# FURTHER STUDY ON (L, M)-FUZZY TOPOLOGIES AND (L, M)-FUZZY NEIGHBORHOOD SYSTEMS

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ABSTRACT. Following the idea of L-fuzzy neighborhood system as introduced by Fu-Gui Shi, and its generalization to (L, M)-fuzzy neighborhood system, the relationship between (L, M)-fuzzy topology and (L, M)-fuzzy neighborhood system will be further studied. As an application of the obtained results, we will describe the initial structures of (L, M)-fuzzy neighborhood subspaces and (L, M)-fuzzy topological product spaces.

### 1. Introduction and Preliminaries

In this paper, based on the idea of (L, M)-fuzzy topological space introduced by T. Kubiak and A. Šostak [6, 7], and the notion of (L, M)-fuzzy neighborhood system as a generalization of *L*-fuzzy neighborhood system of Fu-Gui Shi [10], the relationship between (L, M)-fuzzy topology and (L, M)-fuzzy neighborhood system will be further studied. As an application of the obtained results, we will describe the initial structures of (L, M)-fuzzy neighborhood subspaces and (L, M)-fuzzy topological product spaces.

The following preliminaries will be used throughout this paper, which can be found in [3, 8].

A complete lattice L is called completely distributive, if one of the following conditions hold (the second then follows as a consequence [3]):

(CD1)

$$\bigwedge_{i \in I} \left( \bigvee_{i \in J_i} a_{i,j} \right) = \bigvee_{f \in \prod J_i} \left( \bigwedge_{i \in I} a_{i,f(i)} \right),$$

(CD2)

$$\bigvee_{i\in I} \left(\bigwedge_{i\in J_i} a_{i,j}\right) = \bigwedge_{f\in\prod J_i} \left(\bigvee_{i\in I} a_{i,f(i)}\right),$$

where for each  $i \in I$  and  $j \in J_i$ ,  $a_{i,j} \in L$  and  $f \in \prod J_i$  means that f is a mapping  $f: I \to \bigcup J_i$  such that  $f(i) \in J_i$  for each  $i \in I$ .

An element  $a \neq 0$  in a lattice is called coprime if  $a \leq b \lor c$  implies  $a \leq b$  or  $a \leq c$  for all  $b, c \in L$ . Further, a is said to be join irreducible if  $a = b \lor c$ 

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implies a = b or a = c for all  $b, c \in L$ . The set of all coprime elements (resp. join irreducible elements) of L is denoted by  $\operatorname{Copr}(L)$  (resp. J(L)). It can be verified that if L is distributive, then  $a \in L$  is coprime iff it is join irreducible, which means  $\operatorname{Copr}(L) = J(L)$ . So, for convenience, we usually use J(L) to stand for the set of all coprime elements of L if L is distributive. If L is a completely distributive lattice and  $x \triangleleft \bigvee_{t \in T} y_t$ , then there must be  $t^* \in T$  such that  $x \triangleleft y_{t^*}$  (here  $x \triangleleft a$  means:  $K \subset L, a \leq \bigvee K \Rightarrow \exists y \in K$  such that  $x \leq y$ ). Some more properties of  $\lhd$  can be found in [8].

In the rest of the paper, L and M always denote Hutton algebras. A Hutton algebra L, is a completely distributive lattice with order-reversing involution with the least element 0 and the greatest element 1. Recall that an order-reversing involution ' on L is a map  $(-)': L \longrightarrow L$  such that for any  $a, b \in L$ , the following conditions hold: (1)  $a \leq b$  implies  $b' \leq a'$ . (2) a'' = a. The following properties hold for any subset  $\{b_i: i \in I\} \in L$ : (1)  $(\bigvee_{i \in I} b_i)' = \bigwedge_{i \in I} b'_i$ ; (2)  $(\bigwedge_{i \in I} b_i)' = \bigvee_{i \in I} b'_i$ . We notice that  $L^X$ , the set of all L-subsets of X, is also a Hutton algebra with pointwise order. Its smallest element and the largest element are denoted  $0_X$  and  $1_X$ , respectively. For each  $A \in L^X$ , the L-subset A' is defined A'(x) = (A(x))' for each  $x \in X$ . Clearly,  $J(L^X) = \{x_\lambda : x \in X, \lambda \in J(L)\}$ , where  $x_\lambda$  is defined by  $x_\lambda(y) = \lambda$  if y = x and  $x_\lambda(y) = 0$  otherwise.

**Definition 1.1.** (Kubiak and Šostak [6, 7]) An (L, M)-fuzzy topology on a set X is a map  $\mathcal{T}: L^X \longrightarrow M$  such that

(LMFT1)

$$\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1;$$

(LMFT2)

$$\forall U, V \in L^X, \mathcal{T}(U \wedge V) \ge \mathcal{T}(U) \wedge \mathcal{T}(V);$$

(LMFT3)

$$\forall \{U_j : j \in J\} \subseteq L^X, \mathcal{T}\left(\bigvee_{j \in J} U_j\right) \geqslant \bigwedge_{j \in J} \mathcal{T}(U_j).$$

 $\mathcal{T}(U)$  can be interpreted as the degree to which U is an open L-set,  $\mathcal{T}^*(U) = \mathcal{T}(U')$ will be called the degree of closedness. The pair  $(X, \mathcal{T})$  is called (L, M)-fuzzy topological space. A mapping  $f: X \longrightarrow Y$  from an (L, M)-fuzzy topological space  $(X, \mathcal{T}_1)$  to another (L, M)-fuzzy topological space  $(Y, \mathcal{T}_2)$  is said to be continuous if  $\mathcal{T}_1(f^{\leftarrow}(B)) \geq \mathcal{T}_2(B)$  for each  $B \in L^Y$ . The category of all (L, M)-fuzzy topological spaces and their continuous mappings is denoted by (L, M)-**FTOP**.

The following Definition 1.2 and Lemma 1.3 were introduced by Shi [10] for an L-fuzzy topology and can be easily transformed to an (L, M)-fuzzy topology as follows.

**Definition 1.2.** An (L, M)-fuzzy neighborhood system on a set X is a map  $\mathcal{N} : L^X \longrightarrow M^{J(L^X)}$  satisfying the following conditions:

(LMFN1)

$$\mathcal{N}(1_X)(x_\lambda) = 1, \ \mathcal{N}(0_X)(x_\lambda) = 0 \ \ (\forall \ x_\lambda \in J(L^X));$$

(LMFN2)

$$\mathcal{N}(U)(x_{\lambda}) = 0 \quad (\forall \ U \in L^X, \forall \ x_{\lambda} \in J(L^X), x_{\lambda} \not\leq U);$$

(LMFN3)

$$\mathcal{N}(U \wedge V)(x_{\lambda}) = \mathcal{N}(U)(x_{\lambda}) \wedge \mathcal{N}(V)(x_{\lambda}) \quad (\forall \ U, V \in L^{X}, \forall \ x_{\lambda} \in J(L^{X}));$$

(LMFN4)

$$\mathcal{N}(U)(x_{\lambda}) = \bigvee_{x_{\lambda} \le V \le U} \bigwedge_{y_{\mu} \prec V} \mathcal{N}(V)(y_{\mu}) \text{ (where } \forall U \in L^{X}, x_{\lambda}, y_{\mu} \in J(L^{X})).$$

 $\mathcal{N}(U)(x_{\lambda})$  is called the degree to which  $x_{\lambda}$  belongs to the neighborhood of U. The pair  $(X, \mathcal{N})$  is called (L, M)-fuzzy neighborhood space. A mapping  $f : X \longrightarrow Y$  from an (L, M)-fuzzy neighborhood space  $(X, \mathcal{N}_1)$  to another (L, M)-fuzzy neighborhood space  $(Y, \mathcal{N}_2)$  is said to be continuous if  $\mathcal{N}_2(U)(f^{\rightarrow}(x_{\lambda})) \leq \mathcal{N}_1(f^{\leftarrow}(U))(x_{\lambda})$  for each  $U \in L^Y$  and each  $x_{\lambda} \in J(L^X)$ . The category of all (L, M)-fuzzy neighborhood spaces and their continuous mappings is denoted by (L, M)-**FNS**.

