CATEGORICALLY-ALGEBRAIC TOPOLOGY AND ITS APPLICATIONS

S. A. SOLOVYOV

ABSTRACT. This paper introduces a new approach to topology, based on category theory and universal algebra, and called categorically-algebraic (catalg) topology. It incorporates the most important settings of lattice-valued topology, including poslat topology of S. E. Rodabaugh, (L, M)-fuzzy topology of T. Kubiak and A. Šostak, and M-fuzzy topology on L-fuzzy sets of C. Guido. Moreover, its respective categories of topological structures are topological over their ground categories. The theory also extends the notion of topological system of S. Vickers (and its numerous many-valued modifications of J. T. Denniston, A. Melton and S. E. Rodabaugh), and shows that the categories of catalg topological structures are isomorphic to coreflective subcategories of the categories of catalg topological systems. This extension initiates a new approach to soft topology, induced by the concept of soft set of D. Molodtsov, and currently pursued by various researchers.

1. Introduction

The notion of *L-fuzzy set* introduced by L. A. Zadeh [107] and J. A. Goguen [41] gave a rigid mathematical description of a new kind of uncertainty – *fuzziness*. Many researchers turned to fuzzification of well-known notions, producing new frameworks and often not minding the already existing ones. Nowadays, a working mathematician is confronted with a wide range of fuzzy (also called *many-valued* or *lattice-valued*) theories, dealing with almost every aspect of mathematics, but having no convenient common ground. A good example is the field of *lattice-valued topology*. Inspired by classical topology, it was started by C. L. Chang [14], and continued by J. A. Goguen [42] and R. Lowen [66], every author developing his own ideas, the only common point being the setting of *fixed-basis*, in which the theory is built over an arbitrary but always fixed lattice. This link was soon broken by B. Hutton [54], who introduced *variable-basis* setting, which employed various lattices in one theory. The idea was brought to its completion by S. E. Rodabaugh [82], who called his framework *point-set lattice-theoretic* (*poslat*) *topology* [83], thereby underlining its dependance on sets and lattices. Relying on category theory, it provided a common ground for many of the existing lattice-valued approaches to topology [84, 86]. Based on the algebraic structure of *semi-quantale* though, poslat

theories are not available by definition of settings, which employ other algebras like, e.g., *closure spaces* [5] do.

There exists a different approach to fuzzy topology, initiated by U. Höhle [50], taken up independently by T. Kubiak [60] and A. Šostak [105], and resulting in the theory of (L, M)-*fuzzy topological spaces* [61]. Its variable-basis poslat modification proposed by J. T. Denniston, A. Melton and S. E. Rodabaugh [22] incorporates the poslat topology of the above paragraph, making another step towards unification of the existing many-valued topological frameworks. A modification of (L, M)-fuzzy topology, introduced by C. Guido [45] and called M-*fuzzy topology on an L-fuzzy set*, has no direct counterpart in the setting of [22] at the moment.

The above remarks show a certain success in fighting the diversification of the existing topological theories, suggesting poslat topology as the uniting force. On a closer inspection however, it appears that the approach, based on category theory, provides the standard category-theoretic tools as the only way of intercommunication between different topological settings. To compare, *categorical algebra* in the sense of F. W. Lawvere [62] (the most recent developments are conveniently provided in [2]) defines explicitly the notions of algebraic theory and algebraic theory morphism. Moreover, recently M. Demirci [18] introduced *generalized topology*, motivated by the idea of *generalized lattice-valued set* of N. Nakajima [75], and his approach has not yet been accommodated inside the poslat setting.

Working in the field of lattice-valued topology and inspired by the above discussion, we decided to introduce a new topological framework, which would incorporate the above-mentioned poslat topological theories and their missed setting of closure spaces, M-fuzzy topological L-fuzzy spaces of C. Guido and generalized topology of M. Demirci. Moreover, by analogy with categorical algebra of F. W. Lawvere, we define explicitly the notion of topological theory and topological theory morphism, having two goals in mind:

(1) Construct a common setting for the majority of available (lattice-valued) topological theories.
(2) Provide means of interaction between different topological theories.

It is the main purpose of this paper to address the first goal. The proposed approach is based on category theory and universal algebra and thus is called *categorically-algebraic* (*catalg*) *topology* (cf. *poslat topology* of [83]). The framework extends categorical topology of S. E. Rodabaugh [86] (relying on *categorical fuzzy topology* of P. Eklund [30]), bringing more algebra in play. A particular instance of the new approach, called *retractive lattice-valued topology* and motivated by the above-mentioned framework of C. Guido, provides one of the most general settings for doing lattice-valued topology available at the moment. The new theory uses the same machinery for both crisp and many-valued topology (both are particular instances of catalg topological theories), erasing the border between them. At the bottom of the new framework lies the observation that the majority of the existing topological settings rely on two bedrocks: a ground category, and a variety of algebras (where algebras can have a class of not necessarily finite operations as in, e.g., [8, 80]), which underlies the respective topological structures (in the case

of classical topological spaces, the category **Set** of sets and the variety **Frm** of *frames* [56], respectively). These bedrocks determine the properties of the theory, and, therefore, are the main source of distinction between different settings. Being aware of the importance of the ground category, S. E. Rodabaugh tried to construct a supercategory for all possible varieties of algebraic structures underlying topological theories, which resulted in the concept of semi-quantale. Instead of providing a supercategory, we consider topological theories based in an arbitrary variety of algebras. To obtain a particular topological setting then, one takes the variety in question, which is easily determined by the underlying algebras of the respective topological structures. For example, poslat topology of S. E. Rodabaugh is built over the variety of semi-quantales, the theory of closure spaces is based on the variety of *closure semilattices*, whereas the above-mentioned (L, M)-fuzzy topology and its extension of C. Guido rely on *lattice-valued frames* [99] (which is a particular instance of our extension of the theory of *lattice-valued universal algebra* of A. Di Nola and G. Gerla [24]). On the other hand, generalized topology of M. Demirci, while using the variety of semi-quantales, employs different ground category (stated clearly in this paper). Moreover, motivated by the fact that every variety is itself a category, we show that certain functors between ground categories and varieties, linked by a natural transformation, provide a way of interaction (translation of results) between different topological theories.

An important advantage of a common framework for doing topology is the possibility to obtain results pertaining to every incorporated setting. For example, this paper shows that every catalg topological theory gives rise to a category of topological structures, which is topological over its ground category. As a consequence, we obtain the well-known fact that the categories **Top** of topological spaces and **Cls** of closure spaces are topological. More generally, every poslat topological theory over semi-quantales provides a topological category, and that includes the results of S. E. Rodabaugh [86] and J. T. Denniston *et al.* [22].

To show some applications of the new setting, we consider, firstly, the theory of *pointfree topology*, initiated by D. Papert and S. Papert [77], C. Ehresmann [29], J. R. Isbell [55], and given a coherent statement by P. T. Johnstone [56]. In the manuscript, we employ the setting of S. Vickers [106], who introduced the notion of *topological system* as a common framework for both topological spaces and their underlying algebraic structures – frames (also *locales*), which are the cornerstone of pointfree topology. The concept of topological system has already attracted the attention of several researchers as a possible extension of the theory of many-valued topology [19, 20, 22, 23, 33, 46, 47, 98, 101]. The catalg theory incorporates the notion, providing a common ground for all its lattice-valued extensions, and generalizes the most essential part of the results of S. Vickers, namely, the fact that the category of topological spaces is isomorphic to a full coreflective subcategory of the category of topological systems. As a consequence, we not only obtain the original result of S. Vickers, but also its extension of J. T. Denniston *et al.* [19], the equivalence between the categories of *state property systems* and closure spaces of D. Aerts *et al.* [4, 5] as well as the functor of C. Guido [46], which is a generalization of the *hypergraph functors* of the fuzzy community [51, 59, 81] (see [100]

for the respective results). Moreover, recent results show a close relation between topological systems of S. Vickers and *attachments* of C. Guido [34, 48].

The second application of the proposed catalg topological setting deals with the currently popular approach to uncertainty in mathematics, which is based on the theory of *soft sets* of D. Molodtsov [71], and which incorporates (as claimed by its author) the (*L*-)fuzzy approach of L. A. Zadeh and J. A. Goguen. Successfully continued by P. K. Maji *et al.* [68], the theory initiated "softening" of mathematics. Of particular interest appears to be the area of soft algebra, rapidly progressing at the moment and providing soft versions of the well-known algebraic structures, e.g., *soft group* [6], *soft ring* [65], *soft semiring* [32], *etc.* Moreover, M. Shabir and M. Naz [93] have recently started the theory of *soft topology*, which was immediately taken up in [13, 53, 70] (and partly in [78]). The setting of catalg topological systems of the previous paragraph gives rise to a different approach to soft topology, which bears much analogy with the machinery of soft algebra, and which is outlined in this manuscript in full detail. Moreover, catalg topological systems motivated a new and a more rigid setting for the soft set theory itself introduced recently in [104].

With the number of topological approaches incorporated in the catalg setting and listed in this paper, as well as with some of its applications (mentioned above), we hope to stimulate the interested researchers to develop a unification of modern (lattice-valued) topological settings. It is the purpose of this manuscript to suggest one of the possible approaches. However, there are (at least) two other potential candidates for that role, namely, *monadic topology* of W. Gähler [35, 38] (recently, in collaboration with P. Eklund *et al.* [31]) and $(\mathbb{T}, V)$-*categories* of M. M. Clementino, D. Hofmann and W. Tholen [15]. Both are based in a particular extension of the categorical notion of *monad* [9] (and, therefore, being more demanding than catalg topology), the former called *partially ordered monad*, whereas the latter called *lax extension of monad*. Both employ modifications of complete lattices, the former being motivated by the setting of fuzzy filters [36, 37] (and, therefore, lattice-valued topology), the latter being induced by the category Q-**Rel** of sets (as objects) and Q-valued binary relations (as morphisms), where Q is a *unital quantale* [90] (thereby, extending the category **Rel** of sets and binary relations, but never mentioning explicitly any relationship to lattice-valued sets). Moreover, it can be shown that a particular extension of the monadic approach of W. Gähler provides a setting isomorphic to that of $(\mathbb{T}, V)$-categories [92], so, essentially, one has only one concurrent setting. We will not comment further on the relationships between catalg topology and $(\mathbb{T}, V)$-categories, apart from the remark that while the latter incorporate almost all the algebraic information of topological structures in the categorical notion of monad (apart from that, which is stored in quantales), we use explicitly the concept of variety of universal algebra for that purpose.

In conclusion of this section, we will mention that [103] has recently shown that a concrete category is fibre-small and topological if and only if it is concretely isomorphic to a subcategory of a category of catalg topological structures, which is definable by topological co-axioms. In other words, every fibre-small topological category (and these include most of the categories for (lattice-valued) topology) is incorporated into the theory of catalg topology. The explicit description of the

required topological co-axioms though could be quite complicated in each concrete case, which stimulated us to elaborate in this paper the inclusion of several important (lattice-valued) topological settings into catalg topology in full detail.

The paper uses the tools of category theory and universal algebra, relying more on the former. The necessary categorical background can be found in [1, 49, 67, 69]. For the notions of universal algebra, we recommend [11, 16, 43, 80]. Although we tried to make the paper as much self-contained as possible, some details are still omitted and left for the self-study of the reader.

2. Basic Concepts of Categorically-algebraic Topology

This section introduces basic concepts of *categorically-algebraic* (*catalg*) *topology* and constitutes the core of the paper. The reader should be aware that some parts of the framework have already appeared in the previous manuscripts of the author under the name of *variety-based topology*. By our opinion, the latter term does not properly reflect the underlying theories of the approach and thus, the current paper will stick to catalg denotation. Some internal concepts of the setting still bear the term "variety-based" in their names, to underline that they are motivated by the general structure of varieties. On the whole, we suggest referring the topological stuff based on the category **Set** of sets and maps as variety-based topology, reserving the term categorically-algebraic for more general ground categories.

The new approach was motivated by the *point-set lattice-theoretic* (*poslat*) *topology*, introduced by S. E. Rodabaugh [83] and developed (in strikingly different ways) by P. Eklund, C. Guido, U. Höhle, T. Kubiak, A. Šostak and the initiator himself [30, 45, 52, 60, 61, 84]. The main advantage of the new setting is the fact that catalg framework ultimately removes the border between traditional and lattice-valued developments, producing a theory which brings forward algebraic essence of the whole (not only many-valued) topology. The close relation to the induced poslat setting is underlined by the abbreviation "catalg", which also brings to light the difference in the two theories.
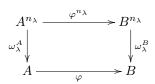
2.1. **Algebraical and Categorical Preliminaries.** This subsection provides those algebraic and categorical preliminaries, which are essential for the understanding of the paper. An experienced reader can easily skip the developments, consulting the subsection for the notations used throughout the paper only.

The cornerstone of the approach is the notion of (*universal*, *general* or *abstract*) *algebra*, which is thought of as a set with a family of operations defined on it, satisfying certain identities, e.g., semigroup, monoid, group and also (that is more important) complete lattice, frame, quantale. In case of *finitary algebras*, i.e., those induced by a set of finitary operations, there are two popular ways to describe their families [11, 16, 43]. The first, algebraic one, uses the concept of *variety* – a class of algebras closed under homomorphic images, subalgebras and direct products. The second, model-theoretic one, is based on the notion of *equational class* – providing a set of identities (equations) and taking precisely those algebras which satisfy all of them. The well-known HSP-theorem of G. Birkhoff [10] says that varieties and

equational classes coincide. Motivated by the algebraic structures used in many-valued topology (where unions are represented as joins), this paper includes infinitary cases as well, extending the algebraic approach of varieties to cover its needs (in the way [8, 80] do), and leaving aside the infinitary model-theoretic machineries of *equationally-definable class* [69] and *equational category* [64, 91].

Definition 2.1.

(1) Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper or empty) class of cardinal numbers. An Ω-*algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, comprising a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (called n_λ-*ary primitive operations* on A). An Ω-*homomorphism* $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that the diagram

$$
\begin{array}{ccc}
A^{n_\lambda} & \xrightarrow{\varphi^{n_\lambda}} & B^{n_\lambda} \\
{\scriptstyle \omega_\lambda^A}\big\downarrow & & \big\downarrow{\scriptstyle \omega_\lambda^B} \\
A & \xrightarrow{\varphi} & B
\end{array}
$$

commutes for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms.

(2) Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*).

(3) Given a variety $\mathbf{A}$, a *reduct* of $\mathbf{A}$ is a pair $(\| - \|, \mathbf{B})$, where $\mathbf{B}$ is a variety such that $\Omega_{\mathbf{B}} \subseteq \Omega_{\mathbf{A}}$ (where $\Omega_{\mathbf{A}}$ and $\Omega_{\mathbf{B}}$ stand for the classes of primitive operations of the varieties $\mathbf{A}$ and $\mathbf{B}$, respectively) and $\mathbf{A} \xrightarrow{\| - \|} \mathbf{B}$ is a concrete functor. The pair $(\mathbf{A}, \| - \|)$ is called an *extension* of $\mathbf{B}$.

Every concrete category of this paper is supposed to be equipped with the underlying functor $| - |$ to its respective ground category, the latter mentioned explicitly in every case. Also a comment is due to item (2) of Definition 2.1, i.e., the author of this paper is unaware of any result that every monomorphism (resp. epimorphism) in the category $\mathbf{Alg}(\Omega)$ has an injective (resp. surjective) underlying map.

The new concepts can be illustrated by several examples, all of which (except the last one) are popular in many-valued topology [46, 86, 88], since their induced categories of lattice-valued structures are topological over their ground categories. The last item in the list was motivated by *closure spaces* and their interrelationships with *state property systems* [4, 5], introduced as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics (the respective variety-based modification of the notion has already been considered in [100]).

Definition 2.2.

(1) Given $\Xi \in \{\bigvee, \bigwedge\}$, a Ξ-*semilattice* is a partially ordered set, which has arbitrary Ξ. $\mathbf{CSLat}(\Xi)$ is the variety of Ξ-semilattices. To give the reader

more intuition, we notice that, e.g., the variety $\mathbf{CSLat}(\bigvee)$ is concretely isomorphic to the category of Eilenberg-Moore algebras for the powerset monad on the category $\mathbf{Set}$ of sets and maps [1, Example 20.5(3)], and, therefore, its objects can be represented as pairs $(X, \mathcal{P}(X) \xrightarrow{\bigvee} X)$, where X is a set, $\mathcal{P}(X)$ is the powerset of X, whereas $\bigvee$ is a map, which assigns to every $S \in \mathcal{P}(X)$ its join $\bigvee S$. On the other hand, following [8, Section 4(3)], we can replace the above-mentioned map $\bigvee$ with a (class-indexed) family of (in general, infinitary) operations $X^\kappa \xrightarrow{\bigvee_\kappa} X$, one for every cardinal number κ, where $\bigvee_\kappa$ maps an element of X^κ (which determines a subset of X with cardinality less or equal than κ) to its join. It follows then that the morphisms of the category $\mathbf{CSLat}(\bigvee)$ are exactly the homomorphisms in the sense of Definition 2.1 w.r.t. the above family of operations. Similar strategy is employed in every item of this definition (its explicit description, however, is left to the reader).

(2) A *semi-quantale* (*s-quantale*) is a $\bigvee$-semilattice equipped with a binary operation $\otimes$ (*multiplication*). $\mathbf{SQuant}$ is the variety of s-quantales.

(3) An s-quantale is called *DeMorgan* provided that it is equipped with an order-reversing involution $(-)'$. $\mathbf{DmSQuant}$ is the variety of DeMorgan s-quantales.

(4) An s-quantale is called *unital* (*us-quantale*) provided that its multiplication has the unit $\mathbb{1}$. $\mathbf{USQuant}$ is the variety of us-quantales.

(5) A *quantale* is an s-quantale whose multiplication is associative and distributes across $\bigvee$ from both sides. $\mathbf{Quant}$ is the variety of quantales.

(6) A *quasi-frame* (*q-frame*) is an s-quantale whose multiplication is $\wedge$. $\mathbf{QFrm}$ is the variety of q-frames.

(7) A *semi-frame* (*s-frame*) is a unital q-frame. $\mathbf{SFrm}$ is the variety of s-frames.

(8) A *frame* is an s-frame which is a quantale. $\mathbf{Frm}$ is the variety of frames.

(9) A *closure semilattice* (*c-semilattice*) is a $\bigwedge$-semilattice, with the singled out bottom element $\bot$. $\mathbf{CSL}$ is the variety of c-semilattices.

Notice that the categories $\mathbf{QFrm}$ and $\mathbf{SFrm}$ (the former is due to C. Guido and the latter is due to S. E. Rodabaugh), having essentially the same objects (complete lattices), differ on morphisms and thus, are different. Also notice that $\mathbf{CSLat}(\bigvee)$ is a reduct of $\mathbf{SQuant}$, $\mathbf{SQuant}$ is a reduct of $\mathbf{USQuant}$ and $\mathbf{DmSQuant}$, $\mathbf{USQuant}$ is a reduct of $\mathbf{SFrm}$, $\mathbf{UQuant}$ is a reduct of $\mathbf{Frm}$, $\mathbf{CSLat}(\bigwedge)$ is a reduct of $\mathbf{CSL}$.

