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FIXED POINTS THEOREMS WITH RESPECT TO FUZZY W-DISTANCE

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Abstract. In this paper, we shall introduce the fuzzy w-distance, then prove a common fixed point theorem with respect to fuzzy w-distance for two mappings under the condition of weakly compatible in complete fuzzy metric spaces.

1. Introduction and Preliminaries

There exists considerable literature of fixed point theory dealing with results on fixed or common fixed points in fuzzy metric space (e.g. [1]-[8], [11]-[13], [18]-[19]). George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [10] which is a special case of probabilistic metric space and proved that the topology introduced by fuzzy metric is Hausdorff. Then Amini and Saadati [1] considered some important topological properties of fuzzy metric spaces. The concept of w-distance in generalized spaces, firstly, introduced by Saadati et.al., [15], they defined probabilistic w-distance and proved some fixed point theorems. Also some extension of w-distance are considered see [16] and [2]. In this paper, using the idea of Saadati et., al., we define fuzzy w-distance and prove a common fixed point theorem with respect to fuzzy w-distance for two mappings under the condition of weakly compatible.

For the sake of completeness, we briefly recall some notions from the theory of fuzzy metric spaces.

Definition 1.1. [17] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2. [5] The triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

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Example 1.3. Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $x, y \in X$. Then $(X, M, *)$ is a fuzzy metric space.

Example 1.4. Let (X, d) be a metric space and ψ be an increasing and continuous function from $\mathbb{R}_+$ into $(0, 1)$ such that $\lim_{t\longrightarrow\infty} \psi(t) = 1$. Four typical examples of these functions are $\psi(x) = \dfrac{x}{x+1}$, $\psi(x) = \sin(\dfrac{\pi x}{2x+1})$, $\psi(x) = 1 - e^{-x}$ and $\psi(x) = e^{\frac{-1}{x}}$. Let $a * b \leq ab$, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = [\psi(t)]^{d(x,y)}$$

for all $x, y \in X$. It is easy to see that $(X, M, *)$ is a fuzzy metric space.

Proof. (FM-1), (FM-2), (FM-3) and (FM-5) of definition 1.2 are obvious. to prove (FM-4), let $x, y, a \in X$ and $t, s > 0$. Then it is easy to show that

$$\begin{aligned} M(x, y, t + s) &= [\psi(t + s)]^{d(x,y)} \\ &\geq [\psi(t + s)]^{d(x,a)+d(a,y)} \\ &= [\psi(t + s)]^{d(x,a)} . [\psi(t + s)]^{d(a,y)} \\ &\geq [\psi(t)]^{d(x,a)} * [\psi(s)]^{d(a,y)} \\ &= M(x, a, t) * M(a, y, s). \end{aligned}$$

□

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

If $(X, M, *)$ is a fuzzy metric space, let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The fuzzy metric space $(X, M, *)$ is said to be *complete* if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 1.5. [6] *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space. Then M is said to be *continuous* on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$. i.e.

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.7. *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Proof. See Proposition 1 of [14]. □

2. Fixed Point Theorems in Fuzzy W-distance

Now, we introduce the concept of fuzzy w-distance and prove many fixed point theorem in fuzzy metric spaces with fuzzy w-distance which are a nice generalization of the known results in metric and ultra fuzzy metric spaces.

Definition 2.1. Let $(X, M, *)$ be a fuzzy metric space. Then a function S : $X \times X \times [0, \infty) \longrightarrow [0, 1]$ is called a fuzzy w-distance on X if the following are satisfied.

(1) $S(x, y, t + s) \geq S(x, z, t) * S(z, y, s)$ for any $x, y, z \in X$ and $t, s > 0$,

(2) for each $x \in X$ and $t > 0$, $S(x, ., t)$ is upper semicontinuous. That is, if there exists a sequence $\{y_n\}$ of X such that $y_n \longrightarrow y$, then

$$\limsup_{n\to\infty} S(x, y_n, t) \leq S(x, y, t),$$

(3) for any $0 < \epsilon < 1$, there exists $0 < \delta < 1$ such that $S(z, x, t) \geq 1 - \delta$ and $S(z, y, s) \geq 1 - \delta$ for all $t, s > 0$ imply $M(x, y, t + s) \geq 1 - \epsilon$.

