FUZZY RELATIONAL MATRIX-BASED STABILITY ANALYSIS FOR FIRST-ORDER FUZZY RELATIONAL DYNAMIC SYSTEMS

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Abstract. In this paper, two sets of sufficient conditions are obtained to ensure the existence and stability of a unique equilibrium point of unforced first-order fuzzy relational dynamical systems by using two different approaches which are both based on the fuzzy relational matrix of the model. In the first approach, the equilibrium point of the system is one of the centers of the related membership functions. In the second approach, the equilibrium point of the system is the origin (the center of the middle membership function) and the behavior of the system, though can be nonlinear, is symmetric around the origin. The results are approved by numerical examples.

1. Introduction

Fuzzy models of dynamic systems are known to be categorized in three main groups, the TSK models, the linguistic models, and the fuzzy relational models. A fuzzy relational model (FRM) can be considered as an extended fuzzy linguistic model. Indeed, in an FRM, a truth degree is assigned to every rule that can be constituted. The collection of such degrees is gathered in a matrix which is representative of the rule base in the model, called the fuzzy relational matrix (FRX). This matrix is composed with model inputs through an appropriate fuzzy relational composition. Thus, fuzzy relational modeling provides a more systematic framework for fuzzy linguistic modeling. See [9] for further information about FRMs.

Fuzzy relational models have been used in various applications, basically in modeling static and dynamic systems, [1, 4, 10, 11]. The focus in this paper is on the equilibrium points of fuzzy relational dynamic systems and their stability. A fuzzy relational dynamic system (FRDS) is indeed an FRM of a dynamic system. The goal of this paper is to obtain some appropriate conditions to ensure the existence and stability of a unique equilibrium point of an FRDS. In the area of linguistic fuzzy systems, several different approaches have been taken to the problem of stability investigation, e.g., frequency domain analysis of fuzzy systems, [13], or investigating the stability of fuzzy models based on petri-nets, [12]. To have an overview of several methods of investigating the stability of linguistic fuzzy models, the reader may refer to [5–8, 12].

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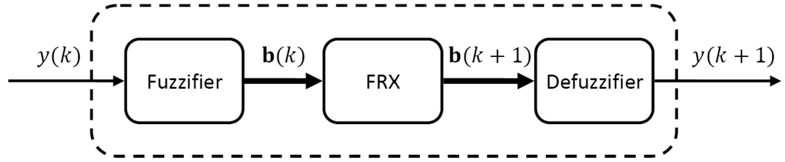

Figure 1. The Block Diagram of an Unforced First-Order FRDS

From the structural and methodological viewpoints, this paper mainly follows the line of [3], where a set of sufficient conditions is obtained for a first-order FRDS with respect to its FRX. It should be noted that in a first-order FRDS, the fuzzy relational equation of the FRM can be written as:

$$\mathbf{b}\,(k+1) = \mathbf{b}\,(k) \circ \mathbf{R}, \tag{1}$$

where $\mathbf{R}$ is a $q \times q$ fuzzy relational matrix and $\mathbf{b}$ is a row vector.

The paper is organized as follows. After this introduction in Section 1, Section 2 represents the model configuration and components as well as a few preliminary concepts from [3]. Main results are developed in Section 3, where the stability issue is addressed by two approaches in two subsections. In the first subsection, the existence of a unique equilibrium point at the center of a linguistic term and its asymptotic stability is studied. The result of this approach is indeed an effective improvement made on the results of [3]. In the second subsection, the focus is on the origin of a system with symmetric behavior around its origin, and a sufficient condition for the asymptotic stability of such systems at the origin is obtained by using the mathematical tools introduced in [2]. Section 4 represents some examples to verify the proposed sets of sufficient conditions. Finally, the paper is concluded with a brief summary of results and suggestions in Section 5.

2. Preliminaries

The block diagram of an unforced first-order FRDS is depicted in Figure 1, where the components types, in this paper, are selected as follows:

- Fuzzifier: standard fuzzifier, see Reminder 2.1.
- Fuzzy Relational Composition: algebraic composition, see Reminder 2.2.
- Defuzzifier: weighted average deffuzifier, see Reminder 2.3.

Reminder 2.1. In [3] the *standard fuzzifier* is defined as a fuzzifier made up of q triangular membership functions with the property of sum-normality, i.e., the sum of the membership function values for every non-fuzzy value is one.

