FIXED POINTS THEOREMS WITH RESPECT TO FUZZY W-DISTANCE

N. SHOBKOLAEI, S. M. VAEZPOUR AND S. SEDGHI

ABSTRACT. In this paper, we shall introduce the fuzzy w-distance, then prove a common fixed point theorem with respect to fuzzy w-distance for two mappings under the condition of weakly compatible in complete fuzzy metric spaces.

1. Introduction and Preliminaries

There exists considerable literature of fixed point theory dealing with results on fixed or common fixed points in fuzzy metric space (e.g. [1]-[8], [11]-[13], [18]-[19]). George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10] which is a special case of probabilistic metric space and proved that the topology introduced by fuzzy metric is Hausdorff. Then Amini and Saadati [1] considered some important topological properties of fuzzy metric spaces. The concept of w-distance in generalized spaces, firstly, introduced by Saadati et.al., [15], they defined probabilistic w-distance and proved some fixed point theorems. Also some extension of w-distance are considered see [16] and [2]. In this paper, using the idea of Saadati et., al., we define fuzzy w-distance and prove a common fixed point theorem with respect to fuzzy w-distance for two mappings under the condition of weakly compatible.

For the sake of completeness, we briefly recall some notions from the theory of fuzzy metric spaces.

Definition 1.1. [17] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous *t*-norm if ([0,1],*) is an Abelian topological monoid with the unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$, for all $a, b, c, d \in [0,1]$.

Two typical examples of continuous *t*-norms are a * b = ab and $a * b = \min\{a, b\}$.

Definition 1.2. [5] The triple (X, M, *) is called a fuzzy metric space if X is an arbitrary non-empty set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, for each $x, y, z \in X$ and t, s > 0,

(FM-1) M(x, y, t) > 0,

- (FM-2) M(x, y, t) = 1 if and only if x = y,
- (FM-3) M(x, y, t) = M(y, x, t),
- (FM-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s),$
- (FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Received: April 2011; Revised: December 2012 and January 2013; Accepted: November 2013 Key words and phrases: Fuzzy w-distance, Fuzzy metric contractive mapping, Complete fuzzy metric space, Common fixed point theorem.

Example 1.3. Let (X, d) be a metric space. Denote a * b = ab for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$. Then (X, M, *) is a fuzzy metric space.

Example 1.4. Let (X, d) be a metric space and ψ be an increasing and continuous function from \mathbb{R}_+ into (0, 1) such that $\lim_{t \to \infty} \psi(t) = 1$. Four typical examples of these functions are $\psi(x) = \frac{x}{x+1}$, $\psi(x) = \sin(\frac{\pi x}{2x+1})$, $\psi(x) = 1 - e^{-x}$ and $\psi(x) = e^{\frac{-1}{x}}$. Let $a * b \leq ab$, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \left[\psi(t)\right]^{d(x, y)}$$

for all $x, y \in X$. It is easy to see that (X, M, *) is a fuzzy metric space. *Proof.* (FM-1), (FM-2), (FM-3) and (FM-5) of definition 1.2 are obvious. to prove (FM-4), let $x, y, a \in X$ and t, s > 0. Then it is easy to show that

$$\begin{split} M(x,y,t+s) &= [\psi(t+s)]^{d(x,y)} \\ &\geq [\psi(t+s)]^{d(x,a)+d(a,y)} \\ &= [\psi(t+s)]^{d(x,a)} \cdot [\psi(t+s)]^{d(a,y)} \\ &\geq [\psi(t)]^{d(x,a)} * [\psi(s)]^{d(a,y)} \\ &= M(x,a,t) * M(a,y,s). \end{split}$$

Let (X, M, *) be a fuzzy metric space. For t > 0, the open ball B(x, r, t) with center $x \in X$ and radius 0 < r < 1 is defined by

$$B(x, r, t) = \{ y \in X : M(x, y, t) > 1 - r \}.$$

If (X, M, *) is a fuzzy metric space, let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist t > 0 and 0 < r < 1 such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each t > 0. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and t > 0, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \ge n_0$. The fuzzy metric space (X, M, *) is said to be *complete* if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists t > 0 and 0 < r < 1 such that M(x, y, t) > 1 - r for all $x, y \in A$.

