FUZZY RISK ANALYSIS BASED ON RANKING OF FUZZY NUMBERS VIA NEW MAGNITUDE METHOD

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Abstract. Ranking fuzzy numbers plays a main role in many applied models in real world and in particular decision-making procedures. In many proposed methods by other researchers may exist some shortcoming. The most commonly used approaches for ranking fuzzy numbers is based on defuzzification method. Many ranking fuzzy numbers cannot discriminate between two symmetric fuzzy numbers with identical core. In 2009, Abbasbandy and Hajjari proposed an approach for ranking normal trapezoidal fuzzy numbers, which computed the magnitude of fuzzy numbers namely "Mag" method. Then Hajjari extended it for non-normal trapezoidal fuzzy numbers and also for all generalized fuzzy numbers. However, these methods have the weakness that we mentioned above. Moreover, the result is not consistent with human intuition in this case. Therefore, we are going to present a new method to overcome the mentioned weakness. In order to overcome the shortcoming, a new magnitude approach for ranking trapezoidal fuzzy numbers based on minimum and maximum points and the value of fuzzy numbers is given. The new method is illustrated by some numerical examples and in particular, the results of ranking by the proposed method and some common and existing methods for ranking fuzzy numbers is compared to verify the advantages of presented method.

1. Introduction

Ranking fuzzy numbers plays a very important role in linguistic decision making and other intuitionistic fuzzy applications. Over the last few decade numerous ranking approaches have been proposed and investigated [37, 23, 24, 12, 42, 43, 13, 5, 3, 15, 29, 28, 27, 20, 41, 39, 18, 32, 35, 44]. Zadeh [51] introduced fuzzy set theory in 1965. Later on the first ranking fuzzy numbers proposed by Jain in 1976 and 1977 [30, 31]. Jain used the concept of maximizing set to order the fuzzy numbers. Jain's method is that the decision maker considers only the right side membership function. A canonical way to extend the natural ordering of real numbers to fuzzy numbers was suggested by Bass and Kwakernaak [7] as early as 1977. In 1979, Baldwin and Guild [6] indicated that these two methods have some disturbing disadvantages. Also, in 1980, Adamo [4] used the concept of α-level set in order to introduce α-preference rule. In 1981 Chang [10] introduced the concept of the preference function of an alternative. Yager [48, 49, 50] proposed four indices and which may be employed for the purpose of ordering fuzzy quantities in $[0, 1]$ and also in 1983 Murakami [38] developed the proposed ranking methods at that time to

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apply for control system. Bortolan and Degani have been compared and reviewed some of these ranking methods [8]. Chen [9] presented ranking fuzzy numbers with maximizing set and minimizing set. Liou and Wang [36] developed a ranking approach based on an integral value index to overcome the shortcomings of Chen's [9] approach. Chen and Hwang [14] thoroughly reviewed the existing the approaches and pointed out some illogical conditions that arise among them. Choobineh [17], Cheng [11] presented an approach for ranking fuzzy numbers by using the distance method. In the paper by Cheng [11], a centroid-based distance method presented. The method utilized the Euclidean distances from the origin to the centroid point of each fuzzy numbers to compare and rank the fuzzy numbers. Chu and Tsao [19] found that the distance method could not rank fuzzy numbers correctly if they are negative and therefore, suggested using the area between centroid point and the origin to rank fuzzy numbers. Abbasbandy and Asady [1] found that Tsao's area method could sometimes lead to counter-intuitive ranking and hence suggested a sign distance. Chen and Tang [16] presented an approach to rank non-normal $p-$norm trapezoidal fuzzy numbers with integral value. In 2011 Kumar et al. [32] showed the weakness of Chen and Tang's method with the help of several counter examples then they presented a new approach in this field. In 2008 Wang and Lee [42] revised Chu and Tsao's method and suggested a new approach for ranking fuzzy numbers based on Chu and Tsao's method in a way similar to original point. However, there is a shortcoming in some situations. Abbasbandy and Hajjari [2] showed a new approach for ranking of trapezoidal fuzzy numbers by magnitude of fuzzy numbers. Wang and Luo [45] proposed and area ranking of fuzzy numbers based on positive and negative ideal points. Kumar et al. [32] offered an approach for ranking generalized exponential fuzzy numbers using an integral value approach. Kumar et al. [35] modified Liou and Wang's [36] approach for ranking of an $L-R$ type generalized fuzzy number. Pani Bushan Rao et al. [39] presented a new method for ranking fuzzy numbers based on the circumcenter of centroids and used an index of optimism to reflect the decision maker's optimistic attitude and also an index of modality that represented the neutrality of the decision maker. Rezvani [40], presented a ranking method for generalized fuzzy numbers with Euclidian distance by the incentre of centroid. Chen and Chen [15] presented an approach for ranking of generalized fuzzy numbers by considering the defuzzified values, then the heights and spread of generalized fuzzy numbers in 2011 Kumar et al. [32] found out that their method is incorrect and presented a new method for ranking of $L-R$ type generalized fuzzy numbers. Recently, Wang and Wang [44] presented a new method so called total orderings defined by using a new concept of upper dense sequence.