Before moving forward, some remarks are due on the many-valued framework employed in the paper. We extend first the concept of lattice-valued set to that of *algebraic set* as follows.

Definition 2.3. Let X be a set and let A be an algebra of some variety $\mathbf{A}$. An $(A\text{-})$*algebraic set* in X is a map $X \xrightarrow{\alpha} |A|$.

The underlying idea of the new setting is simple and is based on a straightforward *algebraization* of the classical frameworks of L. A. Zadeh [107] and J. A. Goguen [41], which can be easily restored through an appropriate variety. Despite that the theory

of algebraic sets is a good research area (the choice of set-theoretic operations is much richer), this paper will not develop the topic off the bounds of its interest.

A word is due to the notations employed throughout the manuscript. Arbitrary varieties are denoted $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ (sometimes with indices), with $\mathbf{S}$ reserved for their subcategories (or the subcategories of their dual categories). The categorical dual of a variety $\mathbf{A}$ is denoted $\mathbf{A}^{op}$, whose objects (resp. morphisms) are called $\mathbf{A}^{op}$-*algebras* (resp. $\mathbf{A}^{op}$-*homomorphisms*). Other categories will employ similar notation for their duals. Following [56], the dual of $\mathbf{Frm}$ is denoted $\mathbf{Loc}$, whose objects are called *locales*. Given an op-algebra A, $\mathbf{S}_A$ stands for the subcategory of $\mathbf{A}^{op}$, with the only morphism the identity $A \xrightarrow{1_A} A$. To underline the variety, which contains the algebra A, we use the notation $\mathbf{S}_A^{\mathbf{A}}$. To distinguish maps (or, more generally, morphisms) and homomorphisms, the former are denoted f, g, h (α, β, γ in case of algebraic sets), reserving φ, ψ, ϕ for the latter. Given an $\mathbf{A}$-homomorphism φ, its respective $\mathbf{A}^{op}$ one is denoted φ^{op} and vice versa.

2.2. **Categorically-algebraic Powerset Theories.** Having the required categorically-algebraic preliminaries in hand, we proceed to the second crucial notion of our approach, which is a mixture of *powerset theories* of [86, Definition 3.5] (see also [85, 88]) and *topological theories* of [1, Exercise 22B].

Definition 2.4. A *variety-based backward powerset theory* (*vbp-theory*) in a category $\mathbf{X}$ (the *ground category* of the theory) is a functor $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ to the dual category of a variety $\mathbf{A}$.

The intuition for the new concept comes from the so-called *image* and *preimage* operators [86], well-known to every working mathematician. Recall that a set map $X \xrightarrow{f} Y$ extends to the respective powersets in two ways: $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$, $f^{\rightarrow}(S) = \{f(x) \mid x \in S\}$ (*image operator*) and $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$, $f^{\leftarrow}(T) = \{x \mid f(x) \in T\}$ (*preimage operator*). The latter map admits a more general setting (recall that $\mathbf{Set}$ stands for the category of sets and maps; also notice that Lemma 2.5 is an immediate consequence of Lemmas 2.9, 2.10).

Lemma 2.5. *Given a variety* $\mathbf{A}$*, every subcategory* $\mathbf{S}$ *of* $\mathbf{A}^{op}$ *induces a functor* $\mathbf{Set} \times \mathbf{S} \xrightarrow{(-)^{\leftarrow}} \mathbf{A}^{op}$*, which is given by* $((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2))^{\leftarrow} = A_1^{X_1} \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} A_2^{X_2}$*, where* $(f,\varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f$*.*

Proof. To show that $A_2^{X_2} \xrightarrow{(f,\varphi)^{\leftarrow}} A_1^{X_1}$ is an $\mathbf{A}$-homomorphism, notice that given $\lambda \in \Lambda_{\mathbf{A}}$ and $\langle \alpha_i \rangle_{n_\lambda} \in (A_2^{X_2})^{n_\lambda}$, for every $x_1 \in X_1$, it follows that

$$((f,\varphi)^{\leftarrow}(\omega_\lambda^{A_2^{X_2}}(\langle \alpha_i \rangle_{n_\lambda})))(x_1) = \varphi^{op} \circ (\omega_\lambda^{A_2^{X_2}}(\langle \alpha_i \rangle_{n_\lambda})) \circ f(x_1) =$$
$$\varphi^{op} \circ (\omega_\lambda^{A_2}(\langle \alpha_i \circ f(x_1) \rangle_{n_\lambda})) = \omega_\lambda^{A_1}(\langle \varphi^{op} \circ \alpha_i \circ f(x_1) \rangle_{n_\lambda}) =$$
$$\omega_\lambda^{A_1}(\langle ((f,\varphi)^{\leftarrow}(\alpha_i))(x_1) \rangle_{n_\lambda}) = (\omega_\lambda^{A_1^{X_1}}(\langle (f,\varphi)^{\leftarrow}(\alpha_i) \rangle_{n_\lambda}))(x_1).$$

To show that the functor preserves composition, notice that given $\mathbf{Set} \times \mathbf{S}$-morphisms $(X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)$ and $(X_2, A_2) \xrightarrow{(g,\psi)} (X_3, A_3)$, for every $\alpha \in$

$A_3^{X_3}$, $(f,\varphi)^\leftarrow \circ (g,\psi)^\leftarrow(\alpha) = (f,\varphi)^\leftarrow(\psi^{op} \circ \alpha \circ g) = \varphi^{op} \circ \psi^{op} \circ \alpha \circ g \circ f = (\psi \circ \varphi)^{op} \circ \alpha \circ g \circ f = (g \circ f, \psi \circ \varphi)^\leftarrow(\alpha) = ((g,\psi) \circ (f,\varphi))^\leftarrow(\alpha)$. $\qquad\square$

Notice that in general the codomain of the functor of Lemma 2.5 is not $\mathbf{S}$, since the latter category is not necessarily closed under coproducts in $\mathbf{A}^{op}$. Also notice that the definition of the functor in question is much dependant on, firstly, the closure of $\mathbf{A}$ under products in its respective category $\mathbf{Alg}(\Omega)$, and, secondly, on $\mathbf{A}$ being a full subcategory of the category $\mathbf{Alg}(\Omega)$.

Following the already accepted powerset operator notations, the functor $\mathbf{Set} \times \mathbf{S}_A \xrightarrow{(-)^\leftarrow} \mathbf{A}^{op}$ (called *fixed-basis approach*, whereas all other cases are referred to as *variable-basis approach*) is denoted $(-)_A^\leftarrow$, omitting the identity morphism 1_A in its definition. The functor of Lemma 2.5 incorporates the majority of approaches to powersets in lattice-valued mathematics. The next example underlines their abundance, and thereby shows the fruitfulness of the common unifying framework.

Example 2.6.

    (1) $\mathbf{Set} \times \mathbf{S_2} \xrightarrow{\mathcal{P}=(-)^\leftarrow} \mathbf{CBAlg}^{op}$, where $\mathbf{CBAlg}$ is the variety of *complete Boolean algebras* (complete, complemented, distributive lattices) and $\mathbf{2} = \{\bot, \top\}$, provides the above-mentioned preimage operator.

    (2) $\mathbf{Set} \times \mathbf{S}_{\mathbb{I}} \xrightarrow{\mathcal{Z}=(-)_{\mathbb{I}}^\leftarrow} \mathbf{DmLoc}$, where $\mathbb{I} = [0,1]$ is the unit interval, provides the fixed-basis fuzzy approach of L. A. Zadeh [107].

    (3) $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_1=(-)_L^\leftarrow} \mathbf{Loc}$ gives the fixed-basis L-fuzzy approach of J. A. Goguen [41]. The setting has been changed to $\mathbf{Set} \times \mathbf{S}_L \xrightarrow{\mathcal{G}_2=(-)_L^\leftarrow} \mathbf{UQuant}^{op}$ in [42].

    (4) $\mathbf{Set} \times \mathbf{S}_A \xrightarrow{\mathcal{S}_{\mathbf{A}}^A=(-)_A^\leftarrow} \mathbf{A}^{op}$ provides the variety-based setting of [97], which unites the three above-mentioned items into one fixed-basis approach.

    (5) $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{R}_1^{\mathbf{S}}=(-)^\leftarrow} \mathbf{DmLoc}$ gives the variable-basis poslat approach of S. E. Rodabaugh [82], generalized to $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{R}_2^{\mathbf{S}}=(-)^\leftarrow} \mathbf{USQuant}^{op}$ in [86] and reduced to $\mathbf{Set} \times \mathbf{Loc} \xrightarrow{\mathcal{R}_3=(-)^\leftarrow} \mathbf{Loc}$ in [22, 23].

    (6) $\mathbf{Set} \times \mathbf{FuzLat} \xrightarrow{\mathcal{E}=(-)^\leftarrow} \mathbf{FuzLat}$ provides the variable-basis approach of P. Eklund [30] (motivated by those of S. E. Rodabaugh [82] and B. Hutton [54]), where $\mathbf{FuzLat}$ is the dual of the variety $\mathbf{HUT}$ of completely distributive DeMorgan frames also called *Hutton algebras* [84].

    (7) $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{S}_{\mathbf{A}}^{\mathbf{S}}=(-)^\leftarrow} \mathbf{A}^{op}$ provides the variety-based setting of [96], which unites the previous two items into one variable-basis approach.

To include the setting of M. Demirci [18], based on *generalized fuzzy sets* of N. Nakajima [75], a more general ground category is needed. We extend the already introduced notations as follows: a product of a family $(A_x)_{x \in X}$ of algebras of a variety $\mathbf{A}$ is denoted $\mathcal{A}^X$ and considered as the set of *choice functions* $X \xrightarrow{\alpha} |\mathcal{A}|$, i.e., maps $X \xrightarrow{\alpha} \bigcup_{x \in X} |A_x|$ such that $\alpha(x) \in |A_x|$ for every $x \in X$.

Definition 2.7. Let X be a set and let $\mathcal{A} = (A_x)_{x \in X}$ be a family of algebras of some variety $\mathbf{A}$. A *generalized* $(\mathcal{A}\text{-})$*algebraic set* in X is a choice function $X \xrightarrow{\alpha} |\mathcal{A}|$.

Definition 2.7 incorporates the respective approach of N. Nakajima [75] through the variety $\mathbf{Frm}$ of frames (called *complete Heyting algebras* in [75]) and also the case of Definition 2.3 through constant families (namely, $A_x = A$ for every $x \in X$). Moreover, the respective change of the ground category, to obtain a new powerset theory, is now straightforward.

Definition 2.8. Given a subcategory $\mathbf{S}$ of a variety $\mathbf{A}$, $\mathbf{Set} \odot \mathbf{S}$ is the category, given by the following data.

    OBJECTS**:** are pairs $(X, \mathcal{A})$, where X is a set, and $\mathcal{A} = (A_x)_{x \in X}$ is a family of $\mathbf{S}$-algebras.

    MORPHISMS**:** $(X, \mathcal{A}) \xrightarrow{(f, \Phi)} (Y, \mathcal{B})$ comprise a map $X \xrightarrow{f} Y$ and a family $\Phi = (\varphi_x)_{x \in X}$ of $\mathbf{S}$-homomorphisms $A_x \xrightarrow{\varphi_x} B_{f(x)}$.

    COMPOSITION OF MORPHISMS**:** $(X, \mathcal{A}) \xrightarrow{(f, \Phi)} (Y, \mathcal{B})$ and $(Y, \mathcal{B}) \xrightarrow{(g, \Psi)} (Z, \mathcal{C})$ is defined by $(g, \Psi) \circ (f, \Phi) = (g \circ f, \Psi \circ \Phi) = (g \circ f, (\psi_{f(x)} \circ \varphi_x)_{x \in X})$.

    IDENTITY**:** on $(X, \mathcal{A})$ is given by $(X, \mathcal{A}) \xrightarrow{(1_X, 1_{\mathcal{A}}) = (1_X, (1_{A_x})_{x \in X})} (X, \mathcal{A})$.

The category of Definition 2.8 has already appeared in [101] (in connection with *non-commutative topology* of C. J. Mulvey and J. W. Pelletier [73, 74]), where some of its properties were investigated. In particular, it is easy to see that the new category extends the product category $\mathbf{Set} \times \mathbf{S}$.

Lemma 2.9. *Given a subcategory $\mathbf{S}$ of a variety $\mathbf{A}^{op}$, there is a (non-full) embedding* $\mathbf{Set} \times \mathbf{S} \xhookrightarrow{\;E\;} \mathbf{Set} \odot \mathbf{S}$, *which is defined by* $E((X, A) \xrightarrow{(f, \varphi)} (Y, B)) = (X, (A)_{x \in X}) \xrightarrow{(f, (\varphi)_{x \in X})} (Y, (B)_{y \in Y})$.

This paper will use the extension of the functor of Lemma 2.5 to the new setting, which will provide an immediate incorporation of the above-mentioned framework of M. Demirci. For the sake of convenience, we introduce an additional notation: given sets S and I, $\langle s_i \rangle_I$ stands for a family of element of S indexed by I.

Lemma 2.10. *Given a variety $\mathbf{A}$, every subcategory $\mathbf{S}$ of $\mathbf{A}^{op}$ induces a functor* $\mathbf{Set} \odot \mathbf{S} \xrightarrow{(-)^{\leftarrow}} \mathbf{A}^{op}$ *given by* $((X, \mathcal{A}) \xrightarrow{(f, \Phi)} (Y, \mathcal{B}))^{\leftarrow} = \mathcal{A}^X \xrightarrow{((f, \Phi)^{\leftarrow})^{op}} \mathcal{B}^Y$, *where* $((f, \Phi)^{\leftarrow}(\alpha))(x) = \varphi_x^{op} \circ \alpha \circ f(x)$.

Proof. To show that $\mathcal{B}^Y \xrightarrow{(f, \Phi)^{\leftarrow}} \mathcal{A}^X$ is a homomorphism, notice that given $\lambda \in \Lambda$ and $\alpha_i \in \mathcal{B}^Y$ for every $i \in n_\lambda$, it follows that $((f, \Phi)^{\leftarrow}(\omega_\lambda^{\mathcal{B}^Y}(\langle \alpha_i \rangle_{n_\lambda})))(x) = \varphi_x^{op} \circ (\omega_\lambda^{\mathcal{B}^Y}(\langle \alpha_i \rangle_{n_\lambda})) \circ f(x) = \varphi_x^{op} \circ \omega_\lambda^{B_{f(x)}}(\langle \alpha_i \circ f(x) \rangle_{n_\lambda}) = \omega_\lambda^{A_x}(\langle \varphi_x^{op} \circ \alpha_i \circ f(x) \rangle_{n_\lambda}) = \omega_\lambda^{A_x}(\langle ((f, \Phi)^{\leftarrow}(\alpha))(x) \rangle_{n_\lambda}) = (\omega_\lambda^{\mathcal{A}^X}(\langle ((f, \Phi)^{\leftarrow}(\alpha)) \rangle_{n_\lambda}))(x)$.

To show that $(-)^{\leftarrow}$ preserves composition, notice that given $\mathbf{Set} \odot \mathbf{S}$-morphisms $(X, \mathcal{A}) \xrightarrow{(f, \Phi)} (Y, \mathcal{B})$, $(Y, \mathcal{B}) \xrightarrow{(g, \Psi)} (Z, \mathcal{C})$ and $\alpha \in \mathcal{C}^Z$, it follows that $(((g, \Psi) \circ (f, \Phi))^{\leftarrow}(\alpha))(x) = ((g \circ f, \Psi \circ \Phi)^{\leftarrow}(\alpha))(x) = \varphi_x^{op} \circ \psi_{f(x)}^{op} \circ \alpha \circ g \circ f(x) = \varphi_x^{op} \circ ((g, \Psi)^{\leftarrow}(\alpha)) \circ f(x) = ((f, \Phi)^{\leftarrow} \circ (g, \Psi)^{\leftarrow}(\alpha))(x)$. $\qquad\square$

An important meta-mathematical consequence of this generalized framework is the fact that inside it, the variable-basis setting of S. E. Rodabaugh becomes "single-basis", i.e., employs constant families of lattices. To distinguish between the two variable-basis settings, we call the extended one *multi-basis approach.*
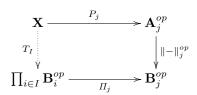
Example 2.11.

(1) $\textbf{Set} \odot \textbf{SQuant}^{op} \xrightarrow{\mathcal{D}=(-)^{\leftarrow}} \textbf{SQuant}^{op}$ provides the multi-basis approach of M. Demirci [18].

(2) $\textbf{Set} \odot \textbf{S} \xrightarrow{\mathcal{S}_{\textbf{A}}^{\odot \textbf{S}}=(-)^{\leftarrow}} \textbf{A}^{op}$ provides the variety-based setting of [101].

The reader should pay attention to the fact that Lemmas 2.5, 2.10 deal with a generalization of the preimage operator, leaving the image one aside. The reason is that the current many-valued analogues of the latter map are $\bigvee$-dependant (e.g., have the form of $(f_A^{\rightarrow}(\alpha))(y) = \bigvee_{f(x)=y} \alpha(x)$ in the fixed-basis case), whereas a general algebra may lack even a partial order.

2.3. **Categorically-algebraic Topological Theories.** The powerset theories in hand, we are ready to introduce our next concept, which is a modification of *composite topological theories* of [102]. The crucial change is that they no longer coincide with the powerset theories of the previous subsection (before moving forward, the reader is advised to recall the construction of *product of categories* [49]).

Definition 2.12. Let $\textbf{X}$ be a category and let $\mathcal{T}_I = ((P_i, (\| - \|_i, \textbf{B}_i)))_{i \in I}$ be a set-indexed family, where for every $i \in I$, $\textbf{X} \xrightarrow{P_i} \textbf{A}_i^{op}$ is a vbp-theory in the category $\textbf{X}$ and $(\| - \|_i, \textbf{B}_i)$ is a reduct of $\textbf{A}_i$. A *composite variety-based topological theory* (*cvt-theory*) *in* $\textbf{X}$ *induced by* $\mathcal{T}_I$ is the functor $\textbf{X} \xrightarrow{T_I = \langle \| - \|_i^{op} \circ P_i \rangle_I} \prod_{i \in I} \textbf{B}_i^{op}$, defined by commutativity of the diagram

$$\begin{array}{ccc} \textbf{X} & \xrightarrow{\ \ P_j\ \ } & \textbf{A}_j^{op} \\ {\scriptstyle T_I} \Big\downarrow & & \Big\downarrow {\scriptstyle \| - \|_j^{op}} \\ \prod_{i \in I} \textbf{B}_i^{op} & \xrightarrow[\ \ \Pi_j\ \ ]{} & \textbf{B}_j^{op} \end{array}$$

for every $j \in I$, where Π_j is the respective projection functor.

