ON MINIMAL REALIZATION OF IF-LANGUAGES: A CATEGORICAL APPROACH

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ABSTRACT. The purpose of this work is to introduce and study the concept of minimal deterministic automaton with IF-outputs which realizes the given IF-language. Among two methods for construction of such automaton presented here, one is based on Myhill-Nerode's theory while the other is based on derivatives of the given IF-language. Meanwhile, the categories of deterministic automata with IF-outputs and IF-languages alongwith a functorial relationship between them are introduced

1. Introduction

The study of fuzzy automata was initiated by Wee [25] and Santos [20] in 1960's after the introduction of fuzzy set theory by Zadeh [26]. Much later, a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) was introduced by Malik, Mordeson and Sen [16, 17]. Somewhat different notions were introduced subsequently in [12, 13]. The usefulness of the concepts of fuzzy automata has been shown in numerous engineering applications such as pattern recognition, clinical monitoring, and also used to model fuzzy discrete event systems (cf., [15, 17, 18, 19]). After the introduction of intuitionistic fuzzy set (a generalization of fuzzy set, which has been found to be highly useful in dealing with vagueness) by Atanassov [2, 3], Jun [10, 11] generalized the concept of fuzzy automata by introducing and studying the concept of an intuitionistic fuzzy finite state machine (At this point, we mention that in several papers, e.g., [7], it has been argued that the use of the term 'intuitionistic' for the concept introduced by Atanassov [2], is inappropriate. Accordingly, we use in this paper the prefix 'IF-' in place of intuitionistic fuzzy, thus for example, an intuitionistic fuzzy set is renamed here as an 'IF-set'. This terminology has already been used in [21, 22, 23, 28]). Interestingly, the usefulness of IF-automata has been shown in the study of social sciences (cf., [5, 6]).

A fairly general definition of a machine in a category has been introduced in the literature by Arbib and Manes [1], which includes as particular cases, the linear systems (used in control theory), the standard automata with outputs, tree automata and stochastic machines. An interesting problem in this context is the realization problem, which says that given a *behaviour*, can we design a machine which realizes it. This problem has been studied in fairly general category-theoretic

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setup by Goguen [8] and Arbib and Manes [1]. In particular, Arbib and Manes have provided a minimal realization in the cases of linear systems, automata with outputs and tree automata. After the introduction of fuzzy languages [14, 24, 27], recently, the similar concept has been studied in [9] for fuzzy languages in algebraic setup. Chiefly inspired from [1, 8], in this paper, we study the concept of minimal realization (i.e., minimal deterministic automaton with IF-outputs) which realizes the given IF-language in category-theoretic setup. We use two concepts for such construction, specifically, one is based on Myhill-Nerode's theory and the other is on the basis of derivatives of the given IF-language. In between, we also introduce the concepts of run map, reachability, observability, category of deterministic automata with IF-outputs and category of IF-languages, and functors between these two categories.

2. Preliminaries

In this section we recall the concept of IF-sets, which is to be used in the next sections. Throughout, I denotes the interval $[0, 1]$ and for a nonempty set X, X^* denotes the free monoid generated by X. We shall denote by e, the identity element of X^*.

Definition 2.1. [2] An **IF-set** A in X is a pair (A_1, A_2) of fuzzy sets in X, i.e., functions $A_1, A_2 : X \to I$, such that $A_1(x) + A_2(x) \le 1; \forall x \in X$.

$(A_1(x)$ and $A_2(x)$, appearing in the above definition, are usually interpreted respectively as the degree of membership and the degree of non-membership of x in A).

Remark 2.2. We shall usually denote the two parts of an IF-set A also as A_1 and A_2 and express A as (A_1, A_2). An IF-set $A = (A_1, A_2)$ in X will frequently be also viewed as a function $A : X \to I \times I$, given by $A(x) = (A_1(x), A_2(x)), x \in X$, with $A_1(x) + A_2(x) \le 1$.

3. Categories of Deterministic Automata with IF-outputs and IF-languages

In this section, we introduce and study the categories of deterministic automata with IF-outputs, and IF-languages. Also, we present a functorial relationship between these two categories.