**Lemma 1.3.** (L, M)-**FTOP** is isomorphic to (L, M)-**FNS**.

*Proof.* Step 1: Define  $\mathcal{N}_{\mathcal{T}}: L^X \longrightarrow M^{J(L^X)}$  by

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) = \bigvee_{x_{\lambda} \le V \le U} \mathcal{T}(V) \quad (\forall U \in L^X, \forall x_{\lambda} \in J(L^X)).$$

Then  $\mathcal{N}_{\mathcal{T}}$  is an (L, M)-fuzzy neighborhood system induced by  $\mathcal{T}$ .

In fact, (LMFN1) and (LMFN2) are easily obtained.

(LMFN3) If  $A \leq B$ , then by the definition of  $\mathcal{N}_{\mathcal{T}}$ , we have

$$\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) \leq \mathcal{N}_{\mathcal{T}}(B)(x_{\lambda}) \ (\forall \ A, B \in L^X, \forall x_{\lambda} \in J(L^X)).$$

Hence

$$\mathcal{N}_{\mathcal{T}}(U \wedge V)(x_{\lambda}) \leq \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \wedge \mathcal{N}_{\mathcal{T}}(V)(x_{\lambda}) \ (\forall \ U, V \in L^{X}, \forall x_{\lambda} \in J(L^{X})).$$

On the other hand, if  $a \triangleleft \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \land \mathcal{N}_{\mathcal{T}}(V)(x_{\lambda})$ , then

$$a \triangleleft \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) = \bigvee_{x_{\lambda} \leq E \leq U} \mathcal{T}(E), \text{ and } a \triangleleft \mathcal{N}_{\mathcal{T}}(V)(x_{\lambda}) = \bigvee_{x_{\lambda} \leq G \leq V} \mathcal{T}(G).$$

Further, there exist E and G such that

$$x_{\lambda} \leq E \leq U, x_{\lambda} \leq G \leq V, \text{and } a \leq \mathcal{T}(E), \ a \leq \mathcal{T}(G).$$

So

$$x_{\lambda} \leq E \wedge G \leq U \wedge V$$
, and  $a \leq \mathcal{T}(E) \wedge \mathcal{T}(G) \leq \mathcal{T}(E \wedge G)$ .

Hence

$$a \leq \mathcal{T}(E \wedge G) \leq \bigvee_{x_{\lambda} \leq M \leq U \wedge V} \mathcal{T}(M) = \mathcal{N}_{\mathcal{T}}(U \wedge V)(x_{\lambda}).$$

This shows

$$\mathcal{N}_{\mathcal{T}}(U \wedge V)(x_{\lambda}) \geq \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \wedge \mathcal{N}_{\mathcal{T}}(V)(x_{\lambda}).$$

(LMFN4) We first show that

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) = \bigwedge_{\mu \triangleleft \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_{\mu}).$$
(1)

By the definition of  $\mathcal{N}_{\mathcal{T}}$ , we can easily obtain

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \leq \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_{\mu})$$

On the other hand, if  $a \triangleleft \bigwedge_{\mu \triangleleft \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_{\mu})$ , then  $a \triangleleft \mathcal{N}_{\mathcal{T}}(U)(x_{\mu}) = \bigvee_{\substack{x_{\mu} \leq G \leq U}} \mathcal{T}(G)$ for each  $\mu \triangleleft \lambda$ . Further, there exists  $G_{x_{\mu}} \in L^X$  such that  $x_{\mu} \leq G_{x_{\mu}} \leq U$  and  $a \leq \mathcal{T}(G_{x_{\mu}})$ . Assuming  $E = \bigvee_{\mu \triangleleft \lambda} G_{x_{\mu}}$ , we have  $x_{\lambda} \leq E \leq U$  and

$$a \leq \bigwedge_{\mu \lhd \lambda} \mathcal{T}(G_{x_{\mu}}) \leq \mathcal{T}(\bigvee_{\mu \lhd \lambda} G_{x_{\mu}}) = \mathcal{T}(E) \leq \bigvee_{x_{\lambda} \leq V \leq U} \mathcal{T}(V) = \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}).$$

This shows

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \ge \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_{\mu}).$$

Now, let  $x_{\lambda} \leq V \leq U$  and  $\mu \lhd \lambda$ , then we have

$$\mathcal{T}(V) \leq \bigwedge_{y_{\mu} \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_{\mu}) \leq \mathcal{N}_{\mathcal{T}}(V)(x_{\mu}) \leq \mathcal{N}_{\mathcal{T}}(U)(x_{\mu}).$$

 $\operatorname{So}$ 

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) = \bigvee_{x_{\lambda} \leq V \leq U} \mathcal{T}(V) \leq \bigvee_{x_{\lambda} \leq V \leq U} \bigwedge_{y_{\mu} \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_{\mu}) \leq \mathcal{N}_{\mathcal{T}}(U)(x_{\mu}).$$

Hence

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \leq \bigvee_{x_{\lambda} \leq V \leq U} \bigwedge_{y_{\mu} \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_{\mu}) \leq \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_{\mu}) = \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}).$$

Therefore,

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) = \bigvee_{x_{\lambda} \leq V \leq U} \bigwedge_{y_{\mu} \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_{\mu}).$$

**Step 2:** Define  $\mathcal{T}_{\mathcal{N}}: L^X \longrightarrow M$  by

$$\mathcal{T}_{\mathcal{N}}(U) = \bigwedge_{x_{\lambda} \triangleleft U} \mathcal{N}(U)(x_{\lambda}) \; (\forall U \in L^X).$$

Then  $\mathcal{T}_{\mathcal{N}}$  is an (L, M)-fuzzy topology induced by  $\mathcal{N}$ . In fact, (LMFT1) is easily obtained from (LMFN1). (LMFT2)  $\forall U, V \in L^X$ ,

$$\mathcal{T}_{\mathcal{N}}(U \wedge V) = \bigwedge_{x_{\lambda} \lhd U \wedge V} \mathcal{N}(U \wedge V)(x_{\lambda}) = \bigwedge_{x_{\lambda} \lhd U \wedge V} \left[ \mathcal{N}(U)(x_{\lambda}) \wedge \ \mathcal{N}(V)(x_{\lambda}) \right]$$

$$\geq \left(\bigwedge_{x_{\lambda} \lhd U} \mathcal{N}(U)(x_{\lambda})\right) \land \left(\bigwedge_{x_{\lambda} \lhd V} \mathcal{N}(V)(x_{\lambda})\right) = \mathcal{T}_{\mathcal{N}}(U) \land \mathcal{T}_{\mathcal{N}}(V).$$
  