Lemma 1.5. [6] Let (X, M, *) be a fuzzy metric space. Then M(x, y, t) is nondecreasing with respect to t, for all x, y in X.

Definition 1.6. Let (X, M, *) be a fuzzy metric space. Then M is said to be *continuous* on $X^2 \times (0, \infty)$ if

$$\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$. i.e.

$$\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7. Let (X, M, *) be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.

Proof. See Proposition 1 of [14].

2. Fixed Point Theorems in Fuzzy W-distance

Now, we introduce the concept of fuzzy w-distance and prove many fixed point theorem in fuzzy metric spaces with fuzzy w-distance which are a nice generalization of the known results in metric and ultra fuzzy metric spaces.

Definition 2.1. Let (X, M, *) be a fuzzy metric space. Then a function $S : X \times X \times [0, \infty) \longrightarrow [0, 1]$ is called a fuzzy w-distance on X if the following are satisfied. (1) $S(x, y, t + s) \ge S(x, z, t) * S(z, y, s)$ for any $x, y, z \in X$ and t, s > 0,

(1) S(x, y, t + s) = S(x, z, t) + S(x, y, s) for any $x, y, z \in \mathbb{N}$ and $t, s \neq s$, (2) for each $x \in X$ and t > 0, S(x, ., t) is upper semicontinuous. That is, if there

(2) for each $x \in X$ and t > 0, S(x, ., t) is upper semicontinuous. That is, if there exists a sequence $\{y_n\}$ of X such that $y_n \longrightarrow y$, then

$$\limsup_{n \to \infty} S(x, y_n, t) \le S(x, y, t),$$

(3) for any $0 < \epsilon < 1$, there exists $0 < \delta < 1$ such that $S(z, x, t) \ge 1 - \delta$ and $S(z, y, s) \ge 1 - \delta$ for all t, s > 0 imply $M(x, y, t + s) \ge 1 - \epsilon$.

Let us give some examples of fuzzy w-distance.

Example 2.2. Every fuzzy metric is a fuzzy w-distance. *Proof.* Let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1 - \delta) * (1 - \delta) \ge 1 - \epsilon$. Then if $M(z, x, t) \ge 1 - \delta$ and $M(z, y, s) \ge 1 - \delta$, we have

$$\begin{array}{rcl} M(x,y,t+s) &\geq & M(z,x,t) \ast M(z,y,s) \\ &\geq & (1-\delta) \ast (1-\delta) \\ &\geq & 1-\epsilon. \end{array}$$

Example 2.3. Let (X, ||.||) be a normed linear space and (X, M, *) be a fuzzy metric space with $M(x, y, t) = \frac{t}{t+||x-y||}$ and a * b = a.b for every $a, b \in [0, 1]$. Then the function $S : X \times X \times [0, \infty) \longrightarrow [0, 1]$ defined by $S(x, y, t) = \frac{t}{t+||x||+||y||}$ for every $x, y \in X, t, s > 0$ is a fuzzy w-distance on X.

Proof. Let $x, y, a \in X$ and t, s > 0. Then it is easy to show that

$$S(x, y, t+s) = \frac{t+s}{t+s+||x||+||y||} \\ \ge \frac{t}{t+||x||+||a||} \cdot \frac{s}{s+||a||+||y||} \\ = S(x, a, t) * S(a, y, s).$$

(2) obviously hold, to prove (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1 - \delta) * (1 - \delta) \ge 1 - \epsilon$. Then if $S(z, x, t) \ge 1 - \delta$ and $S(z, y, t) \ge 1 - \delta$,

we have

106

$$\begin{split} M(x,y,t+s) &= \frac{t+s}{t+s+||x-y||} &\geq \frac{t}{t+||x||} \cdot \frac{s}{s+||y||} \\ &\geq \frac{t}{t+||x||+||z||} \cdot \frac{s}{s+||z||+||y||} \\ &= S(z,x,t) \cdot S(z,y,s) \\ &\geq (1-\delta) \cdot (1-\delta) = (1-\delta) * (1-\delta) \\ &\geq 1-\epsilon. \end{split}$$

By a similar argument we can proof the following examples.

