SOME PROPERTIES OF FUZZY NORM OF LINEAR OPERATORS

M. SAHELI, A. HASANKHANI AND A. NAZARI

Abstract. In the present paper, we study some properties of fuzzy norm of linear operators. At first the bounded inverse theorem on fuzzy normed linear spaces is investigated. Then, we prove Hahn Banach theorem, uniform boundedness theorem and closed graph theorem on fuzzy normed linear spaces. Finally the set of all compact operators on these spaces is studied.

1. Introduction

The concept of fuzzy metric spaces was initially introduced by O. Kaleva and S. Seikkla [5], who proved a fixed point theorem for such spaces. C. Felbin [2] introduced the concept of fuzzy norm and showed that every finite dimensional normed linear space has a completion. J. Xiao and X. Zhu [8] modified the definition of fuzzy norm and studied the topological properties of fuzzy normed linear spaces. In [3] we defined a new norm of on operator and studied some its properties. In [4] we considered the norm of the operator defined in [1] by T. Bag and S.K. Samanta and we studied bounded inverse theorem and compact operators on fuzzy normed linear spaces. In this paper, by an example, we show that the definition of the fuzzy norm of operators was given in [1], does not have fuzzy Felbin's norm of operators. So we consider norm of operator defined in [3] and we study bounded inverse theorem and compact operators on fuzzy normed linear spaces.

Our main result in the present paper, include two parts:

Part I, bounded inverse theorem is one of the basic theorems in classical analysis. We investigate this theorem for fuzzy normed linear spaces. We present a counterexample to show that this theorem is not generally true on fuzzy normed linear spaces. We now define strongly complete and bounded strongly complete fuzzy normed linear spaces, and investigate their properties. In particular, bounded inverse theorem is true for these spaces. Finally, we obtain properties of fuzzy normed linear spaces when the inverse of a given operator is bounded.

Part II, we prove Hahn Banach theorem, uniform boundedness theorem and closed graph theorem in fuzzy normed linear spaces.

Part III, we study some basic properties of compact operators on fuzzy normed linear spaces.

Received: March 2012; Revised: February 2013; Accepted: December 2013

Key words and phrases: Fuzzy norm, Fuzzy normed linear space, Fuzzy bounded linear operator.

2. Preliminaries

Definition 2.1. [8] A mapping $\eta : \mathbf{R} \longrightarrow [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:
(N1) there exists $t_0 \in \mathbf{R}$ such that $\eta(t_0) = 1$.
(N2) for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leq \eta_\alpha^+$ such that the $\alpha-$level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.
The set of all fuzzy real numbers is denoted by $F(\mathbf{R})$. Since each $r \in \mathbf{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbf{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1 & , \quad t = r \\ 0 & , \quad t \neq r, \end{cases}$$

it follows that $\mathbf{R}$ can be embedded in $F(\mathbf{R})$.

Definition 2.2. [5] The arithmetic operations $+$, $-$, $\times$ and $/$ on $F(\mathbf{R}) \times F(\mathbf{R})$ are defined by

$$\begin{aligned} (\eta + \gamma)(t) &= \sup_{t=x+y} (\min(\eta(x), \gamma(y))), \\ (\eta - \gamma)(t) &= \sup_{t=x-y} (\min(\eta(x), \gamma(y))), \\ (\eta \times \gamma)(t) &= \sup_{t=xy} (\min(\eta(x), \gamma(y))), \\ (\eta / \gamma)(t) &= \sup_{t=x/y} (\min(\eta(x), \gamma(y))), \end{aligned}$$

which are special cases of Zadeh's extension principle.

Definition 2.3. [5] The absolute value $|\eta|$ of $\eta \in F(\mathbf{R})$ is defined by

$$|\eta|(t) = \begin{cases} \max(\eta(t), \eta(-t)) & , \quad t \geq 0 \\ 0 & , \quad t < 0. \end{cases}$$

Definition 2.4. [5] Let $\eta \in F(\mathbf{R})$. If $\eta(t) = 0$, for all $t < 0$, then η is called a positive fuzzy real number. The set of all positive fuzzy real numbers is denoted by $F^+(\mathbf{R})$.

Lemma 2.5. [5] *Let* $\eta, \gamma \in F(\boldsymbol{R})$ *and* $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$. *Then*
i) $[\eta + \gamma]_\alpha = [\eta_\alpha^- + \gamma_\alpha^-, \eta_\alpha^+ + \gamma_\alpha^+]$
ii) $[\eta - \gamma]_\alpha = [\eta_\alpha^- - \gamma_\alpha^+, \eta_\alpha^+ - \gamma_\alpha^-]$
iii) $[\eta \times \gamma]_\alpha = [\eta_\alpha^- \gamma_\alpha^-, \eta_\alpha^+ \gamma_\alpha^+]$ *for* $\eta, \gamma \in F^+(\boldsymbol{R})$
iv) $[1/\eta]_\alpha = [\frac{1}{\eta_\alpha^+}, \frac{1}{\eta_\alpha^-}]$ *if* $\eta_\alpha^- > 0$
v) $[|\eta|]_\alpha = [\max(0, \eta_\alpha^-, -\eta_\alpha^+), \max(|\eta_\alpha^-|, |\eta_\alpha^+|)]$.

Lemma 2.6. [5] *Let* $[a^\alpha, b^\alpha]$, $0 < \alpha \leq 1$, *be a family of non-empty intervals. Assume*
a) $[a^{\alpha_1}, b^{\alpha_1}] \supset [a^{\alpha_2}, b^{\alpha_2}]$, *for all* $0 < \alpha_1 \leq \alpha_2$,
b) $[\lim_{k \to -\infty} a^{\alpha_k}, \lim_{k \to \infty} b^{\alpha_k}] = [a^\alpha, b^\alpha]$, *whenever* $\{\alpha_k\}$ *is an increasing sequence in* $(0,1]$ *converging to* α,

c)$-\infty < a^\alpha \leq b^\alpha < +\infty$*, for all* $\alpha \in (0,1]$*.*

Then the family $[a^\alpha, b^\alpha]$ *represents the* α*-level sets of a fuzzy real number* $\eta \in F(\boldsymbol{R})$*. Conversely, if* $[a^\alpha, b^\alpha]$*,* $0 < \alpha \leq 1$*, are the* α*-level sets of a fuzzy number* $\eta \in F(\boldsymbol{R})$*, then the conditions (a),(b) and (c) are satisfied.*

Definition 2.7. [5] Let $\eta, \gamma \in F(\mathbf{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. Define a partial ordering by $\eta \leq \gamma$ if and only if $\eta_\alpha^- \leq \gamma_\alpha^-$ and $\eta_\alpha^+ \leq \gamma_\alpha^+$, for all $\alpha \in (0,1]$. An strict inequality in $F(\mathbf{R})$ is defined by $\eta < \gamma$ if and only if $\eta_\alpha^- < \gamma_\alpha^-$ and $\eta_\alpha^+ < \gamma_\alpha^+$, for all $\alpha \in (0,1]$.

Lemma 2.8. *Let* $\eta \in F(\boldsymbol{R})$*. Then* $\eta \in F^+(\boldsymbol{R})$ *if and only if* $\tilde{0} \leq \eta$*.*

Definition 2.9. [2] Let X be a vector space over $\mathbf{R}$. Assume the mappings $L, R : [0,1] \times [0,1] \longrightarrow [0,1]$ are symmetric and non-decreasing in both arguments, and that $L(0,0) = 0$ and $R(1,1) = 1$. Let $\|.\| : X \longrightarrow F^+(\mathbf{R})$. The quadruple $(X, \|.\|, L, R)$ is called a fuzzy normed linear space (briefly, FNS) with the fuzzy norm $\|.\|$, if the following conditions are satisfied:

(F_1) if $x \neq 0$ then $\inf\limits_{0<\alpha\leq 1} \|x\|_\alpha^- > 0$,

(F_2) $\|x\| = \tilde{0}$ if and only if $x = 0$,

(F_3) $\|rx\| = |\tilde{r}|\|x\|$ for $x \in X$ and $r \in \mathbf{R}$,

(F_4) for all $x, y \in X$,

$(F_4L)\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$ whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$,

$(F_4R)\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$ whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$.