Reminder 2.2. In an FRDS with *algebraic* or *sum-product* fuzzy relational composition, the fuzzy composition is of the form s-t, the t-conorm is the algebraic sum, and the t-norm is the algebraic product, i.e., $s(a, b) = \min(a+b, 1)$ and $t(a, b) = ab$ for $a, b \in [0, 1]$.

Reminder 2.3. The output y of a *weighted average* defuzzifier is calculated as $y = \sum b_i c_i / \sum b_i$, where $\mathbf{b}$ and $\mathbf{c}$ are respectively the input vector and the *centers vector* of the defuzzifier.

Definition 2.4. [3, Definition 3.2] Let Γ_f be the hyperplane of all q-tuples $(x_1, ..., x_q)$ such that:

$$\sum_{j=1}^{q} x_j = 1, \tag{2}$$

where $x_i \in [0, 1]$, for $i = 1, ..., q$. An FRDS described by $\mathbf{b}(k+1) = \mathbf{b}(k) \circ \mathbf{R}$ has the property of *intraplanarity* on Γ_f, if $\mathbf{b}(k+1) \in \Gamma_f$ for every $\mathbf{b}(k) \in \Gamma_f$.

In other words, the FRDS is called *intraplanar* if Γ_f is invariant under the sum-product fuzzy composition.

Definition 2.5. [3, Definition 3.3] A $p \times q$ matrix $\mathbf{R} = [r_{ij}]$ is called *unit-row matrix* if:

$$\sum_{j=1}^{q} r_{ij} = 1 \quad \forall i \in \{1, \ldots, p\}. \tag{3}$$

Lemma 2.6. *[3, Lemma 3.4] Let $\boldsymbol{R} = [r_{ij}]$ be a $q \times q$ relational matrix and "$\circ\boldsymbol{R}$" be sum-product fuzzy composition with $\boldsymbol{R}$, as in $\boldsymbol{b} = \boldsymbol{a} \circ \boldsymbol{R}$. Then the hyperplane Γ_f is invariant under "$\circ\boldsymbol{R}$" if and only if $\boldsymbol{R}$ is a unit-row matrix.*

Throughout this paper: An FRDS with a standard fuzzifier, an algebraic fuzzy relational composition, a weighted average defuzzifier, and a unit-row FRX, is called a *FRDS-SAWU* for the sake of brevity; "$\text{row}_i(\mathbf{M})$" and "$\text{col}_j(\mathbf{M})$" denote respectively the i-th row and the j-th column of a matrix $\mathbf{M}$; c_i is the i-th element of the *centers vector*, i.e., the center of i-th membership function of the $\vee$ and $\wedge$ stand respectively for the *max* and *min* functions; $\underset{zadeh}{\circ}$ and $\underset{algebraic}{\circ}$ denote respectively *max-min* and *sum-product* fuzzy relational compositions.

3. Main Results

The goal of this section is to provide some assessable FRX-based criterions for investigating the stability of an equilibrium point of an FRDS-SAWU. Two sets of sufficient conditions are obtained in this regard which are developed in the two following subsections.

3.1. **First Approach: Convergence to The Center of a Linguistic Term.** In this section, the approach of [3] is directly followed to improve the results therein.

Theorem 3.1. *[3, Theorem 3.8] In a first-order FRDS-SAWU, the output (of the defuzzifier) converges asymptotically globally to c_l, if:*

1. $r_{ll} = 1$,
2. $r_{il} \neq 0, \quad \forall i = 1, \ldots, q.$

Theorem 3.1 represents a set of sufficient conditions for convergence to the center of the l-th linguistic term. In this section, the aim is to alleviate the second condition.

The general fuzzy relational equation of an unforced first-order FRDS (with s-t composition) can be written as:

$$b_j(k+1) = \underset{i}{S}\; t(b_i(k), r_{ij}), \tag{4}$$

where $\underset{i}{S}$ denotes the operation of the t-conorm s on several operands, i.e., $\overset{n}{\underset{i=1}{S}} \lambda_{ij} = s(\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{nj})$ for example. This equation means that all elements of the fuzzy vector in the previous time instant contribute in determining the value of the j-th element of the fuzzy vector in the current time instant. When the element r_{ij} is zero then there is no direct dependence between the i-th element of the previous fuzzy vector and the j-th element of the current one, the fact that has been mentioned in [3]. However, it should be noted that there may be an indirect dependence between these elements, e.g., j-th element depends on the i'-th one and the i'-th element depends on the i-th one. In such a case it takes two time instants for the i-th element to affect the value of j-th element.

