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PONTRYAGIN'S MINIMUM PRINCIPLE FOR FUZZY OPTIMAL CONTROL PROBLEMS

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Abstract. The objective of this article is to derive the necessary optimality conditions, known as Pontryagin's minimum principle, for fuzzy optimal control problems based on the concepts of differentiability and integrability of a fuzzy mapping that may be parameterized by the left and right-hand functions of its α-level sets.

1. **Introduction**

Classical optimal control problems play an increasingly important role in designing of modern systems where the objective is to characterize the control signals that will cause a process to satisfy the economical, social, or physical constraints and at the same time these signals have to minimize some performance criterion. Many authors have studied classical optimal control problems from different view points and the detailed arguments can be found in many textbooks (refer [1] and references therein).

In the past few decades, fuzzy optimal control problems have attracted a great deal of attention and the interest in the filed of fuzzy optimal control theory has increased. A large number of existing schemes of fuzzy optimal control for nonlinear systems are proposed based on the framework of Takagi-Sugeno (T-S) fuzzy model originating from fuzzy identification [15]. Moreover, for most of the T-S modeled nonlinear systems, fuzzy control design is carried out by the aid of the parallel distributed compensation (PDC) approach [16]. Within the framework of T-S fuzzy model, the sufficient conditions for the stability of a fuzzy system is stated in terms of the feasibility of a set of linear matrix inequalities (LMIs) [7, 9]. Recently, a kind of fuzzy optimal control theory as a fuzzy counterpart of stochastic control theory has been established [11] and many results of research have been then reported in the literature (refer [12] and references therein). In [10] based on Banach fixed point theorem, the existence and the uniqueness of solutions and also the controllability of the semilinear fuzzy integrodifferential equations are studied. In [14] the fuzzy control differential equation as the generalization of a fuzzy differential equation is presented and the existence of the solutions of fuzzy differential equation is then discussed. In [13] a fuzzy/approximate necessary optimality condition is derived in the extended Euler-Lagrange form for discrete approximation of the Mayer-type

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problem for semilinear evolution inclusion. However, there exists a large literature dealing with fuzzy optimal control problems in which the rule-based fuzzy controller is implemented to construct the optimal control function.

Although the notion of fuzzy sets is widely spread to various control optimization problems, establishing necessary optimality conditions for fuzzy optimal control problems is seldom available in literatures. Our pervious success of deriving necessary optimality conditions for both fuzzy unconstrained and constrained variational problems [4] lead us to believe that the concepts of differentiability and integrability of a fuzzy mapping, parameterized by the left and right-hand functions of its α-level sets, may be a good way to handle fuzzy optimal control problems. Thus, in this article, we are going to investigate the necessary optimality conditions for fuzzy optimal control problems in the framework of α-level sets of fuzzy state and fuzzy control functions involved in such problems. It seems that, it is a new idea to derive the necessary optimality conditions for fuzzy optimal control problems using the fuzzy differentiability concept introduced in [2].

In this article, we first recall and generalize some fundamental concepts that are key to our discussion, including the concepts of differentiability and integrability of a fuzzy mapping. We then establish the main results concerning the necessary optimality conditions for the fuzzy optimization problems and the fuzzy Pontryagin's minimum principle for the fuzzy optimal control problems in the remaining sections. Finally, we discuss the applicability of the main theorems through an example.

2. **Fundamental Concepts**

In this section, it seems essential to recall some basic notions that we discussed in [4] and furthermore we shall introduce some new concepts concerning fuzzy optimization problems by appealing to some familiar results from the theory of crisp optimization problems.

The fuzzy number $\tilde{a} : R \to [0,1]$ is a mapping with the properties: (i) $\tilde{a}$ is normal, i.e., there exists an $x \in R$ such that $\tilde{a}(x) = 1$; (ii) $\tilde{a}$ is fuzzy convex, i.e., $\tilde{a}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{a}(x), \tilde{a}(y)\}$ for all $\lambda \in [0,1]$, $x, y \in R$; (iii) $\tilde{a}$ is upper semicontinuous, i.e., $\tilde{a}(x_0) \geq \overline{\lim}_{k\to\infty} \tilde{a}(x_k)$ for any $x_k \in R$, as $x_k \to x_0$; (iv) The support of $\tilde{a}$ which is $supp(\tilde{a}) = cl\{x \in R : \ \tilde{a}(x) > 0\}$ is compact.
We denote by $\mathbb{F}$ the set of all fuzzy numbers on R. The α-level set of $\tilde{a} \in \mathbb{F}$, denoted by $\tilde{a}[\alpha]$, is defined by $\tilde{a}[\alpha] = \{x \in R : \ \tilde{a}(x) \geq \alpha\}$ for all $\alpha \in (0,1]$. The 0-level set $\tilde{a}[0]$ is defined as the closure of $\{x \in R : \ \tilde{a}(x) > 0\}$, i.e., $\tilde{a}[0] = cl(supp(\tilde{a}))$.
Obviously, the α-level set $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ is a closed interval in R for all $\alpha \in [0,1]$, where $a^l(\alpha)$ and $a^r(\alpha)$ denote the *left-hand* and *right-hand endpoints* of $\tilde{a}[\alpha]$, respectively. Needless to say that $\tilde{a}$ is a crisp number with value k if its membership function is given by $\tilde{a}(x) = 1$ if $x = k$, and $\tilde{a}(x) = 0$ otherwise. Also we define fuzzy zero as

$$\tilde{0}(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

By the following lemma, from [5], we present some interesting properties associated with $a^l(\alpha)$ and $a^r(\alpha)$ of a fuzzy number $\tilde{a} \in \mathbb{F}$.

Lemma 2.1. *Let $a^l : [0,1] \to R$ and $a^r : [0,1] \to R$ satisfy the conditions:*

C1: $a^l : [0,1] \to R$ *is a bounded increasing function;*

C2: $a^r : [0,1] \to R$ *is a bounded decreasing function;*

C3: $a^l(1) \leq a^r(1)$*;*

C4: $\lim_{\alpha \to k^-} a^l(\alpha) = a^l(k)$ *and* $\lim_{\alpha \to k^-} a^r(\alpha) = a^r(k)$*, for* $0 < k \leq 1$*;*

C5: $\lim_{\alpha \to 0^+} a^l(\alpha) = a^l(0)$ *and* $\lim_{\alpha \to 0^+} a^r(\alpha) = a^r(0)$.

Then $\tilde{a} : R \to [0,1]$ *characterized by* $\tilde{a}(x) = \sup\{\alpha : a^l(\alpha) \leq x \leq a^r(\alpha)\}$ *is a fuzzy number. Also if* $\tilde{a} : R \to [0,1]$ *is a fuzzy number with* $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$*, then functions* $a^l(\alpha)$ *and* $a^r(\alpha)$ *satisfy conditions C1-C5.*

Using the extension principle [8, 17], the binary operation "$\cdot$" in R can be extended to the binary operation "$\odot$" of two fuzzy numbers $\tilde{a}$ and $\tilde{b}$ and it is defined by

$$(\tilde{a} \odot \tilde{b})(z) = \sup_{x \cdot\, y = z} \min\{\tilde{a}(x), \tilde{b}(y)\}.$$

Furthermore from [3], $-\tilde{a}$ is the opposite of the fuzzy number $\tilde{a}$ and characterized by $-\tilde{a}(x) = \tilde{a}(-x)$. In the case that $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$, we have $-\tilde{a}[\alpha] = [-a^r(\alpha), -a^l(\alpha)]$ for all $\alpha \in [0,1]$.
We say that the fuzzy number $\tilde{a}$ is *triangular* if $a^l(1) = a^r(1)$, $a^l(\alpha) = a^l(1) - (1 - \alpha)(a^l(1) - a^l(0))$ and $a^r(\alpha) = a^l(1) + (1 - \alpha)(a^r(0) - a^l(1))$. The triangular fuzzy number $\tilde{a}$ is generally denoted by $\tilde{a} = \langle a^l(0), a^l(1), a^r(0)\rangle$.

