcal{T}(\underline{\top}) = \mathcal{T}(\underline{\perp}) = \top$.
(2) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V) \quad (\forall\, U, V \in L^X)$.
(3) $\mathcal{T}\left(\bigvee_{j \in J} U_j\right) \geq \bigwedge_{j \in J} \mathcal{T}(U_j) \quad (\forall\, \{U_j\}_{j \in J} \subseteq L^X)$.

The pair $(X, \mathcal{T})$ is called an L-fuzzy topological space. $\mathcal{T}(U)$ is called the degree of openness of U, $\mathcal{T}^*(U) = \mathcal{T}(U')$ is called the degree of closedness of U, where U' is the L-complement of U. For any family $\mathcal{U} \subseteq L^X$, $\mathcal{T}(\mathcal{U}) = \bigwedge_{A \in \mathcal{U}} \mathcal{T}(A)$ is called the degree of openness of $\mathcal{U}$.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ. For any $a \in L$, $\underline{a}$ denotes a constant value mapping from X to L, its value is a.

Definition 2.1. [31] An L-fuzzy inclusion on X is a mapping $\widetilde{\subset} : L^X \times L^X \longrightarrow L$ defined by the equality $\widetilde{\subset}(A, B) = \bigwedge\limits_{x \in X} (A'(x) \vee B(x))$.

In this paper, we will write $[A \widetilde{\subset} B]$ instead of $\widetilde{\subset}(A, B)$.

Definition 2.2. [25] Let $a \in L \backslash \{\top\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ in L^X is said to be

(1) an a-shading of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee\limits_{A \in \mathcal{U}} A(x) \nleq a$.

(2) a strong a-shading of G if $\bigwedge\limits_{x \in X} (G'(x) \vee \bigvee\limits_{A \in \mathcal{U}} A(x)) \nleq a$.

Definition 2.3. [25] Let $a \in L \backslash \{\perp\}$ and $G \in L^X$. A subfamily $\mathcal{P}$ in L^X is said to be

(1) an a-remote family of G if for any $x \in X$, it follows that $G(x) \wedge \bigwedge\limits_{B \in \mathcal{P}} B(x) \ngeq a$.

(2) a strong a-remote family of G if $\bigvee\limits_{x \in X} (G(x) \wedge \bigwedge\limits_{B \in \mathcal{P}} B(x)) \ngeq a$.

(3) a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta(G'(x) \vee \bigvee\limits_{A \in \mathcal{P}} A(x))$.

(4) a strong β_a-cover of G if for any $x \in X$, it follows that

$$a \in \beta(\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{P}} A(x))).$$

(5) a Q_a-cover of G if $a \leq \bigwedge\limits_{x \in X} (G'(x) \vee \bigvee\limits_{A \in \mathcal{P}} A(x))$.

Definition 2.4. [28] Let $\mathcal{T}$ be an L-fuzzy topology on X. For any $A \in L^X$, define a mapping $\mathcal{T}_s : L^X \longrightarrow L$ by

$$\mathcal{T}_s(A) = \bigvee_{B \leq A} (\mathcal{T}(B) \wedge \bigwedge_{x_\lambda \prec A} \bigwedge_{x_\lambda \not\leq D \geq B} (\mathcal{T}(D'))').$$

Then $\mathcal{T}_s$ is called the L-fuzzy semiopen operator induced by $\mathcal{T}$, where $\mathcal{T}_s(A)$ can be regarded as the degree to which A is semiopen and $\mathcal{T}_s^*(B) = \mathcal{T}_s(B')$ can be regarded as the degree to which B is semiclosed. For any family $\mathcal{U} \subseteq L^X$, $\mathcal{T}_s(\mathcal{U}) = \bigwedge_{A \in \mathcal{U}} \mathcal{T}_s(A)$ is called the degree of semiopenness of $\mathcal{U}$.