Example 2.4. Let (X, ||.||) be a normed linear space and (X, M, *) be a fuzzy metric space with

$$M(x, y, t) = \begin{cases} \frac{1}{1+||x-y||} & \text{if } 0 < t < 1, \\ \frac{t}{t+||x-y||} & \text{if } t \ge 1, \end{cases}$$

and a * b = a.b for every $a, b \in [0, 1]$. Then the function $S : X \times X \times [0, \infty) \longrightarrow [0, 1]$ defined by $S(x, y, t) = \frac{1}{1+||x||+||y||}$ for every $x, y \in X, t > 0$ is a fuzzy w-distance on X.

Proof. Let $x, y, a \in X$ and t, s > 0. Then it is easy to show that

(2) is obvious. To prove (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1 - \delta) * (1 - \delta) \ge 1 - \epsilon$. Then if $S(z, x, t) \ge 1 - \delta$ and $S(z, y, t) \ge 1 - \delta$, we have $\frac{1}{1+||x||} \ge 1 - \delta$ and $\frac{1}{1+||y||} \ge 1 - \delta$. Hence for every t, s > 0 it is easy to see that

$$M(x, y, t+s) \geq \frac{1}{1+||x||} \cdot \frac{1}{1+||y||} \\ \geq (1-\delta) \cdot (1-\delta) = (1-\delta) * (1-\delta) \\ \geq 1-\epsilon,$$
3).

which prove (3).

Example 2.5. Let (X, M, *) be a fuzzy metric space. Let α be a function from X into [0, 1]. Define $S : X \times X \times [0, \infty) \longrightarrow [0, 1]$ as follows :

$$S(x, y, t) = \alpha(x) * M(x, y, t)$$

for every $x, y \in X, t > 0$. Then S is a fuzzy w-distance on X.

Proof. Let $x, y, a \in X$ and t, s > 0. Then it is easy to show that

$$\begin{array}{lll} S(x,y,t+s) &=& \alpha(x)*M(x,y,t+s) \\ &\geq& (\alpha(x)*\alpha(a))*(M(x,a,t)*M(a,y,s)) \\ &=& S(x,a,t)*S(a,y,s). \end{array}$$

www.SID.ir

(2) is obvious. To prove (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1 - \delta) * (1 - \delta) \ge 1 - \epsilon$. Then if $S(z, x, t) \ge 1 - \delta$ and $S(z, y, s) \ge 1 - \delta$, we have $M(z, x, t) \ge 1 - \delta$ and $M(z, y, s) \ge 1 - \delta$. Hence

$$M(x, y, t+s) \geq M(z, x, t) * M(z, y, s)$$

$$\geq (1-\delta) * (1-\delta)$$

$$\geq 1-\epsilon.$$

The following Lemma plays an important role in the proof of the fixed point theorems, and variational inequalities.

Lemma 2.6. Let (X, M, *) be a fuzzy metric space and let S be a fuzzy w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ be sequences in [0,1] converging to 1 for t > 0, and let $x, y, z \in X, t > 0$. Then the following hold:

(i) If $S(x_n, y, t) \ge \alpha_n(t)$ and $S(x_n, z, t) \ge \beta_n(t)$ for any $n \in \mathbb{N}$, then y = z. In particular, if S(x, y, t) = 1 and S(x, z, t) = 1, then y = z,

(ii) if $S(x_n, y_n, t) \ge \alpha_n(t)$ and $S(x_n, z, t) \ge \beta_n(t)$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z,

(iii) if $S(x_n, x_m, t) \ge \alpha_n(t)$ for any $n, m \in \mathbb{N}$ with m > n, then $\{x_n\}$ is a Cauchy sequence,

(iv) if $S(y, x_n, t) \ge \alpha_n(t)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Proof. We prove parts (ii) and (iii), other parts similarly can be proved. Let $0 < \epsilon < 1$ be given. From the definition of fuzzy w-distance, there exists $0 < \delta < 1$ such that $S(u, v, t) \ge 1 - \delta$ and $S(u, z, t) \ge 1 - \delta$ imply $M(v, z, 2t) \ge 1 - \epsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n(t) \ge 1 - \delta$ and $\beta_n(t) \ge 1 - \delta$, for every $n \ge n_0$. Then we have, for any $n \ge n_0$, that $S(x_n, y_n, t) \ge \alpha_n(t) \ge 1 - \delta$ and $S(x_n, z, t) \ge \beta_n(t) \ge 1 - \delta$ and hence $M(y_n, z, 2t) \ge 1 - \epsilon$. This implies that $\{y_n\}$ converges to z. To prove (iii). Let $0 < \epsilon < 1$ be given. As in the proof of (2), choose $0 < \delta < 1$. Then for any $n, m \ge n_0 + 1$,

$$S(x_{n_0}, x_n, t) \ge \alpha_{n_0}(t) \ge 1 - \delta$$
 and $S(x_{n_0}, x_m, t) \ge \alpha_{n_0}(t) \ge 1 - \delta$

and hence $M(x_n, x_m, 2t) \ge 1 - \epsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. \Box

