FUZZY PROJECTIVE MODULES AND TENSOR PRODUCTS IN FUZZY MODULE CATEGORIES

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Abstract. Let R be a commutative ring. We write $\mathrm{Hom}(\mu_A, \nu_B)$ for the set of all fuzzy R-morphisms from μ_A to ν_B, where μ_A and ν_B are two fuzzy R-modules. We make $\mathrm{Hom}(\mu_A, \nu_B)$ into fuzzy R-module by redefining a function $\alpha : \mathrm{Hom}(\mu_A, \nu_B) \longrightarrow [0, 1]$. We study the properties of the functor $\mathrm{Hom}(\mu_A, -) : FR\text{-Mod} \to FR\text{-Mod}$ and get some unexpected results. In addition, we prove that $\mathrm{Hom}(\xi_p, -)$ is exact if and only if ξ_P is a fuzzy projective R-module, when R is a commutative semiperfect ring. Finally, we investigate tensor product of two fuzzy R-modules and get some related properties. Also, we study the relationships between Hom functor and tensor functor.

1. Introduction

Zadeh [15] introduced the concept of a fuzzy subset μ of a nonempty set X. In 1971, Rosenfeld [12] defined the fuzzy subgroups of a group. Negoita and Ralescu [9] introduced fuzzy module. Pan [10,11] made the $\mathrm{Hom}(\mu_A, \nu_B)$ into a fuzzy module and investigated the properties of the functors $\mathrm{Hom}(\mu_A, -) : FR\text{-Mod} \to FR\text{-Mod}$ and $\mathrm{Hom}(-, \nu_B) : FR\text{-Mod} \to FR\text{-Mod}$. López-Permouth [6,7] gave the definition of the tensor products and studied Mortia theory in fuzzy module categories. Isaac [5] gave an alternate definition for projective L-modules and investigated these fuzzy modules. Chen [3] studied the relation between projective S-acts and Hom functors in the category of S-acts and Liu [8] studied the Hom functors and tensor product functors in the category of fuzzy S-acts. In this paper, we study the properties of Hom functor and tensor functor in fuzzy module categories. We also obtain that a fuzzy R-module ξ_P is projective if and only if $\mathrm{Hom}(\xi_P, -) : FR\text{-Mod} \to FR\text{-Mod}$ is exact.

Hom functors and tensor functors play an important role in ring theory. If R is a ring and ${}_RM_R$ is a bimodule, then $\mathrm{Hom}_R(M, -) : R\text{-Mod} \to R\text{-Mod}$ is left exact and $M \otimes_R - : R\text{-Mod} \to R\text{-Mod}$ is right exact; In addition, $(M \otimes_R -, \mathrm{Hom}_R(M, -))$ is an adjoint pair ([13]). Also, Hom functors and tensor functors play an important role in equivalences of module categories. The concept of projective module over a ring R is a more flexible generalization of the idea of a free module. Various equivalent characterizations of these modules are available. Perhaps the most insightful and certainly the most abstract characterization of a projective R-module M is that

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the functor $\text{Hom}_R(M, -) : R\text{-Mod} \to Ab$ is (right) exact, where Ab is the category of abelian groups ([13]).

Our aim is to develop the fuzzy technology by studying the category of fuzzy R-modules. It is natural to study the two functors and fuzzy projective modules in FR-Mod. This will help us to study equivalences of fuzzy module categories. In this paper, we investigate properties of the two functors in FR-Mod and the relationship of fuzzy projective module and Hom functor. Indeed, we get some similar results in FR-Mod. We also get some unexpected results, e.g., the functor $\text{Hom}(\mu_A, -) : FR\text{-Mod} \to FR\text{-Mod}$ being not left exact in FR-Mod. In section 2, we give some definitions and notations. In section 3, we study Hom functor and fuzzy projective modules. Pan [10] defined fuzzy Hom set. But, when we try to study the relationship of Hom functor and fuzzy projective module, we find it hard to make Γ_{μ_A} a fuzzy morphism. We resolve this problem successfully by redefining a function from $\text{Hom}(\mu_A, \nu_B)$ to $[0, 1]$, which can also make $\text{Hom}(\mu_A, \nu_B)$ into a fuzzy R-module. Also, we get some related properties of Hom functors and prove that $\text{Hom}(\xi_p, -)$ is exact if and only if ξ_P is a fuzzy projective R-module. In section 4, we recall the definition of tensor product in the category of fuzzy R-modules and prove that the tensor product of two fuzzy R-modules exists. We also get some properties of tensor product. In addition, we study the adjointness of Hom functor and tensor functor.

2. Preliminaries

Throughout this paper, let R be a ring with 1. We shall denote a left R-module M by ${}_RM$ and the set of left R-homomorphisms from ${}_RM$ to ${}_RN$ by $\text{Hom}_R(M, N)$. We denote the category of left R-modules by R-Mod and the category of right R-modules will be denoted by Mod-R.

Let A be a left R-module. A function μ_A is called fuzzy (left) R-module, if the map μ from the R-module A to the interval $[0, 1]$ satisfies
1) for all $x, y \in A$, $\mu(x + y) \geq \bigwedge\{\mu(x), \mu(y)\}$,
2) for all $a \in A, r \in R$, $\mu(ra) \geq \mu(a)$,
3) $\mu(0) = 1$.
Similarly, we can define the fuzzy right R-modules. For two fuzzy R-modules μ_A and ν_B, a function $\tilde{f} : \mu_A \longrightarrow \nu_B$ is called fuzzy R-homomorphism, if f is an R-homomorphism and $\nu(f(a)) \geq \mu(a)$ $(\forall a \in A)$.

For simplicity, denote by $\text{Hom}(\mu_A, \nu_B)$ the set of fuzzy R-homomorphisms from μ_A to ν_B.

Let R be a ring. We shall denote by FR-Mod (Mod-FR) the category of fuzzy left (right) R-modules.

A fuzzy R-homomorphism $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$ is called fuzzy split, if there is a fuzzy R-homomorphism $\tilde{t} \in \text{Hom}(\nu_B, \mu_A)$ such that the composition $\tilde{t}\tilde{f} = id$.

A fuzzy R-homomorphism $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$ is called a fuzzy quasi-isomorphism if f is an isomorphism ([1]).