Definition 2.2. (H-difference). Let $\tilde{a}, \tilde{b} \in \mathbb{F}$, where $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$ for all $\alpha \in [0,1]$. By the Hukuhara's idea [6] of introducing the difference operator "$\ominus$", the H-difference is defined by

$$\tilde{a} \ominus \tilde{b} = \tilde{c}, \quad if\ and\ only\ if \quad \tilde{a} = \tilde{b} + \tilde{c}.$$

Obviously, $\tilde{a} \ominus \tilde{a} = \tilde{0}$, and the α-level set of H-difference is

$$(\tilde{a} \ominus \tilde{b})[\alpha] = [a^l(\alpha) - b^l(\alpha), a^r(\alpha) - b^r(\alpha)], \quad \forall \alpha \in [0,1].$$

Definition 2.3. (Partial ordering). Let $\tilde{a}, \tilde{b} \in \mathbb{F}$. We write $\tilde{a} \preceq \tilde{b}$, if $a^l(\alpha) \leq b^l(\alpha)$ and $a^r(\alpha) \leq b^r(\alpha)$ for all $\alpha \in [0,1]$. We also write $\tilde{a} \prec \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and there exists an $\widehat{\alpha} \in [0,1]$ so that $a^l(\widehat{\alpha}) < b^l(\widehat{\alpha})$ or $a^r(\widehat{\alpha}) < b^r(\widehat{\alpha})$. Moreover, $\tilde{a} = \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$. In the other words, $\tilde{a} = \tilde{b}$, if $\tilde{a}[\alpha] = \tilde{b}[\alpha]$ for all $\alpha \in [0,1]$.

In the sequel, we say that $\tilde{a}, \tilde{b} \in \mathbb{F}$ are *comparable* if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$, and *non-comparable* otherwise.

Definition 2.4. (Fuzzy-valued function). The function $\tilde{f} : S \subseteq R \to \mathbb{F}$ is called a fuzzy-valued function if for any $x \in S$, $\tilde{f}(x)$ is a fuzzy number. We also denote $\tilde{f}(x)[\alpha] = [f^l(x,\alpha), f^r(x,\alpha)]$, where $f^l(x,\alpha) = (\tilde{f}(x))^l(\alpha) = \min\{\tilde{f}(x)[\alpha]\}$ and $f^r(x,\alpha) = (\tilde{f}(x))^r(\alpha) = \max\{\tilde{f}(x)[\alpha]\}$. Therefore any fuzzy-valued function $\tilde{f}$ may be understood by $f^l(x,\alpha)$ and $f^r(x,\alpha)$ being respectively a bounded increasing function of α and a bounded decreasing function of α for $\alpha \in [0,1]$. Also it holds $f^l(x,\alpha) \leq f^r(x,\alpha)$ for any $\alpha \in [0,1]$.

Now we are going to introduce some concepts from [2] those play important roles in the fuzzy variational theory.

Definition 2.5. (Continuity of a fuzzy function). We say that $\tilde{f} : S \subseteq R \to \mathbb{F}$ is continuous at $x \in S$, if both $f^l(x,\alpha)$ and $f^r(x,\alpha)$ are continuous functions of $x \in S$, for all $\alpha \in [0,1]$.

Definition 2.6. (Differentiability of a fuzzy function). Suppose that $\tilde{f} : S \subseteq R \to \mathbb{F}$ is fuzzy-valued function with $\tilde{f}(x)[\alpha] = [f^l(x,\alpha), f^r(x,\alpha)]$. If the partial derivatives of $f^l(x,\alpha)$ and $f^r(x,\alpha)$ with respect to $x \in R$ exist and the interval $[\frac{df^l(x,\alpha)}{dx}, \frac{df^r(x,\alpha)}{dx}]$ for $x \in R,\ \alpha \in [0,1]$ defines the α-level set of a fuzzy number, then $\tilde{f}(x)$ is called differentiable and we write

$$\frac{d\tilde{f}(x)}{dx}[\alpha] = [\frac{df^l(x,\alpha)}{dx}, \frac{df^r(x,\alpha)}{dx}],$$

for $x \in R,\ \alpha \in [0,1]$.

Remark 2.7. Using the similar notion as described above, we define the gradient of a fuzzy function $\tilde{f} : S \subseteq R^n \to \mathbb{F}$ as follows: if for each $i = 1, 2, ..., n$, $\frac{\partial \tilde{f}(x)}{\partial x_i}[\alpha] = [\frac{\partial f^l(x,\alpha)}{\partial x_i}, \frac{\partial f^r(x,\alpha)}{\partial x_i}]$, defines the α-level set of a fuzzy number, then the gradient of $\tilde{f}$ at x is

$$\nabla \tilde{f}(x)[\alpha] = (\frac{\partial \tilde{f}(x)}{\partial x_1}[\alpha], ..., \frac{\partial \tilde{f}(x)}{\partial x_n}[\alpha]).$$

From Lemma 2.1, the sufficient conditions that the gradient of $\tilde{f}$ at x exist are: The partial derivatives of $f^l(x,\alpha)$ and $f^r(x,\alpha)$ exist with respect to x_i for $\alpha \in [0,1]$; $\frac{\partial f^l(x,\alpha)}{\partial x_i}$ is a continuous increasing function of α (condition C1); $\frac{\partial f^r(x,\alpha)}{\partial x_i}$ is a continuous decreasing function of α (condition C2); $\frac{\partial f^l(x,1)}{\partial x_i} \leq \frac{\partial f^r(x,1)}{\partial x_i}$ (condition C3).

Definition 2.8. (Integrability of a fuzzy function). We say that $\tilde{f} : S \subseteq R \to \mathbb{F}$ is integrable with respect to x, if both $f^l(x,\alpha)$ and $f^r(x,\alpha)$ are Lebesgue integrable functions of $x \in R$, for all $\alpha \in [0,1]$ and $[\int f^l(x,\alpha)\ dx, \int f^r(x,\alpha)\ dx]$, defines the α-level set of a fuzzy number. We denote the integral of fuzzy function $\tilde{f}$ with respect to x by

$$\int \tilde{f}(x)[\alpha]\ dx = [\int f^l(x,\alpha)\ dx, \int f^r(x,\alpha)\ dx],$$

for $\alpha \in [0,1]$.

Following from Lemma 2.1, the sufficient conditions that the integral of $\tilde{f}$ with respect to x exist are: The Lebesgue integrals of $f^l(x,\alpha)$ and $f^r(x,\alpha)$ exist with respect to x for $\alpha \in [0,1]$; $\int f^l(x,\alpha)\ dx$ is a continuous increasing function of α (condition C1); $\int f^r(x,\alpha)\ dx$ is a continuous decreasing function of α (condition C2); $\int f^l(x,1)\ dx \leq \int f^r(x,1)\ dx$ (condition C3).

Definition 2.9. (Distance measure between fuzzy functions). Consider that $\tilde{f} : S \subseteq R \to \mathbb{F}$ and $\tilde{g} : S \subseteq R \to \mathbb{F}$ are two fuzzy functions. The distance measure between $\tilde{f}$ and $\tilde{g}$ is defined by

$$\begin{aligned} D_{\mathbb{F}}(\tilde{f}(x), \tilde{g}(x)) &= \sup_{0 \leq \alpha \leq 1} H(\tilde{f}(x)[\alpha], \tilde{g}(x)[\alpha]) \\ &= \max\{ \sup_{z \in \tilde{f}(x)[\alpha]} d(z, \tilde{g}(x)[\alpha]),\ \sup_{y \in \tilde{g}(x)[\alpha]} d(\tilde{f}(x)[\alpha], y)\},\ \forall x \in S, \end{aligned} \tag{1}$$

where H is the well-known Hausdorff metric on the family of all nonempty compact subsets of R, and $d(a,B) = \inf_{b\in B} d(a,b)$.
For notational convenience, we define

$$||\tilde{f}(x)||^2_{\mathbb{F}} = D_{\mathbb{F}}(\tilde{f}(x), \tilde{f}(x)), \quad \forall x \in S, \tag{2}$$

for any $\tilde{f} : S \subseteq R \to \mathbb{F}$.

Definition 2.10. (Fuzzy increment). Let $\tilde{x}(.)$ and $\tilde{x}(.) + \delta\tilde{x}(.)$ be fuzzy functions for which the fuzzy functional $\tilde{J}$ is defined. The increment of $\tilde{J}$, denoted by $\Delta\tilde{J}$, is defined as

$$\Delta\tilde{J} := \tilde{J}(\tilde{x} + \delta\tilde{x}) \ominus \tilde{J}(\tilde{x}), \tag{3}$$

where $\delta\tilde{x}(.)$ is known as the variation of $\tilde{x}(.)$.