Lemma 2.10. [8] *Let* $(X, \|.\|, L, R)$ *be an* FNS*.*

(1) If $L \leq \min$*, then* (F_4L) *holds whenever* $\|x + y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$*, for all* $\alpha \in (0,1]$ *and* $x, y \in X$*.*

(2) If $L \geq \min$*, then* $\|x + y\|_\alpha^- \leq \|x\|_\alpha^- + \|y\|_\alpha^-$*, for all* $\alpha \in (0,1]$ *and* $x, y \in X$ *whenever* (F_4L) *holds.*

(3) If $R \geq \max$*, then* (F_4R) *holds whenever* $\|x + y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$*, for all* $\alpha \in (0,1]$ *and* $x, y \in X$*.*

(4) If $R \leq \max$*, then* $\|x + y\|_\alpha^+ \leq \|x\|_\alpha^+ + \|y\|_\alpha^+$*, for all* $\alpha \in (0,1]$ *and* $x, y \in X$ *whenever* (F_4R) *holds.*

In what follows $L(s,t) = \min(s,t)$ and $R(s,t) = \max(s,t)$, for all $s, t \in [0,1]$. We write $(X, \|.\|)$ or simply X when L and R are as above.

The following result is an analogue of the usual triangle inequality.

Theorem 2.11. *In a fuzzy normed linear space* $(X, \|.\|)$*, the condition* (F_4) *is equivalent to*

$$\|x + y\| \leq \|x\| + \|y\|.$$

Proof. The proof is a consequence of Lemmas 2.16 and 2.7. □

Definition 2.12. [8] Let $(X, \|.\|)$ be a FNS.

i) A sequence $\{x_n\} \subseteq X$ is said to be converge to $x \in X$ ($\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$.
ii)A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim_{m,n\to\infty} \|x_n - x_m\|_\alpha^+ = 0$, for all $\alpha \in (0,1]$.

Definition 2.13. [8] Let $(X, \|.\|)$ be a FNS. A subset A of X is said to be complete, if every Cauchy sequence in A converges in A.

Definition 2.14. [8] Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. A function $\varphi : X \longrightarrow Y$ is said to be continuous at $x \in X$, if $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$ whenever $\{x_n\} \subseteq X$ and $\lim_{n\to\infty} x_n = x$.

Definition 2.15. [7] Let X, Y be fuzzy normed linear spaces such that $\sup_{\alpha\in(0,1]} \|x\|_\alpha^+ < +\infty$, for all $x \in X$ and $T : X \longrightarrow Y$ be a linear operator. Then $\|T\|^\sim$ is called a fuzzy norm of T if

$$\|T\|^\sim(t) = \lim_{\alpha\longrightarrow 0^+} \sup_{\|x\|_\alpha^+=1} \|Tx\|(t).$$

Theorem 2.16. [1] *The fuzzy real number* $\|T\|^*$ *defined by*

$$\|T\|^*(t) = \sup\{\alpha \in (0,1] : t \in [\|T\|_\alpha^{*-}, \|T\|_\alpha^{*+}]\},$$

where

$$\|T\|^{*-} = \sup_{x(\neq 0)\in X} \|Tx\|_\alpha^- / \|x\|_\alpha^+ \text{ and } \|T\|^{*+} = \sup_{x(\neq 0)\in X} \|Tx\|_\alpha^+ / \|x\|_\alpha^-$$

is a fuzzy norm of T.

Definition 2.17. [3] Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. Furthermore, let $T : X \longrightarrow Y$ be a linear operator. The operator T is said to be fuzzy bounded, if there is a fuzzy real number η such that

$$\|Tx\| \leq \eta\|x\|, \text{ for all } x \in X.$$

The set of all fuzzy bounded linear operators, $T : X \longrightarrow Y$, is denoted by $B(X, Y)$.

Remark 2.18. The set $B(X, Y)$ is a real vector space.

Definition 2.19. [3] Let $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces and $T : X \longrightarrow Y$ a fuzzy bounded linear operator. We define $\|T\|$ by,

$$[\|T\|]_\alpha = [\sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^- \,,\ \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}], \text{ for all } \alpha \in (0,1].$$

Then $\|T\|$ is called the fuzzy norm of the operator T.

Notation 2.20. We write $\|T\|_\alpha^- = \sup_{\beta<\alpha} \sup_{\|x\|_\beta^- \leq 1} \|Tx\|_\beta^-$ and $\|T\|_\alpha^+ = \inf\{\eta_\alpha^+ : \|Tx\| \leq \eta\|x\|\}$, i.e. $[\|T\|]_\alpha = [\|T\|_\alpha^-, \|T\|_\alpha^+]$, for all $\alpha \in (0,1]$.

Theorem 2.21. [3] *The vector space* $B(X, Y)$ *equipped with the norm defined in definition 2.19 is a fuzzy normed linear space.*

Lemma 2.22. [3] *Let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Then $\|Tx\| \leq \|T\|\|x\|$, for all $x \in X$.*

Theorem 2.23. [3] *Let $(X, \|.\|)$ be a finite dimensional fuzzy normed linear space. Then every linear operator on X is fuzzy bounded.*

Theorem 2.24. [3] *Let $T : X \longrightarrow Y$ be a fuzzy bounded linear operator and $(X, \|.\|)$, $(Y, \|.\|)$ be fuzzy normed linear spaces. Then $\|T\| \leq \eta$ whenever $\|Tx\| \leq \eta\|x\| \quad (\eta \in F(\boldsymbol{R}))$.*

Theorem 2.25. [3] *Let $(Y, \|.\|)$ be a complete fuzzy normed linear space and $(X, \|.\|)$ be a fuzzy normed linear space. Then $B(X, Y)$ is complete fuzzy normed linear space.*

Definition 2.26. [4] Let X be a fuzzy normed linear space. The fuzzy normed linear space X is called a strongly complete, if normed linear spaces $(X, \|.\|_\alpha^+)$, $(X, \|.\|_\alpha^-)$ are complete, for all $\alpha \in (0, 1]$.

Example 2.27. Let X be a complete normed linear space. Define two fuzzy norms $\|x\|_1$ and $\|x\|_2$ as follows:

$[\|x\|_1]_\alpha = [\|x\|, \|x\|]$ and $[\|x\|_2]_\alpha = [\|x\|, \|x\|/\alpha]$, for all $\alpha \in (0, 1]$ and all $x \in X$.

It is easily verified that $(X, \|.\|_1)$ and $(X, \|.\|_2)$ are strongly complete fuzzy normed linear spaces.

Remark 2.28. If X is a strongly complete fuzzy normed linear space, then X is a complete fuzzy normed linear space.

Theorem 2.29. [4] *Let X be a complete fuzzy normed linear space and suppose that for every $\alpha \in (0, 1]$ there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$. Then the normed linear spaces $(X, \|.\|_\alpha^+)$, $(X, \|.\|_\alpha^-)$ are complete spaces, for all $\alpha \in (0, 1]$.*

Theorem 2.30. [4] *Let X be a strongly complete fuzzy normed linear space. Then, for every $\alpha \in (0, 1]$, there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$.*

Corollary 2.31. *Let X be a complete fuzzy normed linear space and suppose that for every $\alpha \in (0, 1]$ there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$. Then X is a strongly complete fuzzy normed linear space.*

Corollary 2.32. *Let X be a complete fuzzy normed linear space. Then X is a strongly complete fuzzy normed linear space if and only if, for all $\alpha \in (0, 1]$, there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$.*

Definition 2.33. [6] A real valued functional p on a vector space X is called subadditive if and only if

$$p(x + y) \leq p(x) + p(y), \text{ for all } x, y \in X,$$

and positive homogeneous if and only if

$$p(\alpha x) = \alpha p(x), \text{ for all } \alpha \geq 0 \text{ in } \mathbf{R} \text{ and } x \in X.$$

Definition 2.34. [6] A real valued functional p on a vector space X is called sublinear functional if p is a subadditive and positive homogeneous.

Definition 2.35. [8] Let $(X, \|.\|)$ be a fuzzy normed linear space and $A \subseteq X$. A is called compact if every fuzzy open cover of A has a finite subcover.

Definition 2.36. [6] Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces. An operator $T : X \longrightarrow Y$ is called a compact linear operator if T is linear and if for every bounded subset M of X, the image $T(M)$ is relatively compact, that is the closure $\overline{T(M)}$ is compact.

Lemma 2.37. [4] *Let $(X, \|.\|)$ be a normed linear space and $M \subseteq X$. If M is compact then $\overline{M}$ is compact.*

3. Some Properties of Fuzzy Norm of Linear Operators

By the next Example, we show that if the condition's $\sup_{\alpha\in(0,1]} \|x\|^+_\alpha < +\infty$ is deleted then the definition of the fuzzy norm of an operator was given in Definition 2.15, does not have fuzzy norm.