A fuzzy R-homomorphism $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$ is called a fuzzy isomorphism, if f is an isomorphism and $\nu\tilde{f}(a) = \mu(a)$ $(\forall a \in A)$.

If $M \in R$-Mod, 0_M represents the fuzzy R-modules $0 : M \longrightarrow [0,1]$ satisfying

$$0(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

While, 1_M represents the fuzzy R-modules $1 : M \longrightarrow [0,1]$ satisfying $1(x) = 1$, for all $x \in M$.

If $\tilde{f} : \mu_A \rightarrow \nu_B$ is a fuzzy R-homomorphism, we define that $\mathrm{Ker}\tilde{f} = \{a \in A|\nu(\tilde{f}(a)) = 1\}$ and that $\mathrm{Im}\tilde{f} = \{\tilde{f}(a)|a \in A\}$. If $f : A \rightarrow B$ is an R-homomorphism and $\mathrm{Ker}f$ is the preimage of $\{0\}$ under f, we have $\mathrm{Ker}f \subseteq \mathrm{Ker}\tilde{f}$. Especially, if $\nu_B = 1_B$, then we have $\mathrm{Ker}\tilde{f} = A$, for all $\tilde{f} \in \mathrm{Hom}_R(\mu_A, \nu_B)$.

Proposition 2.1. *Let R be a ring. If $\tilde{f} \in Hom_R(\mu_A, \nu_B)$, where μ_A and ν_B are two fuzzy R-modules, then we have the following:*
1) $Ker\tilde{f}$ is a submodule of A;
2) Define $\mu^{'} : Ker\tilde{f} \rightarrow [0,1]$ by $\mu^{'}(k) = \mu(k)$, for $k \in Ker\tilde{f}$. Then $\mu^{'}_{Ker\tilde{f}}$ is a fuzzy submodule of μ_A.

Proof. 1) Let s be the zero element of A. Obviously, we have $s \in \mathrm{Ker}\tilde{f}$. Given $a \in \mathrm{Ker}\tilde{f}$ and $r \in R$, then

$$\nu(\tilde{f}(ra)) = \nu(r\tilde{f}(a)) \geq \nu(\tilde{f}(a)) = 1.$$

Hence, we get $ra \in \mathrm{Ker}\tilde{f}$. Particularly, we have $-a \in \mathrm{Ker}\tilde{f}$. If $a, b \in \mathrm{Ker}\tilde{f}$, we can easily get $a + b \in \mathrm{Ker}\tilde{f}$. This proves that $\mathrm{Ker}\tilde{f}$ is a submodule of A.

2) The proof is obvious. □

In R-Mod, the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence if f is a monomorphism, g is an epimorphism and $\mathrm{Im}f = \mathrm{Ker}g$ ([13]). We know that $\mathrm{Ker}f = \{0\}$.

We denote by $\bar{s}$ the fuzzy zero module.

Definition 2.2. Let μ_A, ν_B and η_C be fuzzy R-modules. A short exact sequence is a sequence of the form

$$\bar{s} \longrightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \bar{s}$$

where $\tilde{f}$ is a monomorphism, $\tilde{g}$ is an epimorphism and $\mathrm{Im}\tilde{f} = \mathrm{Ker}\tilde{g}$.

Note that $\mathrm{Ker}\tilde{f}$ is usually larger than $\{0\}$ in Definition 2.2. Hence, the crisp case of the definition is different from the well-known notion of short exact sequence in [13].

If $\eta_C = 1_C$, we get that $\mathrm{Im}\tilde{f} = \mathrm{Ker}\tilde{g} = B$. As $\tilde{f}$ is monic, we can get that $\tilde{f}$ is a quasi-isomorphism.

Let $\{\mu_{i A_i}|i \in I\}$ be a family of fuzzy R-modules. It is well-known that $\coprod_{i\in I} \mu_{i A_i}$ is the coproduct of $\{\mu_{i A_i}|i \in I\}$. The map $\mu : \coprod_{i\in I} A_i \longrightarrow [0,1]$ is defined by putting $\mu((a_i)_{i\in I}) = \bigwedge\{\mu_i(a_i)|i \in I\}$.

Similarly, if $\{\mu_{i A_i} | i \in I\}$ is a family of fuzzy R-modules, the product $\prod_{i\in I} \mu_{i A_i}$ is defined by $\mu((a_i)_{i\in I}) = \bigwedge\{\mu_i(a_i) | i \in I\}$.

A fuzzy R-module δ_P is called projective if and only if for every surjective fuzzy R-homomorphism $\tilde{f} : \mu_A \longrightarrow \nu_B$ and for every fuzzy R-homomorphism $\tilde{g} : \delta_P \longrightarrow \nu_B$, there exists a fuzzy R-homomorphism $\tilde{h} : \delta_P \longrightarrow \mu_A$ such that $\tilde{f}\tilde{h} = \tilde{g}$. If δ_P is projective, then $\delta_P = 0_P$.

Denote by $J(R)$ the radical of a ring R. A ring R is called semiperfect if $R/J(R)$ is semisimple and idempotents lift modulo $J(R)$ ([2]). Let R be a semiperfect ring and let P be a projective R-module, then $P \cong \coprod_{i\in I} Re_i$, where $e_i \in \mathrm{E}(R)$ and $\mathrm{E}(R)$ is the set of idempotent elements of R ([2], Theorem 27.11). Hence we can obtain the following statement.

Proposition 2.3. *Let R be a semiperfect ring. If 0_P is a fuzzy projective R-module, then $0_P \cong \coprod_{i\in I} 0_{Re_i}$, where $e_i \in E(R)$.*

3. Hom Functors and Fuzzy Projective Modules

In the following, we will define a function from $\mathrm{Hom}(\mu_A, \nu_B)$ to $[0, 1]$ and make $\mathrm{Hom}(\mu_A, \nu_B)$ into a fuzzy R-module.