Definition 3.2. Regarding (1) or (4):

- There exists a *route of length one* from the i-th element of $\mathbf{b}$ to its j-th element when $b_j(k+1)$ depends on $b_i(k)$.
- More generally, there exists a *route of length* l_{path} from the i-th element of $\mathbf{b}$ to its j-th element when $b_j(k+1)$ depends on $b_i(k+1-l_{path})$.
- Altogether, there exists a *route* from the i-th element of $\mathbf{b}$ to its j-th element when there exist a delay index d such that $b_j(k+1)$ depends on $b_i(k+1-d)$.

Remark 3.3. Obviously, it takes one time instant for the i-th element to affect the value of the j-th element when there is a route of length one (length-1 route) from the i-th element to the j-th element.

Definition 3.4. Let $\mathbf{R} = [r_{ij}]$ be an $n \times n$ FRX. We define the *indicator function* of an FRX $\mathbf{R}$ as $f_{ind} : [0,1] \to \{0,1\}$, such that:

$$f_{ind}(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases} .$$

An $n \times n$ matrix $\mathbf{B} = [b_{ij}]$ is called the *Boolean matrix* of $\mathbf{R}$ when $b_{ij} = f_{ind}(r_{ij})$.

Remark 3.5. A value of 1 (0) for the element (i,j) of the Boolean matrix $\mathbf{B}$ indicates the existence (nonexistence) of a route of length 1 in $\mathbf{R}$ from the i-th element to the j-th element.

Lemma 3.6. *Let $\boldsymbol{R} = [r_{ij}]$ be an $n \times n$ relational matrix with Boolean matrix $\boldsymbol{B}$. Then:*

a. *There exists a route of length two from the i-th element of the fuzzy vector to its j-th one if and only if:*

$$\mathrm{row}_i\,(\boldsymbol{B}) \underset{zadeh}{\circ} \mathrm{col}_j\,(\boldsymbol{B}) = 1.$$

b. *The number of the routes of length two from the i-th element of the fuzzy vector to its j-th element (n_{path}) can be calculated as:*

$$n_{path} = \mathrm{row}_{\ i}\,(\boldsymbol{B}) \underset{algebraic}{\circ} \mathrm{col}_j\,(\boldsymbol{B})\,.$$

Proof. It should be noted that:

a. $\mathrm{row}_i\,(\mathbf{B}) \underset{zadeh}{\circ} \mathrm{col}_j\,(\mathbf{B}) = 1$ if and only if $\max_{k=1}^{n}\{\min\{b_{ik}, b_{kj}\}\} = 1$, and this holds if and only if $b_{ik} = b_{kj} = 1$, for some k, $1 \leqslant k \leqslant n$.

b. $\text{row}_i(\mathbf{B}) \underset{algebraic}{\circ} \text{col}_j(\mathbf{B}) = n_{path}$ if and only if $\sum_{k=1}^{n} b_{ik}b_{kj} = n_{path}$, and this holds if and only if there are $k_1, \ldots k_{n_{path}}$ such that $1 \leqslant k_r \leqslant n$ and $b_{ik_r} = b_{k_r j} = 1$, for every $1 \leqslant r \leqslant n_{path}$.

So the result follows. □

Remark 3.7. The *product* operator in the composition of Lemma 3.6 (part b) can be replaced by the *min* operator.

Definition 3.8. Let $\mathbf{R} = [r_{ij}]$ be an $n \times n$ relational matrix. We call an $n \times n$ matrix, say $C = [c_{ij}]$, the *l-route-map* of $\mathbf{R}$ if $c_{ij} = 1$ when there exists a route of length l from the i-th element of the fuzzy vector to its j-th one, and $c_{ij} = 0$ when there is no such route. Furthermore we define *$l-$routes-map* as an $n \times n$ matrix, say $F = [f_{ij}]$, such that f_{ij} equals the number of the routes of length l from the i-th element of the fuzzy vector to its j-th one.