Theorem 2.5. [28] *Let $\mathcal{T}$ be an L-fuzzy topology on X and let $\mathcal{T}_s$ be the L-fuzzy semiopen operator induced by $\mathcal{T}$. Then $\mathcal{T}(A) \leq \mathcal{T}_s(A)$ for any $A \in L^X$.*

Definition 2.6. [28, 29] A mapping $f : X \longrightarrow Y$ between two L-fuzzy topological spaces $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ is called

(1) semicontinuous if $\mathcal{T}_2(U) \leq (\mathcal{T}_1)_s(f_L^{\leftarrow}(U))$ holds for any $U \in L^Y$.
(2) irresolute if $(\mathcal{T}_2)_s(U) \leq (\mathcal{T}_1)_s(f_L^{\leftarrow}(U))$ holds for any $U \in L^Y$.
(3) strongly irresolute if $(\mathcal{T}_2)_s(U) \leq \mathcal{T}_1(f_L^{\leftarrow}(U))$ holds for any $U \in L^Y$.

Definition 2.7. [29] Let $(X, \mathcal{T})$ be an L-fuzzy topological space. $G \in L^X$ is said to be L-fuzzy semicompact if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\bigwedge_{A \in \mathcal{U}} \mathcal{T}_s(A) \wedge \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x)).$$

Definition 2.8. [14] Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$.The fuzzy compactness degree $cd_{\mathcal{T}}$ of G is defined as

$$cd_{\mathcal{T}}(G) = \bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}(\mathcal{U}) \to (\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x)))).$$

Theorem 2.9. [21, 25] *Let $f : X \longrightarrow Y$ be a set mapping and $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ is induced by f. Then for any $\mathcal{P} \subseteq L^X$, we have that*

$$\bigwedge_{y \in Y} (f_L^{\rightarrow}(G)'(y) \vee \bigvee_{B \in \mathcal{P}} B(y)) = \bigwedge_{x \in X} (G'(x) \vee \bigvee_{B \in \mathcal{P}} f_L^{\leftarrow}(B)(x)).$$

3. Measures of Fuzzy Semicompactness

In [14], Li and Shi generalized the notion of fuzzy compactness to L-fuzzy topological spaces, and gave the definition of fuzzy compactness degrees in L-fuzzy topological spaces. Based on [14], we will generalize the notion of fuzzy semicompactness to L-fuzzy topological spaces. In order to do this, let us recall fuzzy semicompactness in L-topology [23].

Let $(X, \mathcal{T})$ be an L-topological space and $G \in L^X$. G is fuzzy semicompactness if and only if for every family $\mathcal{U}$ of semiopen L-sets, it follows that

$$\bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} (G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x)).$$

This implies that for every family $\mathcal{U}$ of semiopen L-sets,

$$\left[\left[G\widetilde{\subset}\bigvee\mathcal{U}\right] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[G\widetilde{\subset}\bigvee\mathcal{V}\right]\right] = \top.$$

We know that an L-topology $\mathcal{T}$ can be looked as a special L-fuzzy topology. Therefore, $A \in L^X$ is a semiopen set if and only if $\mathcal{T}_s(A) = \top$ [28]. Thus G is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\mathcal{T}_s(\mathcal{U}) \leq \left[\left[G\widetilde{\subset}\bigvee\mathcal{U}\right] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[G\widetilde{\subset}\bigvee\mathcal{V}\right]\right].$$

Therefore we can naturally generalize the notion of fuzzy semicompactness degrees to L-fuzzy topological spaces as follows:

Definition 3.1. Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. The fuzzy semicompactness degree $scd_{\mathcal{T}}$ of G is defined as

$$scd_{\mathcal{T}}(G) = \bigwedge_{\mathcal{U}\subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \to ([G\widetilde{\subset}\bigvee\mathcal{U}] \to \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [G\widetilde{\subset}\bigvee\mathcal{V}]))$$

$$= \bigwedge_{\mathcal{U}\subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \to (\bigwedge_{x\in X} (G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)) \to \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} (G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)))).$$

Theorem 3.2. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then $scd_{\mathcal{T}}(G) \leq cd_{\mathcal{T}}(G)$.*

Proof. Straightforward. □

Theorem 3.3. *Let $(X, \mathcal{T})$ be an L-topological space and $G \in L^X$. G is fuzzy semicompactness in $(X, \mathcal{T})$ if and only if $scd_{\chi_{\mathcal{T}}}(G) = \top$.*