Theorem 3.1. *Let R be a commutative ring and let μ_A and ν_B be two fuzzy R-modules. Then $Hom(\mu_A, \nu_B)$ is a fuzzy R-module by the function $\alpha : Hom(\mu_A, \nu_B) \to [0, 1]$ defined by*

$$\alpha(\tilde{f}) = \bigwedge\{\nu(\tilde{f}(a)) | a \in A\}.$$

Proof. Assume $r \in R$ and $\tilde{f} \in \mathrm{Hom}(\mu_A, \nu_B)$. Define a function $r \cdot \tilde{f} : \mu_A \longrightarrow \nu_B$ by

$$r \cdot \tilde{f}(a) = r\tilde{f}(a).$$

Then we have

$$\nu(r \cdot \tilde{f}(a)) = \nu(r\tilde{f}(a)) \geq \nu(\tilde{f}(a)) \geq \mu(a).$$

This concludes that $r \cdot \tilde{f} \in \mathrm{Hom}(\mu_A, \nu_B)$. Hence we show that $\mathrm{Hom}(\mu_A, \nu_B)$ is an R-module.

We now have to prove that $\mathrm{Hom}(\mu_A, \nu_B)$ is a fuzzy R-module. Suppose $r \in R, \tilde{f} \in \mathrm{Hom}(\mu_A, \nu_B)$ and $a \in A$. By

$$\nu(r \cdot \tilde{f}(a)) = \nu(r\tilde{f}(a)) \geq \nu(\tilde{f}(a)),$$

we have, $\bigwedge\{\nu(r\cdot\tilde{f}(a)) | a \in A\} \geq \bigwedge\{\nu(\tilde{f}(a)) | a \in A\}$. This concludes that $\alpha(r\cdot\tilde{f}) \geq \alpha(\tilde{f})$.

If $\tilde{f}, \tilde{g} \in \mathrm{Hom}(\mu_A, \nu_B)$, then

$$\begin{aligned} \alpha(\tilde{f} + \tilde{g}) &= \bigwedge\{\nu(\tilde{f}(a) + \tilde{g}(a)) | a \in A\} \\ &\geq \bigwedge\{\bigwedge\{\nu(\tilde{f}(a)), \nu(\tilde{g}(a))\}\} | a \in A\} \\ &\geq \bigwedge\{\bigwedge\{\nu(\tilde{f}(a)) | a \in A\}, \bigwedge\{\nu(\tilde{g}(a)) | a \in A\}\} \\ &= \bigwedge\{\alpha(\tilde{f}), \alpha(\tilde{g})\}. \end{aligned}$$

We obviously have that $\alpha(0) = 1$. Therefore we have shown that $\mathrm{Hom}(\mu_A, \nu_B)$ is a fuzzy R-module. □

For $M \in R$-Mod, the functor $\mathrm{Hom}(M, -)$ is left exact in the R-module category. The situation in FR-Mod is, however, different. The functor $\mathrm{Hom}(\xi_M, -)$ is not left exact.

Example 3.2. Let Z be the integer ring and $A = (6)$. Define $\nu : Z \to [0, 1]$ by

$$\nu(n) = \begin{cases} \frac{1}{10}, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

Define $\eta : Z_6 \to [0, 1]$ by

$$\eta(\bar{k}) = \begin{cases} \frac{1}{2}, & \text{if } \bar{k} \neq \bar{0}, \\ 1, & \text{if } \bar{k} = \bar{0}. \end{cases}$$

Then both ν_A and η_{Z_6} are fuzzy Z-modules. Also, we have a short exact sequence

$$\bar{s} \longrightarrow 0_A \xrightarrow{\tilde{f}} \nu_Z \xrightarrow{\tilde{g}} \eta_{Z_6} \longrightarrow \bar{s},$$

where $\tilde{f}$ is the inclusion homomorphism and $\tilde{g}$ is the epimorphism. We claim that the sequence

$$\bar{s} \longrightarrow \alpha_{\mathrm{Hom}(\nu_Z, 0_A)} \xrightarrow{F\tilde{f}} \beta_{\mathrm{Hom}(\nu_Z, \nu_Z)} \xrightarrow{F\tilde{g}} \gamma_{\mathrm{Hom}(\nu_Z, \eta_{Z_6})}$$

is not exact, where $F = \mathrm{Hom}(\nu_Z, -)$. Define $\tilde{h_1} : \nu_Z \longrightarrow \nu_Z$ by putting $\tilde{h_1}(n) = 6n$ and define $\tilde{h_2} : \nu_Z \longrightarrow \nu_Z$ by putting $\tilde{h_2}(n) = 12n$. We can check that both $\tilde{h_1}$ and $\tilde{h_2}$ are in $\mathrm{Ker}F\tilde{g}$. Hence $|\mathrm{Ker}F\tilde{g}| \geq 2$. Since $\mathrm{Hom}(\nu_Z, 0_A)$ contains only zero morphism, we have $\mathrm{Im}F\tilde{f} \neq \mathrm{Ker}F\tilde{g}$. Then $\mathrm{Hom}(\nu_Z, -)$ is not exact.

We can get the following statement which is similar to Pan's results [10].

Theorem 3.3. *Let R be a commutative ring and let*

$$\bar{s} \longrightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C$$

be an exact sequence, where $\tilde{f}$ is fuzzy split. Then $Hom(\xi_M, -)$ preserves the sequence.

Proof. The proof is similar to that of Theorem 2.38 in [13] or Theorem 2 in [10].

Let $F = \mathrm{Hom}(\xi_M, -)$. We will show that the sequence

$$\bar{s} \longrightarrow \alpha_{\mathrm{Hom}(\xi_M, \mu_A)} \xrightarrow{F\tilde{f}} \beta_{\mathrm{Hom}(\xi_M, \nu_B)} \xrightarrow{F\tilde{g}} \gamma_{\mathrm{Hom}(\xi_M, \eta_C)}$$

is exact.

1) By Theorem 2.38 [13], $\mathrm{Hom}(M, -)$ is left exact. So, it is clear that $F\tilde{f}$ is monic.

2) $\mathrm{Im}F\tilde{f} \subseteq \mathrm{Ker}F\tilde{g}$. Suppose $\tilde{h} \in \mathrm{Im}F\tilde{f}$, then there exists $\tilde{k} \in \mathrm{Hom}(\xi_M, \mu_A)$ such that $\tilde{h} = F\tilde{f}(\tilde{k}) = \tilde{f}\tilde{k}$. Since $\mathrm{Im}\tilde{f} = \mathrm{Ker}\tilde{g}$, we have $\eta(\tilde{g}\tilde{f}\tilde{k}(m)) = 1$ for all $m \in M$. Hence, $\gamma(F\tilde{g}(\tilde{h})) = 1$ and so $\tilde{h} \in \mathrm{Ker}F\tilde{g}$.