Example 3.1. Let $X = \mathbf{R}$(real number). Define two fuzzy real number $\|x\|_1$ and $\|x\|_2$ as follows:

$$[\|x\|_1]_\alpha = [|x|, |x|] \text{ and } [\|x\|_2]_\alpha = [|x|, |x|/\alpha], \text{ for all } \alpha \in (0, 1] \text{ and all } x \in X.$$

It is easily verified that $\|.\|_1$ and $\|.\|_2$ are fuzzy norms on X.

We define $T : (X, \|.\|_2) \longrightarrow (X, \|.\|_1)$ by $Tx = x$, for all $x \in X$. Clearly T is linear. Now we have

$$\|T\|^{\sim+}_\beta = \lim_{\alpha\longrightarrow 0^+} \sup_{\|x\|^+_\alpha=1} \|Tx\|^+_\beta = \lim_{\alpha\longrightarrow 0^+} \sup_{|x|/\alpha=1} |x| = 0, \text{ for all } \beta \in (0, 1].$$

Then $\|T\|^\sim = 0$, hence $T = 0$, which is a contradiction. Thus $\|T\|^\sim$ which is defined in Definition 2.15 is not well defined.

Now by an example, we show that the definition of the fuzzy norm of operators was given in [7], does not have the condition

$$\|Tx\| \leq \|T\|^\sim \|x\| \text{ in general .}$$

Example 3.2. Let $X = C[0, 1]$ be the set of all continuous functions on $[0, 1]$. Consider the norms $\|f\|_1 = \int_0^1 |f(x)|dx$ and $\|f\|_2 = \sup\{|f(x)| : 0 \leq x \leq 1\}$, for all $f \in C[0, 1]$, on $C[0, 1]$. It is easily checked that the norm $\|.\|_2$ is complete and the norm $\|.\|_1$ is not complete on $C[0, 1]$ and $\|f\|_1 \leq \|f\|_2$, for all $f \in C[0, 1]$. We now define fuzzy numbers $\|.\|$ and $\|.\|'$ as follows:

$$[\|f\|]_\alpha = [\|f\|_1, \|f\|_2], \text{ for all } \alpha \in (0, 1] \text{ and } f \in C[0, 1],$$

and

$$[\|f\|']_\alpha = [\|f\|_2, \|f\|_2], \text{ for all } \alpha \in (0, 1] \text{ and } f \in C[0, 1],$$

It is clear that $\|.\|$ and $\|.\|'$ are fuzzy norms on $C[0, 1]$.
Define $T : (X, \|.\|) \longrightarrow (X, \|.\|')$ by $T(f) = f$, for all $f \in X$. We have

$$\lim_{\alpha\to 0^+} \sup_{\|f\|^+_\alpha=1} \|Tf\|^+_\beta = \lim_{\alpha\to 0^+} \sup_{\|f\|_2=1} \|f\|_2 = 1, \text{ for all } \beta \in (0, 1].$$

Hence T is bounded. Now we obtain that

$$\|T\|_{\beta}^{\sim-} = \lim_{\alpha\to 0^+} \sup_{\|f\|_{\alpha}^{+}=1} \|Tf\|_{\beta}^{-} = \lim_{\alpha\to 0^+} \sup_{\|f\|_2=1} \|f\|_2 = 1,$$
$$\|T\|_{\beta}^{\sim+} = \lim_{\alpha\to 0^+} \sup_{\|f\|_{\alpha}^{+}=1} \|Tf\|_{\beta}^{+} = \lim_{\alpha\to 0^+} \sup_{\|f\|_2=1} \|f\|_2 = 1,$$

for all $\beta \in (0,1]$. But we have

$$\|Tf\|_{\alpha}^{-} = \|f\|_2 \not\leq \|f\|_1 = \|T\|_{\alpha}^{\sim-}\|f\|_{\alpha}^{-} \text{ in general .}$$

Now by an example, we show that the definition of the fuzzy norm of operators was given in [1], does not have fuzzy Felbin's norm of operators.

Example 3.3. Let $X = \mathbf{R}$(real number). Define two fuzzy real number $\|x\|_1$ and $\|x\|_2$ as follows:

$$[\|x\|_1]_{\alpha} = [|x|, |x|] \text{ and } [\|x\|_2]_{\alpha} = [|x|, |x|/\alpha], \text{ for all } \alpha \in (0,1] \text{ and all } x \in X.$$

It is easily verified that $\|.\|_1$ and $\|.\|_2$ are fuzzy norms on X.

We define $T : (X, \|.\|_2) \longrightarrow (X, \|.\|_1)$ by $Tx = x$, for all $x \in X$. Clearly T is linear. Now we have

$$\|Tx\|_{1\alpha}^{-}/\|x\|_{2\alpha}^{+} = |x|/(|x|/\alpha) = \alpha$$

and

$$\|Tx\|_{1\alpha}^{+}/\|x\|_{2\alpha}^{-} = |x|/|x| = 1.$$

Thus $[\|T\|^*]_{\alpha} = [\alpha, 1]$, for all $\alpha \in (0,1]$. Hence $\inf_{0<\alpha\leq 1} \|T\|_{\alpha}^{*-} = 0$, so $\|T\|^*$ defined in [1] is not fuzzy Felbin's norm of operators.

4. **Bounded Inverse Theorem**

In this section, the bounded inverse theorem on fuzzy normed linear spaces is studied. In the following example, we show that the bounded inverse theorem in fuzzy normed linear spaces is not valid in general. In this regard, the notion of strongly complete space is defined.

Example 4.1. Let $X = C[0,1]$ be the set of all continuous functions on $[0,1]$. Consider the norms $\|f\|_1 = \int_0^1 |f(x)|dx$ and $\|f\|_2 = \sup\{|f(x)| : 0 \leq x \leq 1\}$, for all $f \in C[0,1]$, on $C[0,1]$. It is easily checked that the norm $\|.\|_2$ is complete and the norm $\|.\|_1$ is not complete on $C[0,1]$ and $\|f\|_1 \leq \|f\|_2$, for all $f \in C[0,1]$. We now define fuzzy numbers $\|.\|$ and $\|.\|'$ as follows:

$$[\|f\|]_{\alpha} = [\|f\|_1, \|f\|_2], \text{ for all } \alpha \in (0,1] \text{ and } f \in C[0,1],$$

and

$$[\|f\|']_{\alpha} = [\|f\|_2, \|f\|_2], \text{ for all } \alpha \in (0,1] \text{ and } f \in C[0,1],$$

It is clear that $\|.\|$ and $\|.\|'$ are fuzzy norms on $C[0,1]$.

Let $\{f_n\}$ be a Cauchy sequence in the fuzzy normed linear space X, i.e.

$$\lim_{m,n\to\infty} \|f_n - f_m\|_2 = 0.$$

Hence $\{f_n\}$ is a Cauchy sequence in the normed linear space $(X, \|.\|_2)$ and since the normed linear space $(X, \|.\|_2)$ is complete, there exists $f \in X$ such that $\{f_n\}$ converges to f, i.e.

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0.$$

Thus, by definition, $\{f_n\}$ converges to f in the fuzzy normed linear space $(X, \|.\|)$. Hence the fuzzy normed linear space X is complete. Similarly, $(X, \|.\|')$ is a complete fuzzy normed linear space. Define $T : (X, \|.\|') \longrightarrow (X, \|.\|)$ by $T(f) = f$, for all $f \in X$. Let $f \in X$. We have

$$\|T(f)\|_1 = \int_0^1 |f|dx \leq \|f\|_2.$$

On the other hand, we have

$$\|T(f)\|_2 = \|f\|_2 \leq \|f\|_2.$$

Hence $\|T(f)\| \leq \|f\|'$ and thus T is a fuzzy bounded linear operator. T is bijective. And $T^{-1} : (X, \|.\|) \longrightarrow (X, \|.\|')$ is defined by $T^{-1}(f) = f$.

Now we show that T^{-1} is not fuzzy bounded. Let

$$f_n(x) = \begin{cases} 4n^2x & , \quad 0 \leq x \leq 1/(2n) \\ 4n^2(x - 1/n) & , \quad 1/(2n) \leq x \leq 1/n \\ 0 & , \quad 1/n \leq x \leq 1, \end{cases}$$

for all $n \in \mathbf{N}$. We have

$$\|f_n\|_1 = \int_0^1 |f_n|dx = 1.$$

Consider

$$\|T^{-1}(f_n)\|_2 = 2n.$$

As $n \longrightarrow \infty$, $\|T^{-1}(f_n)\|_2 \longrightarrow \infty$. Hence T^{-1} is not a fuzzy bounded linear operator.