Among the ranking approaches, Abbasbandy and Asady's [1] sign distance, Liou and Wang's [36] methods and Kumar et al's [32, 35] are commonly used approaches, which are highly cited and have wide applications [15, 16, 33, 34, 21, 26], but there were some shortcomings associated with their ranking approaches. To overcome shortcoming with Liou and Wang's [36] Vincent and Dat [41] revised their approach and presented a novel left, right and total integral value of fuzzy numbers for ranking fuzzy numbers.

In spite of the fact that we know there is no unique and natural order in a family of fuzzy numbers and order is generally chosen with respect to particular applications, however, a ranking method should have some reasonable properties. Some reasonable properties have been discussed in [46, 47]. One of the obvious properties is that if $A \preceq B$ then $-B \preceq -A$. Moreover, most of the above-mentioned methods are counter-intuitive and cannot discriminate between fuzzy numbers such as symmetric triangular fuzzy numbers, and some methods do not agree with human intuition, whereas some methods cannot rank crisp numbers, which are special case of fuzzy numbers.

In 2009, Abbasbandy and Hajjari [2] presented a method for ranking of trapezoidal fuzzy numbers, which measured the magnitude of normal trapezoidal fuzzy numbers so called “Mag” method. Then Hajjari [28] extended this method for non-normal trapezoidal fuzzy numbers and also for all generalized fuzzy numbers. In this method and many other ranking approaches, the symmetric trapezoidal (triangular) fuzzy numbers with identical mode or with identical centroid points have the same ranking order. For example, consider two symmetric triangular fuzzy number $A = (-2, 0, 2)$ and $B = (-1, 0, 1)$ and crisp number $C = (0, 0, 0)$ then we have $Mag(A) = Mag(B) = Mag(C) = 0$ then ranking order is $A \sim B \sim C$. In addition, from some ranking methods such as Cheng's distance [11], Chu and Tsao's [19] area, sign distance and Rezvani's [40] distance, Kumar et al's [32, 35] and Wang and Wang's [44] total orderings, we obtain the same results, which are unreasonable. Intuitively, crisp zero should be larger than B and B larger than C. In other words we expect that $A \prec B \prec C$. Therefore, the result of "Mag" method is not always consistent with human intuition and cannot logically infers ranking order in some situations. To overcome the weakness of “Mag” method and some other defuzzification approaches, we revise “Mag” method and present a new approach that is capable of effectively ranking various types of fuzzy numbers.

The rest of the paper is organized as follows. Section 2 contains the basic definitions and notations use in the remaining parts of the paper. In Section 3, we review “Mag” method [2] and its development [28]. Some examples to state the problem of “Mag” method and new magnitude method "Mag_N"will be given in Section 4. The paper is concluded in Section 5.

2. Preliminaries

In this section, we briefly review some basic concepts of generalized fuzzy numbers and some existing methods for ranking fuzzy numbers. we will identify the name of the number with that of its membership function for simplicity. Throughout this paper, $\mathbb{R}$ stands for the set of all real numbers, E stands for the set of fuzzy numbers, “A” expresses a fuzzy number and $A(x)$ for its membership function, $\forall x \in \mathbb{R}$.