3) $\mathrm{Ker}F\tilde{g} \subseteq \mathrm{Im}F\tilde{f}$. Suppose $\tilde{h} \in \mathrm{Ker}F\tilde{g}$, so that $\gamma(F\tilde{g}(\tilde{h})) = 1$. If $m \in M$, by Theorem 3.1, we have $\eta(\tilde{g}\tilde{h}(m)) = 1$ and $\tilde{h}(m) \in \mathrm{Ker}\tilde{g} = \mathrm{Im}\tilde{f}$. As $\tilde{f}$ is monic, there is a unique $a_m \in A$ with $\tilde{f}(a_m) = \tilde{h}(m)$. Define $\tilde{k} : \xi_M \longrightarrow \mu_A$ by $\tilde{k}(m) = a_m$. Then $F\tilde{f}(\tilde{k}) = \tilde{f}\tilde{k} = \tilde{h}$. We have to show that $\tilde{k}$ is fuzzy. Since $\tilde{f}$ is fuzzy split, there exists a fuzzy morphism $\tilde{t} : \nu_B \longrightarrow \mu_A$ such that $\tilde{t}\tilde{f} = id$. For all $m \in M$, we have

$$\mu(\tilde{k}(m)) = \mu(a_m) = \mu(\tilde{t}\tilde{f}(a_m)) \geq \nu(\tilde{f}(a_m)) = \nu(\tilde{h}(m)) \geq \xi(m).$$

Hence $\tilde{k}$ is fuzzy. □

Theorem 3.4. *Let R be a commutative ring and let*

$$\mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \bar{s}$$

be an exact sequence, where $\tilde{f}$ is fuzzy split. Let $G = Hom(-, \xi_M)$, then we have the following exact sequence

$$\bar{s} \longrightarrow \gamma_{Hom(\eta_C,\xi_M)} \xrightarrow{G\tilde{g}} \beta_{Hom(\nu_B,\xi_M)} \xrightarrow{G\tilde{f}} \alpha_{Hom(\mu_A,\xi_M)}.$$

Proof. The proof is similar to that of Theorem 3.3. □

Now, we study the functor $\mathrm{Hom}(\mu_M, -)$, where $\mu_M = 0_{Re}$ and $e \in \mathrm{E}(R)$.

Define a map

$$\begin{array}{rccc} \Gamma_{\mu_A}: & \alpha_{\mathrm{Hom}(0_{Re},\mu_A)} & \longrightarrow & \mu_{eA} \\ & \tilde{g} & \longmapsto & \tilde{g}(e). \end{array}$$

Lemma 3.5. *Let R be a commutative ring and let $\mu_A \in FR$-Mod. Then Γ_{μ_A} is a fuzzy R-module isomorphism.*

Proof. Assume $ea \in eA$. we define a map $\tilde{f} : 0_{Re} \longrightarrow \mu_{eA}$ by putting $\tilde{f}(re) = rea$. We can easily check that $\tilde{f} \in \alpha_{\mathrm{Hom}(0_{Re},\mu_A)}$ and $\Gamma_{\mu_A}(\tilde{f}) = ea$. It follows that Γ_{μ_A} is a surjective map. If $\tilde{f} \in \alpha_{\mathrm{Hom}(0_{Re},\mu_A)}$, we can see that $\tilde{f}$ is determined by $\tilde{f}(e)$. This shows that Γ_{μ_A} is an injective map.

Let $\tilde{f} \in \mathrm{Hom}(0_{Re}, \mu_A)$. We have $\alpha(\tilde{f}) = \mu(\tilde{f}(e))$ and $\mu\Gamma_{\mu_A} = \alpha$. Thus, we prove that Γ_{μ_A} is a fuzzy isomorphism. □

Proposition 3.6. *Let R be a ring and the following diagram of fuzzy R-modules is commutative:*

$$\begin{array}{ccccccccc} \bar{s} & \rightarrow \mu_A & \xrightarrow{\tilde{f}} & \nu_B & \xrightarrow{\tilde{g}} & \eta_C & \rightarrow & \bar{s} \\ & \tilde{\alpha}\downarrow & & \downarrow\tilde{\beta} & & \downarrow\tilde{\gamma} & & \\ \bar{s} & \rightarrow \chi_L & \xrightarrow{\tilde{h}} & \xi_M & \xrightarrow{\tilde{p}} & \delta_N & \rightarrow & \bar{s}, \end{array}$$

where $\tilde{\alpha}$, $\tilde{\gamma}$ are fuzzy isomorphisms and $\tilde{\beta}$ is a fuzzy quasi-isomorphism. The bottom row is a short exact sequence if and only if so is the top row.

Proof. $\Rightarrow$: If the bottom row is a short exact sequence, we can easily have that $\tilde{g}$ is an epimorphism and $\tilde{f}$ is a monomorphism.

Given $a \in A$, by the commutativity, we have

$$\tilde{\gamma}\tilde{g}\tilde{f}(a) = \tilde{p}\tilde{h}\tilde{\alpha}(a).$$

As $\delta\tilde{p}\tilde{h}\tilde{\alpha}(a) = 1$ and $\tilde{\gamma}$ is a fuzzy isomorphism, then $\eta\tilde{g}\tilde{f}(a) = 1$. This shows that $\mathrm{Im}(\tilde{f}) \subseteq \mathrm{Ker}(\tilde{g})$.

Suppose $b \in \mathrm{Ker}(\tilde{g})$. By the commutativity, we can get $\delta\tilde{p}\tilde{\beta}(b) = 1$. Since $\mathrm{Im}(\tilde{h}) = \mathrm{Ker}(\tilde{p})$, there exist $l \in L$ such that $\tilde{h}(l) = \tilde{\beta}(b)$. Note that $\tilde{\alpha}$ is a fuzzy isomorphism. Then we have $a \in A$ satisfying $\tilde{\alpha}(a) = l$ and so $\tilde{\beta}\tilde{f}(a) = \tilde{h}\tilde{\alpha}(a) = \tilde{\beta}(b)$. Hence, we get $\tilde{f}(a) = b$. This shows that $\mathrm{Ker}(\tilde{g}) \subseteq \mathrm{Im}(\tilde{f})$ and then the top row is a short exact sequence.