Proof. Let $(X, \mathcal{T})$ be an L-topological space. The mapping $\chi_{\mathcal{T}} : L^X \longrightarrow L$ defined by

$$\chi_{\mathcal{T}}(A) = \begin{cases} \top, & A \in \mathcal{T}, \\ \bot, & A \notin \mathcal{T}. \end{cases}$$

is a special L-fuzzy topology. Then $A \in L^X$ is a semiopen set in L-topology $\mathcal{T}$ if and only if $(\chi_{\mathcal{T}})_s(A) = \top$. Thus by the definition of fuzzy semicompactness and the properties of $\to$, we know that G is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$(\chi_{\mathcal{T}})_s(\mathcal{U}) \leq \left[\left[G\widetilde{\subset}\bigvee\mathcal{U}\right] \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[G\widetilde{\subset}\bigvee\mathcal{V}\right]\right].$$

This implies that G is fuzzy semicompactness if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$(\chi_{\mathcal{T}})_s(\mathcal{U}) \to ([G\widetilde{\subset}\bigvee\mathcal{U}] \to \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [G\widetilde{\subset}\bigvee\mathcal{V}]) = \top.$$

By the definition of $scd_{\chi_{\mathcal{T}}}$, the conclusion is hold. □

Theorem 3.4. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. G is L-fuzzy semicompactness in $(X, \mathcal{T})$ if and only if $scd_{\mathcal{T}}(G) = \top$.*

Proof. By the definition of L-fuzzy semicompactness, we know that G is L-fuzzy semicompactness in $(X, \mathcal{T})$ if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$(\mathcal{T}_s(\mathcal{U}) \wedge [G \widetilde{\subset} \bigvee \mathcal{U}]) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}].$$

By the properties of $\to$, we obtain that G is L-fuzzy semicompactness in $(X, \mathcal{T})$ if and only if for every family $\mathcal{U} \subseteq L^X$, it follows that

$$\mathcal{T}_s(\mathcal{U}) \to ([G \widetilde{\subset} \bigvee \mathcal{U}] \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}]) = \top.$$

By the definition of $scd_{\mathcal{T}}$, the conclusion is hold. □

Lemma 3.5. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then $scd_{\mathcal{T}}(G) \geq a$ if and only if for any $\mathcal{U} \subseteq L^X$,*

$$\mathcal{T}_s(\mathcal{U}) \wedge [G \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}].$$

Proof. For any $a \in L$, $scd_{\mathcal{T}}(G) \geq a$, i.e.,

$$\bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \to ([G \widetilde{\subset} \bigvee \mathcal{U}] \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}])) \geq a$$

if and only if for any $\mathcal{U} \subseteq L^X$,

$$\mathcal{T}_s(\mathcal{U}) \to ([G \widetilde{\subset} \bigvee \mathcal{U}] \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}]) \geq a$$

if and only if (by the property (6) of $\to$) for any $\mathcal{U} \subseteq L^X$,

$$(\mathcal{T}_s(\mathcal{U}) \wedge [G \widetilde{\subset} \bigvee \mathcal{U}]) \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}] \geq a$$

if and only if (by the property (1) of $\to$) for any $\mathcal{U} \subseteq L^X$,

$$\mathcal{T}_s(\mathcal{U}) \wedge [G \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}].$$

□

Theorem 3.6. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then $scd_{\mathcal{T}}(G) \geq a$ if and only if for any $\mathcal{P} \subseteq L^X$,*

$$\bigvee_{F \in \mathcal{P}} \mathcal{T}_s^*(F)' \vee (\bigvee_{x \in X} (G(x) \wedge \bigwedge_{F \in \mathcal{P}} F(x))) \vee a' \geq \bigwedge_{\mathcal{H} \in 2^{(\mathcal{P})}} \bigvee_{x \in X} (G(x) \wedge \bigwedge_{F \in \mathcal{H}} F(x)).$$

Proof. It can be easily obtained by Lemma 3.5 and the definition of $\mathcal{T}_s^*$. □

Theorem 3.7. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X$. Then*

$$scd_{\mathcal{T}}(G) = \bigvee \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\}.$$

Proof. By Lemma 3.5, we know that $scd_{\mathcal{T}}(G)$ is an upper bound of

$$\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \tilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\}.$$

Since

$$scd_{\mathcal{T}}(G) = \bigwedge_{\mathcal{U} \subseteq L^X} (\mathcal{T}_s(\mathcal{U}) \to ([G \tilde{\subset} \bigvee \mathcal{U}] \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \Big[G \tilde{\subset} \bigvee \mathcal{V}])),$$

then for every family $\mathcal{U} \subseteq L^X$, we have

$$\begin{aligned} scd_{\mathcal{T}}(G) &\leq \mathcal{T}_s(\mathcal{U}) \to ([G \tilde{\subset} \bigvee \mathcal{U}] \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \tilde{\subset} \bigvee \mathcal{V}]) \\ &= (\mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{\subset} \bigvee \mathcal{U}]) \to \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \tilde{\subset} \bigvee \mathcal{V}]. \end{aligned}$$