In the following Example, we show that there is a complete fuzzy normed linear space that it is not strongly complete fuzzy normed linear space.

Example 4.2. Let X be fuzzy normed linear space in Example 4.1. We showed that the fuzzy normed linear space X is complete. But since $\|.\|_1$ is not complete, the fuzzy normed linear space X is not strongly complete.

Now we study bounded inverse theorem on fuzzy normed linear spaces.

Theorem 4.3. *Let X, Y be a complete fuzzy normed linear spaces, $T : X \longrightarrow Y$ a bijective fuzzy bounded linear operator and, for every $\alpha \in (0, 1]$, $(X, \|.\|_1^-)$, $(X, \|.\|_\alpha^+)$ and $(Y, \|.\|_\alpha^-)$ be Banach spaces. Furthermore, suppose there exists $\alpha_0 \in (0, 1]$ such that*

$$\sup_{0<\beta\leq\alpha_0} \sup_{\|Tx\|_\beta^-\leq 1} \|x\|_\beta^- < \infty.$$

Then $T^{-1} : Y \longrightarrow X$ is a fuzzy bounded linear operator.

Proof. Since T is fuzzy bounded we have $\|Tx\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^+$, then $\|Tx\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^+$. Hence $T : (X, \|.\|_\alpha^+) \longrightarrow (Y, \|.\|_\alpha^-)$ is a bounded linear operator. Since $(X, \|.\|_\alpha^+)$, $(Y, \|.\|_\alpha^-)$ are Banach spaces, by [6, open mapping theorem] $T^{-1} : (Y, \|.\|_\alpha^-) \longrightarrow (X, \|.\|_\alpha^+)$ is bounded; i.e.

$$\sup_{\|y\|_\alpha^-\leq 1} \|T^{-1}y\|_\alpha^+ < +\infty. \tag{1}$$

We now show that $\sup_{0<\beta\leq 1}\sup_{\|Tx\|_\beta^-\leq 1}\|x\|_\beta^- < +\infty$. If

$$\sup_{0<\beta\leq 1}\sup_{\|Tx\|_\beta^-\leq 1}\|x\|_\beta^- = +\infty,$$

then for every $n \in \mathbf{N}$ there exists $\beta_n \in (0,1]$ such that $n \leq \sup_{\|Tx\|_{\beta_n}^-\leq 1}\|x\|_{\beta_n}^-$. Since $\{\beta_n\} \subseteq [0,1]$, there exists a subsequence $\{\beta_{n_k}\}$ such that $\beta_{n_k} \to \beta_0$.
Case1: If $\beta_0 \neq 0$, then there exists $\beta_1 \in (0,1]$ and $N \in \mathbf{N}$ such that $\beta_1 \leq \beta_{n_k} \leq 1$, for all $n_k \geq N$. We have

$$n_k \leq \sup_{\|Tx\|_{\beta_{n_k}}^-\leq 1}\|x\|_{\beta_{n_k}}^- \leq \sup_{\|Tx\|_{\beta_{n_k}}^-\leq 1}\|x\|_1^- \leq \sup_{\|Tx\|_{\beta_1}^-\leq 1}\|x\|_1^-, \text{ for all } n_k \geq N.$$

Hence $\sup_{\|Tx\|_{\beta_1}^-\leq 1}\|x\|_1^- = +\infty$. Since T is fuzzy bounded,

$$\|Tx\|_{\beta_1}^- \leq \|T\|_{\beta_1}^-\|x\|_{\beta_1}^- \leq \|T\|_{\beta_1}^-\|x\|_1^-.$$

Thus $T : (X, \|.\|_1^-) \longrightarrow (Y, \|.\|_{\beta_1}^-)$ is a bounded linear operator. Since $(X, \|.\|_1^-)$ and $(Y, \|.\|_{\beta_1}^-)$ are Banach spaces, by [6, open mapping theorem] $T^{-1} : (Y, \|.\|_{\beta_1}^-) \longrightarrow (X, \|.\|_1^-)$ is a bounded linear operator; i.e. $\sup_{\|y\|_{\beta_1}^-\leq 1}\|T^{-1}y\|_1^- < +\infty$, then $\sup_{\|Tx\|_{\beta_1}^-\leq 1}\|x\|_1^- < +\infty$ which is contradiction.
Case2: If $\beta_0 = 0$, then $\beta_{n_k} \to 0$. Since $\beta_{n_k} \to 0$, there exists $N \in \mathbf{N}$ such that $\beta_{n_k} \leq \alpha_0$, for all $n_k \geq N$. We have

$$n_k \leq \sup_{\|Tx\|_{\beta_{n_k}}^-\leq 1}\|x\|_{\beta_{n_k}}^- \leq \sup_{0<\beta\leq\alpha_0}\sup_{\|Tx\|_\beta^-\leq 1}\|x\|_\beta^-, \text{ for all } n_k \geq N.$$

Hence $\sup_{0<\beta\leq\alpha_0}\sup_{\|Tx\|_\beta^-\leq 1}\|x\|_\beta^- = +\infty$, which is a contradiction by hypothesis. Thus $\sup_{0<\beta\leq 1}\sup_{\|Tx\|_\beta^-\leq 1}\|x\|_\beta^- < +\infty$. Therefore

$$\sup_{0<\beta\leq 1}\sup_{\|y\|_\beta^-\leq 1}\|T^{-1}y\|_\beta^- < +\infty. \tag{2}$$

Hence by (1), (2) and [3, Lemma 5.7] $T^{-1} : Y \longrightarrow X$ is a fuzzy bounded linear operator. □

Corollary 4.4. *Let X, Y be strongly complete fuzzy normed linear spaces and $T : X \longrightarrow Y$ a bijective fuzzy bounded linear operator. Furthermore, suppose there exists $\alpha_0 \in (0,1]$ such that*

$$\sup_{0<\beta\leq\alpha_0}\sup_{\|Tx\|_\beta^-\leq 1}\|x\|_\beta^- < \infty.$$

Then $T^{-1} : Y \longrightarrow X$ is a fuzzy bounded linear operator.

Theorem 4.5. *Let X, Y be a complete fuzzy normed linear spaces and $T : X \longrightarrow Y$ a bijective fuzzy bounded linear operator. Furthermore, suppose there exists $M \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M$, for all $x \in X$ and $\|y\|_\alpha^+/\|y\|_\alpha^- \leq M$, for all $y \in Y$, for all $\alpha \in (0,1]$. Then $T^{-1} : Y \longrightarrow X$ is a fuzzy bounded linear operator.*

Proof. By Corollary 2.32 X and Y are strongly complete fuzzy normed linear spaces. Hence $(X, \|.\|_\alpha^+)$, $(X, \|.\|_\alpha^-)$, $(Y, \|.\|_\alpha^+)$, $(Y, \|.\|_\alpha^-)$ are Banach spaces, for all $\alpha \in (0, 1]$. Since T is fuzzy bounded,

$$\|Tx\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^+, \text{ for all } x \in X.$$

Thus $\|Tx\|_\alpha^- \leq \|T\|_\alpha^- \|x\|_\alpha^+$, for all $x \in X$, and hence $T : (X, \|.\|_\alpha^+) \longrightarrow (Y, \|.\|_\alpha^-)$ is bounded and linear. Since $(X, \|.\|_\alpha^+)$, $(Y, \|.\|_\alpha^-)$ are Banach spaces, by [6, open mapping theorem] $T^{-1} : (Y, \|.\|_\alpha^-) \longrightarrow (X, \|.\|_\alpha^+)$ is a bounded linear operator; i.e.

$$\sup_{\|y\|_\alpha^- \leq 1} \|T^{-1}y\|_\alpha^+ < +\infty. \tag{3}$$

Since T is fuzzy bounded, $\|Tx\|_\alpha^+ \leq \|T\|_\alpha^+ \|x\|_\alpha^+ \leq \|T\|_\alpha^+ \|x\|_\alpha^- \|x\|_\alpha^+ / \|x\|_\alpha^-$. Then by hypothesis $\|Tx\|_\alpha^+ \leq M \|T\|_\alpha^+ \|x\|_\alpha^-$. Hence $T : (X, \|.\|_\alpha^-) \longrightarrow (Y, \|.\|_\alpha^+)$ is a bounded linear operator. Since $(X, \|.\|_\alpha^-)$, $(Y, \|.\|_\alpha^+)$ are Banach spaces, by [6, open mapping theorem] $T^{-1} : (Y, \|.\|_\alpha^+) \longrightarrow (X, \|.\|_\alpha^-)$ is a bounded linear operator; i.e.