The reader could notice that the new concept is never used in the poslat setting of S. E. Rodabaugh, whose framework passes from powerset theories directly to topological theories, which, unlike the setting of this paper, are categories of topological structures. Studying various examples though we have arrived at the conclusion that one more level of abstraction was needed. The idea stems from the observation that the algebraic structure employed by powerset theories is usually richer than that used by topology. For instance, given a set X, the powerset $\mathcal{P}(X)$ of X is a complete Boolean algebra, and a topology τ on X is just a frame, which, however, employs occasionally some algebraic operations of its scope, e.g., infinitary intersections and complementation of sets. The case of closure spaces [4, 5]

mentioned in introduction provides another good example based on c-semilattices (Definition 2.2). This necessitates dropping off a part of unused algebraic structure, on one hand, and preserving it for the future, on the other, resulting in an additional stage between powerset theories and their induced topological structures.

Since a cvt-theory T_I is completely determined by its respective family $\mathcal{T}_I$, we will often use the notation $((P_i, \mathbf{B}_i))_{i \in I}$ instead of T_I. A cvt-theory induced by a singleton family will be denoted T. Moreover, we will employ the shorter T_i for $\| - \|_i^{op} \circ P_i$. A special remark is due to the empty families of powerset theories (one, for every ground category $\mathbf{X}$). In this case, the product category $\prod_{i \in I} \mathbf{B}_i^{op}$ transforms into a singleton terminal category $\mathbf{T} = \bullet \, \circlearrowleft$, with the respective topological theory being the unique functor $\mathbf{X} \xrightarrow{!} \mathbf{T}$.

2.4. **Categorically-algebraic Topological Structures.** All preliminaries in their places, we introduce catalg topology (the use of the standard image operator in the definition of continuity is motivated by purely aesthetic reasons and can be avoided).

Definition 2.13. Let T_I be a cvt-theory in a category $\mathbf{X}$. $\mathbf{CTop}(T_I)$ is the category, concrete over $\mathbf{X}$, given by the following data.

>   OBJECTS**:** (*composite variety-based topological spaces* or T_I-*spaces*) are pairs $(X, (\tau_i)_{i \in I})$, where X is an $\mathbf{X}$-object, and τ_i is a subalgebra of $T_i(X)$ for every $i \in I$ ($(\tau_i)_{i \in I}$ is called *composite variety-based topology* or T_I-*topology* on X).
>
>   MORPHISMS**:** $(X, (\tau_i)_{i \in I}) \xrightarrow{f} (Y, (\sigma_i)_{i \in I})$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that $((T_i f)^{op})^{\rightarrow}(\sigma_i) \subseteq \tau_i$ for every $i \in I$ (*composite variety-based continuity* or T_I-*continuity*).

For the sake of simplicity, $\mathbf{CTop}(T)$ is denoted $\mathbf{Top}(T)$. Moreover, it is precisely the setting of Definition 2.13, which is called in this paper *categorically-algebraic topology*. If the ground category $\mathbf{X}$ has the form $\mathbf{Set} \odot \mathbf{S}$ (Lemma 2.10), the resulting framework is called *variety-based topology*. The new concept was motivated by the multitude of approaches to topological structures in lattice-valued mathematics. Our purpose was to provide a common unifying framework, suitable for exploring interrelations between different topological settings. The composite machinery was inspired by the wish to include *bitopological theories* of S. E. Rodabaugh [87] and T. Kubiak [60]. The next example shows the fruitfulness of the setting.

Example 2.14.

>   (1) $\mathbf{Top}((\mathcal{P}, \mathbf{Frm}))$ is isomorphic to the classical category $\mathbf{Top}$ of topological spaces and continuous maps.
>   (2) $\mathbf{Top}((\mathcal{P}, \mathbf{CSL}))$ is isomorphic to the category $\mathbf{Cls}$ of closure spaces and continuous maps studied by D. Aerts *et al.* [4, 5].
>   (3) $\mathbf{CTop}(((\mathcal{P}, \mathbf{Frm}))_{i \in \{1,2\}})$ is isomorphic to the category $\mathbf{BiTop}$ of bitopological spaces and bicontinous maps of J. C. Kelly [58].
>   (4) $\mathbf{Top}((\mathcal{Z}, \mathbf{Frm}))$ is isomorphic to the category $\mathbb{I}\text{-}\mathbf{Top}$ of fixed-basis fuzzy topological spaces introduced by C. L. Chang [14].

(5) $\textbf{Top}((\mathcal{G}_2, \textbf{UQuant}))$ is isomorphic to the category $L\text{-}\textbf{Top}$ of fixed-basis L-fuzzy topological spaces of J. A. Goguen [42].

(6) $\textbf{Top}((\mathcal{R}_i^{\textbf{S}}, \textbf{USQuant}))$ is isomorphic to the category $\textbf{S-Top}_i$, $i \in \{1, 2\}$ for variable-basis poslat topology of S. E. Rodabaugh [82, 86]. $\textbf{Top}((\mathcal{R}_3, \textbf{Loc}))$ is isomorphic to a somewhat simplified category $\textbf{Loc-Top}$ of J. T. Denniston, A. Melton and S. E. Rodabaugh [19, 20, 22, 23].

(7) $\textbf{CTop}(((\mathcal{R}_1^{\textbf{S}_L}, \textbf{Frm}))_{i \in \{1, 2\}})$ is isomorphic to the category $L\text{-}\textbf{BiTop}$ of fixed-basis L-bitopological spaces of T. Kubiak [60].

(8) $\textbf{CTop}(((\mathcal{R}_2^{\textbf{S}_L}, \textbf{USQuant}))_{i \in \{1, 2\}})$ is isomorphic to the category $L\text{-}\textbf{BiTop}$ of fixed-basis L-bitopological spaces of S. E. Rodabaugh [87].

(9) $\textbf{Top}((\mathcal{E}, \textbf{Frm}))$ is isomorphic to the category $\textbf{FUZZ}$ for variable-basis poslat topology of P. Eklund [30] (motivated by those of S. E. Rodabaugh [82] and B. Hutton [54]).

(10) $\textbf{Top}((\mathcal{S}_{\textbf{A}}^{\textbf{S}_A}, \textbf{A}))$ (resp. $\textbf{Top}((\mathcal{S}_{\textbf{A}}^{\textbf{S}}, \textbf{A})))$ is isomorphic to the fixed- (resp. variable-) basis category $A\text{-}\textbf{Top}$ (resp. $\textbf{S-Top}$) used in the approach to variety-based topology of [97] (resp. [96]).

(11) $\textbf{Top}((\mathcal{D}, \textbf{SQuant}))$ is isomorphic to the category $\textbf{CGTop}$ for multi-basis topology of M. Demirci [18].

(12) $\textbf{Top}((\mathcal{S}_{\textbf{A}}^{\odot\textbf{S}}, \textbf{A}^{op}))$ is isomorphic to the category $\textbf{S-GTop}$, which was used in the approach to variety-based topologies of [101].

Example 2.14 illustrates our claim that the catalg framework ultimately erases the border between traditional and many-valued approaches. Indeed, the framework of powerset theories never distinguishes between the two settings, both being based on essentially the same functor, whose employed algebras come from the same variety. Moreover, in case of an arbitrary variety of algebras, the notion of crispness needs an additional clarification, since the classical case of the two-element Boolean algebra $\textbf{2} = \{\bot, \top\}$ relies not only on the number of elements in the algebra, but also on its certain properties in the variety $\textbf{CBAlg}$ of complete Boolean algebras, e.g., on the facts that it is an initial object [1, Examples 7.2(7)] and also a coseparator [1, Examples 7.18(7)]. As a counterexample, one can mention the category $\textbf{Grp}$ of groups, whose two-element objects share none of the properties.

At the end of this subsection, we would like to notice that the machinery of catalg topology can be employed even in a setting, which (at least, seemingly) has nothing to do with many-valued mathematics.

Example 2.15. The category $A\text{-}\textbf{Top}$ of Example 2.14(10) is precisely the category $\textbf{A}f\textbf{S}et(A)$ of *affine sets over A* of Y. Diers [26, Definition 2.1] (see also [25, 27]).

We notice that the category $A\text{-}\textbf{Top}$ is a kind of simplification of the category $\textbf{A}f\textbf{S}et(A)$. This simplification stems from our use of one-sorted algebras for the algebraic background of the catalg machinery, while Y. Diers employs many-sorted ones, or algebras w.r.t. *varietal theories* in the sense of F. E. J. Linton [64] and also *algebraic theories* of F. W. Lawvere [62]. However, the case of variable-basis as well as more general ground categories in the sense of Definition 2.4 of this paper are never studied in his framework. The motivations of the two settings are also

quite different, since Y. Diers is interested in a generalization of sets equipped with a geometrical structure (*geometrical sets*), mentioning (crisp) topological spaces as one of the examples [26, Example 10.10], whereas our main concern lies in the construction of a convenient common setting for different fuzzifications of topology.

Having introduced the concept of catalg topology, we proceed to a more elaborate approach to the topic.

3. Lattice-valued Categorically-algebraic Topology

Among the examples of the previous section, the reader can find many settings of lattice-valued topology, with one significant exception, i.e., the theory of (L, M)-*fuzzy topologies* of T. Kubiak, A. Šostak [61] and its extension of C. Guido [45] is not incorporated in the approach. Being the second most influential framework in many-valued topology, the setting needs an accommodation in every new unifying theory (J. T. Denniston *et al.* [22] have already done the job for poslat topologies). It is the main purpose of this section to complete the task. Some elements of the required theory have already been presented in [99], being based in the powerset theory $\mathcal{S}_{\mathbf{A}}^{\mathbf{S}}$ of Example 2.6(7) and, therefore, relying on the ground category $\mathbf{Set} \times \mathbf{S}$. This paper extends the setting to an arbitrary ground category $\mathbf{X}$, streamlining the results and removing some superfluous requirements on the way. Strikingly enough, it appears that the new theory restores the framework of many-valuedness, which vanished completely in the previous section.

3.1. **Elements of Lattice-valued Algebra.** To push the theory in the right direction, we need some additional notions related to the already introduced varieties of algebras, namely, the concept of *lattice-valued algebra*. The notion has already appeared in [99], motivated by the concept of *fuzzy group* of A. Rosenfeld [89] and its later generalization of J. M. Anthony and H. Sherwood [7]. The underlying machinery goes in line with the general procedure of J. N. Mordeson and D. S. Malik [72] (also notice the use of the fact that every $\bigvee$-semilattice is actually a complete lattice, i.e., has arbitrary $\bigwedge$).

Definition 3.1. Let $\mathbf{A}$, $\mathbf{L}$ be varieties, let $(\| - \|, \mathbf{CSLat}(\bigvee))$ be a reduct of $\mathbf{L}$ and let $\mathbf{S}$ be a subcategory of $\mathbf{L}$. An $(\mathbf{A}, \mathbf{S})$-*algebra* is a triple (A, μ, L), where A is an $\mathbf{A}$-algebra, L is an $\mathbf{S}$-algebra and $|A| \xrightarrow{\mu} |L|$ is a map such that for every $\lambda \in \Lambda$ and every $\langle a_i \rangle_{n_\lambda} \in A^{n_\lambda}$, it follows that $\bigwedge_{i \in n_\lambda} \mu(a_i) \leq \mu(\omega_\lambda^A(\langle a_i \rangle_{n_\lambda}))$. An $(\mathbf{A}, \mathbf{S})$-*homomorphism* $(A_1, \mu_1, L_1) \xrightarrow{(\varphi, \psi)} (A_2, \mu_2, L_2)$ is an $\mathbf{A} \times \mathbf{S}$-morphism $(A_1, L_1) \xrightarrow{(\varphi, \psi)} (A_2, L_2)$, which satisfies the following lax diagram

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\ \ \varphi\ \ } & A_2 \\
{\scriptstyle \mu_1}\downarrow & \leqslant & \downarrow{\scriptstyle \mu_2} \\
\|L_1\| & \xrightarrow[\|\psi\|]{} & \|L_2\|,
\end{array}
$$

meaning $\psi \circ \mu_1(a) \leq \mu_2 \circ \varphi(a)$ for every $a \in A_1$. $\mathbf{S}$-$\mathbf{A}$ is the category of $(\mathbf{A}, \mathbf{S})$-algebras and $(\mathbf{A}, \mathbf{S})$-homomorphisms, concrete over the category $\mathbf{A} \times \mathbf{S}$.

It is important to underline that A. Di Nola and G. Gerla [24] have already introduced their category $\mathcal{C}(\tau)$ as a "general approach to the theory of fuzzy algebras". When looking into its respective definition in [24], however, one can easily see that $\mathcal{C}(\tau)$ is isomorphic to the subcategory **CLat-Alg**$(\Omega)^=$ of **CLat-Alg**(Ω) (where **CLat** is the variety of complete lattices), with the same objects and whose morphisms $(A_1, \mu_1, L_1) \xrightarrow{(\varphi, \psi)} (A_2, \mu_2, L_2)$ satisfy the identity $\psi \circ \mu_1 = \mu_2 \circ \varphi$. Moreover, [24] started developing the theory of lattice-valued universal algebra, some results of which can be easily extended to our approach. In view of the topological nature of this paper, we will only notice that the category **S-A** of Definition 3.1 provides a more appropriate fuzzification of universal algebra, fuzzifying not only algebras, but also (and that is more important in category theory) their respective homomorphisms.

The reader should also be aware of the fact that the approach of Definition 3.1 is based in the category **Set**(**CSLat**($\bigvee$)) of *lattice-valued sets*, introduced in [95]. In particular, the already obtained results show that **S-A** is indeed a category (closure under composition). One can even consider **S-A** as a concrete category over **Set**(**CSLat**($\bigvee$)), which though being off the topic of the current paper, will be postponed till the subsequent study of the properties of the category **S-A** in [104]. However, one simple result (the proof is thus omitted) can be stated immediately.

Lemma 3.2. *Every* **L***-algebra L gives a (non-full) embedding* $\mathbf{A} \overset{E_L}{\hookrightarrow} \mathbf{S}_L\text{-}\mathbf{A}$, *which is defined by* $E_L(A_1 \xrightarrow{\varphi} A_2) = (A_1, \underline{\top}, L) \xrightarrow{(\varphi, 1_L)} (A_2, \underline{\top}, L)$, *where* $A_i \overset{\underline{\top}}{\Rightarrow} L$ *is the constant map with value* $\top$.

Notice that we can not take an arbitrary element $b \in L$ and use the map $\underline{b}$ in Lemma 3.2, since all nullary operations (which actually are elements of the respective algebra) should be mapped to $\top$.

3.2. **Lattice-valued Categorically-algebraic Topological Structures.** Having introduced lattice-valued varieties, we are ready to define the notion of *lattice-valued categorically-algebraic topology*. In the first step, we generalize slightly the already introduced cvt-theories (Definition 2.12).

Definition 3.3. Let T_I be a cvt-theory in a category **X**, let $(\mathbf{L}_i)_{i \in I}$ be a family of extensions of the variety **CSLat**($\bigvee$), and let $\mathbf{S}_i$ be a subcategory of $\mathbf{L}_i^{op}$ for every $i \in I$. An $\mathbb{L}_I$-valued cvt-theory in **X** induced by T_I and $(\mathbf{S}_i)_{i \in I}$ is the pair $(T_I, \mathbb{L}_I)$, where $\mathbb{L}_I$ is the product category $\prod_{i \in I} \mathbf{S}_i$.

For the sake of simplicity, the category $\mathbb{L}_I$ induced by a single category **S** will be denoted $\mathbb{L}$, identifying the latter two categories. With the notations for cvt-theories in mind, an $\mathbb{L}_I$-valued cvt-theory $(T_I, \mathbb{L}_I)$ will be often denoted $((P_i, \mathbf{B}_i, \mathbf{S}_i^{op}))_{i \in I}$, in order to underline its building blocks. The reader should also pay attention to the fact that we allow not just different underlying lattices (variable-basis framework of S. E. Rodabaugh), but actually different varieties for these lattices to come from.

Definition 3.4. Let $(T_I, \mathbb{L}_I)$ be an $\mathbb{L}_I$-valued cvt-theory in a category **X**. Then $\mathbb{L}_I\mathbf{CTop}(T_I)$ is the category, concrete over $\mathbf{X} \times \mathbb{L}_I$, which is given by the next data.

OBJECTS: ($\mathbb{L}_I$-*valued T_I-spaces*) are triples $(X, (\mathcal{T}_i)_{i \in I}, (L_i)_{i \in I})$, with X an **X**-object, $(L_i)_{i \in I}$ an $\mathbb{L}_I$-object, and $T_i(X) \xrightarrow{\mathcal{T}_i} L_i$ a $(\mathbf{B}_i, \mathbf{S}_i^{op})$-algebra for every $i \in I$ ($(\mathcal{T}_i)_{i \in I}$ is called $\mathbb{L}_I$-*valued T_I-topology* on X).

MORPHISMS: $(X, (\mathcal{T}_i)_{i \in I}, (L_i)_{i \in I}) \xrightarrow{(f, (\psi_i)_{i \in I})} (Y, (\mathcal{S}_i)_{i \in I}, (M_i)_{i \in I})$ are morphisms of the category $\mathbf{X} \times \mathbb{L}_I$ $(X, (L_i)_{i \in I}) \xrightarrow{(f, (\psi_i)_{i \in I})} (Y, (M_i)_{i \in I})$ such that $(T_i(X), \mathcal{T}_i, L_i) \xrightarrow{(T_i f, \psi_i)} (T_i(Y), \mathcal{S}_i, M_i)$ is an $(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op}$-morphism (meaning $\psi_i^{op} \circ \mathcal{S}_i \leqslant \mathcal{T}_i \circ (T_i f)^{op}$) for every $i \in I$ ($\mathbb{L}_I$-*valued T_I-continuity*).

For the sake of simplicity, $\mathbb{L}\mathbf{CTop}(T)$ is denoted $\mathbb{L}\mathbf{Top}(T)$. The main purpose of the new approach is to incorporate the topological framework of U. Höhle, T. Kubiak and A. Šostak, that is done in the next example. The new framework also includes the catalg topology of the previous section as a specific subcase.

Example 3.5.

(1) $\mathbb{L}\mathbf{Top}((\mathcal{S}_{\mathbf{CLat}}^{\mathbf{S}_L}, \mathbf{SFrm}, \mathbf{S}_M^{\mathbf{CDCLat}}))$, where **CLat** is the variety of complete lattices and **CDCLat** is its subcategory of completely distributive lattices, provides a categorical accommodation of the theory of (L, M)-fuzzy topological spaces of T. Kubiak and A. Šostak [61]. To give the reader more intuition, we recall from [61] the definitions of the main building blocks of their theory.