We begin with the following concept of a deterministic automaton with IF-outputs, which is similar to the concept of deterministic finite automaton with vague (final) states in the sense of [4] with the difference that here we use IF-output map instead of vague final states.

Definition 3.1. A **deterministic automaton with IF-outputs** is a 5-tuple $M = (Q, X, \delta, q_0, \beta)$, where

   (i) Q and X are sets called the **state-set** and **input-set**, respectively.
   (ii) $\delta : Q \times X \to Q$ is a map called **transition map**.
   (iii) $\beta : Q \to I \times I$ is an IF-set, also called an **IF-output map**.
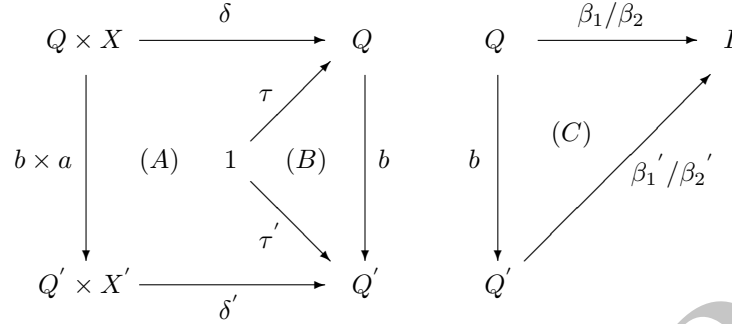   (iv) $q_0 \in Q$ is a fixed state called the **initial state**.

Figure 1. Commuting Diagram of Definition 3.3

Example 3.2. Let Q be the set of integers and $X = \{0, 1, 2, ...\}$. Then $M = (Q, X, \delta, 0, \beta)$ is a deterministic automaton with IF-outputs, where $\delta : Q \times X \to Q$ is a map such that

$$\delta(q, x) = \begin{cases} q + x & \text{if } q > 0 \\ -|q + x| & \text{if } q = 0 \\ q & \text{if } q < 0, x = 0 \\ |q + x| & \text{if } q < 0, x \neq 0 \end{cases}$$

$\forall q \in Q, \forall x \in X$, and $\beta : Q \to I \times I$ is a map such that for all $q \in Q$,

$$\beta_1(q) = \begin{cases} 1, & \text{if } q = 0 \\ \frac{1}{|q|}, & \text{otherwise.} \end{cases}$$

$$\beta_2(q) = \begin{cases} 0, & \text{if } q = 0 \\ 1 - \frac{1}{|q|}, & \text{otherwise.} \end{cases}$$

Definition 3.3. Let $M = (Q, X, \delta, q_0, \beta)$ and $M' = (Q', X', \delta', q'_0, \beta')$ be deterministic automata with IF-outputs. A **homomorphism** $f : M \to M'$ is a pair (a, b), where $a : X \to X'$ and $b : Q \to Q'$ are maps such that the diagrams in Figure 1 commute. In Figure 1 we use 1 for the singleton set $\{0\}$ with $\tau(0) = q_0$ and $\tau'(0) = q'_0$. Also, M' is called the **homomorphic image** of M if f is onto.

Remark 3.4. (i) In Figure 1, the commutativity of Diagram (C) means $(\beta'_1 \circ b)(q) = \beta_1(q)$ and $(\beta'_2 \circ b)(q) = \beta_2(q), \forall q \in Q$.
(ii) Throughout, we will use the notation A_1/A_2 in diagrams to denote an IF-set A. Also, the commutativity of such diagrams will be the same as discussed in (i).