Let us give some examples of fuzzy w-distance.

Example 2.2. Every fuzzy metric is a fuzzy w-distance.

Proof. Let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1-\delta)*(1-\delta) \geq 1 - \epsilon$. Then if $M(z, x, t) \geq 1 - \delta$ and $M(z, y, s) \geq 1 - \delta$, we have

$$\begin{aligned} M(x, y, t + s) &\geq M(z, x, t) * M(z, y, s) \\ &\geq (1 - \delta) * (1 - \delta) \\ &\geq 1 - \epsilon. \end{aligned}$$

□

Example 2.3. Let $(X, ||.||)$ be a normed linear space and $(X, M, *)$ be a fuzzy metric space with $M(x, y, t) = \frac{t}{t+||x-y||}$ and $a * b = a.b$ for every $a, b \in [0, 1]$. Then the function $S : X \times X \times [0, \infty) \longrightarrow [0, 1]$ defined by $S(x, y, t) = \frac{t}{t+||x||+||y||}$ for every $x, y \in X, t, s > 0$ is a fuzzy w-distance on X.

Proof. Let $x, y, a \in X$ and $t, s > 0$. Then it is easy to show that

$$\begin{aligned} S(x, y, t + s) &= \frac{t + s}{t + s + ||x|| + ||y||} \\ &\geq \frac{t}{t + ||x|| + ||a||}.\frac{s}{s + ||a|| + ||y||} \\ &= S(x, a, t) * S(a, y, s). \end{aligned}$$

(2) obviously hold, to prove (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1 - \delta) * (1 - \delta) \geq 1 - \epsilon$. Then if $S(z, x, t) \geq 1 - \delta$ and $S(z, y, t) \geq 1 - \delta$,

we have

$$\begin{aligned} M(x,y,t+s) = \frac{t+s}{t+s+||x-y||} &\geq \frac{t}{t+||x||}.\frac{s}{s+||y||} \\ &\geq \frac{t}{t+||x||+||z||}.\frac{s}{s+||z||+||y||} \\ &= S(z,x,t).S(z,y,s) \\ &\geq (1-\delta).(1-\delta) = (1-\delta)*(1-\delta) \\ &\geq 1-\epsilon. \end{aligned}$$

□

By a similar argument we can proof the following examples.

Example 2.4. Let $(X, ||.||)$ be a normed linear space and $(X, M, *)$ be a fuzzy metric space with

$$M(x,y,t) = \begin{cases} \frac{1}{1+||x-y||} & \text{if } 0 < t < 1, \\ \frac{t}{t+||x-y||} & \text{if } t \geq 1, \end{cases}$$

and $a*b = a.b$ for every $a, b \in [0,1]$. Then the function $S : X \times X \times [0,\infty) \longrightarrow [0,1]$ defined by $S(x,y,t) = \frac{1}{1+||x||+||y||}$ for every $x, y \in X, t > 0$ is a fuzzy w-distance on X.

Proof. Let $x, y, a \in X$ and $t, s > 0$. Then it is easy to show that

$$\begin{aligned} S(x,y,t+s) &= \frac{1}{1+||x||+||y||} \\ &\geq \frac{1}{1+||x||+||a||}.\frac{1}{1+||a||+||y||} \\ &= S(x,a,t)*S(a,y,s). \end{aligned}$$

(2) is obvious. To prove (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1-\delta)*(1-\delta) \geq 1-\epsilon$. Then if $S(z,x,t) \geq 1-\delta$ and $S(z,y,t) \geq 1-\delta$, we have $\frac{1}{1+||x||} \geq 1-\delta$ and $\frac{1}{1+||y||} \geq 1-\delta$. Hence for every $t, s > 0$ it is easy to see that

$$\begin{aligned} M(x,y,t+s) &\geq \frac{1}{1+||x||}.\frac{1}{1+||y||} \\ &\geq (1-\delta).(1-\delta) = (1-\delta)*(1-\delta) \\ &\geq 1-\epsilon, \end{aligned}$$

which prove (3). □

Example 2.5. Let $(X, M, *)$ be a fuzzy metric space. Let α be a function from X into $[0,1]$. Define $S : X \times X \times [0,\infty) \longrightarrow [0,1]$ as follows :

$$S(x,y,t) = \alpha(x) * M(x,y,t)$$

for every $x, y \in X, t > 0$. Then S is a fuzzy w-distance on X.