In order to emphasize that the increment $\Delta\tilde{J}$ depends on the fuzzy functions $\tilde{x}$ and $\delta\tilde{x}$, we may denote $\Delta\tilde{J}$ by $\Delta\tilde{J}(\tilde{x}, \delta\tilde{x})$.

Definition 2.11. (Differentiability of a fuzzy functional). Let the increment of $\tilde{J}$ can be written as

$$\Delta\tilde{J}(\tilde{x}, \delta\tilde{x}) := \delta\tilde{J}(\tilde{x}, \delta\tilde{x}) + \eta(\tilde{x}, \delta\tilde{x}).||\delta\tilde{x}||_{\mathbb{F}}, \tag{4}$$

where $\delta\tilde{J}$ is linear in $\delta\tilde{x}$. We say that $\tilde{J}$ is differentiable on $\tilde{x}$ if for any $\epsilon > 0$,

$$D_{\mathbb{F}}(\eta(\tilde{x}, \delta\tilde{x}), \tilde{0}) < \epsilon, \quad as\ ||\delta\tilde{x}(.)||_{\mathbb{F}} \to 0. \tag{5}$$

3. **The Calculus Variations in Fuzzy Environment**

In this section, we investigate some results to be used in the subsequent discussion. To begin with, let us recall the definition of minimizing function of a fuzzy functional.

Definition 3.1. (Fuzzy relative minimum). A fuzzy functional $\tilde{J}$ with domain $\widetilde{C}[t_0, t_f]$, the class of all fuzzy continuous functions on $[t_0, t_f]$, has a fuzzy relative minimizer $\tilde{x}^* = \tilde{x}^*(t)$, if the increment of $\tilde{J}$ is fuzzy non-negative, that is,

$$\Delta\tilde{J} := \tilde{J}(\tilde{x}) \ominus \tilde{J}(\tilde{x}^*) \succeq \tilde{0}, \tag{6}$$

or equivalently

$$\tilde{J}(\tilde{x}) \succeq \tilde{J}(\tilde{x}^*), \tag{7}$$

for all fuzzy functions $\tilde{x}$ in $\widetilde{C}[t_0, t_f]$.

Notice that the inequality (7) holds if and only if

$$J^l(\tilde{x}, \alpha) \geq J^l(\tilde{x}^*, \alpha),\ and\ J^r(\tilde{x}, \alpha) \geq J^r(\tilde{x}^*, \alpha), \tag{8}$$

for all $\alpha \in [0, 1]$ and all $\tilde{x}'s \in \widetilde{C}[t_0, t_f]$.
Now we are in a position to state a fundamental theorem of the calculus of variations in fuzzy environment.

Theorem 3.2. *(Fuzzy fundamental theorem) Suppose that* $\tilde{x}, \delta\tilde{x} \in \widetilde{C}[t_0, t_f]$ *are fuzzy functions of* $t \in [t_0, t_f]$ *and* $\tilde{J}(\tilde{x})$ *is differentiable fuzzy functional of* $\tilde{x}$. *If* $\tilde{x}^*$ *is a fuzzy minimizer of* $\tilde{J}$, *then the variation of* $\tilde{J}$ *regardless of any boundary conditions must vanish on* $\tilde{x}^*$, *that is,*

$$\delta\tilde{J}(\tilde{x}^*, \delta\tilde{x}) = \tilde{0}, \tag{9}$$

for all admissible $\delta\tilde{x}$ *having the property* $\tilde{x} + \delta\tilde{x} \in \widetilde{C}[t_0, t_f]$.

Proof. It is obvious that equality (9) holds if and only if

$$\delta J^l(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) = 0, \tag{10}$$

$$\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) = 0, \tag{11}$$

for all $\alpha \in [0,1], t \in [t_0, t_f]$ and all admissible $\delta\tilde{x}$ where $\delta\tilde{x}(t)[\alpha] = [\delta x^l(t,\alpha), \delta x^r(t,\alpha)]$.

It is easily observed from Definition 2.11 and sufficiently small $||\delta\tilde{x}||_{\mathbb{F}}$ that $\delta\tilde{J}$ dominates the expression for $\Delta\tilde{J}$. On the other hand, $\tilde{J}(\tilde{x}^*)$ is a fuzzy relative minimum, if

$$\Delta\tilde{J}(\tilde{x}^*) = \Delta\tilde{J}(\tilde{x}^*, \delta\tilde{x}) \succeq \tilde{0}, \tag{12}$$

for all admissible $\delta\tilde{x}$. This implies that

$$\delta\tilde{J}(\tilde{x}^*, \delta\tilde{x}) \succeq \tilde{0}, \tag{13}$$

or equivalently

$$\delta J^l(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) \geq 0, \tag{14}$$

$$\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) \geq 0, \tag{15}$$

for all $\alpha \in [0,1], t \in [t_0, t_f]$ and all admissible $\delta\tilde{x}$.

To complete the proof, we shall show that $\delta J^l = \delta J^r = 0$.

Assume that $\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) > 0$, for all $\alpha \in [0,1], t \in [t_0, t_f]$ and all admissible $\delta\tilde{x}$. Let us take the small enough variation $\delta\tilde{x} = -k^2\delta\tilde{y}$, where k is a non-zero small real number. By Definition 2.11, the linearity of $\delta\tilde{J}$ in $\delta\tilde{x} = -k^2\delta\tilde{y}$ results in

$$\delta\tilde{J}(\tilde{x}^*, -k^2\delta\tilde{y}) = -k^2\delta\tilde{J}(\tilde{x}^*, \delta\tilde{y}),$$

or for all $\alpha \in [0,1]$

$$\delta\tilde{J}(\tilde{x}^*(t), -k^2\delta\tilde{y}(t))[\alpha] = -k^2\delta\tilde{J}(\tilde{x}^*(t), \delta\tilde{y}(t))[\alpha].$$

This implies that for all $\alpha \in [0,1]$

$$\begin{aligned}
&[\delta J^l(\tilde{x}^*(t)[\alpha], -k^2\delta\tilde{y}(t)[\alpha], t, \alpha), \delta J^r(\tilde{x}^*(t)[\alpha], -k^2\delta\tilde{y}(t)[\alpha], t, \alpha)] \\
&\quad = -k^2[\delta J^l(\tilde{x}^*(t)[\alpha], \delta\tilde{y}(t)[\alpha], t, \alpha), \delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{y}(t)[\alpha], t, \alpha)] \\
&\qquad = [-k^2\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{y}(t)[\alpha], t, \alpha), -k^2\delta J^l(\tilde{x}^*(t)[\alpha], \delta\tilde{y}(t)[\alpha], t, \alpha)].
\end{aligned}$$

Consequently, we get

$$\delta J^l(\tilde{x}^*(t)[\alpha], -k^2\delta\tilde{y}(t)[\alpha], t, \alpha) = -k^2\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{y}(t)[\alpha], t, \alpha), \tag{16}$$

$$\delta J^r(\tilde{x}^*(t)[\alpha], -k^2\delta\tilde{y}(t)[\alpha], t, \alpha) = -k^2\delta J^l(\tilde{x}^*(t)[\alpha], \delta\tilde{y}(t)[\alpha], t, \alpha), \tag{17}$$

By the assumption $\delta J^l \geq 0$ for all admissible $\delta\tilde{x}$, the equality (17) gives $-k^2\,\delta J^l \leq 0$, that is, for some admissible $\delta\tilde{x}$ and for all $\alpha \in [0,1], t \in [t_0, t_f]$,
$\delta J^r(\tilde{x}^*(t)[\alpha], \delta\tilde{x}(t)[\alpha], t, \alpha) \leq 0$. This contradicts the assumption that $\tilde{J}(\tilde{x}^*)$ is a fuzzy relative minimum unless $\delta J^r = 0$ is met.