(LMFT3)  $\forall \{E_{j} : j \in J\} \subseteq L^{X},$   
 $\mathcal{T}_{\mathcal{N}}\left(\bigvee_{j \in J} E_{j}\right) = \bigwedge_{x_{\lambda} \lhd \bigvee_{j \in J} E_{j}} \mathcal{N}\left(\bigvee_{j \in J} E_{j}\right)(x_{\lambda}) \geq \bigwedge_{j \in J} \bigwedge_{x_{\lambda} \lhd E_{j}} \mathcal{N}(E_{j})(x_{\lambda}) = \bigwedge_{j \in J} \mathcal{T}_{\mathcal{N}}(E_{j}).$ 

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**Step 3:** We show that

$$\mathcal{N}_{\mathcal{T}\mathcal{N}} = \mathcal{N}.$$

In fact,  $\forall U \in L^X, \forall x_\lambda \in J(L^X)$ , by (LMFN4), we have

$$\mathcal{N}_{\mathcal{T}_{\mathcal{N}}}(U)(x_{\lambda}) = \bigvee_{x_{\lambda} \leq V \leq U} \mathcal{T}_{\mathcal{N}}(V) = \bigvee_{x_{\lambda} \leq V \leq U} \bigwedge_{y_{\mu} \lhd V} \mathcal{N}(V)(y_{\mu}) = \mathcal{N}(U)(x_{\lambda}).$$

Hence  $\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} = \mathcal{N}$ . Step 4: We show that

$$\mathcal{T}(U) = \bigwedge_{x_{\lambda} \triangleleft U} \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \; (\forall U \in L^X) \text{ and } \mathcal{T}_{\mathcal{N}\mathcal{T}} = \mathcal{T}.$$

In fact, for each  $x_{\lambda} \triangleleft U$ ,

$$\mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) = \bigvee_{x_{\lambda} \le V \le U} \mathcal{T}(V) \ge \mathcal{T}(U).$$

Hence,

$$\bigwedge_{C_{\lambda} \triangleleft U} \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \geq \mathcal{T}(U).$$

On the other hand, if  $a \triangleleft \bigwedge_{x_{\lambda} \triangleleft U}^{x_{\lambda} \triangleleft U} \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda})$ , then  $a \triangleleft \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda})$  for each  $x_{\lambda} \triangleleft U$ . Further, there exists  $V_{x_{\lambda}} \in L^X$  such that  $x_{\lambda} \leq V_{x_{\lambda}} \leq U$  and  $a \leq \mathcal{T}(V_{x_{\lambda}})$ . Obviously,  $U = \bigvee_{x_{\lambda} \lhd U} V_{x_{\lambda}}$ . So

$$\mathcal{T}(U) = \mathcal{T}\left(\bigvee_{x_{\lambda} \lhd U} V_{x_{\lambda}}\right) \ge \bigwedge_{x_{\lambda} \lhd U} \mathcal{T}(V_{x_{\lambda}}) \ge a$$

This shows

$$\bigwedge_{x_{\lambda} \triangleleft U} \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \leq \mathcal{T}(U).$$

Hence

$$\mathcal{T}(U) = \bigwedge_{x_{\lambda} \triangleleft U} \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) \; (\forall U \in L^X).$$

Now, by the definition of  $\mathcal{T}_{\mathcal{N}}$ , we have

$$\mathcal{T}_{\mathcal{NT}}(U) = \bigwedge_{x_{\lambda} \triangleleft U} \mathcal{N}_{\mathcal{T}}(U)(x_{\lambda}) = \mathcal{T}(U) \; (\forall U \in L^X).$$

Therefore,  $\mathcal{T}_{\mathcal{NT}} = \mathcal{T}$ .

**Step 5:** If  $f: (X, \mathcal{T}_1) \longrightarrow (Y, \mathcal{T}_2)$  is continuous with respect to (L, M)-fuzzy topologies  $\mathcal{T}_1$  and  $\mathcal{T}_2$ , then

$$\mathcal{T}_1(f^{\leftarrow}(U)) \ge \mathcal{T}_2(U) \; (\forall U \in L^Y).$$

Hence

$$\mathcal{N}_{\mathcal{T}_2}(U)\left(f^{\rightarrow}(x_{\lambda})\right) = \bigvee_{f^{\rightarrow}(x_{\lambda}) \le V \le U} \mathcal{T}_2(V) \le \bigvee_{x_{\lambda} \le f^{\leftarrow}(V) \le f^{\leftarrow}(U)} \mathcal{T}_1(f^{\leftarrow}(V))$$
$$\le \mathcal{N}_{\mathcal{T}_1}(f^{\leftarrow}(U))(x_{\lambda}).$$

Therefore  $f: (X, \mathcal{N}_{\mathcal{T}_1}) \longrightarrow (Y, \mathcal{N}_{\mathcal{T}_2})$  is continuous with respect to (L, M)-fuzzy neighborhood systems  $\mathcal{N}_{\mathcal{T}_1}$  and  $\mathcal{N}_{\mathcal{T}_2}$ .

**Step 6:** If  $f: (X, \mathcal{N}_1) \longrightarrow (Y, \mathcal{N}_2)$  is continuous with respect to (L, M)-fuzzy neighborhood systems  $\mathcal{N}_1$  and  $\mathcal{N}_1$ , then

$$\mathcal{N}_2(V)\left(f^{\rightarrow}(x_{\lambda})\right) \le \mathcal{N}_1(f^{\leftarrow}(V))(x_{\lambda}) \ (\forall V \in L^Y, \forall x_{\lambda} \in J(L^X)).$$

Hence

$$\mathcal{T}_{\mathcal{N}_{2}}(V) = \bigwedge_{y_{\mu} \lhd V} \mathcal{N}_{2}(V)(y_{\mu}) \leq \bigwedge_{f^{\rightarrow}(x_{\lambda}) \lhd V} \mathcal{N}_{2}(V)(f^{\rightarrow}(x_{\lambda})) = \bigwedge_{x_{\lambda} \lhd f^{\leftarrow}(V)} \mathcal{N}_{2}(V)(f^{\rightarrow}(x_{\lambda}))$$
$$\leq \bigwedge_{x_{\lambda} \lhd f^{\leftarrow}(V)} \mathcal{N}_{1}(f^{\leftarrow}(V))(x_{\lambda}) = \mathcal{T}_{\mathcal{N}_{1}}(f^{\leftarrow}(V)).$$

Therefore  $f: (X, \mathcal{T}_{\mathcal{N}_1}) \longrightarrow (Y, \mathcal{T}_{\mathcal{N}_2})$  is continuous with respect to (L, M)-fuzzy topologies  $\mathcal{T}_{\mathcal{N}_1}$  and  $\mathcal{T}_{\mathcal{N}_2}$ .

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## 2. Further Study on (L, M)-fuzzy Topologies and (L, M)-fuzzy Neighborhood Systems

**Theorem 2.1.** Let X be a nonempty set, let  $(Y, \mathcal{T}_Y)$  be an (L, M)-fuzzy topological space, and let  $f: X \longrightarrow Y$  be a mapping. Define  $\mathcal{N}: L^X \longrightarrow M^{J(L^X)}$  as follows:

$$\mathcal{N}(A)(x_{\lambda}) = \mathcal{N}_{\mathcal{T}_{Y}}\left([f^{\rightarrow}(A')]'\right)(f^{\rightarrow}(x_{\lambda})).$$

Then  $\mathcal{N}$  is an (L, M)-fuzzy neighborhood system on X.