Proof. Let $x, y, a \in X$ and $t, s > 0$. Then it is easy to show that

$$\begin{aligned} S(x,y,t+s) &= \alpha(x) * M(x,y,t+s) \\ &\geq (\alpha(x)*\alpha(a)) * (M(x,a,t)*M(a,y,s)) \\ &= S(x,a,t)*S(a,y,s). \end{aligned}$$

(2) is obvious. To prove (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta < 1$ such that $(1-\delta) * (1-\delta) \geq 1-\epsilon$. Then if $S(z,x,t) \geq 1-\delta$ and $S(z,y,s) \geq 1-\delta$, we have $M(z,x,t) \geq 1-\delta$ and $M(z,y,s) \geq 1-\delta$. Hence

$$\begin{aligned} M(x,y,t+s) &\geq M(z,x,t) * M(z,y,s) \\ &\geq (1-\delta) * (1-\delta) \\ &\geq 1-\epsilon. \end{aligned}$$

□

The following Lemma plays an important role in the proof of the fixed point theorems, and variational inequalities.

Lemma 2.6. *Let $(X, M, *)$ be a fuzzy metric space and let S be a fuzzy w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ be sequences in $[0,1]$ converging to 1 for $t > 0$, and let $x, y, z \in X, t > 0$. Then the following hold:*

(i) If $S(x_n, y, t) \geq \alpha_n(t)$ and $S(x_n, z, t) \geq \beta_n(t)$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $S(x,y,t) = 1$ and $S(x,z,t) = 1$, then $y = z$,

(ii) if $S(x_n, y_n, t) \geq \alpha_n(t)$ and $S(x_n, z, t) \geq \beta_n(t)$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z,

(iii) if $S(x_n, x_m, t) \geq \alpha_n(t)$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence,

(iv) if $S(y, x_n, t) \geq \alpha_n(t)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Proof. We prove parts (ii) and (iii), other parts similarly can be proved. Let $0 < \epsilon < 1$ be given. From the definition of fuzzy w-distance, there exists $0 < \delta < 1$ such that $S(u,v,t) \geq 1-\delta$ and $S(u,z,t) \geq 1-\delta$ imply $M(v,z,2t) \geq 1-\epsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n(t) \geq 1-\delta$ and $\beta_n(t) \geq 1-\delta$, for every $n \geq n_0$. Then we have, for any $n \geq n_0$, that $S(x_n, y_n, t) \geq \alpha_n(t) \geq 1-\delta$ and $S(x_n, z, t) \geq \beta_n(t) \geq 1-\delta$ and hence $M(y_n, z, 2t) \geq 1-\epsilon$. This implies that $\{y_n\}$ converges to z. To prove (iii). Let $0 < \epsilon < 1$ be given. As in the proof of (2), choose $0 < \delta < 1$. Then for any $n, m \geq n_0 + 1$,

$$S(x_{n_0}, x_n, t) \geq \alpha_{n_0}(t) \geq 1-\delta \ \text{ and } \ S(x_{n_0}, x_m, t) \geq \alpha_{n_0}(t) \geq 1-\delta$$

and hence $M(x_n, x_m, 2t) \geq 1-\epsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. □

We recall that two maps f and g are said to be weak compatible if they commute at their coincidence point, that is, $fx = gx$ implies that $fgx = gfx$.

Definition 2.7. Define $\Phi = \{\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ | \varphi$ is an integrable mapping such that, for each $0 < \epsilon < 1$, $0 < \int_0^{\epsilon} \varphi(s)ds < 1$, $\int_0^1 \varphi(s)ds = 1\}$, and
$\Psi = \{\psi : (0,1] \rightarrow (0,1] | \psi$ is a continuous and increasing function such that $\psi(a) > a$ for each $a \in (0,1)$ and $\lim_{n\to\infty} \psi^n(a) = 1\}$.