Remark 3.9. Let $\mathbf{R}$ be the FRX of a first-order FRDS with a fuzzy relational composition denoted by "$\circ$" and $\mathbf{B}$ be the Boolean matrix of $\mathbf{R}$ which can be written as $\mathbf{B} = f_{ind}(\mathbf{R})$. Then:

- $\mathbf{B}$ is indeed the 1-route-map of the FRDS.
- The element r_{ij} can be used as a criterion to show the efficacy of the i-th element on the j-th element by the routes of length one; Similarly, The element (i, j) of $\mathbf{R} \circ \mathbf{R}$ can be used as a criterion to show the efficacy of the i-th element on the j-th element by the routes of length two; And so on.
- It is very easy to show that $f_{ind}(\mathbf{R} \circ \mathbf{R}) = \mathbf{B} \underset{zadeh}{\circ} \mathbf{B}$.

Example 3.10. Consider the following FRX ($\mathbf{R}$). The direct routes (routes of length one) of the related FRDS are determined after calculating the related Boolean matrix $\mathbf{B}$.

$$\mathbf{R} = \begin{bmatrix} 0.1 & 0.5 & 0.3 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0.2 & 0 \\ 0 & 0 & 1 & 0.1 & 0.4 \\ 0 & 0 & 0.8 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.3 & 0.1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Also, the existence and the number of indirect routes of length two are indicated by calculating 2-route-map and 2-routes-map as follows:

$$\mathbf{B} \underset{zadeh}{\circ} \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & \boxed{1} & \boxed{1} \\ 0 & 1 & 1 & 1 & \boxed{1} \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 & 1 \end{bmatrix}, \quad \mathbf{B} \underset{algebraic}{\circ} \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 2 & 3 & 3 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix}.$$

Meanwhile, for two classes of fuzzy relational compositions we have:

$$\mathbf{R} \underset{\max-\min}{\circ} \mathbf{R} = \begin{bmatrix} 0.1 & 0.2 & 0.5 & 0.2 & 0.3 \\ 0 & 0.2 & 0.5 & 0.2 & 0.4 \\ 0 & 0 & 1 & 0.3 & 0.4 \\ 0 & 0 & 0.8 & 0.2 & 0.4 \\ 0 & 0 & 0.3 & 0.2 & 0.1 \end{bmatrix},$$

$$\mathbf{R} \underset{\text{sum}-\text{prod}}{\circ} \mathbf{R} = \begin{bmatrix} 0.01 & 0.15 & 0.58 & 0.13 & 0.12 \\ 0 & 0.04 & 0.76 & 0.13 & 0.22 \\ 0 & 0 & 1 & 0.24 & 0.45 \\ 0 & 0 & 0.96 & 0.15 & 0.35 \\ 0 & 0 & 0.24 & 0.09 & 0.04 \end{bmatrix}.$$

For example, consider the element (1,4) of $\mathbf{R}$. From the first element of the fuzzy vector to its fourth element, it is admitted respectively by the 1-route-map, the 2-route-map, and the 2-routes-map that:

- There is no route of length one.
- There is at least one route of length two.
- There are exactly two routes of length two.

Lemma 3.11. *Let $\boldsymbol{R}$ be the FRX of a first-order FRDS and let the $l-$route-map and the $l-$routes-map of $\boldsymbol{R}$ be respectively denoted by $\boldsymbol{B}^l_{zadeh}$ and $\boldsymbol{B}^l_{algebraic}$. Then, the so-called maps can be calculated as follows.*

$$\begin{cases} \boldsymbol{B}^l_{zadeh} = \overbrace{\boldsymbol{B} \underset{zadeh}{\circ} \cdots \underset{zadeh}{\circ} \boldsymbol{B}}^{l} \\ \boldsymbol{B}^l_{algebraic} = \overbrace{\boldsymbol{B} \underset{algebraic}{\circ} \cdots \underset{algebraic}{\circ} \boldsymbol{B}}^{l} \end{cases}$$

Proof. The lemma is easily proved by successive use of Lemma 3.6 in matrix form. □

Definition 3.12. Let $\mathbf{R} = [r_{ij}]$ be the FRX of a first-order FRDS. We define the *complete-route-map* as an $n \times n$ matrix, say $\mathbf{P} = [p_{ij}]$, such that $p_{ij} = 1$ if there is a route (of arbitrary length) from the i-th element of the fuzzy vector to its j-th element, and $p_{ij} = 0$, otherwise.
Also we define the *complete-routes-map* as an $n \times n$ matrix, say $\mathbf{Q} = [q_{ij}]$, where q_{ij} equals the number of the routes from the i-th element of the fuzzy vector to its j-th element, no matter what the length of the routes are.