2.1. Basic Notations and Definitions.

A generalized fuzzy number “A” is a subset of the real line $\mathbb{R}$ with membership function $A(x) : \mathbb{R} \to [0, w]$ such that [25]:

$$A(x)=\begin{cases} L_A(x), & a\le x\le b,\\ \omega, & b\le x\le c,\\ U_A(x), & c\le x\le d,\\ 0, & otherwise, \end{cases} \tag{1}$$

where $0<\omega\le 1$ is a constant, $L_A(x):[a,b]\to[0,\omega]$ and $U_A(x):[c,d]\to[0,\omega]$ are two strictly monotonically and continuous mapping. If $\omega=1$, then A is a normal fuzzy number. If $L_A(x)=\omega(x-a)/(b-a)$, and $U_A(x)=\omega(d-x)/(d-c)$ then it is a trapezoidal fuzzy number and is usually denoted by $A=(a,b,c,d;\omega)$ or $A=(a,b,c,d)$ if $\omega=1$. In particular, when $b=c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $A=(a,b,d;\omega)$ or $A=(a,b,d)$ if $\omega=1$. Therefore, triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers. We show the set of generalized fuzzy numbers by $F_w(\mathbb{R})$ or for simplicity by $F(\mathbb{R})$.

Since $L_A(x)$ and $U_A(x)$ are both strictly monotonically and continuous functions, their inverse functions exist and should be continuous and strictly monotonically. Let $A_L:[0,\omega]\to[a,b]$ and $A_U:[0,\omega]\to[c,d]$ be the inverse functions of $L_A(x)$ and $U_A(x)$, respectively. Then A_L and A_U should be integrable on the close interval $[0,\omega]$. In other words, both $\int_0^\omega A_L(y)dy$ and $\int_0^\omega A_U(y)dy$ should exist. In the case of trapezoidal fuzzy number, the inverse functions A_L and A_U can be analytically expressed as

$$A_L(y)=a+(b-c)y/\omega,\ 0\le y\le\omega, \tag{2}$$

$$A_U(y)=d-(d-c)y/\omega,\ 0\le y\le\omega. \tag{3}$$

The functions $L_A(x)$ and $R_A(x)$ are also called the left and right side of the fuzzy number A, respectively [25].

In this paper, we assume that

$$\int_{-\infty}^{+\infty}A(x)dx<+\infty.$$

A useful tool for dealing with fuzzy numbers are their $\alpha-$cuts. The $\alpha-$cut of a fuzzy number A is non-fuzzy set defined as

$$A_\alpha=\{x\in\mathbb{R}:A(x)\ge\alpha\},$$

for $\alpha\in(0,1]$ and $A_0=cl(\cup_{\alpha\in(0,1]}A_\alpha)$. According to the definition of a fuzzy number, it is seen at once that every $\alpha-$cut of a fuzzy number is closed interval. Hence, for a fuzzy number A, we have $A(\alpha)=[A_L(\alpha),A_U(\alpha)]$ where

$$A_L(\alpha)=\inf\{x\in\mathbb{R}:A(x)\ge\alpha\},$$

$$A_U(\alpha)=\sup\{x\in\mathbb{R}:A(x)\ge\alpha\}.$$

If the left and right sides of the fuzzy number A are strictly monotone, as it is described, A_L and A_U are inverse functions of $L_A(x)$ and $U_A(x)$, respectively.

The set of all elements that have a nonzero degree of membership in $\tilde{A}$ is called the *support* of A, i.e.

$$supp(A)=\{x\in X\mid A(x)>0\}. \tag{4}$$

The set of elements having the largest degree of membership in $\tilde{A}$ is called the *core* of A, i.e.

$$core(A) = \{x \in X \mid A(x) = \sup_{x \in X} A(x)\}. \tag{5}$$

In the following, we will always assume that A is continuous and bounded support $supp(A) = (a, d)$. The strong support of A should be $\overline{supp}\,(A) = [a, d]$.

Definition 2.1. A function $s : [0,1] \longrightarrow [0,1]$ is a reducing function if s is increasing, $s(0) = 0$ and $s(1) = 1$. We say that s is a regular function if $\int_0^1 s(\alpha)\mathrm{d}\alpha = \frac{1}{2}$.