Theorem 2.8. *Let $(X, M, *)$ be a complete fuzzy metric space and S be a fuzzy w-distance. Let f, g be self-mappings on X satisfy the following conditions:*

(i)*$g(X) \subseteq f(X)$ and $f(X)$ is a closed subset of X,*

(ii) *the pair (f, g) are weakly compatible,*

(iii)
$$\int_0^{S(gx,gy,t)} \varphi(s)ds \geq \psi(\int_0^{S(fx,fy,t)} \varphi(s)ds),$$
for each $x, y \in X$ and $t > 0$, where $\varphi \in \Phi$ and $\psi \in \Psi$. If
$$d(t) = \inf\{S(x,y,t)|x,y \in X\} > 0$$
for all $t > 0$, then f, g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (i), choose a point x_1 in X such that $gx_0 = fx_1$. In general there exists a sequence $\{x_n\}$ such that, $gx_n = fx_{n+1}$, for $n = 0, 1, 2, \cdots$. By (iii), we have
$$\begin{aligned}
\int_0^{S(gx_n,gx_{n+m},t)} \varphi(s)ds &\geq \psi(\int_0^{S(fx_n,fx_{n+m},t)} \varphi(s)ds) \\
&= \psi(\int_0^{S(gx_{n-1},gx_{n+m-1},t)} \varphi(s)ds) \\
&\geq \psi^2(\int_0^{S(fx_{n-1},fx_{n+m-1},t)} \varphi(s)ds) \\
&\vdots \\
&\geq \psi^n(\int_0^{S(gx_0,gx_m,t)} \varphi(s)ds) \\
&\geq \psi^n(\int_0^{d(t)} \varphi(s)ds).
\end{aligned}$$
Since $\psi \in \Psi$ by Lemma 2.6, $\{gx_n\}$ is a Cauchy sequence. Since X is complete, $\{gx_n\}$ converges to some point $z \in X$.

Thus, we have
$$\lim_{n\to\infty} M(fx_n, z, t) = \lim_{n\to\infty} M(gx_n, z, t) = 1.$$
Since $f(X)$ is closed, there exists $u \in X$ such that $f(u) = z$. We will prove that $gu = z$.

We have
$$\begin{aligned}
\int_0^{S(gx_n,fu,t)} \varphi(s)ds &= \int_0^{S(gx_n,z,t)} \varphi(s)ds \geq \int_0^{\limsup_{m\to\infty} S(gx_n,gx_m,t)} \varphi(s)ds \\
&= \limsup_{m\to\infty} \int_0^{S(gx_n,gx_{n+m},t)} \varphi(s)ds \\
&\geq \limsup_{m\to\infty} \psi^n(\int_0^{S(gx_0,gx_m,t)} \varphi(s)ds) \\
&\geq \psi^n(\int_0^{d(t)} \varphi(s)ds).
\end{aligned}$$
Hence
$$\liminf_{n\to\infty} \int_0^{S(gx_n,fu,t)} \varphi(s)ds = \liminf_{n\to\infty} \psi^n(\int_0^{d(t)} \varphi(s)ds) = 1.$$
Therefore
$$\lim_{n\to\infty} \int_0^{S(gx_n,fu,t)} \varphi(s)ds = \lim_{n\to\infty} \int_0^{S(fx_n,fu,t)} \varphi(s)ds = 1.$$
On the other hand by (iii) we have,
$$\int_0^{S(gx_n,gu,t)} \varphi(s)ds \geq \psi(\int_0^{S(fx_n,fu,t)} \varphi(s)ds).$$

Since ψ is continuous we have

$$\liminf_{n\to\infty}\int_0^{S(gx_n,gu,t)}\varphi(s)ds \geq \psi(\liminf_{n\to\infty}\int_0^{S(fx_n,fu,t)}\varphi(s)ds) = 1.$$

Thus,

$$\lim_{n\to\infty}\int_0^{S(gx_n,gu,t)}\varphi(s)ds = \lim_{n\to\infty}\int_0^{S(gx_n,fu,t)}\varphi(s)ds = 1.$$

Hence,

$$\lim_{n\to\infty} S(gx_n, fu, t) = \lim_{n\to\infty} S(gx_n, fu, t) = 1.$$

By Lemma 2.6,we have $gu = fu = z$.