By a similar reasoning, we can show that $\delta J^l = 0$. Therefore, if $\tilde{x}^*$ is a minimizer of $\tilde{J}$, it is necessary that

$$\delta\tilde{J}(\tilde{x}^*, \delta\tilde{x}) = \tilde{0},$$

for any admissible $\delta\tilde{x}$. □

4. **Fuzzy Constrained Minimization Problems**

Let $\tilde{x} = \tilde{x}(t)$ be a fuzzy function of $t \in [t_0, t_f] \subseteq R$ and belongs to the class of fuzzy functions with continuous first derivatives with respect to $t \in [t_0, t_f]$. The fuzzy constrained variational problem can now be posed:

$$(FCVP) \qquad Minimize \quad \tilde{J}(\tilde{x}) := \int_{t_0}^{t_f} \tilde{g}(\tilde{x}(t), \dot{\tilde{x}}(t), t)\, dt$$
$$Subject\ to \quad \tilde{f}(\tilde{x}(t), t) = \tilde{0},$$
$$\tilde{x}(t_0) = \tilde{x}_0, \quad \tilde{x}(t_f) = \tilde{x}_f.$$

Here, $\tilde{g}$ and $\tilde{f}$ assign a fuzzy number to the fuzzy points $(\tilde{x}(t), \dot{\tilde{x}}(t), t) \in \mathbb{F}^2 \times R$ and $(\tilde{x}(t), t) \in \mathbb{F} \times R$, respectively, where $\tilde{x}(t)$ and $\dot{\tilde{x}}(t)$ are fuzzy functions of $t \in [t_0, t_f]$. We assume that the integrand $\tilde{g}$ and fuzzy function $\tilde{f}$ have continuous first and second partial derivatives with respect to all of their arguments.

In order to attack this problem, we adopt fuzzy Lagrange multiplier. To begin with, we formulate the *fuzzy augmented functional* as follows:

$$\tilde{J}_a(\tilde{x}, \tilde{\lambda}) := \int_{t_0}^{t_f} \{\tilde{g}(\tilde{x}(t), \dot{\tilde{x}}(t), t) + \tilde{\lambda}(t)\tilde{f}(\tilde{x}(t), t)\}\, dt. \tag{18}$$

Needless to say that if the constraint is satisfied, we observe that $\tilde{J}_a = \tilde{J}$ for any $\tilde{\lambda}$.

In this portion, in order to simplify the result presentations, we limit ourselves to the special case stated in the following assumption. The further extension to the general case will be done in Section 6.

Remark 4.1. To simplify the variational equations, we assume that $J_a^l(\tilde{x}, \tilde{\lambda}, t, \alpha)$ (or $J_a^r(\tilde{x}, \tilde{\lambda}, t, \alpha)$) is stated in terms containing only $x^l(t,\alpha)$ and $\dot{x}^l(t,\alpha)$ (or only $x^r(t,\alpha)$ and $\dot{x}^r(t,\alpha)$). In this case, we may write $J_a^l(x^l(t,\alpha), \lambda^l(t,\alpha), t, \alpha)$ and $J_a^r(x^r(t,\alpha), \lambda^r(t,\alpha), t, \alpha)$ instead of $J_a^l(\tilde{x}, \tilde{\lambda}, t, \alpha)$ and $J_a^r(\tilde{x}, \tilde{\lambda}, t, \alpha)$, respectively.

Now we derive the variation of the fuzzy functional $\tilde{J}_a$ as

$$\delta\tilde{J}_a(\tilde{x}, \delta\tilde{x}, \tilde{\lambda}, \delta\tilde{\lambda})[\alpha] = [\delta J_a^l(x^l, \delta x^l, \lambda^l, \delta\lambda^l, t, \alpha),\ \delta J_a^r(x^r, \delta x^r, \lambda^r, \delta\lambda^r, t, \alpha)], \tag{19}$$

for all $\alpha \in [0,1]$, where

$$\delta J_a^l(x^l, \delta x^l, \lambda^l, \delta\lambda^l, t, \alpha) = \int_{t_0}^{t_f} \{[\frac{\partial g^l}{\partial x^l}(x^l, \dot{x}^l, t, \alpha) + \lambda^l(t,\alpha)(\frac{\partial f^l}{\partial x^l}(x^l, t, \alpha))]\delta x^l$$
$$+[\frac{\partial g^l}{\partial \dot{x}^l}(x^l, \dot{x}^l, t, \alpha)]\delta\dot{x}^l + [f^l(x^l, t, \alpha)]\delta\lambda^l\}\, dt, \tag{20}$$

and

$$\delta J_a^r(x^r,\delta x^r,\lambda^r,\delta\lambda^r,t,\alpha)=\int_{t_0}^{t_f}\{[\frac{\partial g^r}{\partial x^r}(x^r,\dot{x}^r,t,\alpha)+\lambda^r(t,\alpha)(\frac{\partial f^r}{\partial x^r}(x^r,t,\alpha))]\delta x^r$$
$$+[\frac{\partial g^r}{\partial \dot{x}^r}(x^r,\dot{x}^r,t,\alpha)]\delta\dot{x}^r+[f^r(x^r,t,\alpha)]\delta\lambda^r\}\ dt. \tag{21}$$

Here, $x^{l/r},\dot{x}^{l/r},\delta x^{l/r},\lambda^{l/r}$ and $\delta\lambda^{l/r}$ stand for $x^{l/r}(t,\alpha)$, $\dot{x}^{l/r}(t,\alpha)$, $\delta x^{l/r}(t,\alpha)$, $\lambda^{l/r}(t,\alpha)$ and $\delta\lambda^{l/r}(t,\alpha)$.

For the moment, we consider only (20). If we integrate by parts the term containing $\delta\dot{x}^l$ and retain only the terms inside the integral of (20), then we get

$$\delta J_a^l(x^l,\delta x^l,\lambda^l,\delta\lambda^l,t,\alpha)=\int_{t_0}^{t_f}\{[\frac{\partial g^l}{\partial x^l}(x^l,\dot{x}^l,t,\alpha)+\lambda^l(t,\alpha)(\frac{\partial f^l}{\partial x^l}(x^l,t,\alpha))$$
$$-\frac{d}{dt}(\frac{\partial g^l}{\partial \dot{x}^l}(x^l,\dot{x}^l,t,\alpha))]\delta x^l+[f^l(x^l,t,\alpha)]\delta\lambda^l\}\ dt. \tag{22}$$

By Theorem 3.2 we find that the variation (19) must be zero on a fuzzy minimizer $\tilde{x}^*$ and consequently the latter variation must be zero on $\tilde{x}^{*l}$, too. Moreover, the minimizer $\tilde{x}^*$ has to satisfy the constraint

$$f^l(x^{*l},t,\alpha)=0,\quad for\ all\ \alpha\in[0,1],t\in[t_0,t_f]. \tag{23}$$

This relation makes that $\delta\lambda^l$ is removed from the terms inside the integral of (22). It can be seen that if the arbitrary λ^l is chosen such that the coefficient of δx^l in (22) is zero, then the following result is obtained

$$\frac{\partial g^l}{\partial x^l}(x^{*l},\dot{x}^{*l},t,\alpha)+\lambda^{*l}(t,\alpha)(\frac{\partial f^l}{\partial x^l}(x^{*l},t,\alpha))-\frac{d}{dt}(\frac{\partial g^l}{\partial \dot{x}^l}(x^{*l},\dot{x}^{*l},t,\alpha))=0. \tag{24}$$

Now with respect to the *left-hand augmented integrand function* g_a^l defined by

$$g_a^l(x^l,\dot{x}^l,\lambda^l,t,\alpha):=g^l(x^l,\dot{x}^l,t,\alpha)+\lambda^l(t,\alpha)f^l(x^l,t,\alpha), \tag{25}$$

the equation (24) becomes

$$\frac{\partial g_a^l}{\partial x^l}(x^{*l},\dot{x}^{*l},\lambda^{*l},t,\alpha)-\frac{d}{dt}(\frac{\partial g_a^l}{\partial \dot{x}^l}(x^{*l},\dot{x}^{*l},\lambda^{*l},t,\alpha))=0. \tag{26}$$