An (L, M)-*fuzzy topological space* is a tuple $((X, L), \mathcal{T}, M)$, which is also denoted $(X, \mathcal{T})$, and where X is a set, L is a complete lattice, M is a completely distributive lattice, whereas $L^X \xrightarrow{\mathcal{T}} M$ is a map (called (L, M)-*fuzzy topology* on X), which satisfies the following three requirements:
 (a) $\mathcal{T}(\underline{\bot}) = \mathcal{T}(\underline{\top}) = \top$ (recall the notations for constant maps of Lemma 3.2);
 (b) $\mathcal{T}(\alpha) \wedge \mathcal{T}(\beta) \leqslant \mathcal{T}(\alpha \bigwedge \beta)$ for every $\alpha, \beta \in L^X$;
 (c) $\bigwedge_{i \in I} \mathcal{T}(\alpha_i) \leqslant \mathcal{T}(\bigvee \alpha_i)$ for every subset $\{\alpha_i \,|\, i \in I\} \subseteq L^X$.
An (L, M)-*fuzzy continuous map* $(X_1, \mathcal{T}_1) \xrightarrow{f} (X_2, \mathcal{T}_2)$ is a map $X_1 \xrightarrow{f} X_2$ with $\mathcal{T}_2(\alpha) \leqslant \mathcal{T}_1 \circ f_L^{\leftarrow}(\alpha)$ for every $\alpha \in L^{X_2}$ (cf. the functor of Lemma 2.5).

It is easy to see that (L, M)-fuzzy topology is an instance of lattice-valued frames in the sense of Definition 3.1. Moreover, (L, M)-fuzzy continuity is based on a homomorphism of lattice-valued frames.

(2) $\mathbb{L}\mathbf{Top}((\mathcal{R}_3, \mathbf{Frm}, \mathbf{Frm}))$ is isomorphic to the category $\mathbf{Loc\text{-}F^2Top}$ of (lattice-valued)-fuzzy topological spaces of J. T. Denniston *et al.* [22], which provides a variable-basis extension of the above-mentioned approach of T. Kubiak and A. Šostak. More precisely, the frames (notice the change in the varieties employed) L and M are no more fixed, the respective fuzzy continuous morphisms (not maps) $((X_1, L_1), M_1, \mathcal{T}_1) \xrightarrow{(f, \varphi, \psi)} ((X_2, L_2), M_2, \mathcal{T}_2)$ comprising a map $X_1 \xrightarrow{f} X_2$ and frame homomorphisms $L_2 \xrightarrow{\varphi^{op}} L_1$, $M_2 \xrightarrow{\psi^{op}} M_1$ such that $\psi^{op} \circ \mathcal{T}_2(\alpha) \leqslant \mathcal{T}_1 \circ (f, \varphi)^{\leftarrow}(\alpha)$ for every $\alpha \in L_2^{X_2}$.

(3) $\mathbb{L}\mathbf{Top}((\mathcal{P}, \mathbf{Frm}, \mathbf{S}_L^{\mathbf{DMFrm}}))$ provides the approach of U. Höhle [50].

(4) $\mathbb{L}_I\mathbf{CTop}((T_I, \mathbb{L}_I))$, with $\mathbf{S}_i = \mathbf{S}_2^{\mathbf{CSLat}(\bigvee)}$ for every $i \in I$, is isomorphic to the category $\mathbf{CTop}(T_I)$ introduced in Definition 2.13.

In view of Example 3.5, it is worthwhile to underline that the approach of T. Kubiak and A. Šostak (started by U. Höhle) provides an inherently *many-valued* framework for topological structures (requires lattice-valued catalg topology), whereas the setting of S. E. Rodabaugh and his collaborators (started by C. L. Chang, J. A. Goguen and R. Lowen) essentially does not deviate from the classical crisp machinery (can be incorporated into catalg topology). The conclusion looks even more striking, on recalling that the latter framework currently dominates all others in many-valued topology, having numerous papers, studying its developments, under its belt. Motivated by the above observation, we suggest to call the framework of U. Höhle, T. Kubiak and A. Šostak *truly lattice-valued topology*, reserving the standard *lattice-valued topology* for J. T. Denniston, A. Melton and S. E. Rodabaugh. The reader could notice that it is precisely the new approach to lattice-valued topology, which enabled us to draw such an important conclusion. Further advantages of the proposed setting will be seen throughout the paper.

3.3. Retractive lattice-valued Categorically-algebraic Topological Structures.

An attentive reader will recall from introduction that there exists an extension of the theory of (L, M)-fuzzy topology in the form of M-*fuzzy topology on an L-fuzzy set* of C. Guido [45]. Being less developed and, therefore, less popular, his theory subsumes the approach of T. Kubiak and A. Šostak as a particular subcase. The cornerstone of the proposed generalization is the employment of a more sophisticated powerset operator the topology is based upon [17, 44]. Despite the innovations in the underlying machinery, the approach of C. Guido can be easily incorporated into the lattice-valued catalg framework. To begin with, we introduce the notion of *retractive algebra* w.r.t. a given algebra of some variety.

Definition 3.6. Let A be an algebra of a variety $\mathbf{A}$. A subset $R \subseteq A$ is called a *retractive algebra* w.r.t. A provided that R is an $\mathbf{A}$-algebra (but not necessarily a subalgebra of A), and there is an $\mathbf{A}$-homomorphism $A \xrightarrow{\varphi_R} R$, which is a retraction in **Set** with a right inverse the inclusion $|R| \overset{e_R}{\hookrightarrow} |A|$ $(|\varphi_R| \circ e_R = 1_{|R|})$.

An immediate and simple example of the new notion is suggested by the developments of C. Guido [45].

Example 3.7. Given a frame A, every element $a \in A$ provides a retractive algebra w.r.t. A in the form of the *lower set* $\downarrow a = \{b \in A \mid b \leqslant a\}$, which is a frame, but (in general) not a subframe of A (the top element $\top$ does not necessarily belong to the set in question). The respective map $A \xrightarrow{\varphi_{\downarrow a}} \downarrow a$ is given by $\varphi_{\downarrow a}(b) = a \wedge b$, which is a frame homomorphism due to the distributivity property of frames, namely, the fact that $a \wedge (\bigvee S) = \bigvee_{s \in S}(a \wedge s)$ for every subset $S \subseteq A$.

The necessary preliminaries in hand, we proceed to a generalization of the topological setting of Definition 3.4. This step has never been done before by either S. E. Rodabaugh himself, or any of his collaborators.

We begin with the construction of the ground category for the new topological setting, which (as the reader will see) differs significantly from that of Definition 3.4.

Definition 3.8. Let T_I be a cvt-theory in a category $\mathbf{X}$. $\mathbf{CRetX}(T_I)$ is the category, concrete over the category $\mathbf{X}$, which is given by the following data.

    OBJECTS: are pairs $(X, (R_i, \varphi_{R_i})_{i \in I})$, where X is an $\mathbf{X}$-object, whereas for every $i \in I$, R_i is a retractive algebra w.r.t. $T_i(X)$ and $T_i(X) \xrightarrow{\varphi_{R_i}} R_i$ is its corresponding retraction.

    MORPHISMS: $(X, (R_i, \varphi_{R_i})_{i \in I}) \xrightarrow{f} (Y, (S_i, \varphi_{S_i})_{i \in I})$ are $\mathbf{X}$-morphisms $X \xrightarrow{f} Y$ such that for every $i \in I$, the following diagram (in which $|-|$ stands for the forgetful functor to the category $\mathbf{Set}$) commutes

$$
\begin{array}{ccc}
|T_i(Y)| & \xrightarrow{\quad |(T_i f)^{op}| \quad} & |T_i(X)| \\
{\scriptstyle |\varphi_{S_i}|} \downarrow & & \downarrow {\scriptstyle |\varphi_{R_i}|} \\
|S_i| \underset{e_{S_i}}{\hookrightarrow} |T_i(Y)| \xrightarrow[|(T_i f)^{op}|]{} |T_i(X)| \xrightarrow[|\varphi_{R_i}|]{} & & |R_i|,
\end{array}
$$

and, therefore, by the surjectivity of φ_{S_i}, $|S_i| \xrightarrow{|\varphi_{R_i}| \circ |(T_i f)^{op}| \circ e_{S_i}} |R_i|$ is a $\mathbf{B}_i$-homomorphism.

To convince the reader that Definition 3.8 provides a category, we will verify the closure of its morphisms under composition. Given two $\mathbf{CRetX}(T_I)$-morphisms $(X, (R_i, \varphi_{R_i})_{i \in I}) \xrightarrow{f} (Y, (S_i, \varphi_{S_i})_{i \in I}) \xrightarrow{g} (Z, (U_i, \varphi_{U_i})_{i \in I})$, it follows that $\varphi_{R_i} \circ (T_i(g \circ f))^{op} \circ e_{U_i} \circ \varphi_{U_i} = \varphi_{R_i} \circ (T_i f)^{op} \circ (T_i g)^{op} \circ e_{U_i} \circ \varphi_{U_i} = \varphi_{R_i} \circ (T_i f)^{op} \circ e_{S_i} \circ \varphi_{S_i} \circ (T_i g)^{op} \circ e_{U_i} \circ \varphi_{U_i} = \varphi_{R_i} \circ (T_i f)^{op} \circ e_{S_i} \circ \varphi_{S_i} \circ (T_i g)^{op} = \varphi_{R_i} \circ (T_i f)^{op} \circ (T_i g)^{op} = \varphi_{R_i} \circ (T_i(g \circ f))^{op}$ for every $i \in I$. We underline that without any condition on the morphisms involved, we are unable to show the desired closure under composition.

For the sake of simplicity, $\mathbf{CRetX}(T)$ is denoted $\mathbf{RetX}(T)$. It is almost straightforward that the category $\mathbf{CRetX}(T_I)$ provides an (in general, strict) extension of the category $\mathbf{X}$, i.e., the following (easy, and thus, the proof is omitted) result holds.

Lemma 3.9. *There exists a full embedding* $\mathbf{X} \xhookrightarrow{\;E\;} \mathbf{CRetX}(T_I)$ *defined by the formula* $E(X \xrightarrow{f} Y) = (X, (T_i(X), 1_{T_i(X)})_{i \in I}) \xrightarrow{f} (Y, (T_i(Y), 1_{T_i(Y)})_{i \in I})$.

In general, there is no relation between the categories $\mathbf{CRetX}(T_I)$ and $\mathbf{CTop}(T_I)$ (recall Definition 2.13), since the objects of the former need not provide subalgebras of $T_i(X)$, and the objects of the latter need not give retracts of $T_i(X)$.

The new category in hand, we define its respective topological structures.

Definition 3.10. Let $(T_I, \mathbb{L}_I)$ be an $\mathbb{L}_I$-valued cvt-theory in a category $\mathbf{X}$. Then $\mathbb{L}_I\mathbf{CRTop}(T_I)$ is the category, concrete over the product category $\mathbf{CRetX}(T_I) \times \mathbb{L}_I$, which is defined by the following data.

    OBJECTS: ($\mathbb{L}_I$-*valued* T_I-*retractive spaces* or $\mathbb{L}_I$-*valued* T_I-*r-spaces*) are quintuples $(X, (R_i, \varphi_{R_i})_{i \in I}, (\mathfrak{T}_i)_{i \in I}, (L_i)_{i \in I})$, in which $(X, (R_i, \varphi_{R_i})_{i \in I})$ is a $\mathbf{CRetX}(T_I)$-object, $(L_i)_{i \in I}$ is an $\mathbb{L}_I$-object, and $R_i \xrightarrow{\mathfrak{T}_i} L_i$ is a $(\mathbf{B}_i, \mathbf{S}_i^{op})$-algebra for every $i \in I$ $((\mathfrak{T}_i)_{i \in I}$ is called $\mathbb{L}_I$-*valued* T_I-*retractive topology* or $\mathbb{L}_I$-*valued* T_I-*r-topology* on X).

MORPHISMS: $(X, (R_i, \varphi_{R_i})_{i\in I}, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}) \xrightarrow{(f,(\psi_i)_{i\in I})} (Y, (S_i, \varphi_{S_i})_{i\in I},$ $(\mathfrak{S}_i)_{i\in I}, (M_i)_{i\in I})$ are those $\mathbf{CRetX}(T_I) \times \mathbb{L}_I$-morphisms $(X, (R_i, \varphi_{R_i})_{i\in I},$ $(L_i)_{i\in I}) \xrightarrow{(f,(\psi_i)_{i\in I})} (Y, (S_i, \varphi_{S_i})_{i\in I}, (M_i)_{i\in I})$ for which for every $i \in I$, $(R_i, \mathfrak{T}_i, L_i) \xrightarrow{(e_{S_i}^{op}\circ T_i f\circ\varphi_{R_i}^{op}, \psi_i)} (S_i, \mathfrak{S}_i, M_i)$ is an $(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op}$-morphism ($\mathbb{L}_I$-*valued T_I-retractive continuity* or $\mathbb{L}_I$-*valued T_I-r-continuity*).

To convince the reader that Definition 3.10 provides a category, we will verify again the closure under composition. For every two $\mathbb{L}_I$-valued T_I-r-continuous morphisms $(X, (R_i, \varphi_{R_i})_{i\in I}, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}) \xrightarrow{(f,(\psi_i)_{i\in I})} (Y, (S_i, \varphi_{S_i})_{i\in I}, (\mathfrak{S}_i)_{i\in I},$ $(M_i)_{i\in I}) \xrightarrow{(g,(\phi_i)_{i\in I})} (Z, (U_i, \varphi_{U_i})_{i\in I}, (\mathcal{U}_i)_{i\in I}, (N_i)_{i\in I})$, it follows that $(\phi_i \circ \psi_i)^{op} \circ$ $\mathcal{U}_i = \psi_i^{op} \circ \phi_i^{op} \circ \mathcal{U}_i \leqslant \psi_i^{op} \circ \mathfrak{S}_i \circ \varphi_{S_i} \circ (T_i g)^{op} \circ e_{U_i} \leqslant \mathfrak{T}_i \circ \varphi_{R_i} \circ (T_i f)^{op} \circ e_{S_i} \circ \varphi_{S_i} \circ$ $(T_i g)^{op} \circ e_{U_i} = \mathfrak{T}_i \circ \varphi_{R_i} \circ (T_i f)^{op} \circ (T_i g)^{op} \circ e_{U_i} = \mathfrak{T}_i \circ \varphi_{R_i} \circ (T_i(g \circ f))^{op} \circ e_{U_i}$ (the machinery depends heavily on the morphism condition of Definition 3.8).

For the sake of simplicity, the category $\mathbb{L}\mathbf{CRTop}(T)$ will be denoted $\mathbb{L}\mathbf{RTop}(T)$. Using Lemma 3.9, we can show that the new category provides an (in general proper) extension of the category $\mathbb{L}_I\mathbf{CTop}(T_I)$.

Lemma 3.11. *There exists a full embedding* $\mathbb{L}_I\mathbf{CTop}(T_I) \overset{\overline{E}}{\hookrightarrow} \mathbb{L}_I\mathbf{CRTop}(T_I)$

defined by the formula $E((X, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}) \xrightarrow{(f,(\psi_i)_{i\in I})} (Y, (\mathfrak{S}_i)_{i\in I}, (M_i)_{i\in I})) =$ $(X, (T_i(X), 1_{T_i(X)})_{i\in I}, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}) \xrightarrow{(f,(\psi_i)_{i\in I})} (Y, (T_i(Y), 1_{T_i(Y)})_{i\in I}, (\mathfrak{S}_i)_{i\in I},$ $(M_i)_{i\in I})$, *which, taken together with the embedding of Lemma 3.9, makes the following diagram commute (recall that* $|-|$ *stands for the forgetful functor to the ground category)*

$$
\begin{array}{ccc}
\mathbb{L}_I\mathbf{CTop}(T_I) & \overset{\overline{E}}{\hookrightarrow\!\!\!\longrightarrow} & \mathbb{L}_I\mathbf{CRTop}(T_I) \\
{\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\
\mathbf{X} \times \mathbb{L}_I & \underset{E\times 1_{\mathbb{L}_I}}{\hookrightarrow\!\!\!\longrightarrow} & \mathbf{CRetX}(T_I) \times \mathbb{L}_I.
\end{array}
$$

As a consequence, $\mathbb{L}_I\mathbf{CTop}(T_I)$ can be considered as a concrete subcategory of $\mathbb{L}_I\mathbf{CRTop}(T_I)$. The next example justifies the fruitfulness of the new notion. Start with a simple preliminary lemma.

Lemma 3.12. *For a given map* $X \xrightarrow{f} Y$, *a frame L and L-sets $\alpha \in L^X$, $\beta \in L^Y$, the following are equivalent:*

(1) $\varphi_{\downarrow\alpha} \circ f_L^{\leftarrow} \circ e_{\downarrow\beta} \circ \varphi_{\downarrow\beta} = \varphi_{\downarrow\alpha} \circ f_L^{\leftarrow}$;
(2) $\alpha \leqslant \beta \circ f$.

Proof. Straightforward computations show that (1) can be rewritten as $\alpha \wedge (\beta \circ f) \wedge (\gamma \circ f) = \alpha \wedge (\gamma \circ f)$ for every $\gamma \in L^Y$, which, in its turn, is a restatement of $\alpha \wedge (\gamma \circ f) \leqslant \beta \circ f$ for every $\gamma \in L^Y$ (†), which is clearly implied by (2). On the other hand, the constant map $Y \xrightarrow{\top} L$ in (†) implies $\alpha = \alpha \wedge \top \leqslant \beta \circ f$. $\square$

Example 3.13. By Lemma 3.12, we get that the full subcategory $\mathbf{FT}_G$ of the category $\mathbb{L}\mathbf{RTop}((\mathcal{S}_{\mathbf{CLat}}^{\mathbf{S}_L}, \mathbf{SFrm}, \mathbf{S}_M^{\mathbf{CDCLat}}))$ (recall the notations of Example 3.5(1)), comprising all objects of the form $((X, L), (\downarrow \alpha, \varphi_{\downarrow \alpha}), \mathcal{T}, M)$, where α is an element of the L-powerset L^X (recall the developments of Example 3.7), is isomorphic to the category $\mathcal{A}(L, M)$-$\mathbf{TOP}$ of M-fuzzy topological L-fuzzy spaces of C. Guido [45]. Notice the crucial difference from the setting of (L, M)-fuzzy topology of T. Kubiak and A. Šostak (recall Example 3.5(1)), i.e., the employment of a map $\downarrow \alpha \xrightarrow{\mathcal{T}} M$ (in which $\downarrow \alpha$ is a subset of L^X) instead of a map $L^X \xrightarrow{\mathcal{T}} M$, which takes its origin in the standard crisp construction of a subspace of a given topological space.

Lemma 3.11 shows that the approach of C. Guido provides an extension of the setting of T. Kubiak and A. Šostak, which, however, is neglected at the moment. It is one of the goals of this paper, to draw the attention of the reader to the promising research area, which (following Lemma 3.11) provides one of the most general approaches to lattice-valued topology available at the moment.

It is also important to notice that even in the crisp case, the retractive approach provides a convenient framework for doing non-standard topology, e.g., for exploring the properties of *semi-open sets* of N. Levine [63] and their different modifications (for instance, *semi-θ-open sets* of [12]), which are closed under arbitrary set-theoretic unions, but fail to be closed under finite set-theoretic intersections, and, therefore, do not give rise to subalgebras of their respective powerset algebras.