Proposition 3.5. *The class of all deterministic automata with IF-outputs and their homomorphisms form a category under component-wise composition of maps.*

Proof. Let $M = (Q, X, \delta, q_0, \beta)$, $M' = (Q', X', \delta', q'_0, \beta')$ and $M'' = (Q'', X'', \delta'', q''_0, \beta'')$ be deterministic automata with IF-outputs and $f = (a, b) : M \to M'$ and $g = (a', b') : M' \to M''$ be their homomorphisms. To show that the composition $g \circ f = (a' \circ a, b' \circ b) : M \to M''$ is a homomorphism, it is enough to show that the diagrams in Figure 2 commute. The commutativity of Square (A) in Figure 2
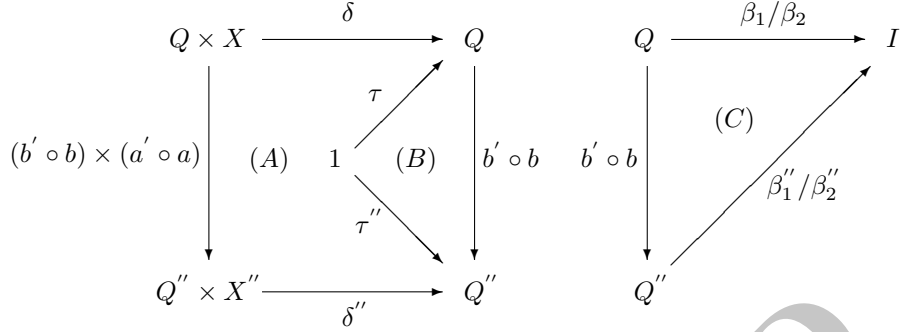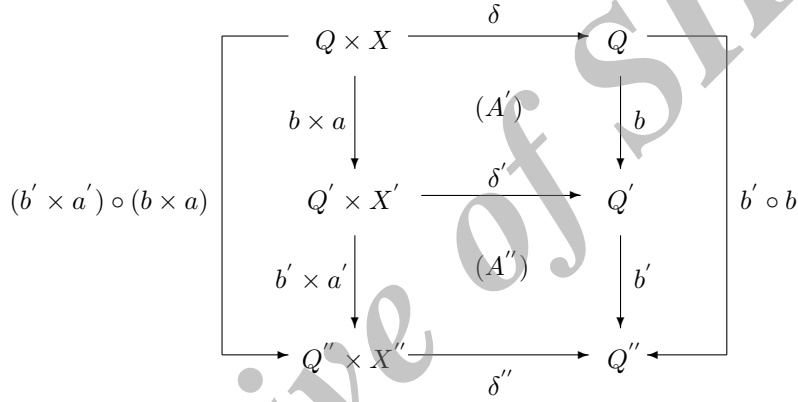
FIGURE 2. Diagram for Composition of Homomorpisms

FIGURE 3. Diagram for Commutativity of Square (A) in Figure 2

follows from the commutativity of both diagrams $(A^{'})$ and $(A^{''})$ in Figure 3. As, for all $(q, x) \in Q \times X$, $((b^{'} \circ b) \circ \delta)(q, x) = b^{'}(b(\delta(q, x))) = b^{'}((\delta^{'} \circ (b \times a))(q, x)) = b^{'}(\delta^{'}(b(q), a(x))) = (\delta^{''} \circ (b^{'} \times a^{'}))(b(q), a(x)) = (\delta^{''} \circ ((b^{'} \times a^{'}) \circ (b \times a)))(q, x)$. Thus $(b^{'} \circ b) \circ \delta = \delta^{''} \circ ((b^{'} \times a^{'}) \circ (b \times a))$.

Also, the commutativity of triangle (B) in Figure 2 follows from the commutativity of both the upper and the lower triangles in Figure 4. As $((b^{'} \circ b) \circ \tau)(0) = (b^{'} \circ b)(\tau(0)) = b^{'}((b \circ \tau)(0)) = b^{'}(\tau^{'}(0)) = (b^{'} \circ \tau^{'})(0) = \tau^{''}(0)$. Thus $(b^{'} \circ b) \circ \tau = \tau^{''}$.