Corollary 3.13. *For a first-order FRDS, the complete-route-map and the complete-routes-map are calculated as follows:*

$$\begin{cases} \text{complete-route-map: } \overset{l}{\underset{i=1}{\vee}} \boldsymbol{B}^i_{zadeh}, \\ \text{complete-routes-map: } \sum_{i=1}^{l} \boldsymbol{B}^i_{algebraic}, \end{cases}$$

where $\boldsymbol{B}^i_{zadeh}$ and $\boldsymbol{B}^i_{algebraic}$ are calculated as in Lemma 3.11.

Theorem 3.14. *For an FRDS-SAWU with an FRX $\boldsymbol{R}$, the output (of the defuzzifier) converges globally asymptotically to c_l if:*

1. *$r_{ll} = 1$ and $r_{ij} < 1$ for all $(i,j) \neq (l,l)$.*
2. *All the elements of the l-th column in the complete-route-map are nonzero.*

Proof. Since we deal with FRDS-SAWU and $r_{ll} = 1$, then (1) can be rewritten as:

$$b_l(k+1) = \sum_{i=1}^{q} b_i(k) r_{il} = b_l(k) + \sum_{i=1,\, i\neq l}^{q} b_i(k) r_{il}.$$

Thus, $\{b_l(k)\}$ is an increasing and bounded sequence and so it is convergent. Indeed, we have:

$$b_l(k+1) - b_l(k) = \sum_{i=1,\, i\neq l}^{q} b_i(k) r_{il},$$

and so:

$$0 = \lim_{k\to\infty} (b_l(k+1) - b_l(k)) = \sum_{i=1,\, i\neq l}^{q} \lim_{k\to\infty} b_i(k) r_{il}.$$

which means:

$$r_{il} \lim_{k\to\infty} b_i(k) = 0, \; \forall i \neq l, \tag{5}$$

because $b_i(k)$ and r_{il} are nonnegative for all i.

By the second condition of the theorem, there is at least one nonzero element in the l-th column of $\mathbf{R}$, other than r_{ll}. Define $I := \{i : r_{il} \neq 0\}$. By (5), we conclude that $\lim_{k\to\infty} b_i(k) = 0$, for every $i \in I - \{l\}$.

Now, suppose that $i \in I^c = \{i : r_{il} = 0\}$. By the second condition of the theorem, there is a route (of unknown length t) from $b_i(k)$ to $b_l(k)$. Consider there is route of length one from $b_i(k)$ to $b_{i_1}(k)$, from $b_{i_1}(k)$ to $b_{i_2}(k)$, and etc., till finally from $b_{i_{t-1}}(k)$ to $b_l(k)$. Clearly, $i_{t-1} \in I$, so $\lim_{k\to\infty} b_{i_{t-1}}(k) = 0$ and $r_{i_{t-2} i_{t-1}} \neq 0$. Thus,

$$b_{i_{t-1}}(k+1) = b_{i_{t-2}}(k) r_{i_{t-2} i_{t-1}} + \sum_{i \neq i_{t-2}} b_i(k) r_{i i_{t-1}}$$

yields $\lim_{k\to\infty} b_{i_{t-2}}(k) = 0$, after the operation of $\lim_{k\to\infty}$ on its both sides, since the terms are nonnegative. By repeating this argument, it is proved that $\lim_{k\to\infty} b_i(k) = 0$ for all $i \in I^c$.

Therefore, $\lim_{k\to\infty} b_i(k) = 0$, for every $i \neq l$, and so the result follows. □

3.2. **Second Approach: Convergence to the Origin.** In this section, a special class of dynamic systems is considered in which the behavior (including the dynamics) of the system is symmetric around the origin and the system is modeled as a first-order FRDS. An inverted pendulum is a well-known example for globally-symmetric behavior around the upright position (considered as the origin). There are also many real-world systems that their behaviors are not globally-symmetric but can be considered locally-symmetric around a desired equilibrium point which serves as the local origin. Fuzzy relational modeling can be used in these cases in a symmetric format, since the fuzzy linguistic modeling is, in essence, a local modeling scheme.

While the goal of this subsection is the same as of the previous subsection, the approach of this subsection is different from the previous subsection. The mathematical foundation of the approach of this subsection is based on an special notion of symmetry for matrices, introduced in [2], which is applied to fuzzy relational matrices in this paper.