4. **Lattice-valued Topology is Topological**

Every new topological theory needs to show some good properties of its developments, in order to convince the researcher of a real usefulness of its provided tools. While working in categorical topology, a promising beginning could be the fact that the induced category of topological structures is topological over its ground category. The result is a folklore [1] for the category $\mathbf{Top}$ of topological spaces and continuous maps. Moreover, much study on the topic has been done by S. E. Rodabaugh, who showed that his poslat approach always provides a topological category [86], developing the theory of s-quantales (Definition 2.2(2)) for the purpose. In collaboration with J. T. Denniston and A. Melton, the result was extended for the category $\mathbf{Loc}$-$\mathbf{F}^2\mathbf{Top}$ [22], which is a variable-basis accommodation of the theory of (L, M)-fuzzy topological spaces of T. Kubiak and A. Šostak. On the other hand, in [102], we have shown that the category $\mathbf{CTop}(T_I)$ is topological over its ground category $\mathbf{X}$, thereby incorporating all the respective results on lattice-valued topology. It is the purpose of this section to do the job for the retractive lattice-valued topology (see the naming convention in the previous section), or, in other words, to show that the category $\mathbb{L}_I\mathbf{CRTop}(T_I)$ is topological over its ground category $\mathbf{CRetX}(T_I) \times \mathbb{L}_I$. This result has yet no analogue in the fuzzy literature, apart from [45], where C. Guido shows that his category $\mathcal{A}(L, M)$-$\mathbf{TOP}$ of M-fuzzy topological L-fuzzy spaces (which, following Example 3.13, is a particular subcategory of some of the categories of the form $\mathbb{L}_I\mathbf{CRTop}(T_I)$) is topological over its ground category.

For the sake of simplicity, we begin with the category $\mathbb{L}\mathbf{RTop}(T)$, extending the result afterwards in a straightforward way to the more complicated composite framework (the reader should recall our notation $|-|$ for the underlying functors of concrete categories, which will be used in this section).

The proof of the desired property will follow the standard path of subbasic continuity. The reader may be well aware of the important result of classical topology, stating that continuity of a map can be checked on the elements of a subbase (for a full discussion of the categorical role of subbase in topology see [84]). The result can be readily extended to our current setting. We begin with a simple lemma.

Lemma 4.1. *Let $(X, (R, \varphi_R), L)$ be a $\mathbf{RetX}(T) \times \mathbb{L}$-object and let $(\mathcal{T}_i)_{i \in I}$ be a family of $\mathbb{L}$-valued T-r-topologies on $(X, (R, \varphi_R), L)$. Then $\bigwedge_{i \in I} \mathcal{T}_i$ is an $\mathbb{L}$-valued T-r-topology on $(X, (R, \varphi_R), L)$, where for every $a \in R$, $(\bigwedge_{i \in I} \mathcal{T}_i)(a) = \bigwedge_{i \in I} \mathcal{T}_i(a)$.*

Proof. Given $\lambda \in \Lambda_{\mathbf{B}}$ and $a_j \in R$ for every $j \in n_\lambda$, one has, $\bigwedge_{j \in n_\lambda}((\bigwedge_{i \in I} \mathcal{T}_i)(a_j)) = \bigwedge_{i \in I} \bigwedge_{j \in n_\lambda} \mathcal{T}_i(a_j) \leqslant \bigwedge_{i \in I} \mathcal{T}_i(\omega_\lambda^R(\langle a_j \rangle_{n_\lambda})) = (\bigwedge_{i \in I} \mathcal{T}_i)(\omega_\lambda^R(\langle a_j \rangle_{n_\lambda}))$. $\square$

Notice that the case of the empty family is included as well, producing the topology $\mathcal{T} \equiv \underline{\top}$. Lemma 4.1 shows that given a $\mathbf{RetX}(T) \times \mathbb{L}$-object $(X, (R, \varphi_R), L)$, its fibre $Fb(X, (R, \varphi_R), L) = \{\mathcal{T} \mid (X, (R, \varphi_R), \mathcal{T}, L) \in \mathbb{L}\mathbf{RTop}(T)\}$ is a $\bigwedge$-semilattice, and that provides an opening for the definition of the concept of *subbase*.

Definition 4.2. Let $(X, (R, \varphi_R), L)$ be a $\mathbf{RetX}(T) \times \mathbb{L}$-object and let $R \xrightarrow{S} L$ be a map. Setting $\langle S \rangle = \bigwedge\{\mathcal{T} \in Fb(X, (R, \varphi_R), L) \mid S \leqslant \mathcal{T}\}$ provides an $\mathbb{L}$-valued T-r-space $(X, (R, \varphi_R), \langle S \rangle, L)$, in which S is called a *subbase* of $\langle S \rangle$.

Definition 4.2 is a direct generalization of the notion of lattice-valued subbase of [84, Definition 3.2.9] and its various poslat modifications. The main purpose of the new definition is contained in the following result.

Proposition 4.3 (Subbasic continuity). *Let $(X_1, (R_1, \varphi_{R_1}), \mathcal{T}_1, L_1), (X_2, (R_2, \varphi_{R_2}), \mathcal{T}_2, L_2)$ be $\mathbb{L}$-valued T-r-spaces such that $\mathcal{T}_2 = \langle S \rangle$ and, moreover, let $|(X_1, (R_1, \varphi_{R_1}), \mathcal{T}_1, L_1)| \xrightarrow{(f, \psi)} |(X_2, (R_2, \varphi_{R_2}), \mathcal{T}_2, L_2)|$ be a $\mathbf{RetX}(T) \times \mathbb{L}$-morphism. Then (f, ψ) is $\mathbb{L}$-valued T-r-continuous iff $\|\psi^{op}\| \circ S \leqslant \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}$.*

Proof. The necessity follows from the definition of $\mathbb{L}$-valued T-r-continuity in Definition 3.10. To show the sufficiency, notice that there exists a $\bigwedge$-preserving map $\|L_1\| \xrightarrow{\|\psi^{op}\|^{\vdash}} \|L_2\|$ (the so-called *right adjoint* to $\|\psi^{op}\|$ in the sense of partially ordered sets [40, Section 0-3]) such that $1_{\|L_2\|} \leqslant \|\psi^{op}\|^{\vdash} \circ \|\psi^{op}\|$ and $\|\psi^{op}\| \circ \|\psi^{op}\|^{\vdash} \leqslant 1_{\|L_1\|}$. Thus, the condition of the theorem implies $S \leqslant \|\psi^{op}\|^{\vdash} \circ \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}$. We show that $\|\psi^{op}\|^{\vdash} \circ \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}$ is an $\mathbb{L}$-valued T-r-topology on $(X_2, (R_2, \varphi_{R_2}), L_2)$. Given $\lambda \in \Lambda_{\mathbf{B}}$ and $a_i \in R_2$ for every $i \in n_\lambda$, $\bigwedge_{i \in n_\lambda}(\|\psi^{op}\|^{\vdash} \circ \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}(a_i)) = \|\psi^{op}\|^{\vdash}(\bigwedge_{i \in n_\lambda} \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}(a_i)) \leqslant \|\psi^{op}\|^{\vdash} \circ \mathcal{T}_1(\omega_\lambda^{R_1}(\langle \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}(a_i) \rangle_{n_\lambda})) = \|\psi^{op}\|^{\vdash} \circ \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}(\omega_\lambda^{R_2}(\langle a_i \rangle_{n_\lambda}))$. The result implies, $\langle S \rangle \leqslant \|\psi^{op}\|^{\vdash} \circ \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}$ and thus, $\|\psi^{op}\| \circ \mathcal{T}_2 = \|\psi^{op}\| \circ \langle S \rangle \leqslant \|\psi^{op}\| \circ \|\psi^{op}\|^{\vdash} \circ \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2} \leqslant \mathcal{T}_1 \circ \varphi_{R_1} \circ (Tf)^{op} \circ e_{R_2}$. $\square$

Proposition 4.3 provides the announced analogue of the classical result on subbasic continuity, which is done in a much more general framework than usual. It also extends a part of [84, Theorem 3.2.13] as well as the respective subbasic machinery of [22]. Having done with the subbasic stuff, we proceed to the next preliminary result on *amnesticity* [1] of the category $\mathbb{L}\mathbf{RTop}(T)$.
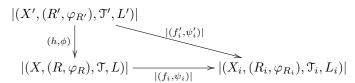
Lemma 4.4. *The category* $\mathbb{L}\mathbf{RTop}(T)$ *is amnestic.*

Proof. If both $|(X,(R,\varphi_R),\mathfrak{T}_1,L)| \xrightarrow{(1_X,1_L)} |(X,(R,\varphi_R),\mathfrak{T}_2,L)|$ and $|(X,(R,\varphi_R),$ $\mathfrak{T}_2,L)| \xrightarrow{(1_X,1_L)} |(X,(R,\varphi_R),\mathfrak{T}_1,L)|$ are $\mathbb{L}$-valued T-r-continuous, then $\mathfrak{T}_2 = \|1_L\| \circ \mathfrak{T}_2 \leqslant \mathfrak{T}_1 \circ \varphi_R \circ (T1_X)^{op} \circ e_R = \mathfrak{T}_1$ implies $\mathfrak{T}_2 \leqslant \mathfrak{T}_1$ and similarly, $\mathfrak{T}_1 \leqslant \mathfrak{T}_2$. $\hspace{1em}\square$

All preliminaries in their places, the desired result is ready to state.

Theorem 4.5. $\mathbb{L}\mathbf{RTop}(T)$ *is topological over* $\mathbf{RetX}(T) \times \mathbb{L}$.

Proof. By [1, Proposition 21.5], it is enough to show that every $| - |$-structured source has an initial lift. Suppose $\mathcal{L} = ((X,(R,\varphi_R),L) \xrightarrow{(f_i,\psi_i)} |(X_i,(R_i,\varphi_{R_i}),\mathfrak{T}_i,$ $L_i)|)_{i \in I}$ is an $| - |$-structured source. Define a map $R \xrightarrow{S} L$ by $S(a) = \bigvee\{\|\psi_i^{op}\| \circ \mathfrak{T}_i(a) \,|\, a = \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i)$ for some $i \in I$ and some $a_i \in R_i\}$ and let $\mathfrak{T} = \langle S \rangle$. One obtains an $\mathbb{L}$-valued T-r-topological space $(X,(R,\varphi_R),\mathfrak{T},L)$ such that $|(X,(R,\varphi_R),\mathfrak{T},L)| \xrightarrow{(f_i,\psi_i)} |(X_i,(R_i,\varphi_{R_i}),\mathfrak{T}_i,L_i)|$ is $\mathbb{L}$-valued T-r-continuous for every $i \in I$. Indeed, for a fixed $i \in I$ and some $a_i \in R_i$, $\mathfrak{T} \circ \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i) \geqslant \bigvee\{\|\psi_j^{op}\| \circ \mathfrak{T}_j(a_j) \,|\, \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i) = \varphi_R \circ (Tf_j)^{op} \circ e_{R_j}(a_j)$ for some $j \in I$ and some $a_j \in R_j\} \geqslant \|\psi_i^{op}\| \circ \mathfrak{T}_i(a_i)$. Altogether, we have built a lift $\tilde{\mathcal{L}} = ((X,(R,\varphi_R),\mathfrak{T},L) \xrightarrow{(f_i,\psi_i)} (X_i,(R_i,\varphi_{R_i}),\mathfrak{T}_i,L_i))_{i \in I}$ of $\mathcal{L}$. The only thing left is verification of its initiality.

Let $\mathcal{L}' = ((X',(R',\varphi_{R'}),\mathfrak{T}',L') \xrightarrow{(f_i',\psi_i')} (X_i,(R_i,\varphi_{R_i}),\mathfrak{T}_i,L_i))_{i \in I}$ be a source in $\mathbb{L}\mathbf{RTop}(T)$ and let $|(X',(R',\varphi_{R'}),\mathfrak{T}',L')| \xrightarrow{(h,\phi)} |(X,(R,\varphi_R),\mathfrak{T},L)|$ be a $\mathbf{RetX}(T) \times \mathbb{L}$-morphism such that the triangle

$$
\begin{array}{ccc}
|(X',(R',\varphi_{R'}),\mathfrak{T}',L')| & & \\
\;\;\;\Big\downarrow{\scriptstyle(h,\phi)} & \searrow^{\;|(f_i',\psi_i')|} & \\
|(X,(R,\varphi_R),\mathfrak{T},L)| & \xrightarrow[\;|(f_i,\psi_i)|\;]{} & |(X_i,(R_i,\varphi_{R_i}),\mathfrak{T}_i,L_i)|
\end{array}
$$

commutes for every $i \in I$. By Proposition 4.3, $\mathbb{L}$-valued T-r-continuity of (h,ϕ) will follow from the inequality $\|\phi^{op}\| \circ S \leqslant \mathfrak{T}' \circ \varphi_{R'} \circ (Th)^{op} \circ e_R$ and that is now the thing to show. Given $a \in R$ such that $a = \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i)$ for some $i \in I$, $\|\phi^{op}\| \circ \|\psi_i^{op}\| \circ \mathfrak{T}_i(a_i) = \|\psi_i'^{op}\| \circ \mathfrak{T}_i(a_i) \leqslant \mathfrak{T}' \circ \varphi_{R'} \circ (Tf_i')^{op} \circ e_{R_i}(a_i) = \mathfrak{T}' \circ \varphi_{R'} \circ (Th)^{op} \circ (Tf_i)^{op} \circ e_{R_i}(a_i) = \mathfrak{T}' \circ \varphi_{R'} \circ (Th)^{op} \circ e_R \circ \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i) = \mathfrak{T}' \circ \varphi_{R'} \circ (Th)^{op} \circ e_R(a)$. It follows that $\|\phi^{op}\| \circ S(a) = \|\phi^{op}\|(\bigvee\{\|\psi_i^{op}\| \circ \mathfrak{T}_i(a_i) \,|\, a = \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i)$ for some $i \in I$ and some $a_i \in R_i\}) = \bigvee\{\|\phi^{op}\| \circ \|\psi_i^{op}\| \circ \mathfrak{T}_i(a_i) \,|\, a = \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i)$ for some $i \in I$ and some $a_i \in R_i\} \leqslant \bigvee\{\mathfrak{T}' \circ \varphi_{R'} \circ$

$(Th)^{op} \circ e_R(a) \,|\, a \,=\, \varphi_R \circ (Tf_i)^{op} \circ e_{R_i}(a_i)$ for some $i \in I$ and some $a_i \in R_i\} \leqslant$ $\mathfrak{T}' \circ \varphi_{R'} \circ (Th)^{op} \circ e_R(a)$. $\hfill\square$

Theorem 4.5 gives rise to one of the main results of the manuscript.

Corollary 4.6. $\mathbb{L}_I\mathbf{CRTop}(T_I)$ *is topological over* $\mathbf{CRetX}(T) \times \mathbb{L}_I$.

Proof. Follows by the application of Theorem 4.5 to every vt-theory T_i. $\hfill\square$

Moreover, we can easily show now that every category of the form $\mathbb{L}_I\mathbf{CTop}(T_I)$ is also topological over its respective ground category. Start with the preliminary result concerning the simplified category $\mathbb{L}\mathbf{Top}(T)$.

Corollary 4.7. $\mathbb{L}\mathbf{Top}(T)$ *is topological over* $\mathbf{X} \times \mathbb{L}$.

Proof. By [1, Proposition 21.30], it is enough to show that the full subcategory $E^{\rightarrow}(\mathbb{L}\mathbf{Top}(T))$ of the category $\mathbb{L}\mathbf{RTop}(T)$, induced by the full embedding of Lemma 3.11, is initially closed in $\mathbb{L}\mathbf{RTop}(T)$. Indeed, given an $|-|$-structured source $\mathcal{L} = ((X, (T(X), 1_{T(X)}), L) \xrightarrow{(f_i, \psi_i)} |(X_i, (T(X_i), 1_{T(X_i)}), \mathfrak{T}_i, L_i)|)_{i \in I}$, the construction of Theorem 4.5 provides a lift $\tilde{\mathcal{L}} = ((X, (T(X), 1_{T(X)}), \mathfrak{T}, L) \xrightarrow{(f_i, \psi_i)} (X_i, (T(X_i), 1_{T(X_i)}), \mathfrak{T}_i, L_i))_{i \in I}$ of $\mathcal{L}$, which clearly lies in $E^{\rightarrow}(\mathbb{L}\mathbf{Top}(T))$. $\hfill\square$

Corollary 4.7 extends [84, Theorem 3.3.9] and its various modifications as well as the above-mentioned proof of topologicity of the category $\mathbf{Loc}\text{-}\mathbf{F^2Top}$ in [22]. Moreover, it gives rise to a more general result on the composite setting.

Corollary 4.8. $\mathbb{L}_I\mathbf{CTop}(T_I)$ *is topological over* $\mathbf{X} \times \mathbb{L}_I$.

Proof. Follows by the application of Corollary 4.7 to every vt-theory T_i. $\hfill\square$

In particular, all categories of Example 3.5 (which include nearly all approaches to many-valued topology) are topological over their ground categories. Moreover, Theorem 4.5 gives rise to the well-known (see, e.g., D. Dikranjan *et al.* [28]) result on the nature of the category $\mathbf{Cls}$ of closure spaces (Example 2.14(2)).

Corollary 4.9. *The construct* $\mathbf{Cls}$ *is topological.*

Meta-mathematically restated, we are doing topology when working in the category $\mathbb{L}_I\mathbf{CTop}(T_I)$. Moreover, the result immediately justifies the claim of S. E. Rodabaugh [86] on the importance of the structure of s-quantale in lattice-valued topology. Indeed, the variety $\mathbf{USQuant}$ provides that lattice-theoretic minimum, which preserves the flavor of classical topology (arbitrary unions and finite intersections). Our variety, $\mathbf{CSL}$ does the job for closure spaces. The discussion raises a challenging and far-reaching problem.

Problem 4.10. For each type of topological structures, find the variety with the minimum conditions on its algebras, to retain the main properties of this structure (*characterizing variety*) (the respective category of lattice-valued catalg spaces (*characterizing category*) is then topological over its ground category).

Having justified the claim on topological nature of the new theory, in the next section, we provide an important application of the new setting to the concept of *topological system* of S. Vickers [106].

5. **Lattice-valued Categorically-algebraic Pointfree Topology**

Up to now, our developed framework was illustrated by examples of either classical or many-valued topological structures. The real push for the new theory came, however, from the quarters of *pointfree topology*. Its history began in 1959, when D. Papert and S. Papert constructed an adjunction between the category **Top** of topological spaces and the dual of the variety **Frm** of frames [77], based in the functor, taking a topological space to its topology and a continuous map to its generated preimage operator on the topologies in question. A more succinct description of the adjunction was given by J. Isbell [55], who introduced the name *locale* for the objects of **Frm**^{op}, suggesting their category **Loc** as a substitute for **Top**. A coherent statement to locale theory was given by P. T. Johnstone [56]. In the next step, S. Vickers [106] used the logic of finite observations to introduce the notion of *topological system*, to get a single framework in which to treat both spaces and locales. He presented the category **TopSys** of topological systems and showed that the category **Loc** (resp. **Top**) is isomorphic to the (resp. co)reflective subcategory of **TopSys**. These crucial properties were based in the *localification* (resp. *spatialization*) procedure for systems, which provided a way to obtain a locale (resp. topological space) from a given topological system.