Finally, the commutativity of triangle (C) in Figure 2 follows from the commutativity of both the upper and the lower triangles in Figure 5. Since, for $i = 1$ and 2; $\beta_i(q) = (\beta_i^{'} \circ b)(q) = \beta_i^{'}(b(q)) = (\beta_i^{''} \circ b^{'})(b(q)) = \beta_i^{''}(b^{'}(b(q))) = \beta_i^{''}((b^{'} \circ b)(q)) = (\beta_i^{''} \circ (b^{'} \circ b))(q)$. Thus $\beta_i = \beta_i^{''} \circ (b^{'} \circ b)$, $i = 1, 2$. □

We shall denote by **DIFA**, the category of deterministic automata with IF-outputs. By abuse of language, we shall also denote the object-class of the category **DIFA** by **DIFA** itself.
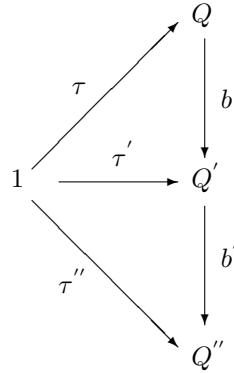
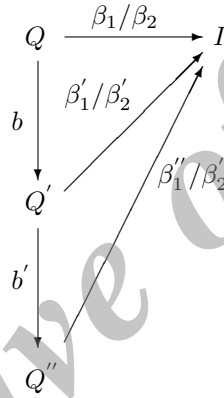FIGURE 4. Diagram for Commutativity of Triangle (B) in Figure 2

FIGURE 5. Diagram for Commutativity of Triangle (C) in Figure 2

Remark 3.6. Let **D** be the class of **DIFA**-objects and **DIFA**-morphisms with the restriction that the first component of each **DIFA**-morphism is onto. Then **D** is also a category. Obviously, it is a subcategory of **DIFA**.

Now, we introduce the concepts of run map, reachability map and observability map of a deterministic automaton with IF-outputs.

Definition 3.7. For $(Q, X, \delta, q_o, \beta) \in$ **DIFA**, the map $\delta : Q \times X \to Q$ can be extended to a map $\delta^* : Q \times X^* \to Q$ such that

   (i) $\delta^*(q, e) = q$,
   (ii) $\delta^*(q, wx) = \delta(\delta^*(q, w), x), \forall q \in Q, w \in X^*$ and $x \in X$,

δ^* is called the **run map** of M.

Example 3.8. In Example 3.2, $(X, +)$ is a free monoid and δ is the run map of M.

Definition 3.9. (a) Let $M = (Q, X, \delta, q_o, \beta) \in$ **DIFA**. The **reachability map** r of M is a map $r : X^* \to Q$ such that

  (i) $r(e) = q_o$, and
  (ii) $r(wx) = \delta(r(w), x), \forall w \in X^*$ and $\forall x \in X$.

(b) M is called **reachable** if r is onto.

Example 3.10. Consider the Example 3.8. The identity map on X^* is reachable.

Remark 3.11. It can be easily seen that

  (i) $r(w) = \delta^*(q_o, w), \forall w \in X^*$.
  (ii) $r(ww') = \delta^*(r(w), w'), \forall w, w' \in X^*$.
  (iii) The reachability of $M = (Q, X, \delta, q_o, \beta) \in$ **DIFA** only tells that for each state $q \in Q$ there exists $w \in X^*$ such that $\delta^*(q_0, w) = q$.

Definition 3.12. Given $M = (Q, X, \delta, q_0, \beta) \in$ **DIFA**, the map f_q given by $(f_q)_1(w) = \beta_1(\delta^*(q, w))$ and $(f_q)_2(w) = \beta_2(\delta^*(q, w)), \forall w \in X^*$, is called the **IF-language accepted by M in state q**.

The IF-language f_{q_0} accepted by M in state q_0 is called the **IF-language** accepted by M.

Remark 3.13. It can be easily seen that $f_{q_0} = \beta \circ r$, i.e., $(f_{q_0})_1 = \beta_1 \circ r$ and $(f_{q_0})_2 = \beta_2 \circ r$.

Definition 3.14. (a) Given $M = (Q, X, \delta, q_0, \beta) \in$ **DIFA**, the map σ given by $\sigma(q) = (\sigma_1(q), \sigma_2(q)), \forall q \in Q$, where $\sigma_1(q) = (f_q)_1$ and $\sigma_2(q) = (f_q)_2$, is called the **observability map**.

(b) M is called observable if σ is one-one.