$$\sup_{\|y\|_\alpha^+ \leq 1} \|T^{-1}y\|_\alpha^- < +\infty.$$

Let $\|T^{-1}\|_\alpha = \sup_{\|y\|_\alpha^+ \leq 1} \|T^{-1}y\|_\alpha^-$, for all $\alpha \in (0, 1]$. We have

$$\|T^{-1}y\|_\alpha^- \leq \|T^{-1}\|_\alpha \|y\|_\alpha^+ \leq \|T^{-1}\|_\alpha \|y\|_\alpha^- \|y\|_\alpha^+ / \|y\|_\alpha^-, \text{ for all } y \in Y.$$

Then by hypothesis

$$\|T^{-1}y\|_\alpha^- \leq \|T^{-1}\|_\alpha \|y\|_\alpha^- M, \text{ for all } y \in Y. \tag{4}$$

Now we have $\sup_{\|y\|_\alpha^+ \leq 1} \|T^{-1}y\|_\alpha^- \leq \sup_{\|y\|_1^+ \leq 1} \|T^{-1}y\|_1^-$, i.e. $\|T^{-1}\|_\alpha \leq \|T^{-1}\|_1$ for all $\alpha \in (0, 1]$. Then by (4), $\|T^{-1}y\|_\alpha^- \leq \|T^{-1}\|_\alpha \|y\|_\alpha^- M \leq \|T^{-1}\|_1 \|y\|_\alpha^- M$, for all $y \in Y$. Hence

$$\sup_{0 < \alpha \leq 1} \sup_{\|y\|_\alpha^- \leq 1} \|T^{-1}y\|_\alpha^- \leq \|T^{-1}\|_1 M < +\infty. \tag{5}$$

Thus by (3), (5) and [3, Lemma 5.7] T^{-1} is a fuzzy bounded linear operator. □

Definition 4.6. Let X be a strongly complete fuzzy normed linear space. By Corollary 2.32, for all $\alpha \in (0, 1]$ there exists $M_\alpha \in \mathbf{R}$ such that $\|x\|_\alpha^+ / \|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$. A strongly complete fuzzy normed linear space X is called bounded, if $\sup M_\alpha < +\infty$.

Corollary 4.7. *Let X , Y be bounded strongly complete fuzzy normed linear spaces and $T : X \longrightarrow Y$ a bijective fuzzy bounded linear operator. Then $T^{-1} : Y \longrightarrow X$ is a fuzzy bounded linear operator.*

Next, we study the fuzzy normed linear spaces when the inverse of a given operator is bounded.

Theorem 4.8. *Let X, Y be fuzzy normed linear spaces, $T : X \longrightarrow Y$ a bijective fuzzy bounded linear operator and $N_\alpha = \sup_{\|x\|_\alpha^- \leq 1} \|Tx\|_\alpha^+ < +\infty$, for all $\alpha \in (0, 1]$. Furthermore, T^{-1} is a fuzzy bounded linear operator. Then for every $\alpha \in (0, 1]$, there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+ / \|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$.*

Proof. Since T^{-1} is a fuzzy bounded linear operator, $\|T^{-1}y\|_\alpha^+ \leq \|T^{-1}\|_\alpha^+\|y\|_\alpha^+$, for all $y \in Y$. Let $x \in X$. T is bijective, there exists $y \in Y$ such that $Tx = y$. We have

$$\|x\|_\alpha^+ = \|T^{-1}y\|_\alpha^+ \leq \|T^{-1}\|_\alpha^+\|y\|_\alpha^+ = \|T^{-1}\|_\alpha^+\|Tx\|_\alpha^+ \leq N_\alpha\|T^{-1}\|_\alpha^+\|x\|_\alpha^-. \quad (6)$$

Let $M_\alpha = N_\alpha\|T^{-1}\|_\alpha^+$. Then by (6), $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$. □

Theorem 4.9. *Let X be a fuzzy normed linear space and $\sup_{\|x\|_\alpha^- \leq 1}\|x\|_\alpha^+ < +\infty$, for all $\alpha \in (0,1]$. Then, for every $\alpha \in (0,1]$, there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$.*

Proof. Define $I : X \longrightarrow X$ by $I(x) = x$. It is clear that I is a bijective fuzzy bounded linear operator and I^{-1} is a fuzzy bounded linear operator. We have

$$\sup_{\|x\|_\alpha^- \leq 1}\|Ix\|_\alpha^+ = \sup_{\|x\|_\alpha^- \leq 1}\|x\|_\alpha^+ < +\infty.$$

Then by Theorem 4.8 for every $\alpha \in (0,1]$ there exists $M_\alpha \in \mathbf{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$. □

Theorem 4.10. *Let X be a fuzzy normed linear space and suppose for every $\alpha \in (0,1]$ there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$. Then $\sup_{\|x\|_\alpha^- \leq 1}\|x\|_\alpha^+ < +\infty$, for all $\alpha \in (0,1]$.*

Proof. By hypothesis we have $\|x\|_\alpha^+ \leq M_\alpha\|x\|_\alpha^-$. Hence $\sup_{\|x\|_\alpha^- \leq 1}\|x\|_\alpha^+ \leq M_\alpha < +\infty$, for all $\alpha \in (0,1]$. □

Corollary 4.11. *Let X be a complete fuzzy normed linear space. The following conditions are equivalent:*

(i) For every $\alpha \in (0,1]$ there exists $M_\alpha \in \boldsymbol{R}$ such that $\|x\|_\alpha^+/\|x\|_\alpha^- \leq M_\alpha$, for all $x \in X$.

(ii) $\sup_{\|x\|_\alpha^- \leq 1}\|x\|_\alpha^+ < +\infty$, for all $\alpha \in (0,1]$.

(iii) X is a strongly complete fuzzy normed linear space.

5. Hahn Banach Theorem

In the Hahn Banach theorem, the object to be extended is a linear functional f which is defined on a subspace Z of a vector space X and has a certain boundedness property which will be formulated in terms of a sublinear functional.

Theorem 5.1. ***Hahn Banach Theorem (Extension of linear functionals).*** *Let X be a real vector space and $\{p_\alpha\}_{\alpha\in(0,1]}$ a family of sublinear functionals on X such that*

$$p_\alpha(x) \leq p_\beta(x), \text{ for all } \alpha \leq \beta \text{ and all } x \in X.$$

Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$f(x) \leq p_\alpha(x), \text{ for all } \alpha \in (0,1] \text{ and all } x \in Z.$$

Then f has a linear extension $\widetilde{f}$ from Z to X satisfying

$$\widetilde{f}(x) \leq p_\alpha(x), \text{ for all } \alpha \in (0,1] \text{ and all } x \in X.$$

Proof. Proceeding stepwise, we shall prove:
(a) The set E of all linear extensions g of f satisfying

$$g(x) \leq p_\alpha(x), \text{ for all } \alpha \in (0,1]$$

on their domains $D(g)$ can be partially ordered and Zorn's Lemma yields a maximal element $\widetilde{f}$ of E.
(b) $\widetilde{f}$ is defined on the entire space X.
(c) An auxiliary relation which was used in (b).
We start with part
(a) Let E be the set of all linear extensions g of f which satisfy the condition

$$g(x) \leq p_\alpha(x), \text{ for all } \alpha \in (0,1] \text{ and all } x \in D(g).$$

Clearly, $E \neq \emptyset$, since $f \in E$. On E we can define a partial ordering by $g \leq h$ that is h is an extension of g. For any chain $C \subseteq E$ we now define $\widetilde{g}$ by

$$\widetilde{g}(x) = g(x) \quad \text{if } x \in D(g)(g \in C).$$

It is clear that $\widetilde{g}$ is an upper bound of C. Since $C \subseteq E$ is arbitrary, Zorn's Lemma implies that E contains a maximal element $\widetilde{f}$. By the definition of E this is a linear extension of f which satisfies

$$\widetilde{f}(x) \leq p_\alpha(x), \text{ for all } \alpha \in (0,1] \text{ and all } x \in D(\widetilde{f}). \tag{7}$$

