THE REMAK-KRULL-SCHMIDT THEOREM ON FUZZY GROUPS

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Abstract. In this paper we study a representation of a fuzzy subgroup μ of a group G, as a product of indecomposable fuzzy subgroups called the components of μ. This representation is unique up to the number of components and their isomorphic copies. In the crisp group theory, this is a well-known Theorem attributed to Remak, Krull, and Schmidt. We consider the lattice of fuzzy subgroups and some of their properties to prove this theorem. We illustrate with some examples.

1. Introduction

The Remak-Krull-Schmidt (RKS) theorem has a long history spread over a century (1900-) starting with a result of Wedderburn. Remak (1911) [9] in his thesis seems to have been the first one to propose the unique decomposition of a group as a product of indecomposable components. Ore [8] unified various forms of this decomposition found in groups and algebras of RKS theorem by 1936. At the same time the Russian Mathematician Kurosh [4] came up with a similar unified theorem. Ore has used the language of modular lattices to derive his unified version and it is now referred to as Kurosh-Ore Theorem in the literature [1]. In order to work out this theorem in the fuzzy group setting, we first need an equivalence relation on the set of all fuzzy subgroups of a group which naturally generalizes the equality of crisp subgroups. This equivalence relation was proposed in the literature by Murali and Makamba [7] and by others [3] and [12]. We call such an equivalence relation *preferential equality* and an equivalence class of fuzzy subgroups, a *preferential fuzzy subgroup*. A precursor to preferential fuzzy subgroups based on α-cuts is the paper by Das [2]. The importance of Remak-Krull-Schmidt Theorem (from now on to be referred to as RKS Theorem) cannot be over-emphasized in group theory since it studies a group by decomposing it into a direct product of indecomposable subgroups. The notion of product of fuzzy groups was first proposed by Sherwood [11] but did not consider any decomposition problems. We use a lattice theoretic approach with preferential fuzzy subgroups to derive a similar representation theorem for fuzzy subgroups. This is an important step in understanding some aspects of fuzzy group theory.

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In section 2 we collect the required preliminary notions such as normal fuzzy subgroup, direct product of fuzzy subgroups and preferential fuzzy subgroups, and fix notations. After recalling some lattice-theoretic concepts, we introduce an important concept of length of a fuzzy subgroup in the lattice of fuzzy subgroups. In section 3 we make clear ideas pertaining to product, decomposable and indecomposable fuzzy subgroups and develop a framework to prove the RKS Theorem on fuzzy subgroups. Finally in section 4 we state the RKS Theorem in a suitable form and prove the theorem using length of a fuzzy subgroup arising out of lattice-theoretic considerations. We provide some examples to illustrate the ideas of the theorem.

2. **Preliminaries**

We use $\mathbf{I} = [\mathbf{0}, \mathbf{1}]$, the real unit interval as a chain with the usual ordering in which $\wedge$ stands for infimum (inf) (or intersection) and $\vee$ stands for supremum (sup)(or union). A *fuzzy subset* of a set G is a mapping $\mu : G \to \mathbf{I}$. The union and intersection of two fuzzy sets are defined using sup and inf pointwise respectively. Throughout this paper we take G to be a group, not necessarily finite and $\{e\}$ to be the trivial subgroup containing only the identity e.
By an α-cut of μ for a real number α in $\mathbf{I}$, we mean a subset $\mu^\alpha = \{x \in G : \mu(x) \geq \alpha\}$ of G. A fuzzy set μ is said to be a *fuzzy subgroup* if $\mu(x.y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$ and $\mu(x) = \mu(x^{-1})$ (see [10],[6],[5]). In this paper we assume $\mu(e) = 1$ and by μ_e we mean the trivial fuzzy subgroup which takes membership value 1 at e and 0 elsewhere. The *core* and *support* of μ are crisp subsets of G defined by $core(\mu) = \{x \in G : \mu(x) = 1\}$ and $supp\,\mu = \{x \in G : \mu(x) \neq 0\}$ respectively.
We denote the set of all fuzzy subgroups of G by $\mathbf{I}^G$ and consider it as a lattice under the natural operation of intersection using $\wedge$. This lattice need not be modular but the set of all normal fuzzy subgroups is a modular lattice, where a *normal fuzzy subgroup* is defined as a fuzzy subgroup μ that satisfies $\mu(ax) = \mu(xa)$ for all $x, a \in G$. We denote the set of all normal fuzzy subgroups of G by $\mathcal{P}$. If μ is a normal fuzzy subgroup, we use the notation $\mathcal{P}(\mu)$ for the lattice of all normal fuzzy subgroups contained in μ, with $0 = \mu_e$ and $1 = \mu$.