Remark 3.15. (i) Note that, for $i = 1, 2; \sigma_i(q_0) = (f_{q_0})_i$, where $(f_{q_0})_i(w) = (\beta_i \circ r)(w) = \beta_i(r(w)) = \beta_i(\delta^*(q_0, w)), \forall w \in X^*$.
(ii) The observability of $M \in$ **DIFA** tells that different IF-languages are assigned to distinct states.

Now, we introduce the concept of IF-languages for arbitrary sets and the category of IF-languages.

Definition 3.16. (a) For a given set X, an **IF-language** is a map $f : X^* \to I \times I$ such that $f_1(w) + f_2(w) \leq 1, \forall w \in X^*$.
(b) A **morphism** between two IF-languages $f : X^* \to I \times I$ and $f' : (X')^* \to I \times I$ is a map $a : X \to X'$ such that the diagram in Figure 6 commutes.

The map a^* in Figure 6, is free extension of the map a in Definition 3.16(b) defined inductively by $a^*(e) = e'$ and $a^*(wx) = a^*(w)a(x), \forall w \in X^*$ and $x \in X$.

Proposition 3.17. *For a given set X, the class of IF-languages and their morphisms form a category.*

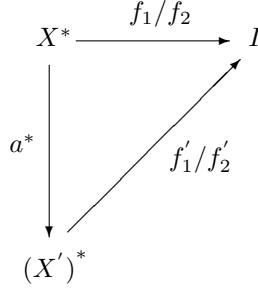*Proof.* The proof is similar to that of Proposition 3.5. $\qquad\square$

$$X^* \xrightarrow{\ f_1/f_2\ } I$$

with a^* downward and f'_1/f'_2 diagonal to I, and $(X')^*$ at bottom.

FIGURE 6. Diagram for Definition 3.16(b)

$$
\begin{array}{ccc}
X^* & \xrightarrow{\quad r \quad} & Q \\
\downarrow{\scriptstyle a^*} & & \downarrow{\scriptstyle b} \\
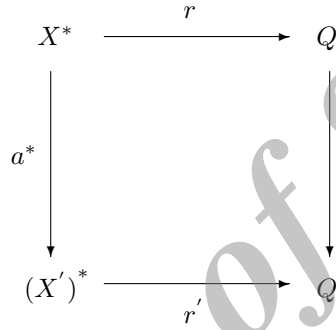(X')^* & \xrightarrow[\quad r' \quad]{} & Q'
\end{array}
$$

FIGURE 7. Commuting Diagram for Proposition 3.19

We shall denote by **IFL**, the category of IF-languages.

Remark 3.18. Let **L** be the class of **IFL**-objects and **IFL**-morphisms with the restriction that the first component of each **IFL**-morphism is onto. Then **L** is also a category. Obviously, it is a subcategory of **IFL**.

Proposition 3.19. *Given* $M = (Q, X, \delta, q_0, \beta)$, $M' = (Q', X', \delta', q'_0, \beta') \in$ **DIFA** *with reachability maps* $r : X^* \to Q$ *and* $r' : (X')^* \to Q'$ *respectively, and* **DIFA**-*morphism* $(a, b) : M \to M'$, *the diagram in Figure 7 commutes.*

Proof. We prove this by induction on the length of strings in X^*. Let $w \in X^*$. For $|w| = 0$, $(b \circ r)(e) = b(r(e)) = b(q_0) = q'_0$ and $(r' \circ a^*)(e) = r'(a^*(e)) = r'(e') = q'_0$. Thus the diagram in Figure 7 commutes for $|w| = 0$. We now assume that the result is true for all strings of length less than or equal to n, i.e., for all $w \in X^*$ such that $|w| \leq n$, $(b \circ r)(w) = (r' \circ a^*)(w)$, i.e., $b(r(w)) = r'(a^*(w))$. Then $(b \circ r)(wx) = b(r(wx)) = b(\delta(r(w), x))$ (from Definition 3.3) $= \delta'(b(r(w)), a(x))$, as (a, b) is a **DIFA**-morphism. Furthermore, $(r' \circ a^*)(wx) = r'(a^*(wx)) = r'(a^*(w)a(x)) = \delta'(b(r(w), a(x)))$ follows from the fact that the result is true for all strings of length less than or equal to n. Hence the diagram in Figure 7 commutes for $|w| = n+1$. $\square$