We recall that two maps f and g are said to be weak compatible if they commute at their coincidence point, that is, fx = gx implies that fgx = gfx.

Definition 2.7. Define $\Phi = \{\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ | \varphi \text{ is an integrable mapping such that, for each <math>0 < \epsilon < 1, 0 < \int_0^{\epsilon} \varphi(s) ds < 1, \int_0^1 \varphi(s) ds = 1\}$, and $\Psi = \{\psi : (0,1] \rightarrow (0,1] | \psi \text{ is a continuous and increasing function such that } \psi(a) > a \text{ for each } a \in (0,1) \text{ and } \lim_{n \to \infty} \psi^n(a) = 1\}.$

Theorem 2.8. Let (X, M, *) be a complete fuzzy metric space and S be a fuzzy w-distance. Let f, g be self-mappings on X satisfy the following conditions:

 $(i)g(X) \subseteq f(X)$ and f(X) is a closed subset of X,

(ii) the pair (f, g) are weakly compatible,

N. Shobkolaei, S. M. Vaezpour and S. Sedghi

(iii)
$$\int_{0}^{S(gx,gy,t)} \varphi(s) ds \ge \psi(\int_{0}^{S(fx,fy,t)} \varphi(s) ds),$$

for each $x, y \in X$ and t > 0, where $\varphi \in \Phi$ and $\psi \in \Psi$. If $d(t) = \inf\{S(x, y, t) | x, y \in X\} > 0$

for all t > 0, then f, g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (i), choose a point x_1 in X such that $gx_0 = fx_1$. In general there exists a sequence $\{x_n\}$ such that, $gx_n = fx_{n+1}$, for $n = 0, 1, 2, \cdots$. By (iii), we have

$$\begin{split} \int_{0}^{S(gx_{n},gx_{n+m},t)} \varphi(s) ds &\geq \psi(\int_{0}^{S(fx_{n},fx_{n+m},t)} \varphi(s) ds) \\ &= \psi(\int_{0}^{S(gx_{n-1},gx_{n+m-1},t)} \varphi(s) ds) \\ &\geq \psi^{2}(\int_{0}^{S(fx_{n-1},fx_{n+m-1},t)} \varphi(s) ds) \\ &\vdots \\ &\geq \psi^{n}(\int_{0}^{S(gx_{0},gx_{m},t)} \varphi(s) ds) \\ &\geq \psi^{n}(\int_{0}^{d(t)} \varphi(s) ds). \end{split}$$

Since $\psi \in \Psi$ by Lemma 2.6, $\{gx_n\}$ is a Cauchy sequence. Since X is complete, $\{gx_n\}$ converges to some point $z \in X$. Thus, we have

$$\lim_{n\to\infty} M(fx_n,z,t) = \lim_{n\to\infty} M(gx_n,z,t) = 1.$$

Since f(X) is closed, there exists $u \in X$ such that f(u) = z. We will prove that gu = z. We have

$$\int_{0}^{S(gx_{n},fu,t)} \varphi(s)ds = \int_{0}^{S(gx_{n},z,t)} \varphi(s)ds \ge \int_{0}^{\limsup_{m\to\infty} S(gx_{n},gx_{m},t)} \varphi(s)ds$$
$$= \limsup_{m\to\infty} \int_{0}^{S(gx_{n},gx_{n+m},t)} \varphi(s)ds$$
$$\ge \limsup_{m\to\infty} \psi^{n} (\int_{0}^{S(gx_{0},gx_{m},t)} \varphi(s)ds)$$
$$\ge \psi^{n} (\int_{0}^{d(t)} \varphi(s)ds).$$