Definition 2.2. [22] If A is a fuzzy number with representation $\big[A_L(\alpha), A_U(\alpha)\big]$ ($\alpha-$cut), and s is a reducing function then the value of A (with respect to s) is defined by

$$Val(A) = \int_0^1 s(\alpha)\big[A_U(\alpha) + A_L(\alpha)\big]\mathrm{d}\alpha. \tag{6}$$

Definition 2.3. [22] If A is a fuzzy number with representation $\big[A_L(\alpha), A_U(\alpha)\big]$ ($\alpha-$cut), and s is a reducing function then the ambiguity of A (with respect to s) is defined by

$$Amb(A) = \int_0^1 s(\alpha)\big[A_U(\alpha) - A_L(\alpha)\big]\mathrm{d}\alpha. \tag{7}$$

2.2. **Arithmetic Operation.** In this subsection, arithmetic operation between two generalized trapezoidal fuzzy numbers, defined on universal set of real numbers $\mathbb{R}$, are reviewed [13].

Let $A_1 = (a_1, b_1, c_1, d_1; \omega_1)$ and $A_2 = (a_2, b_2, c_2, d_2; \omega_2)$ be two generalized trapezoidal fuzzy numbers then

(1) $A_1 \oplus A_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; min(\omega_1, \omega_2))$
(2) $A_1 \ominus A_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; min(\omega_1, \omega_2))$
(3) $\lambda A_1 = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1; \omega_1) & \lambda > 0, \\ (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1; \omega_1) & \lambda < 0. \end{cases}$

2.3. **Ranking Function.** An efficient approach for comparing fuzzy numbers is using of ranking function [30], $R : F(\mathbb{R}) \to \mathbb{R}$, fuzzy numbers defined on set of real numbers, which maps each fuzzy number into real line, where a natural order exist i.e.,

- $A_1 \succ A_2$ iff $R(A_1) > R(A_2)$
- $A_1 \prec A_2$ iff $R(A_1) < R(A_2)$
- $A_1 \sim A_2$ iff $R(A_1) = R(A_2)$.

Now we consider the following reasonable properties for the ordering approaches, see [46, 47].

- P_1 : For an arbitrary finite subset Γ of E and $A_1 \in \Gamma, A_1 \succeq A_1$.
- P_2 : For an arbitrary finite subset Γ of E and $(A_1, A_2) \in \Gamma^2, A_1 \succeq A_2$ and $A_2 \succeq A_1$, we should have $A_1 \sim A_2$.
- P_3 : For an arbitrary finite subset Γ of E and $(A_1, A_2, A_3) \in \Gamma^3, A_1 \succeq A_2$ and $A_2 \succeq A_3$, we should have $A_1 \succeq A_3$.

- P_4 : For an arbitrary finite subset Γ of E and $(A_1, A_2) \in \Gamma^2$, $\inf supp(A_1) > \sup supp(A_2)$, we should have $A_1 \succeq A_2$.
- P_4' : For an arbitrary finite subset Γ of E and $(A_1, A_2) \in \Gamma^2$, $\inf supp(A_1) > \sup supp(A_2)$, we should have $A_1 \succ A_2$.
- P_5 : Let Γ and Γ' be two arbitrary finite subsets of E also A_1 and A_2 are in $\Gamma \cap \Gamma'$.
- P_6 : Let $A_1,\ A_2,\ A_1 + A_3$ and $A_2 + A_3$ be elements of E. If $A_1 \succeq A_2$, then $A_1 + A_3 \succeq A_2 + A_3$.
- P_6' : Let $A_1,\ A_2,\ A_1 + A_3$ and $A_2 + A_3$ be elements of E. If $A_1 \succ A_2$, then $A_1 + A_3 \succ A_2 + A_3$, when $A_3 \neq 0$.

3. A Review on Magnitude Method and its Development

In 2009, Abbasbandy and Hajjari [2] presented a new approach for ranking of trapezoidal fuzzy numbers, which is called "Mag" method. It was given for normal trapezoidal fuzzy numbers and computed the magnitude of a trapezoidal fuzzy number as follows.