Proof. (LMFN1–LMFN2).  $\mathcal{N}(1_X)(x_\lambda) = \mathcal{N}_{\mathcal{T}_Y}(1_Y)(f^{\rightarrow}(x_\lambda)) = 1$ .  $x_\lambda \notin A$ , then  $f^{\rightarrow}(x_\lambda) \notin [f^{\rightarrow}(A')]'$ . In fact, if we have  $f^{\rightarrow}(x_\lambda) \leq [f^{\rightarrow}(A')]'$ , thus

$$x_{\lambda} \leq f^{\leftarrow}[f^{\rightarrow}(x_{\lambda})] \leq f^{\leftarrow}([f^{\rightarrow}(A')]') = [f^{\leftarrow}f^{\rightarrow}(A')]',$$

so  $(x_{\lambda})' \geq f^{\leftarrow} f^{\rightarrow}(A') \geq A'$ . Hence  $x_{\lambda} \leq A$ , which is a contradiction. Therefore,

$$\mathcal{N}(0_X)(x_\lambda) = \mathcal{N}_{\mathcal{T}_Y}([f^{\to}(1_X)]')(f^{\to}(x_\lambda)) = 0$$

and

$$\mathcal{N}(A)(x_{\lambda}) = \mathcal{N}_{\mathcal{T}_{Y}}([f^{\rightarrow}(A')]')(f^{\rightarrow}(x_{\lambda})) = 0 \ (\forall \ x_{\lambda} \nleq A)$$

(LMFN3) For each  $A = A_1 \wedge A_2$ , we have

$$f^{\to}(A') = f^{\to}(A'_1 \lor A'_2) = f^{\to}(A'_1) \lor f^{\to}(A'_2).$$

Hence

$$\mathcal{N}(A_1 \wedge A_2)(x_{\lambda}) = \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}((A_1 \wedge A_2)')]')(f^{\rightarrow}(x_{\lambda}))$$
  
=  $\mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A_1') \vee f^{\rightarrow}(A_2')]')(f^{\rightarrow}(x_{\lambda}))$   
=  $\mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A_1')]')(f^{\rightarrow}(x_{\lambda})) \wedge \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A_2')]')(f^{\rightarrow}(x_{\lambda}))$   
=  $\mathcal{N}(A_1)(x_{\lambda}) \wedge \mathcal{N}(A_2)(x_{\lambda}).$ 

Therefore,  $\mathcal{N}(A_1 \wedge A_2)(x_\lambda) = \mathcal{N}(A_1)(x_\lambda) \wedge \mathcal{N}(A_2)(x_\lambda).$ 

(LMFN4) **Step 1:** We show that

$$\mathcal{N}(A)(x_{\lambda}) = \bigvee_{B \in L^{Y}} \{ \mathcal{N}_{\mathcal{T}_{Y}}(B)(f^{\to}(x_{\lambda})) \mid f^{\leftarrow}(B) \le A \}.$$

If  $f^{\leftarrow}(B) \leq A$ , then  $A' \leq f^{\leftarrow}(B')$  and  $f^{\rightarrow}(A') \leq B'$ , so  $A' \leq f^{\leftarrow}(B')$  and  $B \leq f^{\leftarrow}(B')$  $(f^{\rightarrow}(A'))'$ . Hence

$$\mathcal{N}(A)(x_{\lambda}) = \mathcal{N}_{\mathcal{T}_{Y}}([f^{\rightarrow}(A')]')(f^{\rightarrow}(x_{\lambda})) \ge \mathcal{N}_{\mathcal{T}_{Y}}(B)(f^{\rightarrow}(x_{\lambda})).$$

Therefore,

$$\mathcal{N}(A)(x_{\lambda}) \ge \bigvee_{B \in L^{Y}} \{ \mathcal{N}_{\mathcal{T}_{Y}}(B)(f^{\rightarrow}(x_{\lambda})) \mid f^{\leftarrow}(B) \le A \}$$

On the other hand, let  $B = (f^{\rightarrow}(A'))'$ , we have  $f^{\leftarrow}(B) = (f^{\leftarrow}f^{\rightarrow}(A'))'$ , thus  $(f^{\leftarrow}(B))' = f^{\leftarrow}f^{\rightarrow}(A') > A',$ 

so  $f^{\leftarrow}(B) \leq A$ . Hence

$$\mathcal{N}(A)(x_{\lambda}) = \mathcal{N}_{\mathcal{T}_{Y}}([f^{\rightarrow}(A')]')(f^{\rightarrow}(x_{\lambda}))$$
$$= \mathcal{N}_{\mathcal{T}_{Y}}(B)(f^{\rightarrow}(x_{\lambda})) \le \bigvee_{B \in L^{Y}} \{\mathcal{N}_{\mathcal{T}_{Y}}(B)(f^{\rightarrow}(x_{\lambda})) \mid f^{\leftarrow}(B) \le A\}.$$

**Step 2:** We show that

$$\mathcal{N}(A)(x_{\lambda}) = \bigvee_{x_{\lambda} \leq V \leq A} \bigwedge_{y_{\mu} \triangleleft V} \mathcal{N}(V)(y_{\mu}).$$

By Step 1, let  $a \triangleleft \mathcal{N}(A)(x_{\lambda})$ . Then there exists  $B \in L^{Y}$  satisfying  $f^{\leftarrow}(B) \leq A$  such that  $a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_{\lambda}))$ , since

$$\mathcal{N}_{\mathcal{T}_Y}(B)(f^{\to}(x_\lambda)) = \bigvee_{f^{\to}(x_\lambda) \le V \le B} \bigwedge_{z_t \triangleleft V} \mathcal{N}_{\mathcal{T}_Y}(V)(z_t).$$

So there exists  $V \in L^Y$  satisfying  $f^{\rightarrow}(x_{\lambda}) \leq V \leq B$  such that  $a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(V)(z_t)$  for each  $z_t \triangleleft V$ . Let  $U = f^{\leftarrow}(V)$ , then  $x_{\lambda} \leq U \leq A$  for all  $y_{\mu} \triangleleft U$ . By Step 1, we have

$$a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(V)(f^{\rightarrow}(y_{\mu})) \leq \mathcal{N}(U)(y_{\mu}).$$

Hence  $\mathcal{N}(A)(x_{\lambda}) \leq \bigvee_{x_{\lambda} \leq V \leq A} \bigwedge_{y_{\mu} \triangleleft V} \mathcal{N}(V)(y_{\mu}).$ 

On the other hand, let  $b \in M$  and  $\bigvee_{x_{\lambda} \leq V \leq A} \bigwedge_{y_{\mu} \triangleleft V} \mathcal{N}(V)(y_{\mu}) \nleq b$ . Then there exists  $a \in \alpha(b)$  (where  $\alpha(b)$  is the largest maximal set of b (see [12])) such that  $\bigvee \bigwedge_{x_{\lambda} \leq V \leq A} \bigwedge_{y_{\mu} \triangleleft V} \mathcal{N}(V)(y_{\mu}) \nleq a$ . Further, there exists  $V \in L^{Y}$  such that  $x_{\lambda} \leq V \leq x_{\lambda} \leq x_{\lambda} \leq V \leq x_{\lambda} \leq x_{$ 

A and  $\bigwedge_{y_{\mu} \triangleleft V} \mathcal{N}(V)(y_{\mu}) \nleq a$ , thus  $\mathcal{N}(V)(y_{\mu}) \nleq a \ (\forall y_{\mu} \triangleleft V)$ , and, in particular,  $\mathcal{N}(V)(x_{\gamma}) \nleq a \ (\forall \gamma \triangleleft \lambda)$ . By Step 1 we have