Definition 3.15. [2] Let $\mathbf{R}$ be a $p \times q$ matrix. Then:

- $\mathbf{R}$ is called *centrally symmetric (CS)*, if:

$$r_{i,j} = r_{p+1-i,q+1-j},$$

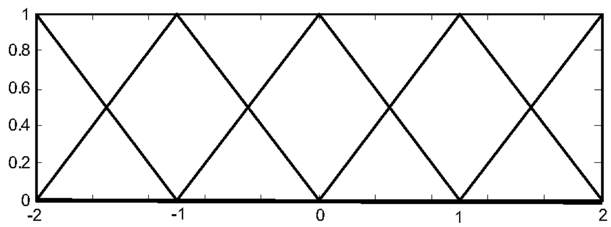

Figure 2. The Membership Functions of an Equally-Spaced Symmetric Standard Fuzzifier with 5 Linguistic Terms

- **R** is called *centrally skew symmetric (CSS)*, if:
$$r_{i,j} = -r_{p+1-i,q+1-j},$$
- **R** is called *row-wise symmetric (RWS)*, if:
$$r_{i,j} = r_{p+1-i,j},$$
- **R** is called *row-wise skew symmetric (RWSS)*, if:
$$r_{i,j} = -r_{p+1-i,j},$$
- **R** is called *column-wise symmetric (CWS)*, if:
$$r_{i,j} = r_{i,q+1-j},$$
- **R** is called *column-wise skew symmetric (CWSS)*, if:
$$r_{i,j} = -r_{i,q+1-j},$$
- **R** is called *plus Symmetric (PS)*, if:
$$\begin{aligned} r_{i,j} &= r_{p+1-i,j} \\ &= r_{i,q+1-j} \\ &= r_{p+1-i,q+1-j}, \end{aligned}$$

for all $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$.

Remark 3.16. Definition 3.15 is obviously valid for vectors as special cases of matrices, where p is equal to 1 (row vector) or q is equal to 1 (column vector).

Definition 3.17. A *symmetric standard fuzzifier* is defined as a standard fuzzifier (as mentioned in Reminder 2.1) in which the non-fuzzy universe of discourse is a symmetric interval in **R** around 0, say $[-x, x]$, and the whole set of membership functions, when depicted, constitute a symmetric graph about the origin (about the vertical axis); See Figure 2 for example.

Definition 3.18. A *symmetric weighted average defuzzifier (SWAD)* is defined as a weighted average defuzzifier (as mentioned in Reminder 2.3) in which the centers vector is CSS.

Lemma 3.19. *Assume a symmetric weighted average defuzzifier with $2m + 1$ centers, where $m \in \mathbb{N}$. If the (fuzzy) input vector of the defuzzifier (say $\boldsymbol{b}$) is CS, then, the (non-fuzzy) output of the defuzzifier (y) equals zero.*

Proof. Let the centers vector and the input vector of the defuzzifier respectively be:

$$\mathbf{c} = \begin{bmatrix} -c_m & \cdots & -c_1 & 0 & c_1 & \cdots & c_m \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} -b_m & \cdots & -b_1 & 0 & b_1 & \cdots & b_m \end{bmatrix},$$

The result follows easily, since:

$$y = \frac{-b_m c_m - \cdots - b_1 c_1 + b_1 c_1 + \cdots + b_m c_m}{b_m + \cdots + b_1 + b_0 + b_1 + \cdots + b_m} = 0.$$

□

Theorem 3.20. *[2, Theorem 2] Let $\boldsymbol{R}$ be a $n \times n$ CS matrix which has n distinct eigenvalues. Then $\boldsymbol{R}^k$ tends to a PS matrix as k grows, if all of the eigenvalues associated with RWSS eigenvectors are located in the unit circle.*

Remark 3.21. In Theorem 3.20, the eigenvectors has been considered to be column vectors, and accordingly, RWSS eigenvectors have been selected to be dealt with. However, in this paper, CSS eigenvectors are selected instead. In this manner, it does not matter for the eigenvectors to be row vector or column vector.

Remark 3.22. The eigenvalues that should be checked in Theorem 3.20 is about the half of all eigenvalues of $\mathbf{R}$. See [2] for more information. Meanwhile, it is worth mentioning that the condition of eigenvalues in this theorem is true for most cases as simulation results admit.

Remark 3.23. The symmetric behavior of a dynamic system leads to a CS FRX when the system is modeled as a first-order FRDS.