Hence

$$\liminf_{n \to \infty} \int_0^{S(gx_n, fu, t)} \varphi(s) ds = \liminf_{n \to \infty} \psi^n (\int_0^{d(t)} \varphi(s) ds) = 1.$$

Therefore

$$\lim_{n \to \infty} \int_0^{S(gx_n,fu,t)} \varphi(s) ds = \lim_{n \to \infty} \int_0^{S(fx_n,fu,t)} \varphi(s) ds = 1.$$

On the other hand by (iii) we have,

$$\int_0^{S(gx_n, gu, t)} \varphi(s) ds \ge \psi(\int_0^{S(fx_n, fu, t)} \varphi(s) ds).$$

Since ψ is continuous we have

$$\min_{u \to \infty} \int_{0}^{S(gx_n, gu, t)} \varphi(s) ds \ge \psi(\liminf_{n \to \infty} \int_{0}^{S(fx_n, fu, t)} \varphi(s) ds) = 1.$$

Thus,

$$\lim_{n\to\infty}\int_0^{S(gx_n,gu,t)}\varphi(s)ds=\lim_{n\to\infty}\int_0^{S(gx_n,fu,t)}\varphi(s)ds=1.$$

Hence,

$$\lim_{n \to \infty} S(gx_n, fu, t) = \lim_{n \to \infty} S(gx_n, fu, t) = 1$$

By Lemma 2.6, we have gu = fu = z.

Since the pair (f,g) are weakly compatible, we have gfu = fgu. It follows that ffu = fgu = gfu = ggu. Now, we prove that gu = ggu. If $S(gu, ggu, t) \neq 1$, then using condition (iii), we get

$$\begin{split} \int_{0}^{S(gu,ggu,t)} \varphi(s) ds & \geq \quad \psi(\int_{0}^{S(fu,fgu,t)} \varphi(s) ds) = \psi(\int_{0}^{S(gu,ggu,t)} \varphi(s) ds) \\ & > \quad \int_{0}^{S(gu,ggu,t)} \varphi(s) ds, \end{split}$$

which is a contradiction. That is S(gu, ggu, t) = 1. Similarly, if $S(gu, gu, t) \neq 1$, then using condition (iii), we get

$$\begin{split} \int_0^{S(gu,gu,t)} \varphi(s) ds & \geq \quad \psi(\int_0^{S(fu,fu,t)} \varphi(s) ds) = \psi(\int_0^{S(gu,gu,t)} \varphi(s) ds) \\ & > \quad \int_0^{S(gu,gu,t)} \varphi(s) ds, \end{split}$$

which is a contradiction. That is, S(gu, gu, t) = 1 and by Lemma 2.6, we have z = gu = ggu = fgu. Thus z is a common fixed point of f and g.

To prove the uniqueness, let y be another common fixed point of f and g. Then y = fy = gy. If $S(z, y, t) \neq 1$, then using condition (iii), we have

$$\int_0^{S(z,y,t)} \varphi(s) ds = \int_0^{S(gz,gy,t)} \varphi(s) ds$$

$$\geq \quad \psi(\int_0^{S(fz,fy,t)} \varphi(s) ds) = \psi(\int_0^{S(z,y,t)} \varphi(s) ds)$$

$$> \quad \int_0^{S(z,y,t)} \varphi(s) ds,$$

which is a contradiction. It follows that S(z, y, t) = 1. Similarly, it follows that S(z, z, t) = 1. By Lemma 2.6, we have z = y. This completes the proof.

Corollary 2.9. Let (X, M, *) be a complete fuzzy metric space and S be a fuzzy w-distance. Let f, g be self-mappings on X satisfying the following conditions:

 $(i)g(X) \subseteq f(X)$ and f(X) is a closed subset of X,

(ii) the pair (f, g) are weakly compatible,

(iii) $S(gx, gy, t) \ge \psi(S(fx, fy, t))$, for each $x, y \in X$ and t > 0, where $\psi \in \Psi$. If

$$d(t) = \inf\{S(x, y, t) | x, y \in X\} > 0$$

for all t > 0, then f, g have a unique common fixed point in X.