By Step 1, we have

$$\mathcal{N}(V)(x_{\gamma}) = \bigvee_{D \in L^{Y}} \{ \mathcal{N}_{\mathcal{T}_{Y}}(D)(f^{\rightarrow}(x_{\gamma})) \mid f^{\leftarrow}(D) \leq V \}.$$

There exists  $D \in L^Y$  such that  $f^{\leftarrow}(D) \leq V$  and  $\mathcal{N}_{\mathcal{T}_Y}(D)(f^{\rightarrow}(x_{\gamma})) \nleq a$ , and therefore  $f^{\leftarrow}(D) \leq A$ . By Lemma 1.3, we have

$$\mathcal{N}_{\mathcal{T}_Y}(D)(f^{\rightarrow}(x_{\lambda})) = \bigwedge_{f^{\rightarrow}(x_{\gamma}) \triangleleft f^{\rightarrow}(x_{\lambda})} \mathcal{N}_{\mathcal{T}_Y}(D)(f^{\rightarrow}(x_{\gamma})) \nleq b.$$

By Step 1, we have  $\mathcal{N}(A)(x_{\lambda}) \leq b$ . Hence  $\mathcal{N}(A)(x_{\lambda}) \geq \bigvee_{x_{\lambda} \leq V \leq A} \bigwedge_{h \triangleleft V} \mathcal{N}(V)(h)$ .  $\Box$ 

**Theorem 2.2.** Let  $\mathcal{N}, \mathcal{T}_Y$  and f be defined as in Theorem 2.1 and define a mapping  $f^{\leftarrow}(\mathcal{T}_Y) : L^X \longrightarrow M$  by

$$f^{\leftarrow}(\mathcal{T}_Y)(A) = \bigwedge_{x_\lambda \triangleleft A} \mathcal{N}(A)(x_\lambda) \ (\forall A \in L^X).$$

Then

(1)  $f^{\leftarrow}(\mathcal{T}_Y)$  is the weakest (L, M)-fuzzy topology on X such that f is continuous.

(2) If  $(Z, \mathcal{T}_Z)$  is an (L, M)-fuzzy topological space and  $g : (Z, \mathcal{T}_Z) \longrightarrow (X, f^{\leftarrow}(\mathcal{T}_Y))$  is a map, then g is continuous iff  $f \circ g$  is continuous.

*Proof.* (1) First, by Lemma 1.3, we know that  $f^{\leftarrow}(\mathcal{T}_Y) = \mathcal{T}_N$  is an (L, M)-fuzzy topology on X. Second, we show that f is continuous, i.e.,  $\mathcal{T}_N(f^{\leftarrow}(A)) \geq \mathcal{T}_Y(A)$  for each  $A \in L^Y$ . In fact, by Lemma 1.3, we can obtain

$$\mathcal{T}_{\mathcal{N}}(f^{\leftarrow}(A)) = \bigwedge_{x_{\lambda} \triangleleft f^{\leftarrow}(A)} \mathcal{N}(f^{\leftarrow}(A))(x_{\lambda}) = \bigwedge_{x_{\lambda} \triangleleft f^{\leftarrow}(A)} \mathcal{N}_{\mathcal{T}_{Y}}([f^{\rightarrow}(f^{\leftarrow}(A'))]')(f^{\rightarrow}(x_{\lambda}))$$
$$\geq \bigwedge_{x_{\lambda} \triangleleft f^{\leftarrow}(A)} \mathcal{N}_{\mathcal{T}_{Y}}(A)(f^{\rightarrow}(x_{\lambda})) = \bigwedge_{f^{\rightarrow}(x_{\lambda}) \triangleleft f^{\rightarrow}f^{\leftarrow}(A)} \mathcal{N}_{\mathcal{T}_{Y}}(A)(f^{\rightarrow}(x_{\lambda}))$$
$$\geq \bigwedge_{f^{\rightarrow}(x_{\lambda}) \triangleleft A} \mathcal{N}_{\mathcal{T}_{Y}}(A)(f^{\rightarrow}(x_{\lambda})) = \mathcal{T}_{Y}(A).$$

Hence f is continuous.

Now, let  $\mathcal{T}_X$  be an (L, M)-fuzzy topology on X such that f is continuous, and let  $A \in L^X$ . If  $B = (f^{\rightarrow}(A'))'$ , then  $f^{\leftarrow}(B) \leq A$ . We only need to show that  $\mathcal{T}_X(A) \geq \mathcal{T}_N(A) \ (\forall A \in L^X)$ . In fact, since  $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$  is continuous, we have that  $f : (X, \mathcal{N}_{\mathcal{T}_X}) \longrightarrow (Y, \mathcal{N}_{\mathcal{T}_Y})$  is continuous, and then for all  $A \in L^X$ , we have

$$\mathcal{N}_{\mathcal{T}_X}(A)(x_{\lambda}) \ge \mathcal{N}_{\mathcal{T}_X}(f^{\leftarrow}(B))(x_{\lambda}) \ge \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_{\lambda}) = \mathcal{N}(A)(x_{\lambda}).$$

For any  $A \in L^X$ , we have

$$\mathcal{T}_X(A) = \mathcal{T}_{\mathcal{N}_{\mathcal{T}_X}}(A) = \bigwedge_{x_\lambda \triangleleft A} \mathcal{N}_{\mathcal{T}_X}(A)(x_\lambda) \ge \bigwedge_{x_\lambda \triangleleft A} \mathcal{N}(A)(x_\lambda) = \mathcal{T}_{\mathcal{N}}(A).$$

So  $\mathcal{T}_X \geq \mathcal{T}_N$ . Hence  $\mathcal{T}_N$  is the weakest (L, M)-fuzzy topology on X such that f is continuous.

(2) If g is continuous, then  $f \circ g$  is continuous. Now, suppose  $f \circ g$  is continuous. we need to show that  $\mathcal{T}_Z(g^{\leftarrow}(A)) \geq \mathcal{T}_N(A) \ (\forall A \in L^X)$ . By Lemma 1.3, we only need to show that

$$\mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(z_{\lambda}) \geq \mathcal{N}_{\mathcal{T}_N}(A)(g^{\rightarrow}(z_{\lambda})) = \mathcal{N}(A)(g^{\rightarrow}(z_{\lambda})) \; (\forall z_{\lambda} \in J(L^Z), \forall A \in L^X).$$
  
In fact, for a  $\mathcal{A}_{\mathcal{N}}(A)(g^{\rightarrow}(z_{\lambda}))$ , there exists  $f^{\leftarrow}(B) \leq A$  such that

In fact, for  $a \triangleleft \mathcal{N}(A)(g^{\rightarrow}(z_{\lambda}))$ , there exists  $f^{\leftarrow}(B) \leq A$  such that

$$a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(g^{\rightarrow}(z_{\lambda}))).$$

Hence

$$a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(g^{\rightarrow}(z_{\lambda}))) \leq \mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(f^{\leftarrow}(B)))(z_{\lambda}) \leq \mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(z_{\lambda}).$$

Therefore,  $\mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(z_{\lambda}) \geq \mathcal{N}(A)(g^{\rightarrow}(z_{\lambda})).$ 

**Theorem 2.3.** Let X be a nonempty set, let  $\{(X_i, \mathcal{T}_i)\}_{i \in I}$  be a collection of (L, M)-fuzzy topological space and let  $f_j : X \longrightarrow X_j$  be a mapping for each  $j \in I$ . Define  $\mathcal{N} : L^X \longrightarrow M^{J(L^X)}$  by

$$\mathcal{N}(A)(x_{\lambda}) = \bigvee_{J \subseteq Ifinite} \left\{ \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_{j}}(A_{j})(f_{j}^{\rightarrow}(x_{\lambda})) \mid \bigwedge_{j \in J} f_{j}^{\leftarrow}(A_{j}) \leq A \right\},$$

where I is an index set. Then

(1)  $\mathcal{N}$  is an (L, M)-fuzzy neighborhood system on X.