$\Leftarrow$: The proof is similar to that of "$\Rightarrow$". □

Lemma 3.7. *Let R be a commutative ring and let*

$$\bar{s} \to \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \to \bar{s}$$

be a short exact sequence of fuzzy R-modules. Let $e \in E(R)$, $\tilde{f}_e = \tilde{f}|_{eA}$ and $\tilde{g}_e = \tilde{g}|_{eB}$. The following sequence

$$\bar{s} \to \mu_{eA} \xrightarrow{\tilde{f}_e} \nu_{eB} \xrightarrow{\tilde{g}_e} \eta_{eC} \to \bar{s}$$

is a short exact sequence.

Proof. Suppose $ec \in eC$. Since $\tilde{g}$ is an epimorphism, there exists $b \in B$ satisfying $\tilde{g}(b) = ec$. Since e is an idempotent, we have $eb \in eB$ and $\tilde{g}_e(eb) = e\tilde{g}(b) = ec$. This proves that $\tilde{g}_e$ is an epimorphism. We can obviously see that $\tilde{f}_e$ is a monomorphism.

We now show that $\mathrm{Im}(\tilde{f}_e) = \mathrm{Ker}(\tilde{g}_e)$. It is obvious that $\mathrm{Im}(\tilde{f}_e) \subseteq \mathrm{Ker}(\tilde{g}_e)$. Suppose $eb \in \mathrm{Ker}(\tilde{g}_e)$. We have an element $a \in A$ satisfying $\tilde{f}(a) = eb$. Hence $ea \in eA$ and $\tilde{f}_e(ea) = \tilde{f}(ea) = e\tilde{f}(a) = eb$, that is, $\mathrm{Ker}(\tilde{g}_e) \subseteq \mathrm{Im}(\tilde{f}_e)$. Thus we get the desired result. □

Lemma 3.8. *Let R be a commutative ring and let $e \in E(R)$. The functor $Hom(0_{Re}, -)$ preserves the sequence*

$$\bar{s} \longrightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \bar{s} \tag{1}$$

of fuzzy R-modules.

Proof. The sequence

$$\bar{s} \longrightarrow \mu_{eA} \xrightarrow{\tilde{f}_e} \nu_{eB} \xrightarrow{\tilde{g}_e} \eta_{eC} \longrightarrow \bar{s}$$

is also a short exact sequence by Lemma 3.7. Consider the following commutative diagram of fuzzy R-modules:

$$\begin{array}{ccccccccc} \bar{s} & \to & \alpha_{\mathrm{Hom}(0_{Re},\mu_A)} & \to & \beta_{\mathrm{Hom}(0_{Re},\nu_B)} & \to & \gamma_{\mathrm{Hom}(0_{Re},\eta_C)} & \to & \bar{s} \\ & & \Gamma_{\mu_A}\downarrow & & \downarrow\Gamma_{\nu_B} & & \downarrow\Gamma_{\eta_C} & & \\ \bar{s} & \to & \mu_{eA} & \to & \nu_{eB} & \to & \eta_{eC} & \to & \bar{s}, \end{array}$$

Note that $\Gamma_{\mu_A}, \Gamma_{\nu_B}$ and Γ_{η_C} are all fuzzy isomorphisms by Lemma 3.5. Using Proposition 3.6, we prove that the top row is a short exact sequence and so get the desired result. □

Lemma 3.9. *Let $\mu_A \in FR$-Mod, $e_i \in E(R)$ for any $i \in I$. Then we have*

$$\alpha_{Hom(\coprod_{i\in I} 0_{Re_i},\mu_A)} \cong \prod_{i\in I} \mu_{e_iA}.$$

Proof. Let

$$k : \mathrm{Hom}(\coprod_{i\in I} Re_i, A) \to \prod_{i\in I} \mathrm{Hom}(Re_i, A)$$

given by

$$f \mapsto (f\lambda_i = f_i)_{i\in I}$$

be the isomorphism, where λ_j is the injection $Re_j \to \coprod_{i\in I} Re_i$.

By the proof of Lemma 3.5, for all $i \in I$, we have $\alpha_i(f_i) = \mu(f_i(e_i))$. Note that $\mu(\sum_{i\in I} f_i(r_ie_i)) = \bigwedge_{i\in I}\{\mu(f_i(r_ie_i))\}$ and $\mu(f_i(r_ie_i)) = \mu(r_if_i(e_i)) \geq \mu(f_i(e_i))$, where $r_i, e_i \in R, i \in I$. For every $f \in \mathrm{Hom}(\coprod_{i\in I} Re_i, A)$, we have

$$\begin{aligned}
\alpha(f) &= \bigwedge\{\mu \circ f((r_ie_i)_{i\in I}) | (r_ie_i)_{i\in I} \in \coprod_{i\in I} Re_i\} \\
&= \bigwedge\{\mu(\sum_{i\in I} f_i(r_ie_i)) | (r_ie_i)_{i\in I} \in \coprod_{i\in I} Re_i\} \\
&= \bigwedge\{\mu(f_i(e_i)) | i \in I\} \\
&\quad (\text{Note that } \{f_i(e_i) | e_i \in \coprod_{i\in I} Re_i\} \subseteq \{\sum_{i\in I} f_i(r_ie_i) | r_ie_i \in \coprod_{i\in I} Re_i\}) \\
&= \bigwedge\{\alpha_i(f_i) | i \in I\} \\
&= \prod_{i\in I} \alpha_i((f_i)_{i\in I}) \\
&= (\prod_{i\in I} \alpha_i) \circ k(f).
\end{aligned}$$

So, we can get

$$\alpha_{\mathrm{Hom}(\coprod_{i\in I} 0_{Re_i}, \mu_A)} \cong \prod_{i\in I} \alpha_{i\,\mathrm{Hom}(0_{Re_i}, \mu_A)}.$$

□

Theorem 3.10. *Let R be a commutative semiperfect ring. Then Hom$(\xi_P, -)$ preserves the short exact sequence (1) of fuzzy R-modules if and only if ξ_P is a fuzzy projective R-module.*

Proof. $\Longrightarrow$: Let $\tilde{g} : \nu_B \to \eta_C$ be a fuzzy epimorphism. We denote that $K = \mathrm{Ker}\tilde{g}$ and $\nu' = \nu|_K$. By Proposition 2.1, we have that ν'_K is a fuzzy submodule of ν_B and so we obtain the following short exact sequence

$$\bar{s} \to \nu_K \xrightarrow{\tilde{\iota}} \nu_B \xrightarrow{\tilde{g}} \eta_C \to \bar{s},$$

where $\tilde{\iota}$ is the inclusion map.