Theorem 3.24. *Let $q = 2m + 1$, where $m \in \mathbb{N}$. Consider an FRDS-SAWU in which:*

- *The $q \times q$ FRX $\boldsymbol{R}$ is CS and has q distinct eigenvalues.*
- *The standard fuzzifier is symmetric with q linguistic terms.*
- *The weighted average defuzzifier is symmetric with q centers (the same as the fuzzifier).*

Then, the output (of the defuzzifier) converges globally asymptotically to the origin if all of the eigenvalues associated with the CSS eigenvectors are located in the unit circle.

Proof. The fuzzy relational equation of the considered model can be written as $\mathbf{b}(k) = \mathbf{b}(k-1)\mathbf{R} = \mathbf{b}(0)\mathbf{R}^k$, since the fuzzy relational composition is of sum-product type and the FRX is a unit-row matrix. Thus, we can write $b_j(k) = \mathbf{b}(0)\,\mathrm{col}_j\left(\mathbf{R}^k\right), \forall j = \{1, \cdots, q\}$. By Theorem 3.20 and Remark 3.21, $\mathbf{R}^k$ tends to a PS matrix as k approaches ∞. Every PS matrix is CWS too. Therefore,

$$\lim_{k\to\infty} \mathrm{col}_j\left(\mathbf{R}^k\right) = \lim_{k\to\infty} \mathrm{col}_{q+1-j}\left(\mathbf{R}^k\right),$$

and so $\lim_{k\to\infty} b_j(k) = \lim_{k\to\infty} b_{q+1-j}(k)$, for all $j = \{1, \cdots, q\}$. Thus, using Lemma 3.19, the final conclusion is made, i.e., $\lim_{k\to\infty} y = 0$. □

4. Examples and Simulation Results

In this section, two examples are provided to validate the proposed sets of sufficient conditions, i.e., Theorem 3.14 and Theorem 3.24.

Example 4.1 (Convergence to the Center of a Linguistic Term)**.** Consider an FRDS-SAWU described by (1). Suppose the FRX **R** as:

$$\mathbf{R} = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 & 0 \\ 0 & 0.8 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2 & 0.8 & 0 \\ 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix}.$$

The conditions of Theorem 3.14 hold for the third linguistic term. Therefore, it can be concluded that the actual output of the dynamic system certainly converges to the center of the third membership function from every arbitrary initial point in the non-fuzzy universe of discourse. This can be observed in Table 1 which shows the evolution of the output fuzzy vector from an initial state for 100 time steps.

It worth mentioning that in Table 1, the sum of the elements of each row is one, as expected. This illustrates the fact that the hyperplane (2) is invariant under the operation "$\circ\mathbf{R}$".

k	$\mathbf{b}(\mathrm{k})$				
0	1	0	0	0	0
1	0.8	0.2	0	0	0
2	0.64	0.32	0.04	0	0
3	0.512	0.384	0.104	0	0
4	0.4096	0.4096	0.1808	0	0
5	0.3277	0.4096	0.2627	0	0
⋮	⋮	⋮	⋮	⋮	⋮
100	0	0	1	0	0

Table 1. Evolution of the Output Fuzzy Vector from $[1, 0, 0, 0, 0]$ to $[0, 0, 1, 0, 0]$ for 100 Time Steps (Example 4.1)

Example 4.2 (Convergence to the Origin)**.** Consider an FRDS-SAWU described by (1). Suppose the FRX **R** as:

$$\mathbf{R} = \begin{pmatrix} 0.3684 & 0.3684 & 0.0789 & 0.0789 & 0.1053 \\ 0.2889 & 0.2000 & 0.2444 & 0.1111 & 0.1556 \\ 0.2167 & 0.1500 & 0.2667 & 0.1500 & 0.2167 \\ 0.1556 & 0.1111 & 0.2444 & 0.2000 & 0.2889 \\ 0.1053 & 0.0789 & 0.0789 & 0.3684 & 0.3684 \end{pmatrix}$$

According to Theorem 3.24, the eigenvalues associated with CSS eigenvectors should be checked. The eigenvectors of **R** (5 column vectors concatenated as a modal matrix **M**) and the corresponding eigenvalues (gathered as a vector **v**) are as follows.