(2) Define a mapping  $\mathcal{T}_{\mathcal{N}}: L^X \longrightarrow M$  as follows:

$$\mathcal{T}_{\mathcal{N}}(A) = \bigwedge_{x_{\lambda} \triangleleft A} \mathcal{N}(A)(x_{\lambda}).$$

Then  $\mathcal{T}_{\mathcal{N}}$  is the weakest (L, M)-fuzzy topology on X such that each  $f_j$  is continuous for each  $j \in I$ , and  $\mathcal{T}_{\mathcal{N}} = \bigvee_{i \in I} f_j^{\leftarrow}(\mathcal{T}_i)$ .

(3) If  $(Z, \mathcal{T}_Z)$  is an (L, M)-fuzzy topological space and  $g : (Z, \mathcal{T}_Z) \longrightarrow (X, \mathcal{T}_N)$ a function, then g is continuous if and only if  $f_j \circ g$   $(j \in I)$  is continuous.

*Proof.* (1) (LMFN1)–(LMFN2) are easily obtained.

(LMFN3) If  $A \leq B$ , then we can easily obtain  $\mathcal{N}(A)(x_{\lambda}) \leq \mathcal{N}(B)(x_{\lambda})$ . Hence

$$\mathcal{N}(A \wedge B)(x_{\lambda}) \leq \mathcal{N}(A)(x_{\lambda}) \wedge \mathcal{N}(B)(x_{\lambda}).$$

On the other hand, suppose that  $a \triangleleft \mathcal{N}(A)(x_{\lambda}) \land \mathcal{N}(B)(x_{\lambda})$ . There exist finite subsets  $J_1, J_2$  of  $I, A_j \in L^{X_j}$   $(\forall j \in J_1), B_j \in L^{X_j}$   $(\forall j \in J_2)$  such that

$$\bigwedge_{j \in J_1} f_j^{\leftarrow}(A_j) \le A, \ \bigwedge_{j \in J_2} f_j^{\leftarrow}(B_j) \le B,$$
$$a \triangleleft \bigwedge_{j \in J_1} \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_{\lambda})), \text{and } a \triangleleft \bigwedge_{j \in J_2} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_{\lambda})).$$

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 $\square$ 

Let  $J = J_1 \cup J_2$ . Taking  $A_j = 1$  ( $\forall j \in J - J_1$ ), we may suppose that  $J = J_1$ , Taking  $B_j = 1$  ( $\forall j \in J - J_2$ ), we may suppose that  $J = J_2$ . Let  $C_j = A_j \wedge B_j$  for every  $j \in J$ . Then  $\bigwedge_{j \in J} f_j^{\leftarrow}(C_j) \leq A \wedge B$  and  $a \leq \bigwedge_{j \in J} \mathcal{N}(C_j)(f_j^{\rightarrow}(x_{\lambda}))$ . Therefore

$$\mathcal{N}(A \wedge B)(x_{\lambda}) \geq \mathcal{N}(A)(x_{\lambda}) \wedge \mathcal{N}(B)(x_{\lambda}).$$

(LMFN4) Suppose that  $a \triangleleft \mathcal{N}(A)(x_{\lambda})$ . Then there exists a finite subset J of I and  $A_j \in L^{X_j}$  ( $\forall j \in J$ ) such that

$$\bigwedge_{j\in J} f_j^{\leftarrow}(A_j) \le A, \quad a \triangleleft \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_\lambda)) \; (\forall j \in J).$$

Since

$$\mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\to}(x_\lambda)) = \bigvee_{f_j^{\to}(x_\lambda) \le B_j \le A_j} \bigwedge_{y_{\mu_j} \triangleleft B_j} \mathcal{N}_{\mathcal{T}_j}(B_j)(y_{\mu_j}),$$

there exists  $f_j^{\rightarrow}(x_{\lambda}) \leq B_j \leq A_j$  such that  $a \triangleleft \bigwedge_{y_{\mu_j} \triangleleft B_j} \mathcal{N}_{\mathcal{T}_j}(B_j)(y_{\mu_j})$ . Let

$$B = \bigwedge_{j \in J} f_j^{\leftarrow}(B_j),$$

then  $x_{\lambda} \leq B \leq A$ . For all  $y_{\mu} \triangleleft B$ , we have

$$a \triangleleft \bigwedge_{y_{\mu_j} \triangleleft B_j} \mathcal{N}_{\mathcal{T}_j}(B_j)(y_{\mu_j}) \leq \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(y_{\mu})) \leq \mathcal{N}(B)(y_{\mu}).$$

Hence

$$\mathcal{N}(A)(x_{\lambda}) \leq \bigvee_{x_{\lambda} \leq B \leq A} \bigwedge_{y_{\mu} \triangleleft B} \mathcal{N}(B)(y_{\mu}).$$

On the other hand, suppose that  $b \in M$  and

$$\bigvee_{x_\lambda \leq B \leq A} \bigwedge_{y_\mu \triangleleft B} \mathcal{N}(B)(y_\mu) \nleq b$$

Then there exists  $a \in \alpha(b)$  such that

$$\bigvee_{x_{\lambda} \leq B \leq A} \bigwedge_{y_{\mu} \triangleleft B} \mathcal{N}(B)(y_{\mu}) \nleq a.$$

Further, there exists  $B \in L^X$  such that  $x_{\lambda} \leq B \leq A$  and  $\bigwedge_{y_{\mu} \triangleleft B} \mathcal{N}(B)(y_{\mu}) \nleq a$ . Hence  $\mathcal{N}(B)(y_{\mu}) \nleq a$  for any  $y_{\mu} \triangleleft B$ . In particular,  $\mathcal{N}(B)(x_{\gamma}) \nleq a$  for each  $\gamma \triangleleft \lambda$ (this is because  $x_{\gamma} \triangleleft x_{\lambda} \leq B \Longrightarrow x_{\gamma} \triangleleft B$ ). By the definition of  $\mathcal{N}$ , there exist finite subsets  $J_1$  of I,  $B_j \in L^{X_j}$  ( $\forall j \in J_1$ ) such that  $\bigwedge_{j \in J_1} f_j^{\leftarrow}(B_j) \leq B$  (thus we have  $\bigwedge_{j \in J_1} f_j^{\leftarrow}(B_j) \leq A$ ) and  $\bigwedge_{j \in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_{\gamma})) \nleq a$ . By (1), we can obtain  $\bigwedge_{j \in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_{\lambda})) = \bigwedge_{j \in J_1} \bigwedge_{\gamma \triangleleft \lambda} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_{\gamma})) = \bigwedge_{\gamma \triangleleft \lambda} \bigwedge_{j \in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_{\gamma})) \not \leq b.$ 

By the definition of  $\mathcal{N}$ , since  $\bigwedge_{j \in J_1} f_j^{\leftarrow}(B_j) \leq A$ , we have  $\mathcal{N}(A)(x_\lambda) \nleq b$ . This shows

$$\bigvee_{x_{\lambda} \leq B \leq A} \bigwedge_{g \triangleleft B} \mathcal{N}(B)(y_{\mu}) \leq \mathcal{N}(A)(x_{\lambda}).$$