Following the scheme of obtaining (23) and (26), and adapting it to the case under consideration involving (21), one can show that on $\tilde{x}^*$

$$f^r(x^{*r},t,\alpha)=0,\quad for\ all\ \alpha\in[0,1],t\in[t_0,t_f], \tag{27}$$

and

$$\frac{\partial g_a^r}{\partial x^r}(x^{*r},\dot{x}^{*r},\lambda^{*r},t,\alpha)-\frac{d}{dt}(\frac{\partial g_a^r}{\partial \dot{x}^r}(x^{*r},\dot{x}^{*r},\lambda^{*r},t,\alpha))=0, \tag{28}$$

where the *right-hand augmented integrand function* g_a^r is defined by

$$g_a^r(x^r,\dot{x}^r,\lambda^r,t,\alpha):=g^r(x^r,\dot{x}^r,t,\alpha)+\lambda^r(t,\alpha)f^r(x^r,t,\alpha). \tag{29}$$

We are now ready to state the necessary conditions that must be satisfied by a fuzzy relative minimizer $\tilde{x}^*$ of (FCVP) as follows:

Theorem 4.2. *Let $\tilde{x}^* = \tilde{x}^*(t)$ be an admissible fuzzy function, i.e., it is twice continuously differentiable fuzzy function. Then, in order that $\tilde{x}^*$ give a relative (local) minimum to the fuzzy functional $\tilde{J}$ in (FCVP), it is necessary that for all $\alpha \in [0,1],\ t \in [t_0, t_f]$*

$$f^l(x^{*l}, t, \alpha) = 0, \tag{30}$$

$$f^r(x^{*r}, t, \alpha) = 0, \tag{31}$$

$$\frac{\partial g_a^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, \lambda^{*l}, t, \alpha) - \frac{d}{dt}\left(\frac{\partial g_a^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, \lambda^{*l}, t, \alpha)\right) = 0, \tag{32}$$

$$\frac{\partial g_a^r}{\partial x^r}(x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha) - \frac{d}{dt}\left(\frac{\partial g_a^r}{\partial \dot{x}^r}(x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha)\right) = 0, \tag{33}$$

where g_a^l and g_a^r are those defined by (25) *and* (29)*, respectively.*

Notice that although the above results are the same as the results obtained perviously in [4], the reasoning used here is quite different.

Let us now introduce a fuzzy variational problem constrained by fuzzy differential equations:

$$(FDCVP) \qquad Minimize \quad \tilde{J}(\tilde{x}) := \int_{t_0}^{t_f} \tilde{g}(\tilde{x}(t), \dot{\tilde{x}}(t), t)\, dt$$

$$Subject\ to \quad \tilde{f}(\tilde{x}(t), \dot{\tilde{x}}(t), t) = \tilde{0},$$

$$\tilde{x}(t_0) = \tilde{x}_0, \quad \tilde{x}(t_f) = \tilde{x}_f.$$

Here, $\tilde{g}$ and $\tilde{f}$ assign a fuzzy number to the fuzzy point $(\tilde{x}(t), \dot{\tilde{x}}(t), t) \in \mathbb{F}^2 \times R$ where $\tilde{x}(t)$ and $\dot{\tilde{x}}(t)$ are fuzzy functions of $t \in [t_0, t_f]$. We assume that the integrand $\tilde{g}$ and fuzzy function $\tilde{f}$ have continuous first and second partial derivatives with respect to all of their arguments.

The reasoning that leads to the equations constitute a set of necessary conditions for $\tilde{x}^*$ to be a fuzzy relative minimizer of (FDCVP) is the same as that used for (FCVP). Therefore necessary conditions that must be satisfied by a minimizer of (FDCVP) can be found as follows:

Theorem 4.3. *Let $\tilde{x}^* = \tilde{x}^*(t)$ be an admissible fuzzy function, i.e., it is twice continuously differentiable fuzzy function. Then, in order that $\tilde{x}^*$ give a relative (local) minimum to the fuzzy functional $\tilde{J}$ in (FDCVP), it is necessary that for all $\alpha \in [0,1],\ t \in [t_0, t_f]$*

$$f^l(x^{*l}, \dot{x}^{*l}, t, \alpha) = 0, \tag{34}$$

$$f^r(x^{*r}, \dot{x}^{*r}, t, \alpha) = 0, \tag{35}$$

$$\frac{\partial g_a^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, \lambda^{*l}, t, \alpha) - \frac{d}{dt}\left(\frac{\partial g_a^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, \lambda^{*l}, t, \alpha)\right) = 0, \tag{36}$$

$$\frac{\partial g_a^r}{\partial x^r}(x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha) - \frac{d}{dt}\left(\frac{\partial g_a^r}{\partial \dot{x}^r}(x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha)\right) = 0, \tag{37}$$

where

$$g_a^l(x^l, \dot{x}^l, \lambda^l, t, \alpha) := g^l(x^l, \dot{x}^l, t, \alpha) + \lambda^l(t, \alpha) f^l(x^l, \dot{x}^l, t, \alpha), \tag{38}$$

and

$$g_a^r(x^r, \dot{x}^r, \lambda^r, t, \alpha) := g^r(x^r, \dot{x}^r, t, \alpha) + \lambda^r(t, \alpha) f^r(x^r, \dot{x}^r, t, \alpha). \tag{39}$$

Remark 4.4. If we regard the fuzzy vector $\tilde{x} = \tilde{x}(t)$ as $[\tilde{x}\vdots\, \tilde{u}]$ in the constraint of (FDCVP), we then may represent the state equation constraints in fuzzy optimal control problems defined latter.

5. **Fuzzy Optimal Control Problems**

In this section, we are interested to apply fuzzy variational approaches to fuzzy optimal control problems to derive necessary conditions for optimal fuzzy control which will be referred to as *fuzzy Pontryagin's minimum principle.*

Let $\tilde{x} = \tilde{x}(t)$ be a fuzzy function of $t \in [t_0, t_f] \subseteq R$ and belonging to the class of fuzzy functions with continuous first derivatives with respect to $t \in [t_0, t_f]$. The fuzzy optimal control problem can now be posed:

$$
\begin{aligned}
(FOCP) \qquad\qquad & Minimize \quad \tilde{J}(\tilde{u}) := \int_{t_0}^{t_f} \tilde{g}(\tilde{x}(t), \tilde{u}(t), t)\, dt \\
& Subject\ to \quad \dot{\tilde{x}}(t) = \tilde{h}(\tilde{x}(t), \tilde{u}(t), t), \\
& \qquad\qquad\qquad \tilde{x}(t_0) = \tilde{x}_0,\ \tilde{x}(t_f) = \tilde{x}_f.
\end{aligned}
$$

Here, $\tilde{g}$ and $\tilde{h}$ assign a fuzzy number to the fuzzy point $(\tilde{x}(t), \tilde{u}(t), t) \in \mathbb{F}^2 \times R$, where the *fuzzy state* $\tilde{x}(t)$ and the *fuzzy control* $\tilde{u}(t)$ are fuzzy functions of t belonging to the specified interval $[t_0, t_f]$. We assume that the integrand $\tilde{g}$ and fuzzy function $\tilde{h}$ have continuous first and second partial derivatives with respect to all of their arguments.

Definition 5.1. (Admissible fuzzy state). We say that $\tilde{x} = \tilde{x}(t)$ is admissible, if it satisfies the endpoints conditions and also is twice continuously differentiable with respect to $t \in [t_0, t_f]$.

In addition, our definition of an *admissible fuzzy control* $\tilde{u} = \tilde{u}(t)$ is that $\tilde{u}$ is not bounded.

Remark that by Definition 5.1, we may cast a (FOCP) into the form of (FDCVP) regardless any boundary conditions and so it is reasonable to expect that necessary conditions for minimizer of (FOCP) have to be the same as that of (FDCVP).

In sequel, to gain the fuzzy Pontryagin's minimum principle as form as the crisp counterpart, we require to convert the fuzzy process

$$\dot{\tilde{x}}(t) = \tilde{h}(\tilde{x}(t), \tilde{u}(t), t), \tag{40}$$

into the form

$$\tilde{f}(\tilde{x}(t), \tilde{u}(t), \dot{\tilde{x}}(t), t) = \tilde{0}, \tag{41}$$

which is in form of a fuzzy differential equation involved in a (FDCVP).