Since the pair (f, g) are weakly compatible , we have $gfu = fgu$. It follows that $ffu = fgu = gfu = ggu$. Now, we prove that $gu = ggu$. If $S(gu, ggu, t) \neq 1$, then using condition (iii), we get

$$\begin{aligned}\int_0^{S(gu,ggu,t)}\varphi(s)ds &\geq \psi(\int_0^{S(fu,fgu,t)}\varphi(s)ds) = \psi(\int_0^{S(gu,ggu,t)}\varphi(s)ds)\\ &> \int_0^{S(gu,ggu,t)}\varphi(s)ds,\end{aligned}$$

which is a contradiction. That is $S(gu, ggu, t) = 1$. Similarly, if $S(gu, gu, t) \neq 1$, then using condition (iii), we get

$$\begin{aligned}\int_0^{S(gu,gu,t)}\varphi(s)ds &\geq \psi(\int_0^{S(fu,fu,t)}\varphi(s)ds) = \psi(\int_0^{S(gu,gu,t)}\varphi(s)ds)\\ &> \int_0^{S(gu,gu,t)}\varphi(s)ds,\end{aligned}$$

which is a contradiction. That is, $S(gu, gu, t) = 1$ and by Lemma 2.6, we have $z = gu = ggu = fgu$. Thus z is a common fixed point of f and g.

To prove the uniqueness, let y be another common fixed point of f and g. Then $y = fy = gy$. If $S(z, y, t) \neq 1$, then using condition (iii), we have

$$\begin{aligned}\int_0^{S(z,y,t)}\varphi(s)ds &= \int_0^{S(gz,gy,t)}\varphi(s)ds\\ &\geq \psi(\int_0^{S(fz,fy,t)}\varphi(s)ds) = \psi(\int_0^{S(z,y,t)}\varphi(s)ds)\\ &> \int_0^{S(z,y,t)}\varphi(s)ds,\end{aligned}$$

which is a contradiction. It follows that $S(z, y, t) = 1$. Similarly, it follows that $S(z, z, t) = 1$. By Lemma 2.6, we have $z = y$. This completes the proof. □

Corollary 2.9. *Let $(X, M, *)$ be a complete fuzzy metric space and S be a fuzzy w-distance. Let f, g be self-mappings on X satisfying the following conditions:*

(i)$g(X) \subseteq f(X)$ *and $f(X)$ is a closed subset of X,*
(ii) *the pair (f, g) are weakly compatible,*
(iii) $S(gx, gy, t) \geq \psi(S(fx, fy, t))$*, for each $x, y \in X$ and $t > 0$, where $\psi \in \Psi$. If*

$$d(t) = \inf\{S(x, y, t)|x, y \in X\} > 0$$

for all $t > 0$, then f, g have a unique common fixed point in X.

Proof. It is enough to set that $\varphi(s) = 1$ in Theorem 2.8. □

Corollary 2.10. *Let $(X, M, *)$ be a complete fuzzy metric space and S be a fuzzy w-distance. Let f, g, h be self-mappings on X satisfy the following conditions:*

(i) h be one to one continuous mapping which commute with f and g

(ii) $hg(X) \subseteq hf(X)$ and $hf(X)$ is a closed subset of X,

(iii) The pair (hf, hg) are weakly compatible,

(iv) $S(hgx, hgy, t) \geq \psi(S(hfx, hfy, t))$, for every $x, y \in X$ and $t > 0$ where $\psi \in \Psi$.

If

$$d(t) = \inf\{S(x, y, t)|x, y \in X\} > 0$$

for all $t > 0$, then f, g, h have a unique common fixed point in X.