Recently, topological systems became of interest for the fuzzy community. Several researchers tried to extend their related results to lattice-valued framework. The most important attempts belong to J. T. Denniston, S. E. Rodabaugh [23] and C. Guido [46]. The drawback of these approaches, however, were significant difficulties (especially in [23]) encountered in the extension of the classical results. The main problem originated in the attempt to embed the category of lattice-valued (that means generalized) topological spaces and their underlying algebraic structures (which, in general, are no more locales) into the classical category of topological systems. The solution was found by J. T. Denniston, A. Melton and S. E. Rodabaugh [19] in the concept of *lattice-valued topological system*. Even this improvement never stimulated them to provide an extension of the localification (resp. spatialization) procedure of S. Vickers, while C. Guido has developed his results to extend the classical membership relation "∈" to many-valued context [46].

To address the above deficiencies, we started the theory of *categorically-algebraic topological systems*. The cornerstone was laid in [98], where the notion of *variety-based topological system* was introduced together with an extension of the spatialization procedure of S. Vickers. Later on, it appeared that the extension of topological systems included a notion from theoretical physics, i.e., the already mentioned *state property systems* of D. Aerts [3]. It soon became clear that his equivalence between the categories of state property systems and closure spaces [4, 5] is a consequence of our generalized spatialization procedure (the topic is discussed in full detail in [100]).

The above-mentioned results based on a restricted setting of the functor of Lemma 2.5, we introduced in [101] a more general approach based on vbp-theories,

making no distinction though between powerset and topological theories. The purpose of this section is to develop the new theory in its full generality. The needed machinery given in the previous section, we proceed to the new notion immediately.

Definition 5.1. Let $(T_I, \mathbb{L}_I)$ be an $\mathbb{L}_I$-valued cvt-theory in a category $\mathbf{X}$. Then $\mathbb{L}_I\mathbf{CTopSys}(T_I)$ is the category, concrete over $\mathbf{X} \times (\prod_{i \in I}(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op})$, which is given by the following data.

OBJECTS: ($\mathbb{L}_I$-*valued composite variety-based topological systems* or $\mathbb{L}_I$-*valued* T_I-*systems*) are the triples $(X, (\kappa_i)_{i \in I}, ((A_i, \mu_i, L_i))_{i \in I})$ such that X is an $\mathbf{X}$-object, $((A_i, \mu_i, L_i))_{i \in I}$ is a $\prod_{i \in I}(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op}$-object, whereas $T_i(X) \xrightarrow{\kappa_i} A_i$ is a $\mathbf{B}_i^{op}$-morphism for every $i \in I$ ($(\kappa_i)_{i \in I}$ is called $\mathbb{L}_I$-*valued composite variety-based satisfaction relation* or $\mathbb{L}_I$-*valued* T_I-*satisfaction relation* on $(X, ((A_i, \mu_i, L_i))_{i \in I})$).

MORPHISMS: $(X, (\kappa_i)_{i \in I}, ((A_i, \mu_i, L_i))_{i \in I}) \xrightarrow{(f,((\varphi_i, \psi_i))_{i \in I})} (Y, (\iota_i)_{i \in I}, ((B_i, \nu_i, M_i))_{i \in I})$ are those morphisms $(X, ((A_i, \mu_i, L_i))_{i \in I}) \xrightarrow{(f,((\varphi_i, \psi_i))_{i \in I})} (Y, ((B_i, \nu_i, M_i))_{i \in I})$ of the product category $\mathbf{X} \times (\prod_{i \in I}(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op})$, for which, for every $i \in I$, the diagram

$$
\begin{array}{ccc}
T_i(X) & \xrightarrow{\;\; T_i f \;\;} & T_i(Y) \\
{\scriptstyle \kappa_i}\downarrow & & \downarrow{\scriptstyle \iota_i} \\
A_i & \xrightarrow{\;\; \varphi_i \;\;} & B_i
\end{array}
$$

commutes ($\mathbb{L}_I$-*valued composite variety-based continuity* or $\mathbb{L}_I$-*valued* T_I-*continuity*).

For convenience sake, the category $\mathbb{L}\mathbf{CTopSys}(T)$ is denoted $\mathbb{L}\mathbf{TopSys}(T)$. The main purpose of the new developments is to provide an analogue of the structure of S. Vickers, suitable for application in the framework of lattice-valued catalg topology. As shows the following example, the presented approach incorporates all (known to the author) many-valued generalizations of the concept.

Example 5.2.

(1) $\mathbb{L}\mathbf{TopSys}((\mathcal{P}, \mathbf{Frm}, \mathbf{S}_2^{\mathbf{CSLat}(\vee)}))$ is isomorphic to the category $\mathbf{TopSys}$ of classical topological systems of S. Vickers [106]. To give more intuition to the reader, we recall the basic definitions from [106].

A *topological system* is a triple (X, κ, A), where X is a set, A is a frame, whereas $A \xrightarrow{\kappa} \|\mathcal{P}(X)\|$ is a frame homomorphism, with $\| - \|$ standing for the reduct to the variety $\mathbf{Frm}$ of frames (recall that the powerset of a set has the structure of a complete Boolean algebra).

A *topological system morphism* (also called a *continuous map*) $(X_1, \kappa_1, A_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, A_2)$ comprises a set map $X_1 \xrightarrow{f} X_2$ and a frame homomorphism $A_2 \xrightarrow{\varphi^{op}} A_1$, making the following diagram commute

$$
\begin{array}{ccc}
A_2 & \xrightarrow{\;\; \varphi^{op} \;\;} & A_1 \\
{\scriptstyle \kappa_2}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\
\|\mathcal{P}(X_2)\| & \xrightarrow{\;\; f^{\leftarrow} \;\;} & \|\mathcal{P}(X_1)\|.
\end{array}
$$

It is easy to see that the translation of the above diagram into the language of the category **Loc** (namely, reversing all the involved arrows) provides an analogue of the diagram of Definition 5.1.

(2) The category $\mathbb{L}\mathbf{TopSys}((\mathcal{R}_3, \mathbf{Frm}, \mathbf{S}_2^{\mathbf{CSLat}(\vee)}))$ is isomorphic to the category **Loc-TopSys** of lattice-valued topological systems of J. T. Denniston *et al.* [22, 19, 20].

(3) The category $\mathbb{L}\mathbf{TopSys}((\mathcal{S}_{\mathbf{Set}}^{\mathbf{S}_K}, \mathbf{Set}, \mathbf{S}_2^{\mathbf{CSLat}(\vee)}))$ is isomorphic to the category $\mathsf{Chu}(\mathbf{Set}, K)$ of *Chu spaces* over a set K of V. Pratt [79] (Chu spaces are called *contexts* in *formal concept analysis* [39]).

(4) The category $\mathsf{Chu}(\mathbf{Set},\mathbf{2})$ provides the category **IntSys** of interchange systems of J. T. Denniston *et al.* [20, 21].

Some remarks are due to the term "$\mathbb{L}_I$-valued T_I-satisfaction relation" of Definition 5.1. In the classical setting of S. Vickers [106] (which is slightly different from the above-mentioned one), a topological system is a triple $(X, A, \models)$, where X is a set, A is a frame and $\models\, \subseteq X \times A$ is a relation called *satisfaction relation* on (X, A), which fulfills certain conditions. In [106, Section 5.6], S. Vickers introduced an alternative definition of systems through the frame map $A \xrightarrow{\kappa} \mathcal{P}(X)$ given by $\kappa(a) = \{x \in X \mid x \models a\}$. The definition was extended to the powerset theory $\mathcal{R}_3$ in J. T. Denniston *et al.* [19] and was taken up by us in [101] and in this manuscript as being more suitable for the catalg developments.

It was already said that the catalg approach to topological systems incorporates state property systems of D. Aerts [3]. The following definition extends the approach to the lattice-valued setting.

Definition 5.3. $\mathbb{L}_I\mathbf{CSP}(T_I)$ is the full subcategory of $\mathbb{L}_I\mathbf{CTopSys}(T_I)$, comprising all $\mathbb{L}_I$-valued T_I-systems $(X, (\kappa_i)_{i \in I}, ((A_i, \mu_i, L_i))_{i \in I})$ ($\mathbb{L}_I$-*valued composite variety-based state property systems* or $\mathbb{L}_I$-*valued T_I-sp-systems*) such that the homomorphism $A_i \xrightarrow{\kappa_i^{op}} T_i(X)$ is injective for every $i \in I$.

For the sake of simplicity, $\mathbb{L}\mathbf{CSP}(T)$ is denoted $\mathbb{L}\mathbf{SP}(T)$. The main reason for introducing the new definition was the desire to obtain another fruitful example for the new theory, which resulted in an additional representation of the concept of lattice-valued closure space [100].

Example 5.4. The category $\mathbb{L}_I\mathbf{SP}((\mathcal{P}, \mathbf{CSL}, \mathbf{S}_2^{\mathbf{CSLat}(\vee)}))$ is isomorphic to the category **SP** of state property systems of D. Aerts [3].

5.1. **Lattice-valued Categorically-algebraic Soft Topology.** The theory of catalg topology has another and (probably) more interesting application, motivated by the concept of *soft set*. Introduced by D. Molodtsov [71], to reduce the difficulties experienced by the existing mathematical approaches to uncertainty, the new theory almost immediately started the process of softening of mathematics, which included the construction of soft analogues of various mathematical structures. In due time, the notions of *soft group* [6], *soft ring* [65], *soft semiring* [32], *soft BCK/BCI-algebra* [57], *etc.* appeared in the literature, extending basic properties of the

classical concepts. The approach can be easily generalized to the notion of *soft algebra* as follows (see [104] for the detailed study of the topic).

Definition 5.5. Let $\mathbf{A}$ ba a variety, let A be an $\mathbf{A}$-algebra and let X be a set. A *soft* ($\mathbf{A}$-)*algebra* over A is a triple $(A, \Vdash, X)$, where $X \xrightarrow{\Vdash} \mathcal{P}(A)$ is a map such that $\Vdash(x)$ is a subalgebra of A for every $x \in X$.

A topologically-minded reader will ask immediately about the theory of *soft topology*. It appears that there exists a straightforward way to apply Definition 5.5 and get the notion of *soft topological space*.

Definition 5.6. Let (X, τ) be a topological space and let L be a frame. A *soft topological space* over L is a triple $((X, \tau), \Vdash, L)$, where $L \xrightarrow{\Vdash} \tau$ is a frame homomorphism.

The reader could see that Definition 5.6 is a particular case of *L-fuzzy locales* of [108, Definition 1.1]. More precisely, while D. Zhang and Y.-M. Liu build their concept over an arbitrary frame A (getting thus a frame homomorphism $L \xrightarrow{\Vdash} A$), we use the frames, which are given by the topologies of topological spaces.

Notice that unlike Definition 5.5, we use a frame L instead of another set, thereby simplifying the definition. The reason for the choice is our wish to incorporate the lattice-valued developments of this paper into soft topology, that is done explicitly in Example 5.8. To achieve the goal, we use the extended notion of topological system to develop the concept of Definition 5.6 in a more general way, e.g., replacing topological spaces and frames with their lattice-valued catalg analogues.

Definition 5.7. Let $(T_I, \mathbb{L}_I)$ be an $\mathbb{L}_I$-valued cvt-theory in a category $\mathbf{X}$. Then $\mathbb{L}_I\mathbf{CSoftTop}(T_I)$ is the category, concrete over $\mathbb{L}_I\mathbf{CTop}(T_I) \times (\prod_{i \in I}(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op})$, which is given by the following data.

  OBJECTS: (*soft $\mathbb{L}_I$-valued T_I-spaces* or *s-$\mathbb{L}_I$-valued T_I-spaces*) are those triples $((X, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}), ((\kappa_i, \varphi_i))_{i\in I}, ((B_i, \nu_i, M_i))_{i\in I})$, for which $(X, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I})$ is an $\mathbb{L}_I$-valued T_I-space, $((B_i, \nu_i, M_i))_{i\in I}$ is an object of the product category $\prod_{i \in I}(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op}$, and $(T_i(X), \mathfrak{T}_i, L_i) \xrightarrow{(\kappa_i, \varphi_i)} (B_i, \nu_i, M_i)$ is an $(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op}$-morphism for every $i \in I$ ($((\kappa_i, \varphi_i))_{i\in I}$ is called *soft $\mathbb{L}_I$-valued T_I-topology* or *s-$\mathbb{L}_I$-valued T_I-topology* on $((X, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}), ((B_i, \nu_i, M_i))_{i\in I})$).

  MORPHISMS:

$$((X, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}), ((\kappa_i, \varphi_i))_{i\in I}, ((B_i, \nu_i, M_i))_{i\in I}) \xrightarrow{((f, (\phi_i)_{i\in I}), ((\xi_i, o_i))_{i\in I})}$$
$$((Y, (\mathcal{S}_i)_{i\in I}, (N_i)_{i\in I}), ((\iota_i, \psi_i))_{i\in I}, ((C_i, \sigma_i, O_i))_{i\in I})$$

  are $\mathbb{L}_I\mathbf{CTop}(T_I) \times (\prod_{i\in I}(\mathbf{S}_i^{op}\text{-}\mathbf{B}_i)^{op})$-morphisms

$$((X, (\mathfrak{T}_i)_{i\in I}, (L_i)_{i\in I}), ((B_i, \nu_i, M_i))_{i\in I}) \xrightarrow{((f, (\phi_i)_{i\in I}), ((\xi_i, o_i))_{i\in I})}$$
$$((Y, (\mathcal{S}_i)_{i\in I}, (N_i)_{i\in I}), ((C_i, \sigma_i, O_i))_{i\in I})$$

such that for every $i \in I$, the following diagram

$$
\begin{array}{ccc}
T_i(X) & \xrightarrow{\ T_i f\ } & T_i(Y) \\
\kappa_i \downarrow & & \downarrow \iota_i \\
B_i & \xrightarrow{\ \xi_i\ } & C_i
\end{array}
$$

commutes (*soft $\mathbb{L}_I$-valued T_I-continuity* or *s-$\mathbb{L}_I$-valued T_I-continuity*).

The reader should be aware that M. Shabir and M. Naz [93] have already initiated the theory of soft topology in a quite obvious way: a soft topology on a set is a family of soft sets over it, which is closed under finite intersections and arbitrary unions. Moreover, F.-G. Shi and B. Pang [94] have recently shown redundancy of (even fuzzy) soft topological spaces. Our approach to soft topology though is more sophisticated, and it is one of the aims of this paper to start the development of the new theory. The next example shows that the new framework incorporates the lattice-valued categorically-algebraic topology of Definition 3.4 (an attentive reader will easily notice that we just take $L = \tau$ in Definition 5.6).

Example 5.8. The (non-full) subcategory **FT** of the category $\mathbb{L}_I\mathbf{CSoftTop}(T_I)$, which comprises all objects $((X, (\mathfrak{T}_i)_{i \in I}, (L_i)_{i \in I}), ((1_{T_i(X)}, 1_{L_i}))_{i \in I}, ((T_i(X), \mathfrak{T}_i, L_i))_{i \in I})$ together with all morphisms

$$
((X, (\mathfrak{T}_i)_{i \in I}, (L_i)_{i \in I}), ((1_{T_i(X)}, 1_{L_i}))_{i \in I},
$$
$$
((T_i(X), \mathfrak{T}_i, L_i))_{i \in I}) \xrightarrow{((f, (\phi_i)_{i \in I}), ((T_i f, \phi_i))_{i \in I})} ((Y, (\mathcal{S}_i)_{i \in I}, (M_i)_{i \in I}),
$$
$$
((1_{T_i(Y)}, 1_{M_i}))_{i \in I}, ((T_i(Y), \mathcal{S}_i, M_i))_{i \in I}),
$$

is isomorphic to the category $\mathbb{L}_I\mathbf{CTop}(T_I)$.

On the other hand, we are still unable to provide a similar incorporation of the extended framework of C. Guido presented in Definition 3.10. This motivates the second open problem of this paper.

Problem 5.9. Is it possible to include the framework of Definition 3.10 into the soft topology?

To convince the reader even more of the usefulness of the catalg topology, in the next section, we are going to provide a relation between $\mathbb{L}_I$-valued T_I-spaces and $\mathbb{L}_I$-valued T_I-systems.

6. Topological Systems Versus Topological Spaces

At the beginning of the previous section, we have mentioned that it is not the notion of topological system itself that provides the main significance of their theory, but the possibility of embedding both the category of topological spaces and the category of their underlying algebraic structures (locales) into the category of topological systems. Moreover, each embedding essentially provides a (co)reflective subcategory. It is the purpose of the current section to extend half of the result to

the lattice-valued catalg setting, i.e., to obtain the respective lattice-valued catalg spatialization procedure and its right adjoint functor.

We begin with the case of the singleton topological theories, extending the achievements later on to the more sophisticated composite setting.

Proposition 6.1. *There is a full embedding* $\mathbb{L}\mathbf{Top}(T) \overset{G}{\hookrightarrow} \mathbb{L}\mathbf{TopSys}(T),$ *given by* $G((X_1, \mathfrak{T}_1, L_1) \xrightarrow{(f,\psi)} (X_2, \mathfrak{T}_2, L_2)) = (X_1, 1_{T(X_1)}, (T(X_1), \mathfrak{T}_1, L_1)) \xrightarrow{(f,(Tf,\psi))} (X_2, 1_{T(X_2)}, (T(X_2), \mathfrak{T}_2, L_2)).$

Proof. The only (not so serious) challenge is verification of fullness. Given an $\mathbb{L}$-valued T-continuous morphism $G(X_1, \mathfrak{T}_1, L_1) \xrightarrow{(f,(\varphi,\psi))} G(X_2, \mathfrak{T}_2, L_2)$, commutativity of the diagram

$$
\begin{array}{ccc}
T(X_1) & \xrightarrow{\ Tf\ } & T(X_2) \\
{\scriptstyle 1_{T(X_1)}}\downarrow & & \downarrow{\scriptstyle 1_{T(X_2)}} \\
T(X_1) & \xrightarrow{\ \varphi\ } & T(X_2)
\end{array}
$$

implies $Tf = \varphi$, and that was to be shown. $\qquad\square$

The reader should be aware that Proposition 6.1 generalizes the respective result of [101] and the embedding of [19, Theorem 64]. Moreover, the embedding in question is not concrete, since the employed categories have different ground categories, namely, $\mathbf{X} \times \mathbb{L}$ for $\mathbb{L}\mathbf{Top}(T)$ and $\mathbf{X} \times (\mathbb{L}^{op}\text{-}\mathbf{B})^{op}$ for $\mathbb{L}\mathbf{TopSys}(T)$.

The next step is more complicated. Firstly, recall from [86] that given a set map $X \xrightarrow{f} Y$ and a $\bigvee$-semilattice L, there is the L-image operator $L^X \xrightarrow{f_L^{\rightarrow}} L^Y$, $(f_L^{\rightarrow}(\alpha))(y) = \bigvee_{f(x)=y} \alpha(x)$. Secondly, recall from [40, Definition I-2.8] that a lattice L is called *completely distributive* provided that it is complete and for every family $\{a_{j,k} \mid j \in J, k \in K(j)\}$ in L, the identity $\bigwedge_{j \in J} \bigvee_{k \in K(j)} a_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} a_{j,f(j)}$ holds, where M is the set of choice maps defined on J with $f(j) \in K(j)$.