In view of the above-mentioned conversion, we define

$$\tilde{f}(\tilde{x}(t), \tilde{u}(t), \dot{\tilde{x}}(t), t) := \tilde{h}(\tilde{x}(t), \tilde{u}(t), t) \ominus \dot{\tilde{x}}(t). \tag{42}$$

Theorem 5.2. *(Fuzzy Pontryagin's minimum principle) Let $\tilde{x}^* = \tilde{x}^*(t)$ be an admissible fuzzy state and assume that $\tilde{u}^* = \tilde{u}^*(t)$ is an admissible fuzzy control.*

Then, in order that $\tilde{u}^*$ give a fuzzy optimal control to the fuzzy functional $\tilde{J}$ in (FOCP), it is necessary that for all $\alpha \in [0,1],\ t \in [t_0, t_f]$

$$\dot{x}^{*l}(t,\alpha) = \frac{\partial \mathbb{H}^l}{\partial \lambda^l}(x^{*l}(t,\alpha), u^{*l}(t,\alpha), \lambda^{*l}(t,\alpha), t, \alpha), \tag{43}$$

$$\dot{x}^{*r}(t,\alpha) = \frac{\partial \mathbb{H}^r}{\partial \lambda^r}(x^{*r}(t,\alpha), u^{*r}(t,\alpha), \lambda^{*r}(t,\alpha), t, \alpha), \tag{44}$$

$$\dot{\lambda}^{*l}(t,\alpha) = -\frac{\partial \mathbb{H}^l}{\partial x^l}(x^{*l}(t,\alpha), u^{*l}(t,\alpha), \lambda^{*l}(t,\alpha), t, \alpha), \tag{45}$$

$$\dot{\lambda}^{*r}(t,\alpha) = -\frac{\partial \mathbb{H}^r}{\partial x^r}(x^{*r}(t,\alpha), u^{*r}(t,\alpha), \lambda^{*r}(t,\alpha), t, \alpha), \tag{46}$$

$$0 = \frac{\partial \mathbb{H}^l}{\partial u^l}(x^{*l}(t,\alpha), u^{*l}(t,\alpha), \lambda^{*l}(t,\alpha), t, \alpha), \tag{47}$$

$$0 = \frac{\partial \mathbb{H}^r}{\partial u^r}(x^{*r}(t,\alpha), u^{*r}(t,\alpha), \lambda^{*r}(t,\alpha), t, \alpha), \tag{48}$$

where fuzzy function $\tilde{\mathbb{H}}$, called fuzzy Hamiltonian function, is defined by

$$\tilde{\mathbb{H}}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t) := \tilde{g}(\tilde{x}(t), \tilde{u}(t), t) + \tilde{\lambda}(t)\tilde{h}(\tilde{x}(t), \tilde{u}(t), t), \tag{49}$$

with the α-level set

$$\tilde{\mathbb{H}}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t)[\alpha] = [\mathbb{H}^l(x^l, u^l, \lambda^l, t, \alpha),\ \mathbb{H}^r(x^r, u^r, \lambda^r, t, \alpha)]. \tag{50}$$

Proof. Let us first formulate the following fuzzy augmented functional by adjoining the constraining relation to $\tilde{J}$ of (FOCP) that yields

$$\tilde{J}_a(\tilde{u}) := \int_{t_0}^{t_f} \tilde{g}_a(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), \dot{\tilde{x}}(t), t)\ dt, \tag{51}$$

where

$$\tilde{g}_a(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), \dot{\tilde{x}}(t), t) := \tilde{g}(\tilde{x}(t), \tilde{u}(t), t) + \tilde{\lambda}(t)(\tilde{h}(\tilde{x}(t), \tilde{u}(t), t) \ominus \dot{\tilde{x}}(t)), \tag{52}$$

and $\tilde{\lambda}$ is the so-called fuzzy Lagrange multiplier.

On the extremal $\tilde{u}^*$, the variation of $\tilde{J}_a$ must be zero, that is, $\delta\tilde{J}_a(\tilde{u}^*) = \tilde{0}$. This admits for all $\alpha \in [0,1]$,

$$\delta J_a^l(u^{*l}, \alpha) = 0, \tag{53}$$

and

$$\delta J_a^r(u^{*r}, \alpha) = 0. \tag{54}$$

In the remaining of the proof we will ignore the similar arguments and thus we consider only (53).

Now, the variation of J_a^l is determined as

$$\begin{aligned} 0 &= \delta J_a^l(u^{*l}, \alpha) \\ &= \int_{t_0}^{t_f} \{[\frac{\partial g_a^l}{\partial x^l}(x^{*l}, u^{*l}, \lambda^{*l}, \dot{x}^{*l}, t, \alpha) - \frac{d}{dt}(\frac{\partial g_a^l}{\partial \dot{x}^l}(x^{*l}, u^{*l}, \lambda^{*l}, \dot{x}^{*l}, t, \alpha))]\delta x^l \\ &\quad + [\frac{\partial g_a^l}{\partial u^l}(x^{*l}, u^{*l}, \lambda^{*l}, \dot{x}^{*l}, t, \alpha)]\delta u^l + [\frac{\partial g_a^l}{\partial \lambda^l}(x^{*l}, u^{*l}, \lambda^{*l}, \dot{x}^{*l}, t, \alpha)]\delta \lambda^l\}\ dt. \end{aligned} \tag{55}$$

With respect to the fuzzy augmented integrand function $\tilde{g}_a$ defined in (52), the latter equation can be written

$$0 = \int_{t_0}^{t_f} \{[\frac{\partial g^l}{\partial x^l}(x^{*l}, u^{*l}, t, \alpha) + \lambda^{*l}(t, \alpha)(\frac{\partial h^l}{\partial x^l}(x^{*l}, u^{*l}, t, \alpha)) - \frac{d}{dt}(-\lambda^{*l}(t, \alpha))]\delta x^l + [\frac{\partial g^l}{\partial u^l}(x^{*l}, u^{*l}, t, \alpha) + \lambda^{*l}(t, \alpha)(\frac{\partial h^l}{\partial u^l}(x^{*l}, u^{*l}, t, \alpha))]\delta u^l + [h^l(x^{*l}, u^{*l}, t, \alpha) - \dot{x}^{*l}(t, \alpha)]\delta\lambda^l\} \ dt. \quad (56)$$

Since the constraint must be satisfied by the extremal $\tilde{u}^*$, we find that

$$\dot{x}^{*l}(t, \alpha) = h^l(x^{*l}, u^{*l}, t, \alpha), \quad (57)$$

and hence the coefficient of $\delta\lambda^l$ in (56) is zero.

In addition, the arbitrary fuzzy Lagrange multiplier λ^{*l} can be chosen such that the coefficient of δx^l does not appear in the above integral. Thus,

$$\dot{\lambda}^{*l}(t, \alpha) = -[\frac{\partial g^l}{\partial x^l}(x^{*l}, u^{*l}, t, \alpha) + \lambda^{*l}(t, \alpha)(\frac{\partial h^l}{\partial x^l}(x^{*l}, u^{*l}, t, \alpha))]. \quad (58)$$

There exists still a term inside the integral (56) to deal with. Since the equality (56) must be satisfied, we get

$$0 = \frac{\partial g^l}{\partial u^l}(x^{*l}, u^{*l}, t, \alpha) + \lambda^{*l}(t, \alpha)(\frac{\partial h^l}{\partial u^l}(x^{*l}, u^{*l}, t, \alpha)). \quad (59)$$

Using the *left-hand Hamiltonian function* $\mathbb{H}^l(x^l, u^l, \lambda^l, t, \alpha)$ defined in (50), the equations (57)-(59) can be written more compactly as follows:

$$\dot{x}^{*l}(t, \alpha) = \frac{\partial \mathbb{H}^l}{\partial \lambda^l}(x^{*l}(t, \alpha), u^{*l}(t, \alpha), \lambda^{*l}(t, \alpha), t, \alpha), \quad (60)$$

$$\dot{\lambda}^{*l}(t, \alpha) = -\frac{\partial \mathbb{H}^l}{\partial x^l}(x^{*l}(t, \alpha), u^{*l}(t, \alpha), \lambda^{*l}(t, \alpha), t, \alpha), \quad (61)$$

$$0 = \frac{\partial \mathbb{H}^l}{\partial u^l}(x^{*l}(t, \alpha), u^{*l}(t, \alpha), \lambda^{*l}(t, \alpha), t, \alpha), \quad (62)$$

for all $\alpha \in [0, 1], \ t \in [t_0, t_f]$.