For an arbitrary trapezoidal fuzzy number $A = (a, b, c, d)$ with parametric form $\big[A_L(\alpha), A_U(\alpha)\big]$, the magnitude of the trapezoidal fuzzy number A is

$$Mag(A) = \frac{1}{2}\Big[\int_0^1 \big(A_L(\alpha) + A_U(\alpha) + b + c\big)s(\alpha)\mathrm{d}\alpha\Big], \tag{8}$$

where the function $s(x)$ is a non-negative and increasing function on $[0, 1]$ with $s(0) = 0$, $s(1) = 1$ and $\int_0^1 s(x)\mathrm{d}x = \frac{1}{2}$. Obviously, function $s(x)$ can be considered as a weighting function. For more details we refer the reader to [2]. Then Hajjari [28] extended this method for non-normal trapezoidal fuzzy numbers and also for all generalized fuzzy numbers.

Let A be a fuzzy number with parametric form $A = \big(A_L(\alpha), A_U(\alpha)\big)$. Then the developed "Mag" ranking method can be written as

$$Mag(A) = \frac{1}{2}\Big[\int_0^1 \big(A_L(\alpha) + A_U(\alpha) + (A_L(1) + A_U(1)\big)s(\alpha)\mathrm{d}\alpha\Big], \tag{9}$$

Suppose $A = (a, b, c, d; \omega)$ is a non-normal trapezoidal fuzzy number with $\alpha-$cut representation $\big[A_L(\alpha), A_U(\alpha)\big]$. Consequently, from equation (2) and equation (3) we have

$$A_L(\alpha) = \frac{\alpha(b-a)}{\omega} + a,$$
$$A_U(\alpha) = \frac{\alpha(c-d)}{\omega} + d.$$

Replacing to formula (8) the following will be obtained

$$Mag(A) = \frac{(3\omega^2 + 2)(b + c)}{12\omega} + \frac{(3\omega - 2)(a + d)}{12\omega}. \tag{10}$$

It is clear that for normal trapezoidal fuzzy numbers the formula (8) reduces to

$$Mag(A) = \frac{5}{12}(b + c) + \frac{1}{12}(a + d). \tag{11}$$

It is clear that the developed method has all properties, as the previous one. For more detail, we refer the reader to [28].

4. Problem and Revision of Magnitude Approach

In this section, we first give two simple examples to show the weakness of "Mag" method [2] and some of defuzzification methods such as Cheng's distance [11], Chu and Tsao's [19], Abbasbandy'sign distance [2], Kumar et al's [32, 35], Rezvani's approach [40] and Wang and Wang's total ordering [44]. For example, consider the crisp number $A = (0, 0, 0)$ and two symmetric fuzzy numbers $B = (-1, 0, 1)$ *and* $C = (-2, 0, 2)$ Figure 2. By applying "Mag" method it will be obtained $Mag(A) = Mag(B) = Mag(C) = 0$, then the ranking order is $A \sim B \sim C$. Moreover, from Wang and Wang's total ordering [44] for any (optimistic, moderate and pessimistic) decision maker the results is $A \prec B \prec C$, which is unacceptable. By using "RM" method [32] we have the following:

a) For pessimistic decision maker, with $\alpha = 0$ ranking order is $C \prec B \prec A$.
b) For moderate decision maker, with $\alpha = 0.5$ ranking order is $A \sim B \sim C$.
c) For optimistic decision maker, with $\alpha = 1$ ranking order is $A \prec B \prec C$.

According to obtained results the ranking order is reasonable only from the point of pessimistic decision maker. We could have the same discussion for comparing the crisp number $A = (1, 1, 1)$ and two symmetric fuzzy numbers $B = (0, 1, 2)$ *and* $C = (-1, 1, 3)$. Intuitively, the results are unreasonable and it is not consistence with human intuition.

4.1. **New Magnitude Method.** In order to overcome the weakness of "Mag" method and some other defuzzification methods that deal with this problem, this part proposes a new approach for ranking fuzzy numbers based on the value of fuzzy numbers, minimum and maximum points of supports, which is called new magnitude method.