By the property (1) of $\to$, we obtain that for every family $\mathcal{U} \subseteq L^X$,

$$\mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{\subset} \bigvee \mathcal{U}] \wedge scd_{\mathcal{T}}(G) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \tilde{\subset} \bigvee \mathcal{V}],$$

thus

$$scd_{\mathcal{T}}(G) \in \{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G \tilde{\subset} \bigvee \mathcal{U}] \wedge a \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \tilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\}.$$

Therefore, the conclusion is hold. □

In order to write simply, for any mapping $\mathcal{T} : L^X \longrightarrow L$, denote $\mathcal{T}_b = \{A \in L^X : \mathcal{T}(A) \geq b\}$.

Theorem 3.8. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X, a \in L \backslash \{\bot\}$. The following conditions are equivalent:*

(1) $scd_{\mathcal{T}}(G) \geq a$.

(2) For any $b \in P(L)$, $b \not\geq a$,each strong b-shading $\mathcal{U}$ of G with $\mathcal{T}_s(\mathcal{U}) \not\leq b$ has a finite subfamily $\mathcal{V}$ which is a strong b-shading of G.

(3) For any $b \in P(L)$, $b \not\geq a$,each strong b-shading $\mathcal{U}$ of G with $\mathcal{T}_s(\mathcal{U}) \not\leq b$,there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $r \in \alpha^*(b)$ such that $\mathcal{V}$ is an r-shading of G.

(4) For any $b \in P(L)$, $b \not\geq a$,each strong b-shading $\mathcal{U}$ of G with $\mathcal{T}_s(\mathcal{U}) \not\leq b$,there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $r \in \alpha^*(b)$ such that $\mathcal{V}$ is a strong r-shading of G.

(5) For any $b \in M(L)$, $b \not\leq a'$,each strong b-remote family $\mathcal{P}$ of G with $\mathcal{T}^*_s(\mathcal{P}) \not\leq b'$ has a finite subfamily $\mathcal{H}$ which is a strong b-remote family of G.

(6) For any $b \in M(L)$, $b \not\leq a'$,each strong b-remote family $\mathcal{P}$ of G with $\mathcal{T}^*_s(\mathcal{P}) \not\leq b'$, there exists a finite subfamily $\mathcal{H}$ of $\mathcal{P}$ and $r \in \beta^*(b)$ such that $\mathcal{H}$ is an r-remote family of G.

(7) For any $b \in M(L)$, $b \not\leq a'$,each strong b-remote family $\mathcal{P}$ of G with $\mathcal{T}^*_s(\mathcal{P}) \not\leq b'$, there exists a finite subfamily $\mathcal{H}$ of $\mathcal{P}$ and $r \in \beta^*(b)$ such that $\mathcal{H}$ is a strong r-remote family of G.

(8) For any $b \leq a$,$r \in \beta(b)$, $b, r \neq \bot$,each Q_b-cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily $\mathcal{V}$ which is a Q_r-cover of G.

(9) For any $b \leq a$,$r \in \beta(b)$, $b, r \neq \bot$,each Q_b-cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily $\mathcal{V}$ which is a strong β_r-cover of G.

(10) For any $b \leq a$,$r \in \beta(b)$, $b, r \neq \perp$,each Q_b-cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily $\mathcal{V}$ which is a β_r-cover of G.

(11) For any $b \leq a$,$r \in \beta(b)$, $b, r \neq \perp$,each strong β_b-cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily $\mathcal{V}$ which is a Q_r-cover of G.

(12) For any $b \leq a$,$r \in \beta(b)$, $b, r \neq \perp$,each strong β_b-cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily $\mathcal{V}$ which is a strong β_r-cover of G.

(13) For any $b \leq a$,$r \in \beta(b)$, $b, r \neq \perp$,each strong β_b-cover $\mathcal{U} \subseteq (\mathcal{T}_s)_b$ of G has a finite subfamily $\mathcal{V}$ which is a β_r-cover of G.