We recall the notions of product and quotient fuzzy subgroups used in the literature. Suppose μ and ν are two fuzzy subgroups of G. Then the *product* $\mu\nu$ has the membership function defined by $\mu\nu(x) = \sup\{\mu(a) \wedge \nu(b) : x = ab\}$ for all $x \in G$, and $\mu\nu$ is a fuzzy subgroup provided one of the two fuzzy subgroups is normal in G. A product $\mu\nu$ is called a direct product, denoted by $\mu \otimes \nu$, if both μ and ν are fuzzy normal and $\mu \wedge \nu = \mu_e$. Now assume ν is a normal fuzzy subgroup and $\nu \leq \mu$. Then a *fuzzy quotient subgroup* μ modulo ν, denoted by μ/ν is defined by $\mu/\nu : G/supp\,\nu \to \mathbf{I}$ where $(\mu/\nu)(x\,supp\,\nu) = \sup\{\mu(a) : a\,supp\,\nu = x\,supp\,\nu\}$ for all $x \in G$. It is easy to check that μ/ν is a fuzzy subgroup of $G/supp\,\nu$. Further we can view μ/ν as a fuzzy subset of G by lifting the mapping μ/ν to G through the surjective homomorphism from $G \to G/supp\,\nu$.
We also need the concept of preferential equality which we proceed to define now. We recall from [7] that an *equivalence relation of two fuzzy subgroups* $\sim$ on $\mathbf{I}^G$ is

defined as $\mu \sim \nu$ if and only if the following two conditions are satisfied:

$$\begin{array}{ll} \text{(i)} & \forall x, y \in G,\ \mu(x) > \mu(y) \quad \text{if and only if} \quad \nu(x) > \nu(y) \\ \text{(ii)} & \mu(x) = 0 \quad \text{if and only if} \quad \nu(x) = 0. \end{array} \tag{1}$$

We refer to the above equivalence relation as *preferential equality* of fuzzy sets and denote it by $\mu \sim \nu$, and the equivalence class containing μ is called a preferential fuzzy subgroup and is denoted by $[\mu]$. Two fuzzy subgroups are said to be *distinct* if they are not preferentially equal.
More general than the above equivalence is the notion of isomorphism which we now define as it is required in later discussions. Suppose μ and ν are two fuzzy subgroups of G. Then μ is said to be *fuzzy isomorphic* to ν, written as $\mu \simeq \nu$, if there exists an isomorphism f between supports such that $\mu(x) = c\nu(x)$ for all $x \in supp(\mu) \setminus \{e\}$ and for some constant $0 < c \leq 1$.

3. Length and Decomposability

In this section we define and study two related ideas pertaining to fuzzy subgroups, one is the lattice theoretic concept of length, of a Lattice and that of a point, as defined in Cohn [1], and the other is the well-known theoretic concept of decomposability.

Suppose L is a lattice. Two elements x_1 and x_2 are said to be $x-$*related* or simply *related* in L if they have a common complement x in L in the usual sense of complement in a lattice. Clearly the relation is symmetric but in general neither transitive nor reflexive. We say ℓ is the *length* of L if ℓ is the supremum of the number of non-trivial intervals in any chain of L. In particular a lattice is of *finite length* if the lengths of its chains are bounded. The length $\ell(a)$ of a point $a \in L$ is the length of the interval $[0, a]$. It is easy to show that

$$\ell(a) + \ell(b) = \ell(a \wedge b) + \ell(a \vee b) \quad \forall a, b \in L. \tag{2}$$

We now define the length of the lattice $\mathcal{P}(\mu)$ of all normal fuzzy subgroups contained in μ. The *length of a chain* of fuzzy subgroups $\mu_1 \geq \mu_2 \geq \cdots$ is the number of non-trivial distinct quotients of the form μ_1/μ_2, μ_2/μ_3, $\cdots$.
The *length of a quotient* μ/ν is the supremum of the number of distinct non-trivial quotients in any chain of fuzzy subgroups between μ and ν. We write $\ell(\mu/\nu)$ for the length of μ/ν .
The quotient μ/ν is of finite length in case $\ell(\mu/\nu)$ is finite.

Definition 3.1. The length of a fuzzy subgroup μ is $\ell(\mu) = \ell(\mu/\mu_e)$ and the length of $\mathcal{P}(\mu)$ is the length of μ.