Definition 4.1. Suppose $A_1, A_2, A_3, ..., A_n$ are trapezoidal fuzzy numbers as $A_i = (a_i, b_i, c_i, d_i)$. We define $D(A_i)$ as follows:

$$D(A_i) = (a_i - x_{\min}) + |\frac{1}{2}(b_i + c_i) + val(A_i)| + (x_{\max} - d_i) \tag{12}$$

such that $x_{\min} = \inf S,\ x_{\max} = \sup S,\ S = \bigcup_{i=1}^{n} S_i,$ and $S_i = \{x|A_i(x) > 0\}$.

Remark 4.2. For an arbitrary fuzzy number A_i we have $D(A_i) \geq 0$.

Definition 4.3. Let $\gamma(A) : E \longrightarrow \{-1, 1\}$ be a function defined as follows:

$$\gamma(A) = sign\Big[\int_0^1 \big(A_L(\alpha) + A_U(\alpha)\big)\mathrm{d}\alpha\Big], \tag{13}$$

where

$$\gamma(A) = \begin{cases} 1, & if\ sign\int_0^1 \big(A_L(\alpha) + A_U(\alpha)\big)\mathrm{d}\alpha \geq 0, \\ \\ -1, & if\ sign\int_0^1 \big(A_L(\alpha) + A_U(\alpha)\big)\mathrm{d}\alpha < 0. \end{cases}$$

Remark 4.4. If $\inf\{x : x \in supp(A)\} > 0$ then $\gamma(\cdot) = 1$.

Remark 4.5. If $\inf\{x : x \in supp(A)\} < 0$ then $\gamma(\cdot) = -1$.

Definition 4.6. Suppose $A_1, A_2, A_3, ..., A_n$ are trapezoidal fuzzy numbers, which will be considered as $A_i = (a_i, b_i, c_i, d_i)$. The new magnitude of the fuzzy number A_i will be denoted by $Mag_N(A_i)$ and defined as follows:

$$Mag_N(A_i) = \gamma(A) \cdot D(A_1) \tag{14}$$

If A_i and A_j are two fuzzy numbers, then their ranking order is defined as follows:

- $A_i \succ A_j$ iff $Mag_N(A_i) > Mag_N(A_j)$
- $A_i \prec A_j$ iff $Mag_N(A_i) < Mag_N(A_j)$
- $A_i \sim A_j$ iff $Mag_N(A_i) = Mag_N(A_j)$.

Remark 4.7. Suppose there is an opposite of fuzzy number $A_i = (a_i, b_i, c_i, d_i)$, denoted by $-A_i = (-d_i, -c_i, -b_i, -a_i)$ then $Mag_N(-A_i) = -Mag_N(A_i)$

Remark 4.8. For any trapezoidal fuzzy numbers $A_i, A_j,\ A_i \preceq A_j$, by Mag_N if and only if $-A_j \preceq -A_i$.

Remark 4.9. The new magnitude Mag_N, has the properties $P_1, P_2, ..., P_6'$.

Now we re-consider two previous examples to show that the revised method can overcome the problem of "Mag" method.

Example 4.10. Let $A = (0, 0, 0)$ be and $B = (-1, 0, 1)$ and $C = (-2, 0, 2)$ two symmetric fuzzy, numbers indicated in Figure 1. From new magnitude method (Mag_N) we get that $x_{\min} = -2,\ x_{\max} = 2$ then $Mag_N(A) = 2,\ Mag_N(B) = 1$ and $Mag_N(C) = 0$. Therefore, the ranking order is $C \prec B \prec A$, which is consist of human intuition. Obviously, the ranking order obtained from the proposed method is more reasonable than the outcome obtained by "Mag" method, revised sign distance, Rezvani's distance and Cheng's approach.

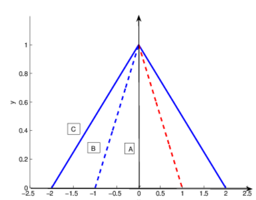

FIGURE 1. Fuzzy Numbers A, B and C in Example 4.10

Example 4.11. Comparing the crisp number $A = (1, 1, 1)$ and two symmetric fuzzy numbers $B = (0, 1, 2)$ and $C = (-1, 1, 3)$ (See Figure 2). It is clear that $x_{\min} = -1,\ x_{\max} = 3$ then by applying "Mag_N" it will be obtained $Mag_N(A) = 6,\ Mag_N(B) = 4$ and $Mag_N(C) = 2$. Therefore, the ranking order is $C \prec B \prec A$. Since, the crisp number should be stronger than these triangular fuzzy numbers, this will be happend using the new method.