Again, following the scheme of obtaining (60)-(62) and adapting it to the case under consideration involving (54), one may show that for all $\alpha \in [0, 1], \ t \in [t_0, t_f]$

$$\dot{x}^{*r}(t, \alpha) = \frac{\partial \mathbb{H}^r}{\partial \lambda^r}(x^{*r}(t, \alpha), u^{*r}(t, \alpha), \lambda^{*r}(t, \alpha), t, \alpha), \quad (63)$$

$$\dot{\lambda}^{*r}(t, \alpha) = -\frac{\partial \mathbb{H}^r}{\partial x^r}(x^{*r}(t, \alpha), u^{*r}(t, \alpha), \lambda^{*r}(t, \alpha), t, \alpha), \quad (64)$$

$$0 = \frac{\partial \mathbb{H}^r}{\partial u^r}(x^{*r}(t, \alpha), u^{*r}(t, \alpha), \lambda^{*r}(t, \alpha), t, \alpha), \quad (65)$$

where the *right-hand Hamiltonian function* $\mathbb{H}^r(x^r, u^r, \lambda^r, t, \alpha)$ is defined in (50). □

6. **Generalization**

Without restrictions imposed by Remark 4.1, this section is devoted to extend Theorem 4.2 for the general case where both $J_a^l(\tilde{x},\tilde{\lambda},t,\alpha)$ and $J_a^r(\tilde{x},\tilde{\lambda},t,\alpha)$ in (19) are considered to be in terms containing $x^l(t,\alpha)$, $\dot{x}^l(t,\alpha)$, $x^r(t,\alpha)$ and $\dot{x}^r(t,\alpha)$. In this case, we may write $J_a^l(x^l,x^r,\lambda^l,t,\alpha)$ and $J_a^r(x^l,x^r,\lambda^r,t,\alpha)$ instead of $J_a^l(\tilde{x},\tilde{\lambda},t,\alpha)$ and $J_a^r(\tilde{x},\tilde{\lambda},t,\alpha)$, respectively, where $x^{l/r},\dot{x}^{l/r},\lambda^{l/r}$ stand for $x^{l/r}(t,\alpha),\dot{x}^{l/r}(t,\alpha),\lambda^{l/r}(t,\alpha)$.

This assumption makes variations (20) and (21) become

$$\begin{aligned}&\delta J_a^l(x^l,\delta x^l,x^r,\delta x^r,\lambda^l,\delta\lambda^l,t,\alpha)=\\&\int_{t_0}^{t_f}\{[\frac{\partial g^l}{\partial x^l}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)+\lambda^l(t,\alpha)(\frac{\partial f^l}{\partial x^l}(x^l,x^r,t,\alpha))]\delta x^l\\&+[\frac{\partial g^l}{\partial x^r}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)+\lambda^l(t,\alpha)(\frac{\partial f^l}{\partial x^r}(x^l,x^r,t,\alpha))]\delta x^r\\&+[\frac{\partial g^l}{\partial \dot{x}^l}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)]\delta\dot{x}^l+[\frac{\partial g^l}{\partial \dot{x}^r}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)]\delta\dot{x}^r\\&+[f^l(x^l,x^r,t,\alpha)]\delta\lambda^l\}\ dt,\end{aligned}\tag{66}$$

and

$$\begin{aligned}&\delta J_a^r(x^l,\delta x^l,x^r,\delta x^r,\lambda^l,\delta\lambda^l,t,\alpha)=\\&\int_{t_0}^{t_f}\{[\frac{\partial g^r}{\partial x^l}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)+\lambda^r(t,\alpha)(\frac{\partial f^r}{\partial x^l}(x^l,x^r,t,\alpha))]\delta x^l\\&+[\frac{\partial g^r}{\partial x^r}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)+\lambda^r(t,\alpha)(\frac{\partial f^r}{\partial x^r}(x^l,x^r,t,\alpha))]\delta x^r\\&+[\frac{\partial g^r}{\partial \dot{x}^l}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)]\delta\dot{x}^l+[\frac{\partial g^r}{\partial \dot{x}^r}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)]\delta\dot{x}^r\\&+[f^r(x^l,x^r,t,\alpha)]\delta\lambda^r\}\ dt.\end{aligned}\tag{67}$$

Now, we consider only (66). If we integrate by parts the terms containing $\delta\dot{x}^l$ and $\delta\dot{x}^r$ and retain only the terms inside the integral of (66), then it results in

$$\begin{aligned}&\delta J_a^l(x^l,\delta x^l,x^r,\delta x^r,\lambda^l,\delta\lambda^l,t,\alpha)=\\&\int_{t_0}^{t_f}\{[\frac{\partial g^l}{\partial x^l}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)+\lambda^l(t,\alpha)(\frac{\partial f^l}{\partial x^l}(x^l,x^r,t,\alpha))\\&\qquad-\frac{d}{dt}(\frac{\partial g^l}{\partial \dot{x}^l}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha))]\delta x^l\\&+[\frac{\partial g^l}{\partial x^r}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha)+\lambda^l(t,\alpha)(\frac{\partial f^l}{\partial x^r}(x^l,x^r,t,\alpha))\\&\qquad-\frac{d}{dt}(\frac{\partial g^l}{\partial \dot{x}^r}(x^l,\dot{x}^l,x^r,\dot{x}^r,t,\alpha))]\delta x^r\\&+[f^l(x^l,x^r,t,\alpha)]\delta\lambda^l\}\ dt.\end{aligned}\tag{68}$$

Using the reasoning that led to (23) and (26), one can easily obtain for all $\alpha\in[0,1],t\in[t_0,t_f]$ the following results:

$$f^l(x^{*l},x^{*r},t,\alpha)=0,\tag{69}$$

and

$$\frac{\partial g_a^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha) - \frac{d}{dt}(\frac{\partial g_a^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha)) = 0, \quad (70)$$

$$\frac{\partial g_a^l}{\partial x^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha) - \frac{d}{dt}(\frac{\partial g_a^l}{\partial \dot{x}^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha)) = 0. \quad (71)$$

Here, g_a^l is defined by

$$g_a^l(x^l, \dot{x}^l, x^r, \dot{x}^r, \lambda^l, t, \alpha) := g^l(x^l, \dot{x}^l, x^r, \dot{x}^r, t, \alpha) + \lambda^l(t, \alpha) f^l(x^l, x^r, t, \alpha). \quad (72)$$

Similar conclusions follow by concerning (67) and hence in general case the generalized form of Theorem 4.2 can be stated as:

Theorem 6.1. *Let $\tilde{x}^* = \tilde{x}^*(t)$ be an admissible fuzzy function, i.e., it is twice continuously differentiable fuzzy function. Then, in order that $\tilde{x}^*$ give a relative (local) minimum to the fuzzy functional $\tilde{J}$ in (FCVP), it is necessary that for all* $\alpha \in [0,1],\ t \in [t_0, t_f]$

$$f^l(x^{*l}, x^{*r}, t, \alpha) = 0, \quad (73)$$

$$f^r(x^{*l}, x^{*r}, t, \alpha) = 0, \quad (74)$$

$$\frac{\partial g_a^l}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha) - \frac{d}{dt}(\frac{\partial g_a^l}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha)) = 0, \quad (75)$$

$$\frac{\partial g_a^l}{\partial x^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha) - \frac{d}{dt}(\frac{\partial g_a^l}{\partial \dot{x}^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*l}, t, \alpha)) = 0, \quad (76)$$

$$\frac{\partial g_a^r}{\partial x^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha) - \frac{d}{dt}(\frac{\partial g_a^r}{\partial \dot{x}^l}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha)) = 0, \quad (77)$$

$$\frac{\partial g_a^r}{\partial x^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha) - \frac{d}{dt}(\frac{\partial g_a^r}{\partial \dot{x}^r}(x^{*l}, \dot{x}^{*l}, x^{*r}, \dot{x}^{*r}, \lambda^{*r}, t, \alpha)) = 0, \quad (78)$$

where

$$g_a^l(x^l, \dot{x}^l, x^r, \dot{x}^r, \lambda^l, t, \alpha) := g^l(x^l, \dot{x}^l, x^r, \dot{x}^r, t, \alpha) + \lambda^l(t, \alpha) f^l(x^l, x^r, t, \alpha),$$
$$g_a^r(x^l, \dot{x}^l, x^r, \dot{x}^r, \lambda^r, t, \alpha) := g^r(x^l, \dot{x}^l, x^r, \dot{x}^r, t, \alpha) + \lambda^r(t, \alpha) f^r(x^l, x^r, t, \alpha).$$

7. Illustrative Example

Let us now illustrate the determination of the fuzzy optimal control for the following (FOCP) by applying the fuzzy Pontryagin's minimum principle derived in Theorem 5.2.