(b) We now show that $D(\widetilde{f})$ is equal to X. Suppose that this is false. Then we can choose $y_1 \in X - D(\widetilde{f})$ and consider the subspace Y_1 of X spanned by $D(\widetilde{f})$ and y_1. A functional g_1 on Y_1 is defined by

$$g_1(y + \lambda y_1) = f(y) + \lambda c, \tag{8}$$

where c is any constant. It is clear that g_1 is a proper extension of $\widetilde{f}$. Consequently, if we prove that $g_1 \in E$, by showing that

$$g_1(x) \leq p_\alpha(x), \text{ for all } \alpha \in (0,1] \text{ and all } x \in D(g_1), \tag{9}$$

this will contradict the maximality of $\widetilde{f}$ so that $D(\widetilde{f}) \neq X$ is false and so $D(\widetilde{f}) = X$ is true.
Accordingly, we finally show that g_1 with a suitable c in (8) satisfies (9). We consider y and z in $D(\widetilde{f})$. From (7) we obtain

$$\begin{aligned} \widetilde{f}(y) - \widetilde{f}(z) = \widetilde{f}(y-z) &\leq p_\alpha(y-z) \\ &= p_\alpha(y + y_1 - y_1 - z) \\ &\leq p_\alpha(y + y_1) + P_\alpha(-y_1 - z), \end{aligned}$$

for all $\alpha \in (0,1]$. Now we have

$$-p_\alpha(-y_1 - z) - \widetilde{f}(z) \leq p_\alpha(y + y_1) - \widetilde{f}(y), \text{ for all } \alpha \in (0,1]. \tag{10}$$

Let $\alpha \le \beta$. Hence by (10) we have

$$\begin{aligned} -p_\beta(-y_1 - z) - \widetilde{f}(z) &\le -p_\alpha(-y_1 - z) - \widetilde{f}(z) \\ &\le p_\alpha(y + y_1) - \widetilde{f}(y) \\ &\le p_\beta(y + y_1) - \widetilde{f}(y). \end{aligned} \tag{11}$$

Suppose that $\alpha_n = 1/n$. By (10) there exists $c_n \in \mathbf{R}$ such that

$$\sup_{z \in D(\widetilde{f})} (-p_{\alpha_n}(-y_1 - z) - \widetilde{f}(z)) \le c_n \le \inf_{y \in D(\widetilde{f})} (p_{\alpha_n}(y + y_1) - \widetilde{f}(y)), \tag{12}$$

for all $\alpha \in (0, 1]$. Hence BY (11) and (12)

$$\sup_{z \in D(\widetilde{f})} (-p_{\alpha_1}(-y_1 - z) - \widetilde{f}(z)) \le c_n \le \inf_{y \in D(\widetilde{f})} (p_{\alpha_1}(y + y_1) - \widetilde{f}(y)), \text{ for all } n \in \mathbf{N}.$$

Then there is a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ such that $c_{n_k} \longrightarrow c$. Let $\alpha \in (0, 1]$, then there exists $N_1 > 0$ such that $1/n_k \le \alpha$, for all $n_k > N_1$. Thus by (11)

$$\sup_{z \in D(\widetilde{f})} (-p_\alpha(-y_1 - z) - \widetilde{f}(z)) \le c_{n_k} \le \inf_{y \in D(\widetilde{f})} (p_\alpha(y + y_1) - \widetilde{f}(y)).$$

Hence

$$\sup_{z \in D(\widetilde{f})} (-p_\alpha(-y_1 - z) - \widetilde{f}(z)) \le c \le \inf_{y \in D(\widetilde{f})} (p_\alpha(y + y_1) - \widetilde{f}(y)), \tag{13}$$

for all $\alpha \in (0, 1]$. So

$$-p_\alpha(-y_1 - z) - \widetilde{f}(z) \le c \le p_\alpha(y + y_1) - \widetilde{f}(y), \text{ for all } z, y \in D(\widetilde{f}), \tag{14}$$

for all $\alpha \in (0, 1]$.
Now we prove (9). Let $\lambda < 0$. By (14) we have

$$-p_\alpha(-y_1 - (1/\lambda)y) - \widetilde{f}((1/\lambda)y) \le c,$$

multiplication by $-\lambda > 0$ gives

$$\lambda p_\alpha(-y_1 - (1/\lambda)y) + \widetilde{f}(y) \le -\lambda c.$$

From this and (8), using $y + \lambda y_1$, we obtain the desired inequality

$$g_1(x) = \widetilde{f}(y) + \lambda c \le -\lambda p_\alpha(-y_1 - (1/\lambda)y) = p_\alpha(\lambda y_1 + y) = p_\alpha(x), \text{ for all } \alpha \in (0, 1].$$

For $\lambda > 0$ by (14) we have

$$c \le p_\alpha((1/\lambda)y + y_1) - \widetilde{f}((1/\lambda)y),$$

multiplication by $\lambda > 0$ gives

$$\lambda c \le \lambda p_\alpha((1/\lambda)y + y_1) - \widetilde{f}(y) = p_\alpha(x) - \widetilde{f}(y).$$

From this and (8),

$$g_1(x) = \widetilde{f}(y) + \lambda c \le p_\alpha(x), \text{ for all } \alpha \in (0, 1].$$

□

Theorem 5.2. ***Hahn Banach Theorem (Extension of linear functionals).*** *Let X be a real vector space and $\{p_\alpha\}_{\alpha\in(0,1]}$ a family of sublinear functionals on X such that*

$$p_\beta(x) \le p_\alpha(x),\ \textit{for all } \alpha \le \beta \textit{ and all } x \in X.$$

Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$f(x) \le p_\alpha(x),\ \textit{for all } \alpha \in (0,1] \textit{ and all } x \in Z.$$

Then f has a linear extension $\widetilde{f}$ from Z to X satisfying

$$\widetilde{f}(x) \le p_\alpha(x),\ \textit{for all } \alpha \in (0,1] \textit{ and all } x \in X.$$

Proof. This is similar to the proof of Theorem 5.1 □

Theorem 5.3. ***Hahn Banach Theorem (Generalized).*** *Let X be a real vector space and $\{p_\alpha\}_{\alpha\in(0,1]}$ a family of real valued functionals on X which are subadditive and for every scalar λ satisfy*

$$p_\alpha(\lambda x) = |\lambda| p_\alpha(x),\ \textit{for all } \alpha \in (0,1] \textit{ and all } x \in X.$$

And

$$p_\alpha(x) \le p_\beta(x),\ \textit{for all } \alpha \le \beta \textit{ and all } x \in X.$$

Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$|f(x)| \le p_\alpha(x),\ \textit{for all } \alpha \in (0,1] \textit{ and all } x \in Z. \tag{15}$$

Then f has a linear extension $\widetilde{f}$ from Z to X satisfying

$$|\widetilde{f}(x)| \le p_\alpha(x),\ \textit{for all } \alpha \in (0,1] \textit{ and all } x \in X.$$

Proof. (15) implies $f(x) \le p_\alpha(x)$, for all $\alpha \in (0,1]$ and all $x \in Z$. Hence by Theorem 5.1 there is a linear extension $\widetilde{f}$ from Z to X such that

$$\widetilde{f}(x) \le p_\alpha(x),\ \text{for all } \alpha \in (0,1] \text{ and all } x \in X.$$

We have

$$-\widetilde{f}(x) = \widetilde{f}(-x) \le p_\alpha(-x) = p_\alpha(x),\ \text{for all } \alpha \in (0,1] \text{ and all } x \in X.$$

Hence

$$|\widetilde{f}(x)| \le p_\alpha(x),\ \text{for all } \alpha \in (0,1] \text{ and all } x \in X.$$

□

Theorem 5.4. ***Hahn Banach Theorem (Generalized).*** *Let X be a real vector space and $\{p_\alpha\}_{\alpha\in(0,1]}$ a family of real valued functionals on X which are subadditive and for every scalar λ satisfy*

$$p_\alpha(\lambda x) = |\lambda| p_\alpha(x),\ \textit{for all } \alpha \in (0,1] \textit{ and all } x \in X.$$

And

$$p_\beta(x) \le p_\alpha(x),\ \textit{for all } \alpha \le \beta \textit{ and all } x \in X.$$

Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$|f(x)| \le p_\alpha(x),\ \textit{for all } \alpha \in (0,1] \textit{ and all } x \in Z.$$

Then f has a linear extension $\widetilde{f}$ from Z to X satisfying

$$|\widetilde{f}(x)| \leq p_\alpha(x), \text{ for all } \alpha \in (0,1] \text{ and all } x \in X.$$