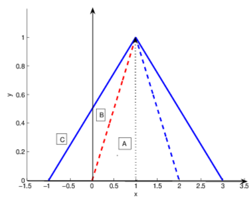

FIGURE 2. Fuzzy Numbers A, B and C in Example 4.11

Example 4.12. Figure 3 presents three normal triangular fuzzy numbers $A = (1, 3, 5)$, $B = (2, 3, 4)$, $C = (1, 4, 6)$, and their images i.e. $-A = (-5, -3, -1)$, $-B = (-4, -3, -2)$, $-C = (-6, -4, -1)$.

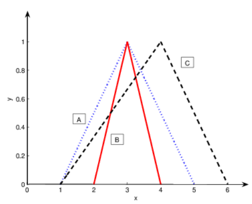

FIGURE 3. Fuzzy Numbers A, B and C in Example 4.12

Vincent and Dat [41] found that Liou and Wang's [36] approach has shortcoming. They indicated the ranking orders for A, B and C are $A \prec B \sim C$ for $\alpha = 0$, $B \prec A \prec C$ for $\alpha = 1$ and $A \sim B \prec C$ for $\alpha = 0.5$. However, the ranking order of the images of these fuzzy numbers as $-B \succ -A \sim -C$ for $\alpha = 0$, $-A \prec -B \sim -C$ for $\alpha = 1$ and $-A \sim -B \succ -C$ for $\alpha = 0.5$. Obviously, Liou and Wang's approach inconsistently rank fuzzy numbers and their images. From Vincent and Dat's integral values method the results are $A \prec B \sim C$ for $\alpha = 0$, $B \prec A \prec C$ for $\alpha = 1$ and $A \sim B \prec C$ for $\alpha = 0.5$, and their images as $-B \succ -A \succ -C$ for $\alpha = 0$, $-A \prec -B \sim -C$ for $\alpha = 1$ and $-A \sim -B \succ -C$ for $\alpha = 0.5$. Using revised sign distance the ranking order is $A \sim B \prec C$ for both $p = 1$ and $p = 2$. Moreover, for the images results are $-A \sim -B \prec C$ for both $p = 1$ and $p = 2$. From proposed method for triangular fuzzy numbers A, B and C we obtain that $Mag_N(A) = 7$, $Mag_N(B) = 9$ and $Mag_N(C) = 7.83$. In addition, for the images we have $Mag_N(-A) = -7$, $Mag_N(-B) = -9$ and $Mag_N(-C) = -7.83$. Therefore, the results are $A \prec C \prec B$ and $-B \prec -C \prec -A$, respectively. This example shows the strong discrimination power of the new magnitude for ranking fuzzy numbers and its advantages.

Example 4.13. Consider the four fuzzy numbers $A = (-4, 0, 4), B = (-2, 0, 2), C = (0, 1, 1.9)$ and $D = (0, 2, 2.9)$ shown in Figure 4.

By using new magnitude we get that $x_{\min} = -4$ and $x_{\max} = 4$. Then $Mag_N(A) = 0$, $Mag_N(B) = 4$, $Mag_N(C) = 8.0833$ and $Mag_N(D) = 8.9167$. Hence the ranking order is $A \prec B \prec C \prec D$, which is logically reasonable. However, many ranking

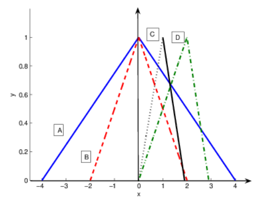

FIGURE 4. Fuzzy Numbers A, B, C and D in Example 4.13

approaches prove that two symmetric fuzzy numbers A and B have the same order. To compare with other ranking methods, the proposed ranking approach can overcome the shortcoming of the inconsistency of many approaches in ranking fuzzy numbers such as those are shown in Table 1. This example also shows the strong discrimination power of the proposed ranking approach and its advantages.