Proof. It is enough to set that $\varphi(s) = 1$ in Theorem 2.8.

www.SID.ir

Corollary 2.10. Let (X, M, *) be a complete fuzzy metric space and S be a fuzzy w-distance. Let f, g, h be self-mappings on X satisfy the following conditions:

(i) h be one to one continuous mapping which commute with f and g

(ii) $hg(X) \subseteq hf(X)$ and hf(X) is a closed subset of X,

(iii) The pair (hf, hg) are weakly compatible,

(iv) $S(hgx, hgy, t) \ge \psi(S(hfx, hfy, t))$, for every $x, y \in X$ and t > 0 where $\psi \in \Psi$.

$$d(t)=\inf\{S(x,y,t)|x,y\in X\}>0$$

for all t > 0, then f, g, h have a unique common fixed point in X.

Proof. By Corollary 2.9, hf and hg have a unique common fixed point $z \in X$. Since h is one to one, from hfz = hgz = z, it follows that fz = gz. We claim that gz = z. If $S(z, gz, t) \neq 1$, then using condition (iii) and hggz = g(hgz) = gz we have,

$$\begin{array}{lll} S(z,gz,t) &=& S(hgz,hggz,t)\\ &\geq& \psi(S(hfz,hfgz,t)) = \psi(S(z,fz,t)) = \psi(S(z,gz,t))\\ &>& S(z,gz,t) \end{array}$$

which is a contradiction. That is S(z, gz, t) = 1. Similarly, if $S(z, z, t) \neq 1$ then

$$S(z,z,t)=S(hgz,hgz,t)\geq\psi(S(hfz,hfz,t))\geq\psi(S(z,z,t))>S(z,z,t),$$

which is a contradiction. Thus S(z, z, t) = 1. and by Lemma 2.6, we have z = gz = fz.

We recall that, self-mapping T has property P if fixed point set $F(T) \neq \emptyset$, implies $F(T^n) = F(T)$, for each $n \in \mathbb{N}$. For more details see [9].

Corollary 2.11. Let (X, M, *) be a complete fuzzy metric space and S be a fuzzy w-distance. Let g be a self-mapping on X satisfy the following conditions:

i) $S(gx, gy, t) \ge \psi(S(x, y, t))$, for every $x, y \in X$ and t > 0, where $\psi \in \Psi$. If

$$d(t) = \inf\{S(x, y, t) | x, y \in X\} > 0$$

for all t > 0, then g have a unique common fixed point in X. Moreover, g has property P.

Proof. By Corollary 2.9, if set f = I, the identity map, then g has a fixed point. Therefore, $F(g^m) \neq \emptyset$, for each positive integer $m \ge 1$. Fix a positive integer n > 1and let $z \in F(g^n)$. We claim that gz = z. If $S(z, gz, t) \neq 1$, then using (i) we have

$$S(z,gz,t) = S(g^n z, g^{n+1}z, t) \ge \psi^n(S(z,gz,t),$$

which is a contradiction. That is, S(z, gz, t) = 1. Similarly, if $S(z, z, t) \neq 1$, then

$$S(z, z, t) = S(g^n z, g^n z, t) \ge \psi^n(S(z, z, t)),$$

which is a contradiction. Thus S(z, z, t) = 1. By Lemma 2.6, we have z = gz. Therefore, g has property P.

If

Example 2.12. Let (X, M, *) be a fuzzy metric space, where X = [0, 1], $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ with t-norm defined by a * b = a.b, for all $a, b \in [0, 1]$. Let $S(x, y, t) = e^{-\frac{y}{t}}$ for all t > 0 and $x, y \in X$. Define self-maps f and g on X as follows:

$$gx = \frac{x^2}{2} \quad , fx = x$$

for any $x \in X$.

First we show that S is a fuzzy w-distance on X. For all $x, y, a \in X$ and t, s > 0, we have

$$S(x, y, t+s) = e^{-\frac{y}{t+s}} \ge e^{-\frac{z}{t}} \cdot e^{-\frac{y}{s}} = S(x, z, t) * S(z, y, s).$$

(2) is obvious. To show (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta = \epsilon < 1$. Then $S(z, x, t) \ge 1 - \delta$ and $S(z, y, s) \ge 1 - \delta$. Hence

$$M(x, y, t+s) = e^{-\frac{|x-y|}{t+s}} \ge e^{-\frac{y}{t}} \ge 1-\delta = 1-\epsilon.$$

Also, (f,g) is weakly compatible. If $\psi(a) = \sqrt{a}$, it is easy to see that

$$S(gx, gy, t) \ge \psi(S(fx, fy, t)).$$

It follows that all conditions in Corollary 2.9 are hold, and z = 0 is a unique common fixed point of f and g.