$$\mathbf{M}=\begin{pmatrix} 0.2279 & 0.1583 & 0.1138 & 0.3514 & 0.1486 \\ 0.1857 & 0.4047 & -0.5903 & 0.3367 & -0.3367 \\ 0.1728 & -1.1259 & 0.9531 & 0 & 0 \\ 0.1857 & 0.4047 & -0.5903 & -0.3367 & 0.3367 \\ 0.2279 & 0.1583 & 0.1138 & -0.3514 & -0.1486 \end{pmatrix}$$
$$\mathbf{v}=\begin{pmatrix} 1.0000 & 0.0688 & -0.0173 & 0.3909 & -0.0389 \end{pmatrix}$$

The conditions of Theorem 3.24 are satisfied. Note that one of the eigenvalues equals 1 and is not in the unit circle but that eigenvalue is not related to a CSS eigenvector. Therefore, we expect that the output of the system converges to the origin from any arbitrary initial condition. Let us, for example, run the system with initial condition $y(0) = -2$ which is equivalent to the fuzzy vector $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, according to the fuzzifier specified in Figure 2. Table 2 shows the non-fuzzy and fuzzy outputs in some instances of time. It can be observed in this table that the output fuzzy vector $\mathbf{b}(k)$ tends to a CS vector (CWS row vector) and the non-fuzzy output tends to zero.

k	$y(k)$	$\mathbf{b}(k)$				
0	-2.000	1.0000	0	0	0	0
1	-0.816	0.3684	0.3684	0.0789	0.0789	0.1053
2	-0.318	0.2826	0.2383	0.1678	0.1364	0.1748
3	-0.124	0.2490	0.2059	0.1725	0.1656	0.2070
4	-0.049	0.2361	0.1935	0.1728	0.1778	0.2197
8	-0.001	0.2281	0.1858	0.1728	0.1855	0.2277
9	-0.000	0.2280	0.1857	0.1728	0.1856	0.2279

TABLE 2. Evolution of the System Output
(Crisp and Fuzzy Outputs) for 10 Time Steps (Example 4.2)

5. Conclusion

In this paper, some sets of sufficient conditions were obtained to conclude the global asymptotic stability of an equilibrium point of an unforced first-order fuzzy relational dynamic systems.

First, by introducing the concept of direct and indirect paths, the result of [3] was improved by obtaining a less conservative set of sufficient conditions. Second, a new set of sufficient conditions was obtained for dynamic systems with symmetric behavior about the equilibrium point by using the mathematical results of [2]. The intraplanarity property, introduced in [3], has been used in both approaches of this paper.

In view of future works and based on the methods and the results of this paper, we present some suggestions as follows.

Concerning the analytical derivation of the conditions, the results may be improved by obtaining yet less conservative sets of sufficient conditions (for the FRM with the configuration of this paper or any other configurations).

In view of possible applications of the proposed conditions, it should be noted that the derived sets of sufficient conditions might be used as an extra property of the FRX of an FRM/FRDS structure, both in plant identification procedures (to obtain more reliable plant models) and in controller tuning procedures (to obtain more reliable controllers).

Furthermore, knowing the fact that the results of this paper are derived for first-order fuzzy relational dynamic systems, an important step ahead is to generalize the results of this paper to higher-order fuzzy relational dynamic systems [1] which can handle more complex systems.

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FUZZY RELATIONAL MATRIX-BASED STABILITY ANALYSIS FOR FIRST-ORDER FUZZY RELATIONAL DYNAMIC SYSTEMS

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تحلیل پایداری سیستم های دینامیکی رابطه ای فازی مرتبه اول بر مبنای ماتریس رابطه ای فازی

چکیده. در این مقاله، دو دسته شرایط کافی متفاوت برای تضمین وجود و پایداری یک نقطه تعادل یکتا در سیستم‌های رابطه‌ای فازی مرتبه اول بدون ورودی ارائه شده است. این دو دسته شرایط کافی حاصل دو رویکرد متفاوت به تحلیل مساله است که وجه اشتراک آنها این است که هر دو رویکرد برمبنای بررسی ماتریس رابطه‌ای فازی سیستم مورد نظر هستند. در رویکرد اول، نقطه تعادل سیستم، می‌تواند مرکز یکی از توابع عضویت باشد. در رویکرد دوم، نقطه تعادل سیستم، مبدا و یا به عبارتی مرکز تابع عضویت میانی بوده و رفتار سیستم (هرچند غیرخطی باشد) در اطراف مبدا (به صورت محلی) متقارن می‌باشد. نتایج هر دو رویکرد با مثال عددی راستی‌آزمایی شده است.