Ranking approach	A	B	C	D	Result
Cheng [11]	0.0	0.0	0.5	1.0	$A \sim B \prec C \prec D$
Chu and Tsao [19]	0.0	0.0	0.5	1.0	$A \sim B \prec C \prec D$
Wang et al.s [43]	0.0	0.0	1.4	3.2	$A \sim B \prec C \prec D$
Asady's [5]	0.0	0.0	1.4	3.2	$A \sim B \prec C \prec D$
Abbasbandy and Hajjari [2]	0.0	0.0	1.0	2.0	$A \sim B \prec C \prec D$
Kumar et al. [32]					
($\alpha = 1$)	2	1	1.45	2.45	$B \prec C \prec A \prec D$
($\alpha = 0.5$)	0	0	0.975	1.725	$A \sim B \prec C \prec D$
($\alpha = 0$)	-2	-1	0.5	1	$A \prec B \prec C \prec D$
Rezvani [40]	0.4	0.4	1.6	2.0	$A \sim B \prec C \prec D$
Wang and Wang[44]					
($\alpha = 1$)					$A \sim B \prec C \prec D$
($\alpha = 0.5$)					$C \sim D \prec B \prec A$
($\alpha = 0.25$)					$C \sim D \prec B \prec A$
Proposed approach ("Mag_N")	0.0	4.0	8.0833	8.9167	$A \prec B \prec C \prec D$

TABLE 1. Comparative Results of Example 4.13

Example 4.14. Consider the data used in [18] i.e. the three normal fuzzy numbers $A = (5, 6, 7)$, $B = (5.9, 6, 7)$ and $C = (6, 6, 7)$ as shown in Figure 5.

According to Definitions 4.1, 4.3 and 4.6 we get that $Mag_N(A) = 9$, $Mag_N(B) = 10$ and $Mag_N(C) = 10.2$. It is clear that the ranking order is $A \prec B \prec C$, which is consistent with the ranking obtained by other approaches [43, 5, 41, 9, 1, 2]. Table 2 summarized the results obtained by different methods. Note that the ranking $A \succ B \succ C$ obtained by Cheng [9] is thought of as unreasonable and not consistent with human intuition.

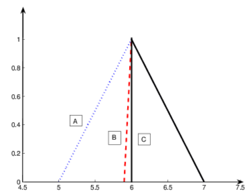

FIGURE 5. Fuzzy Numbers $A,\ B,\ C$ and D in Example 4.14

Ranking approach	A	B	C	Result
Cheng [11]	6.021	6.349	6.752	$A \succ B \succ C$
Chen [9]	0.500	0.571	0.583	$A \prec B \prec C$
Wang et al.s approach [43]	0.25	0.571	0.583	$A \prec B \prec C$
Asady's revision [5]	0.25	0.571	0.583	$A \prec B \prec C$
Abbasbandy and Asady [1]				
$(P = 1)$	6.12	12.45	12.5	$A \prec B \prec C$
$(P = 2)$	8.52	8.82	8.85	$A \prec B \prec C$
$(P = 3)$	6.0	6.075	6.083	$A \prec B \prec C$
Vincent and Dat [41] $(\alpha = 1)$				$A \sim B \sim C$
Vincent and Dat approach [41] $(\alpha = 0,\ \alpha = 0.5)$				$A \prec B \prec C$
Proposed approach ("Mag_N")	0.0	4.0	8.0833	$A \prec B \prec C$

TABLE 2. Comparative Results of Example 4.14

5. Conclusions

In spite of many ranking methods, no one can rank fuzzy numbers with human intuition consistently in all cases. Here, we pointed out the weakness of "Mag" method and also some other defuzzification approaches for ranking fuzzy numbers. Many ranking fuzzy numbers cannot discriminate between two symmetric fuzzy numbers with identical core. The paper herein presents several comparative examples to illustrate the validity and advantages of proposed ranking method. It shows that the ranking order obtained by the proposed approach is more consistent with human intuitions than existing methods. Furthermore, the new method is capable of effectively ranking various types of fuzzy numbers and overcome to weakness of previous one.

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