(2) By Lemma 1.3, it is obvious that  $\mathcal{T}_{\mathcal{N}}$  is an (L, M)-fuzzy topology on X. In order to prove that  $f_j : (X, \mathcal{T}_{\mathcal{N}}) \longrightarrow (X_j, \mathcal{T}_j)$  is continuous, i.e.,

$$\mathcal{T}_{\mathcal{N}}(f_{j}^{\leftarrow}(A_{j})) \geq \mathcal{T}_{j}(A_{j}) = \mathcal{T}_{\mathcal{N}_{\mathcal{T}_{j}}}(A_{j}) \; (\forall A_{j} \in L^{X_{j}}, \forall j \in I),$$

we need to prove that  $f_j : (X, \mathcal{N}) \longrightarrow (X_j, \mathcal{N}_{\mathcal{T}_j})$  is continuous. In fact,  $\forall x_\lambda \in J(L^X), A_j \in L^{X_j}$ , by the definition of  $\mathcal{N}$ ,

$$\mathcal{N}(f_j^{\leftarrow}(A_j))(x_{\lambda}) \ge \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_{\lambda})).$$

By Theorem 2.2, we have  $\mathcal{T}_{\mathcal{N}} \geq f_j^{\leftarrow}(\mathcal{T}_j) \ (\forall j \in I)$ . Hence

$$\mathcal{T}_{\mathcal{N}} \ge \mathcal{T}^{\star} = \bigvee_{j \in I} f_j^{\leftarrow}(\mathcal{T}_j).$$

On the other hand, suppose that for every  $x_{\lambda} \in J(L^X)$ ,  $A_j \in L^{X_j}$  and every finite subset  $J \subseteq I$  and  $\bigwedge_{j \in J} f_j^{\leftarrow}(A_j) \leq A$ . We have that

$$\mathcal{N}_{\mathcal{T}^{\star}}(A)(x_{\lambda}) \geq \mathcal{N}_{\mathcal{T}^{\star}}(\bigwedge_{j \in J} f_{j}^{\leftarrow}(A_{j}))(x_{\lambda})$$
$$= \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}^{\star}}(f_{j}^{\leftarrow}(A_{j}))(x_{\lambda})$$
$$\geq \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_{j}}(A_{j})(f_{j}^{\rightarrow}(x_{\lambda})).$$

By the definition of  $\mathcal{N}$ , we have  $\mathcal{N}_{\mathcal{T}^{\star}} \geq \mathcal{N}$ . Further, by Lemma 1.3, we have

$$\mathcal{T}^{\star} = \mathcal{T}_{\mathcal{N}_{\mathcal{T}^{\star}}} \geq \mathcal{T}_{\mathcal{N}}.$$

Therefore  $\mathcal{T}_{\mathcal{N}} = \bigvee_{j \in I} f_j^{\leftarrow}(\mathcal{T}_j).$ 

Now, since  $f_j: (X, \mathcal{T}_N) \longrightarrow (X_j, \mathcal{T}_j)$  is continuous, suppose that  $\delta$  is an (L, M)-fuzzy topology on X such that  $f_j: (X, \delta) \longrightarrow (X_j, \mathcal{T}_j)$  is continuous for each  $j \in I$ . By Theorem 2.2, we have  $\delta \geq f_j^{\leftarrow}(\mathcal{T}_j)$  for each  $j \in I$ , and therefore  $\delta \geq \mathcal{T}^* = \bigvee_{j \in I} f_j^{\leftarrow}(\mathcal{T}_j)$ .

(3) Necessity is straightforward. Suppose that  $f_j \circ g$  is continuous for each  $j \in I$ . We show that  $g: (Z, \mathcal{N}_{\mathcal{T}_Z}) \longrightarrow (X, \mathcal{N})$  is continuous i.e.

$$\mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(x_{\lambda}) \geq \mathcal{N}(A)(g^{\rightarrow}(x_{\lambda})) \; (\forall x_{\lambda} \in J(L^X), \forall A \in L^X).$$

In fact, suppose that  $a \triangleleft \mathcal{N}(A)(g^{\rightarrow}(x_{\lambda}))$ . Then there exists a finite subset J of I such that  $\bigwedge_{j \in J} f_j^{\leftarrow}(A_j) \leq A$  and  $a \triangleleft \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(A_j)((f_j \circ g)^{\rightarrow}(x_{\lambda}))$ . If  $B = \bigwedge_{j \in J} f_j^{\leftarrow}(A_j)$ ,

then  $g^{\leftarrow}(B) \leq g^{\leftarrow}(A)$  and

$$a \triangleleft \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_{j}}(A_{j})(f_{j} \circ g)^{\rightarrow}(x_{\lambda}))$$

$$\leq \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_{Z}}((f_{j} \circ g)^{\leftarrow}(A_{j}))(x_{\lambda})$$

$$= \mathcal{N}_{\mathcal{T}_{Z}}(g^{\leftarrow}(\bigwedge_{j \in J} f_{j}^{\leftarrow}(A_{j}))(x_{\lambda})$$

$$= \mathcal{N}_{\mathcal{T}_{Z}}(g^{\leftarrow}(B))(x_{\lambda}) \leq \mathcal{N}_{\mathcal{T}_{Z}}(g^{\leftarrow}(A))(x_{\lambda}).$$

### 3. Subspaces and Product Spaces

**Theorem 3.1.** Let  $(Y, \mathcal{N}_Y)$  be an (L, M)-fuzzy neighborhood system, let X be a subset of Y, and let  $id_Y|_X : X \longrightarrow Y$  be its respective embedding. Define  $\mathcal{N}|_X : L^X \longrightarrow M^{J(L^X)}$  as follows:

$$\mathcal{N}|_X(A)(x_\lambda) = \mathcal{N}_{\mathcal{T}_{\mathcal{N}_Y}} \left( \left[ (id_Y|_X)^{\rightarrow} (A') \right]' \right) \left( (id_Y|_X)^{\rightarrow} (x_\lambda) \right)$$
$$= \mathcal{N}_Y \left( \left[ (id_Y|_X)^{\rightarrow} (A') \right]' \right) (x_\lambda).$$

Then  $\mathcal{N}|_X$  is an (L, M)-fuzzy neighborhood system on X.

*Proof.* The proof of Theorem 3.1 is easily obtained from Theorem 2.1.  $\Box$ 

**Definition 3.2.** If  $\mathcal{N}|_X$  be defined as in Theorem 3.1, then the pair  $(X, \mathcal{N}|_X)$  is called a subspace of  $(Y, \mathcal{N}_Y)$ .