Example 7.1. Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}) := \int_0^1 \tilde{u}^2(t)\ dt,$$

subject to

$$\dot{\tilde{x}}(t) = \tilde{u}(t) - \langle 0, 1, 3\rangle \tilde{x}(t), \quad t \in [0,1],$$

with boundary conditions

$$\tilde{x}(0) = 1 = \langle 1, 1, 1\rangle,\ \tilde{x}(1) = 0 = \langle 0, 0, 0\rangle.$$

Solution. The first step is to form the fuzzy Hamiltonian

$$\tilde{\mathbb{H}}(\tilde{x}(t), \tilde{u}(t), \tilde{\lambda}(t), t) := \tilde{u}^2(t) + \tilde{\lambda}(t)[\tilde{u}(t) - \langle 0, 1, 3\rangle \tilde{x}(t)], \tag{79}$$

and hence the α-level set of $\tilde{\mathbb{H}}$ is characterized by

$$\mathbb{H}^l(x^l, u^l, \lambda^l, t, \alpha) = u^{l^2}(t,\alpha) + \lambda^l(t,\alpha)[u^l(t,\alpha) - (3-2\alpha)x^l(t,\alpha)], \tag{80}$$

and

$$\mathbb{H}^r(x^r, u^r, \lambda^r, t, \alpha) = u^{r^2}(t,\alpha) + \lambda^r(t,\alpha)[u^r(t,\alpha) - (\alpha)x^r(t,\alpha)]. \tag{81}$$

By applying Theorem 5.2 to (80), we find the necessary conditions for optimality as follows:

$$\dot{x}^l(t,\alpha) = \frac{\partial \mathbb{H}^l}{\partial \lambda^l} = u^l(t,\alpha) - (3-2\alpha)x^l(t,\alpha), \tag{82}$$

$$\dot{\lambda}^l(t,\alpha) = -\frac{\partial \mathbb{H}^l}{\partial x^l} = (3-2\alpha)\lambda^l(t,\alpha), \tag{83}$$

$$0 = \frac{\partial \mathbb{H}^l}{\partial u^l} = 2u^l(t,\alpha) + \lambda^l(t,\alpha). \tag{84}$$

We begin with solving differential equation (83)

$$\dot{\lambda}^l(t,\alpha) - (3-2\alpha)\lambda^l(t,\alpha) = 0.$$

The above differential equation is linear with constant coefficients, for a fixed $\alpha \in [0,1]$. Hence, by virtue of the classical differential equation theory, we may solve it analytically for fixed $\alpha \in [0,1]$ to arrive at

$$\lambda^{*l}(t,\alpha) = k_1 e^{(3-2\alpha)t}.$$

Substituting $\lambda^{*l}(t,\alpha)$ into (1) gives

$$u^{*l}(t,\alpha) = -\frac{k_1}{2} e^{(3-2\alpha)t}.$$

If equation (82) is solved for $u^{*l}(t,\alpha)$, then we obtain

$$x^{*l}(t,\alpha) = -\frac{k_1}{4(3-2\alpha)} e^{(3-2\alpha)t} + k_2 e^{-(3-2\alpha)t}. \tag{85}$$

Constants of integration k_1, k_2 might be given by the boundary conditions

$$1 = x^{*l}(0,\alpha) = k_2 - \frac{k_1}{4(3-2\alpha)},$$

$$0 = x^{*l}(1,\alpha) = k_2 e^{-(3-2\alpha)} - \frac{k_1}{4(3-2\alpha)} e^{(3-2\alpha)},$$

hence,

$$k_1 = \frac{2(3-2\alpha)e^{-(3-2\alpha)}}{\sinh(3-2\alpha)}, \quad k_2 = 1 + \frac{e^{-(3-2\alpha)}}{2\sinh(3-2\alpha)}. \tag{86}$$

Using the constants (86) and applying simple calculations, we obtain for all $\alpha \in [0,1], t \in [0,1]$ that

$$x^{*l}(t,\alpha) = \frac{\sinh(1-t)(3-2\alpha)}{\sinh(3-2\alpha)}, \tag{87}$$

and

$$u^{*l}(t,\alpha) = \frac{-(3-2\alpha)e^{-(1-t)(3-2\alpha)}}{\sinh(3-2\alpha)}. \tag{88}$$

By (87) and (88), it is not hard to verify that $x^{*l}(t,\alpha)$ and $u^{*l}(t,\alpha)$ are continuous increasing functions of α (condition C1 of Lemma 2.1).

Again, following the same arguments as above and considering (81), it is apparent that

$$x^{*r}(t,\alpha)=\frac{\sinh(1-t)\alpha}{\sinh\alpha}, \tag{89}$$

and

$$u^{*r}(t,\alpha)=\frac{-\alpha e^{-(1-t)\alpha}}{\sinh\alpha}. \tag{90}$$

By (89) and (90), one can easily show that $x^{*r}(t,\alpha)$ and $u^{*r}(t,\alpha)$ are continuous decreasing functions of α (condition C2 of Lemma 2.1).

Moreover, we observe that for all $0\le t\le 1$,

$$\begin{aligned}
x^{*l}(t,1)&=\frac{\sinh(1-t)(3-2(1))}{\sinh(3-2(1))},\\
x^{*r}(t,1)&=\frac{\sinh(1-t)(1)}{\sinh(1)},\\
u^{*l}(t,1)&=\frac{-(3-2(1))e^{-(1-t)(3-2(1))}}{\sinh(3-2(1))},\\
u^{*r}(t,1)&=\frac{-(1)e^{-(1-t)(1)}}{\sinh(1)}.
\end{aligned}$$

Thus, $x^{*l}(t,1)\le x^{*r}(t,1)$ and $u^{*l}(t,1)\le u^{*r}(t,1)$ (condition C3 of Lemma 2.1).

Therefore, the α-level sets of fuzzy numbers: optimal fuzzy state $\tilde{x}^*$ and optimal fuzzy control $\tilde{u}^*$ are characterized, respectively, by

$$\begin{aligned}
\tilde{x}^*(t)[\alpha] &= [x^{*l}(t,\alpha),x^{*r}(t,\alpha)]\\
&= [\frac{\sinh(1-t)(3-2\alpha)}{\sinh(3-2\alpha)},\ \frac{\sinh(1-t)\alpha}{\sinh\alpha}],
\end{aligned}$$

and

$$\begin{aligned}
\tilde{u}^*(t)[\alpha] &= [u^{*l}(t,\alpha),u^{*r}(t,\alpha)]\\
&= [\frac{-(3-2\alpha)e^{-(1-t)(3-2\alpha)}}{\sinh(3-2\alpha)},\ \frac{-\alpha e^{-(1-t)\alpha}}{\sinh\alpha}].
\end{aligned}$$

8. **Conclusion**

In this article, we established the necessary optimality conditions for the fuzzy optimization problems and the fuzzy Pontryagin's minimum principle for the fuzzy optimal control problems using the concepts of differentiability and integrability of a fuzzy mapping, parameterized by the left and right-hand functions of its α-level sets, together with the concept of fuzzy variation. By an example, we summarized and highlighted the main feature of the article.

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PONTRYAGIN'S MINIMUM PRINCIPLE FOR FUZZY OPTIMAL CONTROL PROBLEMS

B. FARHADINIA

اصل کمینه پونتریاگین برای مسایل کنترل بهینه فازی

چکیده. دراین مقاله شرایط بهینگی لازم که به عنوان اصل کمینه پونتریاگین شناخته می شود برای مسایل کنترل بهینه فازی ارایه می گردد. اساس این شرایط بر مبنای مفاهیم مشتقپذیری و انتگرالپذیری نگاشت فازی بنا نهاده شده که خود این مفاهیم با استفاده از توابع چپ و راست مجموعه های α -برش پارامتری شده اند.