In Theorem 3.8 (8)-(13), if we replace $b, r \neq \perp$ and $r \in \beta(b)$ with $b \in M(L)$ and $r \in \beta^*(b)$, then the conclusions are still right.

Theorem 3.9. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G \in L^X, a \in L\backslash\{\perp\}$. If for any $c, d \in L$, $\beta(c \wedge d) = \beta(c) \wedge \beta(d)$. Then the following conditions are equivalent:*

(1) $scd_{\mathcal{T}}(G) \geq a$.

(2) For any $b \in \beta(a)$, $b \neq \perp$, each strong β_b-cover $\mathcal{U}$ of G with $b \in \beta(\mathcal{T}_s(\mathcal{U}))$ has a finite subfamily $\mathcal{V}$ which is a Q_b-cover of G.

(3) For any $b \in \beta(a)$, $b \neq \perp$, each strong β_b-cover $\mathcal{U}$ of G with $b \in \beta(\mathcal{T}_s(\mathcal{U}))$ has a finite subfamily $\mathcal{V}$ which is a strong β_b-cover of G.

(4) For any $b \in \beta(a)$, $b \neq \perp$, each strong β_b-cover $\mathcal{U}$ of G with $b \in \beta(\mathcal{T}_s(\mathcal{U}))$ has a finite subfamily $\mathcal{V}$ which is a β_b-cover of G.

4. Properties of Fuzzy Semicompactness Degrees

Theorem 4.1. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G, H \in L^X$. Then $scd_{\mathcal{T}}(G \vee H) \geq scd_{\mathcal{T}}(G) \wedge scd_{\mathcal{T}}(H)$.*

Proof. By Theorem 3.7 we have

$$\begin{aligned} scd_{\mathcal{T}}(G \vee H) &= \bigvee\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [(G \vee H)\tilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [(G \vee H)\tilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} \\ &= \bigvee\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\subset} \bigvee \mathcal{U}] \wedge [H\tilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} ([G\tilde{\subset} \bigvee \mathcal{V}] \wedge [H\tilde{\subset} \bigvee \mathcal{V}]), \forall \mathcal{U} \subseteq L^X\} \\ &\geq \bigvee\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G\tilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G\tilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} \wedge \bigvee\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [H\tilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [H\tilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} = scd_{\mathcal{T}}(G) \wedge scd_{\mathcal{T}}(H). \end{aligned}$$

□

Theorem 4.2. *Let $(X, \mathcal{T})$ be an L-fuzzy topological space and $G, H \in L^X$. Then $scd_{\mathcal{T}}(G \wedge H) \geq scd_{\mathcal{T}}(G) \wedge \mathcal{T}_s^*(H)$.*

Proof. By Theorem 3.7 we have

$$\begin{aligned} scd_{\mathcal{T}}(G \wedge H) &= \bigvee\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [(G \wedge H)\widetilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [(G \wedge H)\widetilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} \\ &= \bigvee\{a \in L : \mathcal{T}_s(\mathcal{U}) \wedge [G\widetilde{\subset}(H' \vee \bigvee \mathcal{U})] \wedge a \\ &\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [G\widetilde{\subset}(H' \vee \bigvee \mathcal{V})], \forall \mathcal{U} \subseteq L^X\} \\ &\geq \bigvee\{a \wedge \mathcal{T}_s^*(H) : \mathcal{T}_s(\mathcal{U}) \wedge [G\widetilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [G\widetilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} = scd_{\mathcal{T}}(G) \wedge \mathcal{T}_s^*(H). \end{aligned}$$

□

Corollary 4.3. *Let* $(X, \mathcal{T})$ *be an* L*-fuzzy topological space and* $G \in L^X$*. Then* $scd_{\mathcal{T}}(G) \geq scd_{\mathcal{T}}(\underline{\top}) \wedge \mathcal{T}_s^*(G)$*.*

Theorem 4.4. *Let* $(X, \mathcal{T}_1), (X, \mathcal{T}_2)$ *be two* L*-fuzzy topological spaces and satisfy* $\mathcal{T}_1 \leq \mathcal{T}_2$*,* $G \in L^X$*. Then* $scd_{\mathcal{T}_2}(G) \leq scd_{\mathcal{T}_1}(G)$*.*