Proof. This is similar to the proof of Theorem 5.3 □

Theorem 5.5. ***Hahn Banach Theorem (Fuzzy normed spaces)***. *Let f be a fuzzy bounded linear functional on a subspace Z of a fuzzy normed space X. Then there exists a linear functional $\widetilde{f}$ on X which is an extension of f to X and has the same norm,*

$$\|\widetilde{f}\| = \|f\|.$$

Proof. Useing Theorems 5.3 and 5.4, we have

$$|f(x)| \leq \|f\|_\alpha^- \|x\|_\alpha^-, \text{ for all } x \in Z \text{ and all } \alpha \in (0,1].$$

Now we define

$$p_\alpha(x) = \|f\|_\alpha^- \|x\|_\alpha^-, \text{ for all } x \in X \text{ and all } \alpha \in (0,1].$$

Using Theorem 5.3, there exists a linear functional $\widetilde{f}$ on X which is an extension of f and satisfies

$$|\widetilde{f}(x)| \leq \|f\|_\alpha^- \|x\|_\alpha^-, \text{ for all } x \in X \text{ and all } \alpha \in (0,1].$$

We have

$$\|f\|_\alpha^- \|x\|_\alpha^- \leq \|f\|_\alpha^+ \|x\|_\alpha^+, \text{ for all } x \in X \text{ and all } \alpha \in (0,1].$$

Thus

$$|\widetilde{f}(x)| \leq \|f\|_\alpha^+ \|x\|_\alpha^+, \text{ for all } x \in X \text{ and all } \alpha \in (0,1].$$

So we obtain that

$$\|\widetilde{f}\| \leq \|f\|.$$

Since $\widetilde{f}$ is an extension of f, we have $\|f\| \leq \|\widetilde{f}\|$, and so the proof is completed. □

Uniform boundedness theorem states that if X is a Banach space and a sequence of operators T_n is bounded at every point $x \in X$, then the sequence is uniformly bounded.

Theorem 5.6. ***Uniform Boundedness Theorem.*** *Let $\{T_n\}$ be a sequence of fuzzy bounded linear operators $T_n : X \longrightarrow Y$ from a strongly complete fuzzy normed linear space X into a fuzzy normed linear space Y such that*

$$\|T_n x\| \leq \eta_x, \text{ for all } n \in N,$$

where η_x is a fuzzy real number. Then there is a fuzzy real number η such that

$$\|T_n\| \leq \eta, \text{ for all } n \in N.$$

Proof. Since $T_n : X \longrightarrow Y$ is a linear operator and $(X, \|.\|_\alpha^+)$ a Banach space, for any $\alpha \in (0,1]$, it follows from Theorem 4.7.3 of [3] that there exist a real number d_α such that

$$\|T_n\|_\alpha^+ \leq d_\alpha, \text{ for all } \alpha \in (0,1].$$

We define $a_\alpha = \inf\{d_\beta : \beta \in (0,1]\}$ and $b_\alpha = \inf\{d_\beta : 0 < \beta \le \alpha\}$, for all $\alpha \in (0,1]$. We have that the closed interval $[a_\alpha, b_\alpha]$ is nested, for any $\alpha \in (0,1]$. So there is a fuzzy real number η such that

$$[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+] = [\sup_{0<\beta<\alpha} a_\beta, \inf_{0<\beta<\alpha} b_\beta], \text{ for all } \alpha \in (0,1].$$

Now we obtain that $\|T_n\|_\alpha^- \le \|T_n\|_1^+ \le \|T_n\|_\gamma^+ \le d_\gamma$, for all $\gamma \in (0,1]$. Hence $\|T_n\|_\alpha^- \le a_\beta$, for all $\beta \in (0,1]$. Thus $\|T_n\|_\alpha^- \le \eta_\alpha^-$, for all $\alpha \in (0,1]$. Moreover, we have $\|T_n\|_\alpha^+ \le \|T_n\|_\beta^+ \le d_\beta$, for all $\beta \le \alpha$. So $\|T_n\|_\alpha^+ \le b_\alpha$, for all $\alpha \in (0,1]$. Since

$$\|T_n\|_\alpha^+ \le b_\alpha \le b_\beta, \text{ for all } \beta < \alpha,$$

it follows that $\|T_n\|_\alpha^+ \le \eta_\alpha^+$, for all $\alpha \in (0,1]$. Hence $\|T_n\| \le \eta$. □

Theorem 5.7. *Let X be strongly complete fuzzy normed linear space and Z a closed subspace of X. Then Z is a strongly complete fuzzy normed linear space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in a normed linear space $(Z, \|.\|_\alpha^+)$. Z is a subspace of X, hence $\{x_n\}$ is a Cauchy sequence in a normed linear space $(X, \|.\|_\alpha^+)$. Since X is a strongly complete fuzzy normed linear space, there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$. Let $\alpha \le \beta$. We have $\|x_n - x\|_\beta^+ \le \|x_n - x\|_\alpha^+$. Hence $\lim_{n\to\infty} \|x_n - x\|_\beta^+ = 0$. Let $\beta \le \alpha$. Now we have

$$\|x_n - x\|_\beta^- \le \|x_n - x\|_\alpha^- \le \|x_n - x\|_\alpha^+.$$

Thus $\lim_{n\to\infty} \|x_n - x\|_\beta^- = 0$. By Theorem 4.7. in [8], we obtain that

$$\|x_n - x\|_\beta^+ \le M_\beta \|x_n - x\|_\beta^-,$$

hence $\lim_{n\to\infty} \|x_n - x\|_\beta^+ = 0$. So $\lim_{n\to\infty} \|x_n - x\|_\beta^+ = 0$, for all $\beta \in (0,1]$. Then $\lim_{n\to\infty} x_n = x$. Since Z is a closed subspace of X it follows that $x \in Z$. Thus the normed space $(Z, \|.\|_\alpha^+)$ is complete. similarly, the normed space $(Z, \|.\|_\alpha^-)$ is complete. Hence Z is a strongly complete fuzzy normed linear space. □

Definition 5.8. (Closed linear operator). Let X and Y be fuzzy normed linear spaces and $T : D(T) \longrightarrow Y$ a linear operator with domain $D(T) \subseteq X$. Then T is called a closed linear operator if its graph

$$G(T) = \{(x,y) : x \in D(T), y = Tx\}$$

is closed in the normed space $X \times Y$, where the two algebraic operation of a vector space in $X \times Y$ are defined as usual, that is

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \; c(x,y) = (cx, cy),$$

and the norm on $X \times Y$ is defined by

$$\|(x,y)\| = \|x\| + \|y\|.$$

Theorem 5.9. ***Closed Graph Theorem****. Let X and Y be strongly complete fuzzy normed linear spaces and $T : D(T) \longrightarrow Y$ a closed linear operator, where $D(T) \subseteq X$. Then if $D(T)$ is closed in X, the operator T is bounded.*

Proof. It is clear that $X \times Y$ with the norm defined in Definition 5.8, is strongly complete. Since $G(T) \subseteq X \times Y$ and $D(T) \subseteq X$ are closed, it follows from Theorem 5.7 that $G(T)$ and $D(T)$ are strongly complete fuzzy normed linear spaces. We consider the mapping

$$P : G(T) \longrightarrow D(T)$$
$$(x, Tx) \longmapsto x \quad .$$

P is linear, bounded and bijective. Since $\sup_{0<\beta\leq 1} \sup_{\|P(x,Tx)\|_\beta^- \leq 1} \|x\|_\beta^- = 1 < \infty$, it follows from Corollary 4.4 that P^{-1} is bounded. Now we have

$$\|Tx\| \leq \|Tx\| + \|x\| = \|P^{-1}(x)\| \leq \|P^{-1}\|\|x\|, \text{ for all } x \in D(T).$$

Hence T is bounded. □

6. Compact Linear Operators on Fuzzy Normed Linear Spaces

In this section, the compact operators on fuzzy normed linear spaces are studied. We know that all ordinary compact operators on normed linear spaces are bounded. But the following example shows that this is not true for compact operators on fuzzy normed linear spaces.