Proof. By Corollary 2.9, hf and hg have a unique common fixed point $z \in X$. Since h is one to one, from $hfz = hgz = z$, it follows that $fz = gz$. We claim that $gz = z$. If $S(z, gz, t) \neq 1$, then using condition (iii) and $hggz = g(hgz) = gz$ we have,

$$\begin{aligned} S(z, gz, t) &= S(hgz, hggz, t) \\ &\geq \psi(S(hfz, hfgz, t)) = \psi(S(z, fz, t)) = \psi(S(z, gz, t)) \\ &> S(z, gz, t) \end{aligned}$$

which is a contradiction.That is $S(z, gz, t) = 1$. Similarly, if $S(z, z, t) \neq 1$ then

$$S(z, z, t) = S(hgz, hgz, t) \geq \psi(S(hfz, hfz, t)) \geq \psi(S(z, z, t)) > S(z, z, t),$$

which is a contradiction. Thus $S(z, z, t) = 1$. and by Lemma 2.6,we have $z = gz = fz$. □

We recall that, self-mapping T has property P if fixed point set $F(T) \neq \emptyset$, implies $F(T^n) = F(T)$, for each $n \in \mathbb{N}$. For more details see [9].

Corollary 2.11. *Let $(X, M, *)$ be a complete fuzzy metric space and S be a fuzzy w-distance. Let g be a self-mapping on X satisfy the following conditions:*

ı) $S(gx, gy, t) \geq \psi(S(x, y, t))$, for every $x, y \in X$ and $t > 0$, where $\psi \in \Psi$. If

$$d(t) = \inf\{S(x, y, t)|x, y \in X\} > 0$$

for all $t > 0$, then g have a unique common fixed point in X. Moreover, g has property P.

Proof. By Corollary 2.9, if set $f = I$, the identity map, then g has a fixed point. Therefore, $F(g^m) \neq \emptyset$, for each positive integer $m \geq 1$. Fix a positive integer $n > 1$ and let $z \in F(g^n)$. We claim that $gz = z$. If $S(z, gz, t) \neq 1$, then using (i) we have

$$S(z, gz, t) = S(g^n z, g^{n+1} z, t) \geq \psi^n(S(z, gz, t),$$

which is a contradiction. That is, $S(z, gz, t) = 1$. Similarly, if $S(z, z, t) \neq 1$, then

$$S(z, z, t) = S(g^n z, g^n z, t) \geq \psi^n(S(z, z, t),$$

which is a contradiction. Thus $S(z, z, t) = 1$. By Lemma 2.6, we have $z = gz$. Therefore, g has property P. □

Example 2.12. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 1]$, $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ with t-norm defined by $a * b = a.b$, for all $a, b \in [0, 1]$. Let $S(x, y, t) = e^{-\frac{y}{t}}$ for all $t > 0$ and $x, y \in X$. Define self-maps f and g on X as follows:

$$gx = \frac{x^2}{2} \quad , fx = x$$

for any $x \in X$.

First we show that S is a fuzzy w-distance on X. For all $x, y, a \in X$ and $t, s > 0$, we have

$$S(x, y, t+s) = e^{-\frac{y}{t+s}} \geq e^{-\frac{z}{t}}.e^{-\frac{y}{s}} = S(x, z, t) * S(z, y, s).$$

(2) is obvious. To show (3), let $0 < \epsilon < 1$ be given, we can choose $0 < \delta = \epsilon < 1$. Then $S(z, x, t) \geq 1 - \delta$ and $S(z, y, s) \geq 1 - \delta$. Hence

$$M(x, y, t+s) = e^{-\frac{|x-y|}{t+s}} \geq e^{-\frac{y}{t}} \geq 1 - \delta = 1 - \epsilon.$$

Also, (f, g) is weakly compatible. If $\psi(a) = \sqrt{a}$, it is easy to see that

$$S(gx, gy, t) \geq \psi(S(fx, fy, t)).$$

It follows that all conditions in Corollary 2.9 are hold, and $z = 0$ is a unique common fixed point of f and g.

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FIXED POINTS THEOREMS WITH RESPECT TO FUZZY W- DISTANCE

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قضایای نقاط ثابت نسبت به دبلیو فاصله فازی

چکیده. در این مقاله ابتدا به معرفی دبلیو فاصله فازی می پردازیم سپس قضیه نقطه ثابت مشترک نسبت به W- فاصله فازی را برای دو نگاشت که در شرط مقایسه پذیر ضعیف صدق می کنند در فضا های متریک فازی اثبات می کنیم.