Corollary 4.5. *Let* $(X, \mathcal{T})$ *be an* L*-fuzzy topological space and let* $\mathcal{B}$ *be a base or subbase* [7, 8, 34] *of* $\mathcal{T}$*,* $G \in L^X$*. Then* $scd_{\mathcal{T}}(G) \leq scd_{\mathcal{B}}(G)$*.*

Theorem 4.6. *Let* $f : X \longrightarrow Y$ *be a set mapping,* $\mathcal{T}_1$ *be an* L*-fuzzy topology on* X*,* $\mathcal{T}_2$ *be an* L*-fuzzy topology on* Y*, and* $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ *be an* L*-fuzzy strong irresolute mapping. Then for any* $G \in L^X$*,* $cd_{\mathcal{T}_1}(G) \leq scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G))$*.*

Proof. For any $G \in L^X$, we have

$$\begin{aligned} scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G)) &= \bigvee\{a \in L : (\mathcal{T}_2)_s(\mathcal{U}) \wedge [f_L^{\rightarrow}(G)\widetilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [f_L^{\rightarrow}(G)\widetilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} \\ &\geq \bigvee\{a \in L : \mathcal{T}_1(f_L^{\leftarrow}(\mathcal{U})) \wedge [G\widetilde{\subset} \bigvee f_L^{\leftarrow}(\mathcal{U})] \wedge a \\ &\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [G\widetilde{\subset} \bigvee f_L^{\leftarrow}(\mathcal{V})], \forall \mathcal{U} \subseteq L^X\} \geq cd_{\mathcal{T}_1}(G). \end{aligned}$$

□

Theorem 4.7. *Let* $f : X \longrightarrow Y$ *be a set mapping,* $\mathcal{T}_1$ *be an* L*-fuzzy topology on* X*,* $\mathcal{T}_2$ *be an* L*-fuzzy topology on* Y*, and* $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ *be an* L*-fuzzy irresolute mapping. Then for any* $G \in L^X$*,* $scd_{\mathcal{T}_1}(G) \leq scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G))$*.*

Proof. For any $G \in L^X$, we have

$$\begin{aligned} scd_{\mathcal{T}_2}(f_L^{\rightarrow}(G)) &= \bigvee\{a \in L : (\mathcal{T}_2)_s(\mathcal{U}) \wedge [f_L^{\rightarrow}(G)\widetilde{\subset} \bigvee \mathcal{U}] \wedge a \\ &\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [f_L^{\rightarrow}(G)\widetilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} \\ &\geq \bigvee\{a \in L : (\mathcal{T}_1)_s(f_L^{\leftarrow}(\mathcal{U})) \wedge [G\widetilde{\subset} \bigvee f_L^{\leftarrow}(\mathcal{U})] \wedge a \\ &\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} [G\widetilde{\subset} \bigvee f_L^{\leftarrow}(\mathcal{V})], \forall \mathcal{U} \subseteq L^X\} \geq scd_{\mathcal{T}_1}(G). \end{aligned}$$

□

Theorem 4.8. *Let $f : X \longrightarrow Y$ be a set mapping, $\mathcal{T}_1$ be an L-fuzzy topology on X, $\mathcal{T}_2$ be an L-fuzzy topology on Y, and $f : (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$ be an L-fuzzy semicontinuous mapping. Then for any $G \in L^X$, $scd_{\mathcal{T}_1}(G) \leq cd_{\mathcal{T}_2}(f_L^{\rightarrow}(G))$.*

Proof. For any $G \in L^X$, we have

$$\begin{aligned} cd_{\mathcal{T}_2}(f_L^{\rightarrow}(G)) = & \bigvee\{a \in L : \mathcal{T}_2(\mathcal{U}) \wedge [f_L^{\rightarrow}(G) \widetilde{\subset} \bigvee \mathcal{U}] \wedge a \\ & \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [f_L^{\rightarrow}(G) \widetilde{\subset} \bigvee \mathcal{V}], \forall \mathcal{U} \subseteq L^X\} \\ & \geq \bigvee\{a \in L : (\mathcal{T}_1)_s(f_L^{\leftarrow}(\mathcal{U})) \wedge [G \widetilde{\subset} \bigvee f_L^{\leftarrow}(\mathcal{U})] \wedge a \\ & \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} [G \widetilde{\subset} \bigvee f_L^{\leftarrow}(\mathcal{V})], \forall \mathcal{U} \subseteq L^X\} \geq scd_{\mathcal{T}_1}(G). \end{aligned}$$

□

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