We now define the idea of an *indecomposable* fuzzy subgroup that is central to RKS Theorem.

Definition 3.2. Let μ be a fuzzy subgroup of G. Then μ is said to be *indecomposable* if μ is not μ_e and if $\mu \simeq \mu_1 \otimes \mu_2$ then μ_1 or μ_2 is μ_e. We say μ is *decomposable* if it is not indecomposable.

Remark 3.3. We collect some easy consequnces of the definition of length in context of fuzzy subgroups. Firstly if μ is of finite length, then $\ell(\mu) \geq \ell(\mu/\nu)+\ell(\nu)$. Secondly if $\mathcal{P}(\mu)$ is of finite length, the support of μ satisfies the ascending chain condition (ACC) as well as the descending chain condition (DCC), and further if $\nu_1 \leq \nu_2 \leq \cdots \leq \mu$ is a normal chain in $\mathcal{P}(\mu)$, then there are only finitely many distinct quotients of the form ν_{i+1}/ν_i. Thirdly if μ is of finite length and has a decomposition $\mu_1 \otimes \mu_2$ then both μ_1 and μ_2 are of finite length. The next theorem is important in the sense that it relates the concept of finite length to indecomposability.

Theorem 3.4. Suppose μ is a non-trivial fuzzy subgroup of G of finite length. Then μ is a direct product of a finite number of indecomposable fuzzy subgroups.

Proof. If μ itself is indecomposable then there is nothing to prove. So suppose μ is decomposable, say, $\mu = \nu_1 \otimes \mu_2$ where each component is non-trivial. If both components are indecomposable, there is nothing to prove. If one of the components is decomposable then write as a product of two non-trivial fuzzy subgroups. Continuing this process inductively, we arrive at $\mu = \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_{n-1} \otimes \mu_n$, where each ν_i is indecomposable. This gives rise to a descending chain $\mu \geq \mu_2 \geq \cdots \geq$, where each μ_i is normal in μ. Since $\ell(\mu)$ is finite, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then the quotient μ_n/μ_{n+1} is preferentially equal to one of the quotients that has appeared previously. The support of μ has the property of DCC, and hence $\mu_{n+1} = \mu_e$. Hence $\mu_n = \nu_n$ which is indecomposable. This completes the proof. □

We conclude this section by recalling two properties pertaining to lengths which are easily proved. Therefore they are stated without proofs as Remarks:

Remark 3.5. 1. Let μ_1 and μ_2 be related in $\mathcal{P}(\mu)$. Then they have the same length, that is, $\ell(\mu_1) = \ell(\mu_2)$.
2. If $\nu_1 w = \mu_1 w$ where $\mu_1 \wedge w = \mu_e$ and ν_1 is related to μ_1, then $\ell(\nu_1 \wedge w) = 0$. Hence $\nu_1 \wedge w = \mu_e$.

4. Remak-Krull-Schmidt Theorem

In this section we state and prove the central theorem of the paper on the decomposition of fuzzy subgroups of finite length similar to a theorem on subgroups due to Remak, Krull and Schimdt.

Theorem 4.1. Suppose μ is fuzzy subgroup of G such that the length of $\mathcal{P}(\mu)$ is finite and

$$\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_m = \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n, \tag{3}$$

where each of the fuzzy subgroups μ_i for $1 \leq i \leq m$ and ν_j for $1 \leq j \leq n$ is indecomposable.

Then

(I) each μ_i for $1 \le i \le m$ is related to some ν_j for $1 \le j \le n$;
(II) $m = n$. For each $r = 1, \cdots, n-1$, after re-indexing
$\mu \simeq \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_r \otimes \nu_{r+1} \otimes \cdots \otimes \nu_n$;
(III) each μ_i is isomorphic to ν_j for some $1 \le i \le m$ and $1 \le j \le n$; (4)

Proof. First we set a convenient notation. μ_i' denotes the product $\prod_{k=1;k\neq i}^{m} \mu_k$ for any $1 \le i \le m$. Similarly for ν_i'. Secondly by modular law

$$w \le \prod_{i=1}^{m} (w\mu_i' \wedge \mu_i) \text{ for any } w \in \mathcal{P}(\mu). \tag{5}$$

Thirdly, we prove that each

$$\mu_i \text{ is } \mu_i'\text{-related to some } \nu_j \tag{6}$$

using induction on the length $\ell(\mu)$.