Example 6.1. Let $X = l^1$. Define fuzzy numbers $\|.\|$ and $\|.\|_0$ by

$$[\|x\|]_\alpha = [\sup_{n\geq 0} |a_n|, \sum_{n=1}^{\infty} |a_n|] \text{ and } [\|x\|_0]_\alpha = [\sum_{n=1}^{\infty} |a_n|, \sum_{n=1}^{\infty} |a_n|],$$

where $x = \{a_n\}$, for all $\alpha \in (0, 1]$. It is clear that $\|.\|$ and $\|.\|_0$ are fuzzy norms on X. We define $T : (X, \|.\|) \to (X, \|.\|_0)$ by $T\{a_n\} = \{a_n/n\}$, for all $\{a_n\} \in X$. It is easily checked that T is a compact linear operator.
now we show that T is not fuzzy bounded. If T is fuzzy bounded, then there exists a fuzzy real number $\eta > 0$ such that $\|Tx\|_0 \leq \eta\|x\|$, for all $x \in X$ and hence $\|Tx\|_{0\alpha}^- \leq \eta_\alpha^- \|x\|_\alpha^-$, for all $x \in X$ and all $\alpha \in (0, 1]$. However,

$$\|T\{a_n\}\|_{0\alpha}^- / \|\{a_n\}\|_\alpha^- = \|\{a_n/n\}\|_{0\alpha}^- / \sup_{n\geq 0} |a_n| = (\sum_{n=1}^{\infty} |a_n/n|)/ \sup_{n\geq 0} |a_n| \leq \eta_\alpha^-. \quad (16)$$

Let

$$a_n = \begin{cases} 1, & n = 1, ..., k \\ 0, & n > k. \end{cases}$$

Then $\{a_n\} \in X$ and by (16) we have $\sum_{n=1}^{k} 1/n \leq \eta_\alpha^-$, for all $k \in \mathbf{N}$. As $k \to \infty$ then $\eta_\alpha^- = +\infty$, which is a contradiction.

Theorem 6.2. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces and $T : X \longrightarrow Y$ a linear operator. Then T is compact if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Tx_n\}$ in Y which has a convergent subsequence.*

Proof. This is similar to the proof of[6, Theorem 8.1.3] □

Theorem 6.3. *Let $(X, \|.\|)$ and $(Y, \|.\|)$ be fuzzy normed linear spaces and $T : X \longrightarrow Y$ a linear operator.*
(i) If T is bounded and $\dim T(X) < \infty$, then the operator T is compact.
(ii) If $\dim X < \infty$ the operator T is compact.

Proof. (i) Let $\{x_n\}$ be a bounded sequence in X and $y_n = Tx_n$. Since T is bounded by Lemma 2.22, $\|Tx\| \le \|T\|\|x\|$, for all $x \in X$, hence $\|Tx_n\|_\alpha^+ \le \|T\|_\alpha^+ \|x_n\|_\alpha^+$, for all $\alpha \in (0, 1]$ and $n \in \mathbf{N}$.
Since $T : (X, \|.\|_1^+) \longrightarrow (Y, \|.\|_1^+)$ is a bounded linear operator and $\dim T(X) < \infty$, by [6, Theorem 8.1.4 and Theorem 8.1.3] there is a $y \in Y$ and a subsequence $\{y_{1n}\}$ of $\{y_n\}$ such that $\lim_{n\to\infty} \|y_{1n} - y\|_1^+ = 0$.
Since $T : (X, \|.\|_{1/2}^+) \longrightarrow (Y, \|.\|_{1/2}^+)$ is a bounded linear operator and $\dim T(X) < \infty$, by [6, Theorem 8.1.4 and Theorem 8.1.3] there is a $y' \in Y$ and a subsequence $\{y_{2n}\}$ of $\{y_{1n}\}$ such that $\lim_{n\to\infty} \|y_{2n} - y'\|_{1/2}^+ = 0$. Since $\|y_{2n} - y'\|_1^+ \le \|y_{2n} - y'\|_{1/2}^+$ it follows that $\lim_{n\to\infty} \|y_{2n} - y'\|_1^+ = 0$. On the other hand, $\lim_{n\to\infty} \|y_{2n} - y\|_1^+ = 0$. Hence $y = y'$.
Similarly, there is a subsequence $\{y_{k+1n}\}$ of $\{y_{kn}\}$ such that $\lim_{n\to\infty} \|y_{k+1n} - y\|_{1/k+1}^+ = 0$, for all $k \in \mathbf{N}$. Consider the subsequence $\{y_{nn}\}$ of $\{y_n\}$. We shell show that $\lim_{n\to\infty} \|y_{nn} - y\|_\alpha^+ = 0$, for all $\alpha \in (0, 1]$.
Let $\alpha \in (0, 1]$ and $0 < \varepsilon$ be fixed. There is a $N_1 \in \mathbf{N}$ such that $1/n < \alpha$, for all $N_1 \le n$.
We have $\lim_{n\to\infty} \|y_{N_1 n} - y\|_{1/N_1}^+ = 0$. Hence there exists $N_2 \in \mathbf{N}$ such that $\|y_{N_1 n} - y\|_{1/N_1}^+ < \varepsilon$, for all $N_2 \le n$.
Let $N = \max\{N_1, N_2\}$. We now have $\|y_{nn} - y\|_\alpha^+ \le \|y_{nn} - y\|_{1/N_1}^+ < \varepsilon$, for all $N \le n$ and the proof is complete by Definition 2.36.

(ii) Since $\dim X < \infty$, by Theorem 2.23, T is bounded. Moreover $\dim T(X) \le \dim X < \infty$. Hence by (i), T is compact. □

Theorem 6.4. *Let $\{T_n\} \in B(X, Y)$ be a sequence of compact linear operators from a fuzzy normed linear space $(X, \|.\|)$ into a complete fuzzy normed linear space $(Y, \|.\|)$. If $\{T_n\}$ is uniformly operator convergent, say, $\|T_n - T\| \longrightarrow 0$, then the limit operator T is compact.*

Proof. This is similar to the proof of [6, Theorem 8.1.5] □

7. Conclusion

As the idea of the fuzzy norm on a linear space is relatively new, the systematic development of the subject fuzzy functional analysis has just started. In this paper, we have dealt with fuzzy normed linear space and have started studying fuzzy operator theory. An attempt has been made to give a consistent definition of a fuzzy bounded linear operator by rectifying some defects in its previous definition given by Bag and Samanta [1], Xiao and Zhu [7]. Bounded Inverse Theorem and compact linear operators on fuzzy normed linear spaces are studied. This will open up the possibility of studying fuzzy operator theory which would have a wide range of applicability of fuzzy functional analysis.

References

[1] T. Bag and S. K. Samanta, *Fuzzy bounded linear operators in Felbin's type fuzzy normed linear spaces*, Fuzzy Sets and Systems, **159** (2008), 685-707.

[2] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets and Systems, **48** (1992), 239-248.
[3] A. Hasankhani, A. Nazari and M. Saheli, *Some properties of fuzzy hilbert spaces and norm of operators*, Iranian Journal of Fuzzy Systems, **7(3)** (2010), 129-157.
[4] A. Hasankhani, A. Nazari and M. Saheli, *Bounded inverse theorem and compact linear operators on fuzzy normed linear spaces*, Ital. J. Pure Appl. Math., accepted for publication.
[5] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems, **12** (1984), 215-229.
[6] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley and Sons, New York, 1978.
[7] J. Xiao and X. Zhu, *Fuzzy normed space of operators and its completeness*, Fuzzy Sets and Systems, **133** (2003), 389-399.
[8] J. Xiao and X. Zhu, *On linearly topological structure and property of fuzzy normed linear space*, Fuzzy Sets and Systems, **125** (2002), 153-161.

M. Saheli*, Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran
E-mail address: `saheli@ mail.vru.ac.ir`

A. Hasankhani, Department of Mathematics, Islamic Azad University, Kerman Branch, Kerman, Iran
E-mail address: `abhasan@mail.uk.ac.ir`

A. Nazari, Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran
E-mail address: `nazari@ mail.uk.ac.ir`

*Corresponding author

Iranian Journal of Fuzzy Systems Vol.11, No. 2 (2014)

SOME PROPERTIES OF FUZZY NORM OF LINEAR OPERATORS

M. SAHELI, A. HASANKHANI AND A. NAZARI

بعضی خواص نرم فازی عملگرهای خطی

چکیده. در این مقاله، بعضی خواص نرم فازی عملگرهای خطی مطالعه شده است. ابتدا قضیه کرانداری معکوس مورد بررسی قرار گرفته است. در ادامه قضیه هان-باناخ، قضیه کرانداری یکنواخت و قضیه گراف بسته روی فضاهای نرمدار فازی اثبات گردیده است. در نهایت مجموعه عملگرهای فشرده روی این فضاها مطالعه شده اند.