Proposition 6.2. *If the underlying lattices of $\mathbb{L}$ are completely distributive, then there is a functor* $\mathbb{L}\mathbf{TopSys}(T) \xrightarrow{\mathrm{Spat}} \mathbb{L}\mathbf{Top}(T), \mathrm{Spat}((X_1, \kappa_1, (B_1, \mu_1, L_1)) \xrightarrow{(f,(\varphi,\psi))} (X_2, \kappa_2, (B_2, \mu_2, L_2))) = (X_1, (\kappa_1^{op})_{L_1}^{\rightarrow}(\mu_1), L_1) \xrightarrow{(f,\psi)} (X_2, (\kappa_2^{op})_{L_2}^{\rightarrow}(\mu_2), L_2).$

Proof. It will be enough to verify correctness of the functor on both objects and morphisms. To see that $(X_i, (\kappa_i^{op})_{L_i}^{\rightarrow}(\mu_i), L_i)$ is an $\mathbb{L}$-valued T-system notice that given $\lambda \in \Lambda_{\mathbf{B}}$ and $b_j \in T(X_i)$ for $j \in n_\lambda$,

$$
\bigwedge_{j \in n_\lambda} ((\kappa_i^{op})_{L_i}^{\rightarrow}(\mu_i))(b_j) = \bigwedge_{j \in n_\lambda} \bigvee_{\kappa_i^{op}(a_i)=b_j} \mu_i(a_i) \overset{(\dagger)}{=} \bigvee_{f \in M} \bigwedge_{j \in n_\lambda} \mu_i(f(j)) \leqslant
$$

$$
\bigvee_{f \in M} \mu_i(\omega_\lambda^{B_i}(\langle f(j) \rangle_{n_\lambda})) \leqslant \bigvee_{f \in M} ((\kappa_i^{op})_{L_i}^{\rightarrow}(\mu_i))(\kappa_i^{op}(\omega_\lambda^{B_i}(\langle f(j) \rangle_{n_\lambda}))) =
$$

$$
\bigvee_{f \in M} ((\kappa_i^{op})_{L_i}^{\rightarrow}(\mu_i))(\omega_\lambda^{T(X_i)}(\langle \kappa_i^{op} \circ f(j) \rangle_{n_\lambda})) = \bigvee_{f \in M} ((\kappa_i^{op})_{L_i}^{\rightarrow}(\mu_i))(\omega_\lambda^{T(X_i)}(\langle b_j \rangle_{n_\lambda})) \leqslant
$$

$$
((\kappa_i^{op})_{L_i}^{\rightarrow}(\mu_i))(\omega_\lambda^{T(X_i)}(\langle b_j \rangle_{n_\lambda})),
$$

where (†) uses the assumption on complete distributivity.

To check correctness on morphisms, we show laxity of the following diagram

$$
\begin{array}{ccc}
T(X_2) & \xrightarrow{\;(Tf)^{op}\;} & T(X_1) \\
{\scriptstyle (\kappa_2^{op})_{L_2}^{\rightarrow}(\mu_2)}\Big\downarrow & \leqslant & \Big\downarrow{\scriptstyle (\kappa_1^{op})_{L_1}^{\rightarrow}(\mu_1)} \\
\|L_2\| & \xrightarrow[\;\|\psi^{op}\|\;]{} & \|L_1\|.
\end{array}
$$

Given $b_2 \in T(X_2)$, we get $(\|\psi^{op}\| \circ (\kappa_2^{op})_{L_2}^{\rightarrow}(\mu_2))(b_2) = \|\psi^{op}\|(\bigvee_{\kappa_2^{op}(a_2)=b_2} \mu_2(a_2)) = \bigvee_{\kappa_2^{op}(a_2)=b_2} \|\psi^{op}\| \circ \mu_2(a_2) \overset{(\dagger\dagger)}{\leqslant} \bigvee_{\kappa_2^{op}(a_2)=b_2} \mu_1 \circ \varphi^{op}(a_2) = \bigvee S_2$, where (††) uses the fact that $\|\psi^{op}\| \circ \mu_2 \leqslant \mu_1 \circ \varphi^{op}$. On the other hand, $(\kappa_1^{op})_{L_1}^{\rightarrow}(\mu_1) \circ (Tf)^{op}(b_2) = \bigvee_{\kappa_1^{op}(a_1)=(Tf)^{op}(b_2)} \mu_1(a_1) = \bigvee S_1$. Lastly, given $a_2 \in S_2$, $\kappa_2^{op}(a_2) = b_2$ implies $(Tf)^{op}(b_2) = (Tf)^{op} \circ \kappa_2^{op}(a_2) = \kappa_1^{op} \circ \varphi^{op}(a_2)$ and, therefore, $\varphi^{op}(a_2) \in S_1$. It follows that $S_2 \subseteq S_1$, which implies $\bigvee S_2 \leqslant \bigvee S_1$, and that was to be shown. $\qquad\square$

After two preliminary propositions, the main result of this section can be stated as the following theorem.

Theorem 6.3. *The functor* Spat *is a right-adjoint-left-inverse to the functor G.*

Proof. For the first claim, we show that every $\mathbb{L}$-valued T-system $(X, \kappa, (B, \mu, L))$ has a G-co-universal arrow, i.e., an $\mathbb{L}$-valued T-continuous morphism $G\,\mathrm{Spat}(X, \kappa, (B, \mu, L)) \xrightarrow{\;\varepsilon\;} (X, \kappa, (B, \mu, L))$ such that for every $\mathbb{L}$-valued T-continuous morphism $G(Y, \mathcal{S}, M) \xrightarrow{\;(f,(\varphi,\psi))\;} (X, \kappa, (B, \mu, L))$, there is a unique $\mathbb{L}$-valued T-continuous morphism $(Y, \mathcal{S}, M) \xrightarrow{\;(g,\phi)\;} \mathrm{Spat}(X, \kappa, (B, \mu, L))$ making the following triangle commute:

$$
\begin{array}{ccc}
G(Y, \mathcal{S}, M) & & \\
{\scriptstyle G(g,\phi)}\Big\downarrow & \searrow^{(f,(\varphi,\psi))} & \\
G\,\mathrm{Spat}(X, \kappa, (B, \mu, L)) & \xrightarrow[\;\varepsilon\;]{} & (X, \kappa, (B, \mu, L)).
\end{array}
$$

In the first step, define the required morphism ε by $G\,\mathrm{Spat}(X, \kappa, (B, \mu, L)) \xrightarrow{\;\varepsilon\;} (X, \kappa, (B, \mu, L)) = (X, 1_{T(X)}, (T(X), (\kappa^{op})_L^{\rightarrow}(\mu), L)) \xrightarrow{\;(1_X, (\kappa, 1_L))\;} (X, \kappa, (B, \mu, L))$. To show that ε is $\mathbb{L}$-valued T-continuous, notice that commutativity of the diagram

$$
\begin{array}{ccc}
T(X) & \xrightarrow{\;1_{T(X)}\;} & T(X) \\
{\scriptstyle 1_{T(X)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \kappa} \\
T(X) & \xrightarrow[\;\kappa\;]{} & B
\end{array}
$$

leaves the laxity of the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;\kappa^{op}\;} & T(X) \\
{\scriptstyle \mu}\Big\downarrow & \leqslant & \Big\downarrow{\scriptstyle (\kappa^{op})_L^{\rightarrow}(\mu)} \\
\|L\| & \xrightarrow[\;\|1_L\|\;]{} & \|L\|.
\end{array}
$$

as the only point unverified. The desired result how ever is the consequence of the fact that given $a \in A$, $((\kappa^{op})_L^{\rightarrow}(\mu))(\kappa^{op}(a)) = \bigvee_{\kappa^{op}(b)=\kappa^{op}(a)} \mu(b) \geqslant \mu(a) = \mu \circ \|1_L\|(a)$.

In the second step, we consider an $\mathbb{L}$-valued T-continuous morphism $(Y, 1_{T(Y)},$ $(T(Y), \mathcal{S}, M)) \xrightarrow{(f,(\varphi,\psi))} (X, \kappa, (B, \mu, L))$ and define $(Y, \mathcal{S}, M) \xrightarrow{(g,\phi)} \mathrm{Spat}(X, \kappa, (B,$ $\mu, L)) = (Y, \mathcal{S}, M) \xrightarrow{(f,\psi)} (X, (\kappa^{op})_L^{\rightarrow}(\mu), L)$. To show that the morphism is $\mathbb{L}$-valued T-continuous, we notice that given $b \in T(X)$, it follows that $\|\psi^{op}\| \circ$ $((\kappa^{op})_L^{\rightarrow}(\mu))(b) = \|\psi^{op}\|(\bigvee_{\kappa^{op}(a)=b} \mu(a)) = \bigvee_{\kappa^{op}(a)=b} \|\psi^{op}\| \circ \mu(a) \overset{(\dagger)}{\leqslant} \bigvee_{\kappa^{op}(a)=b} \mathcal{T} \circ$ $\varphi^{op}(a) \overset{(\dagger\dagger)}{=} \bigvee_{\kappa^{op}(a)=b} \mathcal{T} \circ (Tf)^{op} \circ \kappa^{op}(a) = \bigvee_{\kappa^{op}(a)=b} \mathcal{T} \circ (Tf)^{op}(b) \leqslant \mathcal{T} \circ (Tf)^{op}(b)$, where $(\dagger)$ uses the fact that $(B, \mu, L) \xrightarrow{(\varphi^{op}, \psi^{op})} (T(Y), \mathcal{S}, M)$ is a $(\mathbf{B}, \mathbf{S}^{op})$-homomorphism and $(\dagger\dagger)$ relies on commutativity of the diagram in Definition 5.1.

In the third step, we show commutativity of the above-mentioned triangle. Verification is simple, since $\varepsilon \circ G(g, \phi) = (1_X, (\kappa, 1_L)) \circ (f, (Tf, \psi)) = (f, (\kappa \circ Tf, \psi)) \overset{(\dagger\dagger\dagger)}{=}$ $(f, (\varphi, \psi))$, where $(\dagger\dagger\dagger)$ employs the fact that $(f, (\varphi, \psi))$ is $\mathbb{L}$-valued T-continuous.

In the forth (and the last) step, we show uniqueness of (g, ϕ). Suppose there exists another $\mathbb{L}$-valued T-continuous morphism (g', ϕ'), making the above-mentioned triangle commute. It follows that $(f, (\varphi, \psi)) = \varepsilon \circ G(g', \phi') = (1_X, (\kappa, 1_L)) \circ (g', (Tg', \phi')) = (g', (\kappa \circ Tg', \phi'))$ implying $(g', \phi') = (f, \psi)$.

The only challenge of the last claim on $\mathrm{Spat}\, G = 1_{\mathbb{L}\mathbf{Top}(T)}$ is the identity on objects and that follows from the fact that given an $\mathbb{L}$-valued T-space $(X, \mathcal{T}, L)$, $\mathrm{Spat}\, G(X, \mathcal{T}, L) = (X, ((1_{T(X)})_L^{\rightarrow}(\mathcal{T})), L) = (X, \mathcal{T}, L)$. $\square$

Corollary 6.4. *The category $\mathbb{L}\mathbf{Top}(T)$ is isomorphic to a full coreflective subcategory of the category $\mathbb{L}\mathbf{TopSys}(T)$.*

The reader should pay attention to the important fact that our previous approaches of [98, 101] allowed to get a stronger result in the sense that the category of topological spaces appeared to be a (regular mono)-coreflective subcategory of the category of topological systems. The current extended framework does not have the property, since, in general, $\mathbb{L}$-valued T-satisfaction relation κ on $(X, (B, \mu, L))$ need not provide a surjective homomorphism $B \xrightarrow{\kappa^{op}} T(X)$ and, therefore, ε in Theorem 6.3 need not be a monomorphism. Turning back to the composite setting, we can formulate the concluding result of this section.

Theorem 6.5.

(1) *There exists a full embedding* $\mathbb{L}_I\mathbf{CTop}(T_I) \overset{G_I}{\hookrightarrow} \mathbb{L}_I\mathbf{CTopSys}(T_I)$ *defined by the formula*

$$G((X, (\mathcal{T}_i)_{i \in I}, (L_i)_{i \in I}) \xrightarrow{(f,(\psi_i)_{i \in I})} (Y, (\mathcal{S}_i)_{i \in I}, (M_i)_{i \in I})) =$$
$$(X, (1_{T(X)})_{i \in I}, ((T(X), \mathcal{T}_i, L_i))_{i \in I}) \xrightarrow{(f,((Tf,\psi_i)_{i \in I})}$$
$$(Y, (1_{T(Y)})_{i \in I}, ((T(Y), \mathcal{S}_i, M_i))_{i \in I}).$$

(2) *Suppose that the underlying lattices of $\mathbb{L}_I$ are completely distributive. Then there exists a functor $\mathbb{L}_I\mathbf{CTopSys}(T_I) \xrightarrow{\mathrm{Spat}_I} \mathbb{L}_I\mathbf{CTop}(T_I)$ defined by*

$$\mathrm{Spat}_I((X, (\kappa_i)_{i\in I}, ((B_i, \mu_i, L_i))_{i\in I}) \xrightarrow{(f,((\varphi_i,\psi_i))_{i\in I})} (Y, (\iota_i)_{i\in I}, ((C_i, \nu_i, M_i))_{i\in I})) =$$

$$(X, ((\kappa_i^{op})_{L_i}^{\rightarrow}(\mu_i))_{i\in I}, (L_i)_{i\in I}) \xrightarrow{(f,\psi)} (Y, ((\iota_i^{op})_{M_i}^{\rightarrow}(\nu_i))_{i\in I}, (M_i)_{i\in I}).$$

(3) *The functor Spat_I is a right-adjoint-left-inverse to the functor G_I.*

(4) *The category $\mathbb{L}_I\mathbf{CTop}(T_I)$ is isomorphic to a full coreflective subcategory of the category $\mathbb{L}_I\mathbf{CTopSys}(T_I)$.*

Theorem 6.5 is a lattice-valued catalg analogue of the spatialization procedure of S. Vickers [106, Theorem 5.3.4], restoring a significant part of the classical setting.

7. **Conclusion**

Motivated by the diversity of approaches to many-valued topology currently available in the literature, in this paper, we introduced a new topological framework, which incorporated in itself the most important many-valued topological settings (including poslat topology of S. E. Rodabaugh, (L, M)-fuzzy topology of T. Kubiak and A. Šostak, and M-fuzzy topological L-fuzzy spaces of C. Guido) as well as the classical theory of closure spaces. As a consequence, it appeared that the framework of S. E. Rodabaugh actually does not differ much from the classical crisp approach, in the sense that his categorically-algebraic machinery follows the path of the standard crisp one. On the other hand, the theories of T. Kubiak, A. Šostak and C. Guido deviate significantly from the classical setting, providing an inherently many-valued framework. Based on these observations, we proposed to call their settings *truly lattice-valued topology*, reserving the term *lattice-valued topology* for the approach of S. E. Rodabaugh. As the main achievement, we showed that all the categories of topological structures generated by truly lattice-valued topology are topological over their ground categories, thereby incorporating both the classical and many-valued results on the topic.

The new framework was also relevant to pointfree topology through a generalization of topological systems of S. Vickers. In particular, we have extended his system spatialization procedure, which made a topological space from a topological system. Such an extension provides an important step in the theory of generalized topological systems, namely, it shows that the crucial property of the original concept (a convenient extension of topological spaces) still holds in the generalized setting. However, it is precisely the system framework of this paper, which shows many potential applications of their theory to non-topological areas (e.g., to the field of state property systems of D. Aerts *et al.* [4, 5]). There still exists another part of the classical theory though which is never touched in this paper, i.e., the localification procedure, which gives the machinery for converting a topological system into a locale. Thus, we can postulate the third open problem of this manuscript.

Problem 7.1. What will be the localification procedure for lattice-valued catalg topological systems?

The paper also provided the catalg approach to topology of C. Guido, which clearly appeared to extend the more known and already much cited topological setting of T. Kubiak and A. Šostak. The main drawback of the framework of C. Guido is its relative unpopularity among the researchers. It is one of the aims of the current paper, to promote the theory. In particular, we showed that the retractive topology of Definition 3.8, motivated by the above-mentioned setting of C. Guido, is one of the most general approaches to lattice-valued topology currently available in the fuzzy community. Moreover, its respective categories are topological over their ground categories (Corollary 4.6). In view of this result, the next problem (or, better, research proposal) springs into mind immediately.

Problem 7.2. Develop the theory of retractive lattice-valued topology up to the extent of its currently more popular poslat and (L, M)-fuzzy analogues. In particular, study the properties of the category $\mathbb{L}_I\mathbf{CRTop}(T_I)$ of $\mathbb{L}_I$-valued T_I-retractive spaces, which is introduced in Definition 3.10.

The paper has also introduced an alternative approach to soft topology motivated by the well-known concept of soft set of D. Molodtsov, thereby starting a completely new approach to topological structures. This induces the next, certainly non-trivial and, moreover, far-reaching, problem.

Problem 7.3. Develop the theory of soft topology. In particular, explore the properties of the category $\mathbb{L}_I\mathbf{CSoftTop}(T_I)$ of Definition 5.7.

The above concept of soft topology was motivated by the notion of soft algebra of Definition 5.5. On the other hand, Definition 3.1 introduced the concept of lattice-valued algebra. Comparing the two notions, the categorically-minded reader will miss soft algebra homomorphisms. The next definition fills the gap. For the sake of convenience, we reverse the notation for soft algebras from $(A, \Vdash, X)$ to $(X, \Vdash, A)$.

Definition 7.4. Let $\mathbf{A}$ ba a variety and let $(X_1, \Vdash_1, A_1)$, $(X_2, \Vdash_2, A_2)$ be soft $\mathbf{A}$-algebras. A *soft* ($\mathbf{A}$-)*algebra homomorphism* $(X_1, \Vdash_1, A_1) \xrightarrow{(f, \varphi)} (X_2, \Vdash_2, A_2)$ is a $\mathbf{Set} \times \mathbf{A}$-morphism $(X_1, A_1) \xrightarrow{(f, \varphi)} (X_2, A_2)$, which satisfies the following lax diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f} & X_2 \\
{\scriptstyle\Vdash_1}\downarrow & \leqslant & \downarrow{\scriptstyle\Vdash_2} \\
\mathcal{P}(A_1) & \xrightarrow[\varphi^{\rightarrow}]{} & \mathcal{P}(A_2),
\end{array}
$$

meaning $\varphi^{\rightarrow} \circ \Vdash(x) \subseteq \Vdash_2 \circ f(x)$ for every $x \in X_1$. $\mathbf{SoftA}$ is the category of soft $\mathbf{A}$-algebras and soft $\mathbf{A}$-algebra homomorphisms, concrete over the category $\mathbf{Set} \times \mathbf{A}$.

The reader should notice that the inducing concepts of soft group, soft ring, soft semiring, *etc.* use commutativity of the diagram of Definition 7.4. Our main motivation for the extension of the notion of homomorphism is the following easy but rather important result.