(I) Suppose $\ell(\mu) = 1$. Hence any quotient μ/ν is preferentially equal to μ/μ_e or μ/μ. So if $\mu = \mu_1 \otimes \mu_2 = \nu_1 \otimes \nu_2$ where the μ_1, μ_2, ν_1 and ν_2 are all indecomposable, then either μ_1 or μ_2 is μ_e and similarly ν_1 or ν_2 is μ_e. Hence (I) is true in this case. Assume the relation in (6) is true for a fuzzy subgroup w of length less than $\ell(\mu)$. Two cases arise.

Case 1: $supp\, \mu_1\nu_j' \neq supp\, \mu$ for some j.
Let $w_j = \nu_j \wedge \mu_1\nu_j'$, and $w = w_1 w_2 \cdots w_n$. Then $w_j \le \nu_j$ for all j; hence $w_j \wedge w_j' \le \nu_j \wedge \nu_j' = \mu_e$. Therefore $w = w_1 \otimes w_2 \otimes \cdots \otimes w_n$. If $w_j = \nu_j$ for all j then $\nu_j = \nu_j \wedge \mu_1\nu_j'$ showing that $\nu_j \le \mu_1\nu_j'$. So $\mu = \nu_j\nu_j' \le \mu_1\nu_j$. Therefore $\mu = \mu_1\nu_j'$ for all j, a contradiction. Hence there exists an i such that $w_i \neq \nu_i$. It follows that $supp\, w_j \neq supp\, \nu_j$. Hence the length of ν_j is not equal to length of w_j and ν_j/w_j is not isomorphic μ_e.
If $\nu_j/w_j \sim w'/w_j$ for some $w' \le w$, then $supp\, \nu_j = supp\, w'$, hence w' is indecomposable. However $w' = w_j \otimes (w' \wedge w_j')$. Therefore $w_j = w'$, and this contradicts the fact that ν_j/w_j is not isomorphic to μ_e. Hence we conclude that the length $\ell(w)$ of w is less than the length $\ell(\mu)$ of μ. We apply the Mathematical Induction on $\ell(w)$ as follows:
By the inequality in equation (5) we get $\mu_1 \le w$. Therefore it is clear that $w = \mu_1 \otimes (w \wedge \mu_1')$. Since the length of μ is finite, each w_i can be decomposed into a direct product of a finite number of indecomposable fuzzy subgroups. Suppose $w_i = \eta_i \otimes \gamma_i$ is a decomposition of w_i into indecomposable components, for each $i = 1, 2, \cdots, n$. Then $w = \eta_1 \otimes \gamma_1 \otimes \eta_2 \otimes \gamma_2 \otimes \cdots \otimes \eta_n \otimes \gamma_n$. By induction μ_1 is $w \wedge \mu_1'$-related to η_i or γ_i for some i. We may assume without loss of generality that μ_1 is $w \wedge \mu_1'$-related to η_1. Then $w = \mu_1 \otimes (w \wedge \mu_1') = \eta_1 \otimes (w \wedge \mu_1')$. Now $\eta_1\mu_1' = \eta_1(w \wedge \mu_1')\mu_1' = \mu_1(w \wedge \mu_1')\mu_1' = \mu$. Since μ_1 and η_1 are related, they have the same length and the length of $\eta_1 \wedge \mu_1'$ is 0 by the Remark 3.5. Therefore $\eta_1 \wedge \mu_1' = \mu_e$ so that $\mu = \eta_1 \otimes \mu_1'$. Now $\eta_1 \le w_1 \le \mu_1$, therefore $\nu_1 = \eta_1 \otimes (\nu_1 \wedge \eta_1')$. Since ν_1 is indecomposable, $\nu_1 = \eta_1 = w_1$ and $\nu_1 \wedge \eta_1' = \mu_e$. Thus ν_1 is $w \wedge \mu_1'$-related to μ_1. Therefore $\nu_1 \le \mu_1\nu_1'$ because $\nu_1 = w_1 = \mu_1\nu_1' \wedge \nu_1$. Hence $\mu \le \mu_1\nu_1'$. Therefore

$\mu = \mu_1\nu_1'$.