Since $\mathrm{Hom}(\xi_P, -)$ preserves the sequence, $\mathrm{Hom}(\xi_P, -)$ preserves the epimorphism φ. Hence, ξ_P is a fuzzy projective R-module.

$\Longleftarrow$: Since ξ_P is a fuzzy projective R-module, we have $\xi_P \cong \coprod_{i\in I} 0_{Re_i}$, where $e_i \in \mathrm{E}(R)$. Let $\bar{s} \longrightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \bar{s}$ be a short exact sequence, then the sequence

$$\bar{s} \longrightarrow \prod_{i\in I} \mu_{e_iA} \longrightarrow \prod_{i\in I} \nu_{e_iB} \longrightarrow \prod_{i\in I} \eta_{e_iC} \longrightarrow \bar{s}$$

is also a short exact sequence by Lemma 3.7. Using Lemma 3.9, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\bar{s} \to & \alpha_{\mathrm{Hom}(\coprod_{i\in I} 0_{Re_i}, \mu_A)} & \to & \beta_{\mathrm{Hom}(\coprod_{i\in I} 0_{Re_i}, \nu_B)} & \to & \gamma_{\mathrm{Hom}(\coprod_{i\in I} 0_{Re_i}, \eta_C)} & \to \bar{s} \\
& \downarrow & & \downarrow & & \downarrow & \\
\bar{s} \to & \prod_{i\in I} \mu_{e_iA} & \to & \prod_{i\in I} \nu_{e_iB} & \to & \prod_{i\in I} \eta_{e_iC} & \to \bar{s}.
\end{array}$$

Since the bottom row is a short exact sequence, the top row is also a short exact sequence by Proposition 3.6. □

4. **Tensor Products**

In this section, R is not necessarily commutative. In [6], López-Permouth gave the definition of tensor product of two fuzzy modules. If A is an R-module, $s(A)$, $s({}_ZA)$ and $s({}_RA)$ represnt respectively the set of all grade functions on A, the set of all group grade functions on A, and the set of all R-module grade functions on A. Given two complete lattices $L_1 \subseteq L_2$, for $l_0 \in L_2$, the closure of l_0 in L_1 is the infimum in L_1 of the set $\{l \in L_1 | l_0 \leq l\}$.

We denote by $\mathcal{C}$ the category of complete lattices with all order-preserving mappings. A homomorphism $f : X \to Y$ of $\mathcal{C}$ preserves infima if and only if there exists a homomorphism $g : Y \to X$ in $\mathcal{C}$ such that $y \leq f(x) \Leftrightarrow g(y) \leq x$, for all $x \in X$ and $y \in Y$ [14]. If $f : M \to N$ is a group homomorphism (R-homomorphism), let $s(f) : s(N) \to s(M)$ be given by $s(f)(\beta)(m) = \beta(f(m))$, for $\beta \in s(N), m \in M$. Define $t(A) = s(A)$. The map $t(f) : t(M) \to t(N)$ satisfies that for all $\alpha \in t(M)$ and $\beta \in t(N)$, $s(f)(\beta) \geq \alpha$ if and only if $\beta \geq t(f)(\alpha)$. As the referee points out, $s(p)$ and $t(p)$ is a Galois connection between $s(N)$ and $s(M)$ in the sense of Definition 0-3.1 in [4], thereby yielding the closure operator $s(p) \circ t(p)$ on $s(M)$ in the sense of Definition 0-3.8 in [4].

Let $M \in$ Mod-R, $N \in R$-Mod. If $p : M \times N \longrightarrow M \otimes N$ is the tensor product of M and N, the fuzzy map $\mu \otimes \nu$ on $M \otimes N$ is defined to be the closure in $s({}_ZM \otimes N)$ of $t(p)(\mu \times \nu) \in s(M \otimes N)$. Since $s({}_ZM \otimes N) \subseteq s(M \otimes N)$, $\mu \otimes \nu$ is the smallest group grade function on ${}_ZM \otimes N$ larger than $t(p)(\mu \times \nu)$[7].

Let μ_A and ν_B be two fuzzy R-modules and F be a free abelian group with basis $A \times B = \{(a, b)|a \in A, b \in B\}$. We define $\mu_A \times \nu_B : A \times B \to [0, 1]$ by $\mu \times \nu(a, b) = \bigvee\{\mu(a), \nu(b)\}$ and $\mu \times \nu(\sum(a_i, b_i)) = \bigwedge\{\bigvee\{\mu(a_i), \nu(b_i)\}|i \in I\}$, Then we have defined a fuzzy membership function on F. We shall prove the tensor product of two fuzzy R-modules exists in this situation which is different from [6].

Let $A \in$ Mod-R, $B \in R$-Mod and C be an abelian group. A map $f : A \times B \longrightarrow C$ is called biadditive provided that for all $a \in A, r \in R$ and $b \in B$,

1) $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$;
2) $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$;
3) $f(ar, b) = f(a, rb)$.

Let $A \in$ Mod-R, $B \in R$-Mod and G be an abelian group. Let $\varphi : A \times B \longrightarrow G$ be biadditive. A pair (G, φ) is a tensor product of A and B if, for every biadditive $\psi : A \times B \longrightarrow H$, where H is an abelian group, there is a unique homomorphism $\theta \in \text{Hom}(G, H)$ satisfying $\theta\varphi = \psi$ [13]. Define $A \otimes B = F/S$, where S is the subgroup generated by elements of forms $(a_1 + a_2, b) - (a_1, b) - (a_2, b), (a, b_1 + b_2) - (a, b_1) - (a, b_2)$, and $(ar, b) - (a, rb)$. Then $A \otimes B$ is the tensor product of A and B ([13]).

As mentioned above, the tensor product of two modules is an abelian group. We will study this problem based on the above situation. So, the tensor product of two fuzzy modules should be a fuzzy abelian group.