Lemma 7.5. *Given a variety* $\mathbf{A}$ *and an extension* $\mathbf{L}$ *of the variety* $\mathbf{CSLat}(\bigvee)$*, every subcategory* $\mathbf{S}$ *of* $\mathbf{L}$ *gives the functor* $\mathbf{S}\text{-}\mathbf{A} \xrightarrow{F} \mathbf{SoftA}$ *with* $F((A_1, \mu_1, L_1) \xrightarrow{(\varphi, \psi)} (A_2, \mu_2, L_2)) = (|L_1|, \Vvdash_1, A_1) \xrightarrow{(|\psi|, \varphi)} (|L_2|, \Vvdash_2, A_2)$*, where* $|L_i| \xrightarrow{\Vvdash_i} \mathcal{P}(A_i)$*,* $\Vvdash_i(b) = \{a \in A_i \,|\, b \leqslant \mu_i(a)\}$*. The functor* F *is faithful, but not dense. If the underlying functor of* $\mathbf{L}$ *is an embedding, then* F *is an embedding.*

Proof. Correctness of the functor on objects follows from Definition 3.1. Correctness on morphisms is the consequence of the fact that given $b_1 \in L_1$ and $a_2 \in \varphi^{\to} \circ \Vvdash_1(b_1)$, $a_2 = \varphi(a_1)$ for some $a_1 \in \Vvdash_1(b_1)$ and thus, $b_1 \leqslant \mu_1(a_1)$. It follows that $\psi(b_1) \leqslant \psi \circ \mu_1(a_1) \leqslant \mu_2 \circ \varphi(a_1)$, implying $a_2 = \varphi(a_1) \in \Vvdash_2 \circ \psi(b_1)$.

The functor F is clearly faithful. The non-density claim is the consequence of the fact that given an algebra A, the soft algebra $(\varnothing, \Vvdash, A)$ is isomorphic to no object from the image of F.

For the embedding property, notice that given two $(\mathbf{A}, \mathbf{S})$-algebras (A_1, μ_1, L_1) and (A_2, μ_2, L_2) such that $F(A_1, \mu_1, L_1) = F(A_2, \mu_2, L_2)$, it follows that $A_1 = A = A_2$, $|L_1| = X = |L_2|$ and $\Vvdash_1 = \Vvdash = \Vvdash_2$. In general, one cannot proceed farther, but the condition of the lemma gives $L_1 = L = L_2$. Then $\mu_1(a) = \bigvee\{b \in L \,|\, a \in \Vvdash(b)\} = \mu_2(a)$ (use the representation theorem for fuzzy sets of, e.g., [76]). $\square$

Notice that the requirement on the variety $\mathbf{L}$ of Lemma 7.5 almost never holds, since essentially it replaces the category $\mathbf{L}$ with a subcategory of the category $\mathbf{Set}$, i.e., makes one to deal with sets instead of algebras. Still, it follows from the lemma that the theory of soft algebras provides a certain extension of the theory of lattice-valued algebras, motivating the last open problem of this paper.

Problem 7.6. What are the properties of the functor of Lemma 7.5? In particular, is it (co)adjoint?

All the open problems posed in this manuscript will be addressed in our further research on the topic.

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and concrete categories: the joy of cats*, Dover Publications (Mineola, New York), 2009.

[2] J. Adámek, J. Rosický, and E. M. Vitale, *Algebraic theories. a categorical introduction to general algebra*, Cambridge University Press, 2011.

[3] D. Aerts, *Foundations of quantum physics: a general realistic and operational approach*, Int. J. Theor. Phys., **38(1)** (1999), 289-358.

[4] D. Aerts, E. Colebunders, A. van der Voorde and B. van Steirteghem, *State property systems and closure spaces: a study of categorical equivalence*, Int. J. Theor. Phys., **38(1)** (1999), 359-385.

[5] D. Aerts, E. Colebunders, A. van der Voorde and B. van Steirteghem, *On the amnestic modification of the category of state property systems*, Appl. Categ. Struct., **10(5)** (2002), 469-480.

[6] H. Aktaş and N. Çağman, *Soft sets and soft groups*, Inf. Sci., **177(13)** (2007), 2726-2735.

[7] J. M. Anthony and H. Sherwood, *Fuzzy groups redefined*, J. Math. Anal. Appl., **69(1)** (1979), 124-130.

[8] B. Banaschewski and E. Nelson, *Tensor products and bimorphisms*, Canad. Math. Bull., **19(4)** (1976), 385-402.

[9] M. Barr and C. Wells, *Toposes, triples and theories*, Repr. Theory Appl. Categ., **2005(12)** (2005), 1-288.

[10] G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Phil. Soc., **31(4)** (1935), 433-454.

[11] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics, Springer-Verlag, **78** (1981).

[12] M. Caldas, S. Jafari and R. K. Saraf, *Semi-θ-open sets and new classes of maps*, Bull. Iran. Math. Soc., **31(2)** (2005), 37-52.

[13] N. Çağman, S. Karataş and S. Enginoglu, *Soft topology*, Comput. Math. Appl., **62(1)** (2011), 351-358.

[14] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24(1)** (1968), 182-190.

[15] M. M. Clementino, D. Hofmann and W. Tholen, *One setting for all: Metric, topology, uniformity, approach structure*, Appl. Categ. Struct., **12(2)** (2004), 127-154.

[16] P. M. Cohn, *Universal Algebra*, D. Reidel Publ. Comp., 1981.

[17] C. De Mitri and C. Guido, *Some remarks on fuzzy powerset operators*, Fuzzy Sets Syst., **126(2)** (2002), 241-251.

[18] M. Demirci, *Pointed semi-quantales and lattice-valued topological spaces*, Fuzzy Sets Syst., **161(9)** (2010), 1224-1241.

[19] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Lattice-valued topological systems*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, (2009), 24-31.

[20] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Lattice-valued predicate transformers and interchange systems*, Abstracts of the 31st Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, (2010), 31-40.

[21] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Formal concept analysis and lattice-valued interchange systems*, Abstracts of the 32nd Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, (2011), 41-47.

[22] J. T. Denniston, A. Melton and S. E. Rodabaugh, *Interweaving algebra and topology: Lattice-valued topological systems*, Fuzzy Sets Syst., **192** (2012), 58-103.

[23] J. T. Denniston and S. E. Rodabaugh, *Functorial relationships between lattice-valued topology and topological systems*, Quaest. Math., **32(2)** (2009), 139-186.

[24] A. Di Nola and G. Gerla, *Lattice valued algebras*, Stochastica, **11(2-3)** (1987), 137-150.

[25] Y. Diers, *Categories of algebraic sets*, Appl. Categ. Struct., **4(2-3)** (1996), 329-341.

[26] Y. Diers, *Affine algebraic sets relative to an algebraic theory*, J. Geom., **65(1-2)** (1999), 54-76.

[27] Y. Diers, *Topological geometrical categories*, J. Pure Appl. Algebra, **168(2-3)** (2002), 177-187.

[28] D. Dikranjan, E. Giuli and A. Tozzi, *Topological categories and closure operators*, Quaest. Math., **11(3)** (1988), 323-337.

[29] C. Ehresmann, *Gattungen von lokalen Strukturen*, Jahresber. Dtsch. Math.-Ver., (German), **60(1)** (1957), 49-77.

[30] P. Eklund, *Categorical fuzzy topology*, Ph.D. thesis, Åbo Akademi, 1986.

[31] P. Eklund, M. A. Galán and W. Gähler, *Partially ordered monads for monadic topologies, rough sets and Kleene algebras*, Electron. Notes Theor. Comput. Sci., **225** (2009), 67-81.

[32] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings*, Comput. Math. Appl., **56(10)** (2008), 2621-2628.

[33] A. Frascella, *Attachment and topological systems in varieties of algebras*, Ph.D. thesis, Department of Mathematics "Ennio De Giorgi", University of Salento, Italy, 2011.

[34] A. Frascella, C. Guido and S. Solovyov, *Dual attachment pairs in categorically-algebraic topology*, Appl. Gen. Topol., **12(2)** (2011), 101-134.

[35] W. Gähler, *Monadic topology – a new concept of generalized topology*, Recent Developments of General Topology and its Applications, International Conference in Memory of Felix Hausdorff (1868 - 1942) (W. Gähler, ed.), Akademie-Verlag, (1992), 136-149.

[36] W. Gähler, *The general fuzzy filter approach to fuzzy topology I*, Fuzzy Sets Syst., **76(2)** (1995), 205-224.

[37] W. Gähler, *The general fuzzy filter approach to fuzzy topology II*, Fuzzy Sets Syst., **76(2)** (1995), 225-246.

[38] W. Gähler, *General topology – the monadic case, examples, applications*, Acta Math. Hung., **88(4)** (2000), 279-290.

[39] B. Ganter and R. Wille, *Formale begriffsanalyse. mathematische grundlagen*, Berlin: Springer, 1996.

[40] G. Gierz, K. H. Hofmann and etc., *Continuous lattices and domains*, Cambridge University Press, 2003.

[41] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl., **18(1)** (1967), 145-174.

[42] J. A. Goguen, *The fuzzy Tychonoff theorem*, J. Math. Anal. Appl., **43(3)** (1973), 734-742.

[43] G. Grätzer, *Universal Algebra*, 2nd ed., Springer, 2008.

[44] C. Guido, *The subspace problem in the traditional point-set context of fuzzy topology*, Quaest. Math., **20(3)** (1997), 351-372.

[45] C. Guido, *Powerset operators based approach to fuzzy topologies on fuzzy sets*, Topological and Algebraic Structures in Fuzzy Sets. A Handbook of Recent Developments in the Mathematics of Fuzzy Sets (S. E. Rodabaugh and E. P. Klement, eds.), Kluwer Academic Publishers, (2003), 401-413.

[46] C. Guido, *Fuzzy points and attachment*, Fuzzy Sets Syst., **161(16)** (2010), 2150-2165.

[47] C. Guido and V. Scarciglia, *L-topological spaces as spaces of points*, Fuzzy Sets Syst., **173(1)** (2011), 45-59.

[48] C. Guido and S. Solovyov, *Topological systems versus attachment relations*, Quaest. Math., **37(4)** (2014), 455-484.

[49] H. Herrlich and G. E. Strecker, *Category theory*, 3rd ed., Sigma Series in Pure Mathematics, Heldermann Verlag, **1** (2007).

[50] U. Höhle, *Upper semicontinuous fuzzy sets and applications*, J. Math. Anal. Appl., **78(2)** (1980), 659-673.

[51] U. Höhle, *A note on the hypergraph functor*, Fuzzy Sets Syst., **131(3)** (2002), 353-356.

[52] U. Höhle and A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Kluwer Academic Publishers, (1999), 123-272.

[53] S. Hussain and B. Ahmad, *Some properties of soft topological spaces*, Comput. Math. Appl., **62(11)** (2011), 4058-4067.

[54] B. Hutton, *Products of fuzzy topological spaces*, Topology Appl., **11(1)** (1980), 59-67.

[55] J. R. Isbell, *Atomless parts of spaces*, Math. Scand., **31** (1972), 5-32.

[56] P. T. Johnstone, *Stone spaces*, Cambridge University Press, 1982.

[57] Y. B. Jun, *Soft BCK/BCI-algebras*, Comput. Math. Appl., **56(5)** (2008), 1408-1413.

[58] J. C. Kelly, *Bitopological spaces*, Proc. Lond. Math. Soc., **13(III)** (1963), 71-89.

[59] W. Kotzé and T. Kubiak, *Fuzzy topologies of Scott continuous functions and their relation to the hypergraph functor*, Quaest. Math., **15(2)** (1992), 175-187.

[60] T. Kubiak, *On fuzzy topologies*, Ph.D. thesis, Adam Mickiewicz University, Poznań, Poland, 1985.

[61] T. Kubiak and A. Šostak, *Foundations of the theory of (L, M)-fuzzy topological spaces*, Abstracts of the 30th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, (2009), 70-73.

[62] F. W. Lawvere, *Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories*, Repr. Theory Appl. Categ., **2004(5)** (2004), 1-121.

[63] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Am. Math. Mon., **70** (1963), 36-41.

[64] F. E. J. Linton, *Some aspects of equational categories*, Proc. Conf. Categor. Algebra, La Jolla, (1965), 84–94.

[65] X. Liu, D. Xiang, J. Zhan and K. P. Shum, *Isomorphism theorems for soft rings*, Algebra Colloq., **19(4)** (2012), 649-656.

[66] R. Lowen, *Fuzzy topological spaces and fuzzy compactness*, J. Math. Anal. Appl., **56(3)** (1976), 621-633.

[67] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer-Verlag, 1998.

[68] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Comput. Math. Appl., **45(4-5)** (2003), 555-562.

[69] E. G. Manes, *Algebraic theories*, Springer-Verlag, 1976.

[70] W. K. Min, *A note on soft topological spaces*, Comput. Math. Appl., **62(9)** (2011), 3524-3528.

[71] D. Molodtsov, *Soft set theory – first results*, Comput. Math. Appl., **37(4-5)** (1999), 19-31.

[72] J. N. Mordeson and D. S. Malik, *Fuzzy commutative algebra*, Singapore: World Scientific, 1998.

[73] C. J. Mulvey and J. W. Pelletier, *On the quantisation of points*, J. Pure Appl. Algebra, **159(2)** (2001), 231-295.

[74] C. J. Mulvey and J. W. Pelletier, *On the quantisation of spaces*, J. Pure Appl. Algebra, **175(1-3)** (2002), 289-325.

[75] N. Nakajima, *Generalized fuzzy sets*, Fuzzy Sets Syst., **32(3)** (1989), 307-314.

[76] C. V. Negoita and D. A. Ralescu, *Applications of fuzzy sets to systems analysis*, Interdisciplinary Systems Research, Birkhäuser Verlag, **11** (1975).

[77] D. Papert and S. Papert, *Sur les treillis des ouverts et les paratopologies*, Semin. de Topologie et de Geometrie differentielle Ch. Ehresmann 1 (1957/58), **1** (1959), 1-9.

[78] B. Pazar Varol, A. Shostak and H. Aygün, *Categories related to topology viewed as soft sets*, Proceedings of the 7th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT-2011) and LFA-2011, Atlantis Press, **1** (2011), 883-890.

[79] V. Pratt, *Chu spaces*, School on Category Theory and Applications (Coimbra, 1999), 39-100, Textos Mat. Sér. B, **21**, Univ. Coimbra, Coimbra, 1999.

[80] G. Richter, *Kategorielle Algebra*, Akademie-Verlag, 1979.

[81] S. E. Rodabaugh, *The Hausdorff separation axiom for fuzzy topological spaces*, Topology Appl., **11(3)** (1980), 319-334.

[82] S. E. Rodabaugh, *A categorical accommodation of various notions of fuzzy topology*, Fuzzy Sets Syst., **9(1)** (1983), 241-265.

[83] S. E. Rodabaugh, *Point-set lattice-theoretic topology*, Fuzzy Sets Syst., **40(2)** (1991), 297-345.

[84] S. E. Rodabaugh, *Categorical foundations of variable-basis fuzzy topology*, Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (U. Höhle and S. E. Rodabaugh, eds.), Kluwer Academic Publishers, (1999), 273-388.

[85] S. E. Rodabaugh, *Necessary and sufficient conditions for powersets in Set and Set × C to form algebraic theories*, Abstracts of the 26th Linz Seminar on Fuzzy Set Theory, Johannes Kepler Universität, Linz, (2005), 89-97.

[86] S. E. Rodabaugh, *Relationship of algebraic theories to powerset theories and fuzzy topological theories for lattice-valued mathematics*, Int. J. Math. Math. Sci., **2007** (2007), 1-71.

[87] S. E. Rodabaugh, *Functorial comparisons of bitopology with topology and the case for redundancy of bitopology in lattice-valued mathematics*, Appl. Gen. Topol., **9(1)** (2008), 77-108.

[88] S. E. Rodabaugh, *Relationship of algebraic theories to powersets over objects in Set and Set×C*, Fuzzy Sets Syst., **161(3)** (2010), 453-470.

[89] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., **35(3)** (1971), 512-517.

[90] K. I. Rosenthal, *Quantales and their applications*, Addison Wesley Longman, 1990.

[91] J. Rosický, *Equational categories*, Cah. Topol. Géom. Différ., **22(1)** (1981), 85–95.

[92] G. J. Seal, *A Kleisli-based approach to lax algebras*, Appl. Categ. Structures, **17(1)** (2009), 75-89.

[93] M. Shabir and M. Naz, *On soft topological spaces*, Comput. Math. Appl., **61(7)** (2011), 1786-1799.

[94] F.-G. Shi and B. Pang, *Redundancy of fuzzy soft topological spaces*, J. Intell. Fuzzy Syst., **27(4)** (2014), 1757-1760.

[95] S. Solovyov, *On the category* **Set**(**JCPos**), Fuzzy Sets Syst., **157(3)** (2006), 459-465.

[96] S. Solovyov, *Categorical frameworks for variable-basis sobriety and spatiality*, Math. Stud. (Tartu), **4** (2008), 89-103.

[97] S. Solovyov, *Sobriety and spatiality in varieties of algebras*, Fuzzy Sets Syst., **159(19)** (2008), 2567-2585.

[98] S. Solovyov, *Variable-basis topological systems versus variable-basis topological spaces*, Soft Comput., **14(10)** (2010), 1059-1068.

[99] S. Solovyov, *Fuzzy algebras as a framework for fuzzy topology*, Fuzzy Sets Syst., **173(1)** (2011), 81-99.

[100] S. Solovyov, *On a generalization of the concept of state property system*, Soft Comput., **15(12)** (2011), 2467-2478.

[101] S. Solovyov, *Categorical foundations of variety-based topology and topological systems*, Fuzzy Sets Syst., **192** (2012), 176-200.

[102] S. Solovyov, *Composite variety-based topological theories*, Fuzzy Sets Syst., **195** (2012), 1-32.

[103] S. Solovyov, *Categorically-algebraic topology versus universal topology*, Fuzzy Sets Syst., **227** (2013), 25-45.

[104] S. Solovyov, *Lattice-valued soft algebras*, Soft Comput., **17(10)** (2013), 1751-1766.

[105] A. P. Šostak, *On a fuzzy topological structure*, Rend. Circ. Mat. Palermo, II. Ser. Suppl., **11** (1985), 89-103.

[106] S. Vickers, *Topology via Logic*, Cambridge University Press, 1989.

[107] L. A. Zadeh, *Fuzzy sets*, Inf. Control, **8(3)** (1965), 338-365.

[108] D. Zhang and Y.-M. Liu, *L-fuzzy version of Stone's representation theorem for distributive lattices*, Fuzzy Sets Syst., **76(2)** (1995), 259-270.

Sergey A. Solovyov, Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technicka 2896/2, 616 69 Brno, Czech Republic and Institute of Mathematics and Computer Science, University of Latvia, Raina bulvaris 29, LV-1459 Riga, Latvia

E-mail address: `solovjovs@fme.vutbr.cz, sergejs.solovjovs@lumii.lv`