Case 2: $supp\,\mu_1\nu_j' = supp\,\mu$ for all j.
Suppose $\nu_j\mu_1' \neq \mu$ for all the $j's$. Then applying case 1, with $\mu_i's$ and $\nu_i's$ interchanged, we see that ν_j is related to some μ_j. So we may replace ν_1 by μ_1', ν_2 by μ_2', and so on until we reach a certain j such that $\nu_j' = \mu$. Without loss of generality we may assume $\nu_1' = \mu$. By an argument similar to case 1, $\mu = \nu_1\mu_1'$ which contradicts our assumption. Therefore there will exist a j such that $\nu_j\mu_1' = \mu$, say for $j = 1$. So $\nu_1\mu_1' = \mu = \mu_1\mu_1'$.
Now $\nu_1\mu_1'/\mu_1' = \mu_1\mu_1'/\mu_1'$, and so $\nu_1/(\mu_1' \wedge \nu_1) \simeq \mu_1$ by the second isomorphism theorem [5]. Therefore length of μ_1 is equal to the length of $\nu_1/(\mu_1' \wedge \nu_1)$. Now

$$\ell(\nu_1) \geq \ell(\nu_1/(\mu_1' \wedge \nu_1)) + \ell(\nu_1 \wedge \mu_1') = \ell(\mu_1) + \ell(\nu_1 \wedge \mu_1') \tag{7}$$

So $\ell(\nu_1) - \ell(\mu_1) \geq 0$. By Case 2, $supp\,\mu_1\nu_1' = supp\,\mu$ which is equal to $supp\,\nu_1\nu_1'$. By applying exactly the same arguments as in the first part of case 2 above, we get $\ell(\mu_1) - \ell(\nu_1) \geq 0$. Therefore $\ell(\mu_1) = \ell(\nu_1)$ implying $\ell(\nu_1 \wedge \mu_1') = 0$ by relation (7), which in turn implies $\nu_1 \wedge \mu_1' = \mu_e$. We now conclude that $\mu = \nu_1 \otimes \mu_1' = \mu_1 \otimes \mu_1'$. This concludes the proof of part (I).

(II) Suppose ν_i is ν_i'-related to μ_i for each $i = 1, 2, \cdots, n$. Then by part **(I)**, $\mu = \mu_1 \otimes \nu_1' = \mu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_n$. Inductively for $r \geq 2$ suppose $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{r-1} \otimes \nu_r \otimes \nu_{r+1} \otimes \cdots \otimes \nu_n$. Since ν_r is ν_r'-related to μ_r then $\nu_r \otimes \nu_r' = \mu_r \otimes \nu_r'$. Therefore $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_{r-1} \otimes \mu_r \otimes \nu_{r+1} \otimes \cdots \otimes \nu_n$. Suppose $n > m$. Then $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_m \otimes \nu_{m+1} \otimes \cdots \otimes \nu_n$. By part **(I)** of the theorem, $\nu_{m+1} = \mu_e = \cdots = \nu_n$ which implies $n \leq m$ by indecomposability of ν_i's. By symmetry m must be less than or equal to n. Hence $n = m$.

(III) Suppose μ_i is μ_i'-related to ν_i after some re-indexing the subscripts of the μ's and the ν's. So $\mu_1 \otimes \mu_1' = \mu = \nu_1 \otimes \mu_1'$. Hence by the second isomorphism theorem, $(\mu_1 \otimes \mu_1')/\mu_1' \simeq \mu_1$ and $(\nu_1 \otimes \mu_1')/\mu_1' \simeq \nu_1$. So $\mu_1 \simeq \nu_1$.
This concludes the proof of the theorem. □

Example 4.2. Let $G = \mathbb{Z}_6$. Now if μ is a normal fuzzy subgroup of G, we define a μ_1 by $\mu_1 = \mu$ on $\mathbb{Z}_2$ and zero elsewhere, and μ_2 by $\mu_2 = \mu$ on $\mathbb{Z}_3$ and zero elsewhere. It is straightforward to check that $\mu = \mu_1 \otimes \mu_2$ where μ_1 and μ_2 are indecomposable normal fuzzy subgroups.

The above example is a simple illustration of RKS Theorem. One would require a computational tool such GAP or Cayley or other software to produce a non-trivial example. This would take us too far from the goal of the paper.

5. Conclusion

As pointed out in the introduction, Ore set the RKS theorem in the modular lattices. As normal fuzzy subgroups form a modular lattice, we have generalized the RKS theorem to fuzzy groups using Ore's ideas. Thus we got the fuzzy version of the RKS theroem. It is not difficult to derive the Kurosh-Ore version of decomposition since the theorem involves irreducibility and irredundancy of product representation rather than direct product. Since the uniquness of decomposition is central to algebra in general and in particular to group theory, the above generalization is worthwhile in fuzzy group theory.

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