**Theorem 3.3.** 
$$\mathcal{N}|_X(A) = \bigvee \{\mathcal{N}_Y(D)(x_\lambda) \mid D|_X = A\} \ (\forall A \in L^X).$$
  
*Proof.* Let  $[(id_Y|_X)^{\rightarrow}(A')]' = C$ , we have  $C|_X = A$ . By Theorem 3.1,  
 $\mathcal{N}|_X(A)(x_\lambda) = \mathcal{N}_Y ([(id_Y|_X)^{\rightarrow}(A')]')(x_\lambda) = \mathcal{N}_Y(C)(x_\lambda)$   
 $\leq \bigvee \{\mathcal{N}_Y(D)(x_\lambda) \mid D|_X = A\}.$ 

On the other hand, by the proof of Theorem 2.1 (see (LMFN4)) and Theorem 3.1,

$$\mathcal{N}|_X(A)(x_{\lambda}) = \bigvee \{ \mathcal{N}_{\mathcal{T}_{\mathcal{N}_Y}}(B)(x_{\lambda}) \mid (id_Y|_X)^{\leftarrow}(B) \le A \}$$
$$= \bigvee \{ \mathcal{N}_Y(B)(x_{\lambda}) \mid (id_Y|_X)^{\leftarrow}(B) \le A \} \ge \bigvee \{ \mathcal{N}_Y(D)(x_{\lambda}) \mid D|_X = A \}$$

**Definition 3.4.** For any set X, let  $\{(X_j, \mathcal{T}_j)\}_{j \in I}$  be a family of (L, M)-**FTOP**objects, let  $X = \prod_{j \in I} X_j$ , and let  $p_j : X \longrightarrow X_j$  be the *j*-th projection. The product (L, M)-fuzzy topology on X, denoted by  $\prod_{j \in I} \mathcal{T}_j$ , is the weakest (L, M)fuzzy topology on X such that  $p_j$  is continuous. The pair  $(X, \prod_{j \in I} \mathcal{T}_j)$  is called the product space of  $\{(X_j, \mathcal{T}_j)\}_{j \in I}$ . **Theorem 3.5.** (1) If  $\mathcal{T} = \prod_{j \in I} \mathcal{T}_j$ , then  $\mathcal{T} = \bigvee_{j \in I} p_j^{\leftarrow}(\mathcal{T}_j)$ .

(2) If  $(Y, \mathcal{T}_Y)$  is an (L, M)-fuzzy topological space, then a mapping  $g: Y \longrightarrow X$ is continuous if and only if  $p_j \circ g$  ( $\forall j \in I$ ) is continuous. (3)  $\forall x_{\lambda} \in J(L^X), \forall A \in L^X$  and every index set I, we have

$$\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) = \bigvee_{J \subseteq I finite} \left\{ \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_{j}}(A_{j})(p_{j}^{\rightarrow}(x_{\lambda})) \mid \bigwedge_{j \in J} p_{j}^{\leftarrow}(A_{j}) \leq A \right\}.$$

(4) If J is a finite subset of I and  $A = \prod_{j \in I} A_j$ , and  $A_j = 1$  when  $j \notin J$ , then

$$\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) = \bigwedge_{j \in I} \mathcal{N}_{\mathcal{T}}(A_j)(p_j^{\to}(x_{\lambda})), \quad \mathcal{T}(A) = \bigwedge_{j \in J} \mathcal{T}_j(A_j).$$

*Proof.* By  $\mathcal{N} = \mathcal{N}_{\mathcal{T}_{\mathcal{N}}}$  and Theorem 2.3, we can easily obtain (1)–(3).

(4) We first show that

$$\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) = \bigwedge_{j \in I} \mathcal{N}_{\mathcal{T}}(A_j)(p_j^{\rightarrow}(x_{\lambda})).$$

It is obvious when  $A = 1_X$  or  $A = 0_X$ . Without loss of generality, we assume  $A \neq 1_X$  and  $A \neq 0_X$ . We also assume that  $A_j \neq 1$  for each  $j \in J$  (if not, then we have  $\mathcal{N}_{\mathcal{T}}(A_j)(p_j^{\rightarrow}(x_{\lambda})) = 1$ ). By the definition of  $\mathcal{N}_{\mathcal{T}}$ , it is obvious that

$$\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) \ge \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\to}(x_{\lambda})).$$

On the other hand, let  $J_1$  be a finite subset of I, and let  $B_j \in L^{X_j}$   $(\forall j \in J_1)$ be such that  $B = \bigwedge_{j \in J_1} p_j^{\leftarrow}(B_j) \leq A$ . By  $A = \prod_{j \in I} A_j = \bigwedge_{j \in J} p_j^{\leftarrow}(A_j)$ , we have  $J \subseteq J_1$ and  $B_j \leq A_j \ (\forall j \in J)$ . Hence,

$$\bigwedge_{j \in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(p_j^{\rightarrow}(x_{\lambda})) \leq \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(B_j)(p_j^{\rightarrow}(x_{\lambda}))$$
$$\leq \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_{\lambda}))$$

(by the definition of  $\mathcal{N}_{\mathcal{T}}$ )  $\leq \mathcal{N}_{\mathcal{T}}(A)(x_{\lambda})$ .

Therefore,  $\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) = \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\to}(x_{\lambda})).$ Now, since  $p_j: (X, \mathcal{T}) \longrightarrow (X_j, \mathcal{T}_j) \ (\forall j \in J)$  is continuous, we have

$$\mathcal{T}(A) = \mathcal{T}(\bigwedge_{j \in J} p_j^{\leftarrow}(A_j)) \ge \bigwedge_{j \in J} \mathcal{T}(p_j^{\leftarrow}(A_j)) \ge \bigwedge_{j \in J} \mathcal{T}_j(A_j).$$

(2)

In order to prove  $\mathcal{T}(A) = \bigwedge_{j \in J} \mathcal{T}_j(A_j)$ , we need to show that  $\mathcal{T}(A) \leq \bigwedge_{j \in J} \mathcal{T}_j(A_j)$ . If  $\mathcal{T}(A) \not\leq \bigwedge_{j \in J} \mathcal{T}_j(A_j)$ , then there exists  $j_0 \in J$  such that  $\mathcal{T}(A) \not\leq \mathcal{T}_{j_0}(A_{j_0})$ . By Lemma 1.3, we can obtain

$$\mathcal{T}(A) = \bigwedge_{x_{\lambda} \triangleleft A} \mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) \text{ and } \mathcal{T}_{j_{0}}(A_{j_{0}}) = \bigwedge_{y_{\mu_{j_{0}}} \triangleleft A_{j_{0}}} \mathcal{N}_{\mathcal{T}_{j_{0}}}(A_{j_{0}})(y_{\mu_{j_{0}}}).$$

Hence  $\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) \nleq \mathcal{T}_{j_0}(A_{j_0})$  for each  $x_{\lambda} \triangleleft A$ . Further, there exists  $y_{\mu_{j_0}} \triangleleft A_{j_0}$  such that  $\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) \nleq \mathcal{N}_{\mathcal{T}_{j_0}}(A_{j_0})(y_{\mu_{j_0}})$ . However, by (2), we have

$$\mathcal{N}_{\mathcal{T}}(A)(x_{\lambda}) = \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_{j}}(A_{j})(p_{j}^{\rightarrow}(x_{\lambda})) \leq \mathcal{N}_{\mathcal{T}_{j_{0}}}(A_{j_{0}})(y_{\mu_{j_{0}}}),$$

which is a contradiction.

### 4. Conclusions

In this paper, the relationship between (L, M)-fuzzy topology and (L, M)-fuzzy neighborhood system is further studied, and the initial structures of (L, M)-fuzzy neighborhood subspaces and (L, M)-fuzzy topological product spaces are given. Similarly, we can also give the initial structures of (L, M)-fuzzy topological subspaces and (L, M)-fuzzy neighborhood product spaces.

The construction of initial structures in the category of (L, M)-fuzzy topological spaces through those in the category of (L, M)-fuzzy neighborhood systems really looks rather interesting; the fact that the two categories are isomorphic, however, enables researchers to substitute one of them with the other, to find a solution of a complicated problem.

The related topic of (L, M)-fuzzy topological spaces will be studied further in our subsequent papers (e.g. (L, M)-fuzzy topological groups and (L, M)-fuzzy topological vector spaces), involving, possibly, product of the latter.

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