References

- M. Amini and R. Saadati, Topics in fuzzy metric spaces, J. Fuzzy Math., 11(4) (2003), 765-768.
- [2] Y. J. Cho, R. Saadati and S. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput. Math. Appl., 61(4) (2011), 1254-1260.
- [3] Y. J. Cho, Fixed points in fuzzy metric spaces, J. Fuzzy Math., 5 (1997), 949-962.
- [4] J. X. Fang, On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and Systems, 46 (1992), 107-113.
- [5] A. George and P. Veeramani, On some result in fuzzy metric space, Fuzzy Sets and Systems, 64 (1994), 395-399.
- [6] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27 (1988), 385-389.
- [7] V. Gregori and A. Sapena, On fixed-point theorem in fuzzy metric spaces, Fuzzy Sets and Systems, 125 (2002), 245-252.
- [8] O. Hadžić and E. Pap, Fixed point theory in probabilistic metric spaces, Kluwer Academic Publishers, Dordrecht, 2001.
- [9] G.S. Jeong and B.E. Rhoades, Maps for which $F(T) = F(T^n)$, Fixed Point Theory Appl., 6 (2006), 72-105.
- [10] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica, 11 (1975), 326-334.
- [11] D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, Fuzzy Sets and Systems, 158 (2007), 915 -921.
- [12] D. Miheţ, Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces, Fuzzy Sets and Systems, 159 (2008), 739-744.
- [13] S. N. Mishra, S. N. Sharma and S. L. Singh, Common fixed points of maps in fuzzy metric spaces, Internat. J. Math. Math. Sci., 17 (1994), 253-258.
- [14] J. L. Rodríguez and S. Ramaguera, The Hausdorff fuzzy metric on compact sets, Fuzzy Sets and Systems, 147 (2004), 273-283.
- [15] R. Saadati, D. O'Regan, S. M. Vaezpour and J. K. Kim, Generalized distance and common fixed point theorems in Menger probabilistic metric spaces, Bull. Iran. Math. Soc., 35(2) (2009), 97-117.

- [16] R. Saadati, S.M. Vaezpour and Lj.B. Ciric, Generalized distance and some common fixed point theorems, J. Comput. Anal. Appl., 12(1) (2010), 157-162.
- [17] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math., 10 (1960), 313-334.
- [18] S. Sedghi, N. Shobe and I. Altun, A fixed fuzzy point for fuzzy mappings in complete metric spaces, Math. Commun., **13(2)** (2008), 289-294.
- [19] T. Žikić-Došenović, A common fixed point theorem for compatible mappings in fuzzy metric spaces using implicit relation, Acta Math. Hungar, 125(4) (2009), 357-368.

NABI SHOBKOLAEI*, DEPARTMENT OF MATHEMATICS, ISLAMIC AZAD UNIVERSITY, SCIENCE AND RESEARCH BRANCH, 14778 93855 TEHRAN, IRAN

E-mail address: nabi_shobe@yahoo.com

S. MANSOUR VAEZPOUR, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, AMIRKABIR UNIVERSITY OF TECHNOLOGY, 424 HAFEZ AVENUE, TEHRAN 15914, IRAN. *E-mail address:* vaez@aut.ac.ir

Shaban Sedghi, Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr , Iran

E-mail address: sedghi_gh@yahoo.com

*Corresponding Author

Iranian Journal of Fuzzy Systems Vol.11, No. 2 (2014)

FIXED POINTS THEOREMS WITH RESPECT TO FUZZY W- DISTANCE

N. SHOBKOLAEI, S. M. VAEZPOUR AND S. SEDGHI

قضایای نقاط ثابت نسبت به دبلیو فاصله فازی

چکیده. در این مقاله ابتدا به معرفی دبلیو فاصله فازی می پردازیم سپس قضیه نقطه ثابت مشترک نسبت به W- فاصله فازی را برای دو نگاشت که در شرط مقایسه پذیر ضعیف صدق می کنند در فضا های متریک

فازى اثبات مي كنيم.