Definition 4.1. [6] Let $\mu_A \in$ Mod-FR, $\nu_B \in FR$-Mod and let η_C be a fuzzy abelian group, a map $\tilde{f} : \mu_A \times \nu_B \longrightarrow \eta_C$ is called fuzzy biadditive if, f is biadditive and for all $\sum(a_i, b_i) \in A \times B$,

$$\eta\tilde{f}(\sum(a_i, b_i)) \geq (\mu \times \nu)(\sum(a_i, b_i)).$$

Definition 4.2. [6] Let $\mu_A \in$ Mod-FR and $\nu_B \in FR$-Mod. A pair $(\chi_G, \tilde{\varphi})$ is called a tensor product of μ_A and ν_B if, for every fuzzy biadditive $f : \mu_A \times \nu_B \longrightarrow \rho_H$, where ρ_H is a fuzzy abelian group, there is a unique fuzzy map $\tilde{\theta} \in \mathrm{Hom}(\chi_G, \rho_H)$ such that $\tilde{\theta}\tilde{\varphi} = f$.

Theorem 4.3. *Let $\mu_A \in$ Mod-FR and $\nu_B \in FR$-Mod. The tensor product of the two fuzzy R-modules μ_A and ν_B exists and it is unique up to isomorphism.*

Proof. Let $\tilde{\varphi} : A \times B \longrightarrow A \otimes B$ be the tensor product of the right R-module A and the left R-module B. We can define a map $\mu \otimes \nu : A \otimes B \longrightarrow [0, 1]$ by putting

$$\mu \otimes \nu(\sum(a_i \otimes b_i)) = \bigvee\{\mu \times \nu(\sum(a_i^{'}, b_i^{'}))| \sum(a_i^{'} \otimes b_i^{'}) = \sum(a_i \otimes b_i)\}.$$

It is easily check that $\tilde{\varphi}$ is fuzzy biadditive. Let η_H be a fuzzy abelian group and $\tilde{\psi} : (\mu \times \nu)_{A \times B} \longrightarrow \eta_H$ be fuzzy biadditive. By the definition of tensor product in module category, there is a unique homomorphism $\tilde{\theta} : A \otimes B \longrightarrow H$ such that $\tilde{\theta}\tilde{\varphi} = \tilde{\psi}$. We now have to show that for every $\sum(a_i \otimes b_i) \in A \otimes B$,

$$\eta\tilde{\theta}(\sum(a_i \otimes b_i)) \geq (\mu \otimes \nu)(\sum(a_i \otimes b_i)).$$

Suppose $\sum(a_i^{'} \otimes b_i^{'}) = \sum(a_i \otimes b_i) \in A \otimes B$. We have

$$\begin{aligned}
\eta\tilde{\theta}(\sum(a_i^{'} \otimes b_i^{'})) &= \eta(\sum \tilde{\theta}(a_i^{'} \otimes b_i^{'})) \geq \bigwedge \eta\tilde{\theta}(a_i^{'} \otimes b_i^{'}) \\
&= \bigwedge \eta\tilde{\theta}\tilde{\varphi}(a_i^{'}, b_i^{'}) = \bigwedge \eta\tilde{\sigma}(a_i^{'}, b_i^{'}) \\
&\geq \bigwedge \mu \times \nu(a_i^{'}, b_i^{'}) = \mu \times \nu(\sum(a_i^{'}, b_i^{'})).
\end{aligned}$$

This concludes that $\eta\tilde{\theta}(\sum(a_i \otimes b_i)) \geq (\mu \otimes \nu)(\sum(a_i \otimes b_i))$ and hence $((\mu \otimes \nu)_{A \otimes B}, \tilde{\varphi})$ is the tensor product of μ_A and ν_B.

It is obvious that the tensor product is unique up to isomorphism. □

For simplicity, we shall write $\mu_A \otimes \nu_B$ for the tensor product of μ_A and ν_B instead of $(\mu \otimes_R \nu)_{A \otimes_R B}$.

If $A \in R$-Mod, we have $R \otimes_R A \cong A$. Similarly, if $A \in$ Mod-R, then we have $A \otimes_R R \cong A$. Using this fact, we can get the following statements.

Proposition 4.4. *Let $\mu_A \in FR$-Mod. We have $0_R \otimes \mu_A \cong \mu_A$.*

Proposition 4.5. *Let $\mu_A \in$ Mod-FR. We have $\mu_A \otimes 0_R \cong \mu_A$.*

For $M \in R$-Mod ($M \in$ Mod-R), it is well-known that the functor $- \otimes_R M$ $(M_R \otimes -)$ is right exact. In FR-Mod, we can easily get the following results.

Proposition 4.6. *Let $\xi_M \in FR$-Mod. Then $\xi_M \otimes -$ preserves epimorphisms in FR-Mod.*

Proof. Let $\mu_A \xrightarrow{\tilde{g}} \nu_B \to \bar{s}$ be an epic. Since $M \otimes -$ is right exact, we have an epimorphism

$$\xi_M \otimes \mu_A \xrightarrow{1_M \otimes \tilde{g}} \xi_M \otimes \nu_B \to \bar{s}.$$

Now, we only need to show that $1_M \otimes \tilde{g}$ is a fuzzy homomorphism. For every $\sum m_i \otimes a_i \in \xi_M \otimes \mu_A$, we have

$$\begin{aligned}
& (\xi \otimes \nu)(1_M \otimes \tilde{g})(\sum m_i \otimes a_i) \\
= \;& (\xi \otimes \nu)(\sum m_i \otimes \tilde{g}(a_i)) \\
= \;& \bigvee \{\xi \times \nu(\sum (m_i^{'}, b_i^{'})) | \sum m_i^{'} \otimes b_i^{'} = \sum m_i \otimes \tilde{g}(a_i)\} \\
\geq \;& \bigvee \{\xi \times \mu(\sum (m_i^{'}, a_i^{'})) | \tilde{g}(a_i^{'}) = b_i^{'} \text{ and } \sum m_i^{'} \otimes b_i^{'} = \sum m_i \otimes \tilde{g}(a_i)\} \\
\geq \;& \bigvee \{\xi \times \mu(\sum (m_i^{''}, a_i^{''})) | \sum m_i^{''} \otimes a_i^{''} = \sum m_i \otimes a_i\} \\
= \;& \xi \otimes \mu(\sum m_i \otimes a_i).
\end{aligned}$$

Hence, we get the desired results. □

Proposition 4.7. *Let $\xi_M \in FR$-Mod. Then $- \otimes \xi_M$ preserves epimorphisms in Mod-FR.*

Proof. The proof is similar to that of Proposition 4.6. □

Now, we study the relationships between Hom functor and tensor functor.

Theorem 4.8. *Let R be a commutative ring. Then there are quasi-isomorphisms*

$$\tau : \gamma_{Hom(\nu_B \otimes \mu_A, \eta_C)} \cong_Q \alpha_{Hom(\mu_A, \beta_{Hom(\nu_B, \eta_C)})};$$
$$\tau^{'} : \gamma_{Hom(\mu_A \otimes \nu_B, \eta_C)} \cong_Q \alpha_{Hom(\mu_A, \beta_{Hom(\nu_B, \eta_C)})}.$$

Proof. We only prove the first quasi-isomorphism. Firstly, we recall the definition of τ in Theorem 2.11 [13]. Suppose $f \in \mathrm{Hom}(\nu_B \otimes \mu_A, \eta_C)$, $a \in A$ and $b \in B$, define $f_a : \nu_B \to \eta_C$ by $f_a(b) = f(b \otimes a)$ and define $\bar{f} : \mu_A \to \mathrm{Hom}(\nu_B, \eta_C)$ by $\bar{f}(b) = f_a$. Then $\tau : \gamma_{\mathrm{Hom}(\nu_B \otimes \mu_A, \eta_C)} \to \alpha_{\mathrm{Hom}(\mu_A, \beta_{\mathrm{Hom}(\nu_B, \eta_C)})}$ is defined by $\tau(f) = \bar{f}$.

So, we have to show that f_a, $\bar{f}$ and τ are fuzzy morphisms.

1) f_a is a fuzzy homomorphism. For $b \in B$, we have

$$\eta(f_a(b)) = \eta(f(b \otimes a)) \geq \nu \otimes \mu(b \otimes a) \geq \nu \times \mu(b, a) = \bigvee \{\nu(b), \mu(a)\} \geq \nu(b).$$

2) $\bar{f}$ is a fuzzy homomorphism. For $a \in A$, we have

$$\begin{aligned}
\beta(\bar{f}(a)) &= \beta(f_a) = \bigwedge \{\eta(f_a(b)) | b \in B\} = \bigwedge \{\eta(f(b \otimes a)) | b \in B\} \\
&\geq \bigwedge \{\bigvee \{\nu(b), \mu(a)\} | b \in B\} \geq \mu(a).
\end{aligned}$$

3) τ is a fuzzy homomorphism. For $f \in \mathrm{Hom}_R(B \otimes A, C)$ we have

$$\begin{aligned}
\alpha(\tau(f)) &= \alpha(\bar{f}) = \bigwedge \{\beta(\bar{f}(a)) | a \in A\} = \bigwedge \{\beta(f_a) | a \in A\} \\
&= \bigwedge \{\bigwedge \{\eta(f_a(b)) | b \in B\} | a \in A\} \\
&= \bigwedge \{\bigwedge \{\eta f(b \otimes a) | b \in B\} | a \in A\} \\
&= \bigwedge \{\eta f(b \otimes a) | b \in B, a \in A\}.
\end{aligned}$$

While, $\gamma(f) = \bigwedge\{\eta f(\Sigma b_i \otimes a_i) | b_i \in B, a_i \in A\}$. So, we get $\alpha(\tau(f)) \geq \gamma(f)$. □

5. Conclusion

In this paper, we make $\mathrm{Hom}(\mu_A, \nu_B)$ into fuzzy R-module by redefining a function $\alpha : \mathrm{Hom}(\mu_A, \nu_B) \longrightarrow [0, 1]$. Then we get some related properties of Hom functor. We also obtain that $\mathrm{Hom}(\xi_p, -)$ is exact if and only if ξ_P is a fuzzy projective R-module. In addition, we prove that the tensor product of two fuzzy R-modules exists. We also get some properties of tensor product. Finally, we study the adjointness of Hom functor and tensor functor. The obtained results generalized the related theory in R-Mod.

To conclude, we would like to notice that there are several open questions on the topic of the paper, some of which we provide below. 1) Is the functor $\mathrm{Hom}(-, \xi_M)$ left exact? 2) What are the relationships between a fuzzy injective R-module ξ_M and the functor $\mathrm{Hom}(-, \xi_M)$? 3) Can we define fuzzy module structures in such a way that these functors do become left exact? 4) If R is not a commutative semiperfect ring, can we get Theorem 3.9? 5) Is the tensor product in fuzzy module category associative? 6) Are τ and τ' in Theorem 4.8 fuzzy isomorphisms? Our future work on this topic will focus on these questions. Our obtained results may help in the study of the homological properties of FR-Mod and equivalences of fuzzy module categories.

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FUZZY PROJECTIVE MODULES AND TENSOR PRODUCTS IN FUZZY MODULE CATEGORIES

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مدولهای پروژکتیو فازی وضربهای تنوری در رسته مدولهای فازی

چکیده. فرض کنید R یک حلقه جابجایی باشد. مجموعه تمام $R-$ همریختی ها از μ_Aبه ν_B، که μ_A و ν_B دو$R-$ مدول فازی می باشند را با (ν_B ، μ_A) Hom نشان می دهیم. با تعـریف دوباره ، $\alpha : \text{Hom}(\mu_A , \nu_B) \rightarrow [0,1]$ ، $\text{Hom}(\mu_A , \nu_B)$ را به یک $R-$ مدول فازی تبدیل می کنیم. خواص تابعگون $\text{Hom}(\mu_A , -) : FR\text{-Mod} \rightarrow FR\text{-Mod}$ را مطالعه ونتایج غیر منتظره ای را بدست می آوریم . بعلاوه ، ثابت می کنیم $\text{Hom}(\xi_p , -)$ دقیق است اگر وتنها اگر ξ_p یک $R-$ مدول پروژکتیو فازی باشد، هنگامی که R یک حلقه نیم کامل جابجایی است . نهایتاً ، ضرب تنوری دو $R-$ مدول فازی را مورد بررسی قرار داده ونتایجی در این رابطه بدست می آوریم . همچنین ، رابطه بین تابعگون Hom وضرب تنوری را مطالعه می کنیم .