Proposition 3.20. *Let* $E :$ **DIFA** $\to$ **IFL** *such that* $E(M) = f_{q_o}, \forall M \in$ **DIFA** *and for all* **DIFA**-*morphism* $(a, b) : M \to M', E(a, b) = a$. *Then* E *is a functor.*
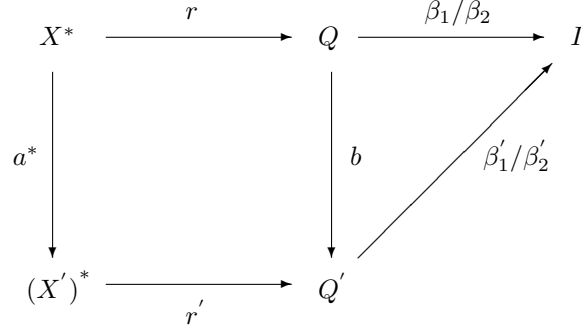
$$X^* \xrightarrow{\ r\ } Q \xrightarrow{\ \beta_1/\beta_2\ } I$$

with a^*, b, β_1'/β_2', r', $(X')^*$, Q'

FIGURE 8. Commuting Diagram of Proposition 3.20

$$X^* \xrightarrow{\ \beta_1 \circ r/\beta_2 \circ r\ } I$$

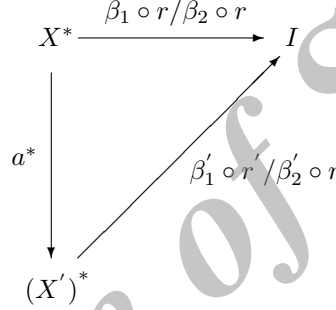with a^*, $\beta_1' \circ r'/\beta_2' \circ r'$, $(X')^*$

FIGURE 9. Commuting Diagram Equivalent to Diagrams in Figure 8

Proof. We will only show that $'a'$ is an **IFL**-morphism. Since (a, b) is a **DIFA**-morphism each of the diagram in Figure 8 commutes. The diagram in Figure 8 lead us to the commutativity of the diagram in Figure 9. Also, as $f_{q_o} = (\beta_1 \circ r, \beta_2 \circ r)$ and $f'_{q_o} = (\beta_1' \circ r', \beta_2' \circ r')$, the diagram in Figure 9 states that $'a'$ is indeed an **IFL**-morphism. $\square$

Inspired from Proposition 3.20 and Remarks 3.11, 3.15, we now introduce the following concept of minimal realization of a given IF-language.

Definition 3.21. $M \in \textbf{DIFA}$ is a **realization** of an IF-language f if $E(M) = f$. Further the realization M is **minimal** if M is both reachable and observable.

4. Minimal Realization of IF-language

In order to present two approaches to construct minimal deterministic automaton with IF-outputs for the given IF-language, we divide this section into two subsections.

4.1. **Minimal Realization Based on Myhill-Nerode's Theory.** This subsection is towards the construction of minimal realization of the given IF-language based on Myhill-Nerode's theory. We begin with the following.

Proposition 4.1. *Given an IF-language $f : X^* \to I \times I$ there exists a deterministic automaton with IF-outputs N_f, which realizes f.*

Proof. Define a relation $\approx$ on X^* by $w_1 \approx w_2$ iff $f_1(w_1w) = f_1(w_2w)$ and $f_2(w_1w) = f_2(w_2w)$, $\forall w \in X^*$. Then $\approx$ is an equivalence relation on X^*. Now, let $Q_f = X^*/\approx = \{[w] : w \in X^*\}$, where $[w] = \{w' \in X^* : w \approx w'\}$, and define the maps δ_f and β_f as:

$$\delta_f : Q_f \times X \to Q_f \text{ such that } \delta_f([w], x) = [wx], \text{ and}$$
$$\beta_f : Q_f \to I \times I \text{ such that } (\beta_f)_1([w]) = f_1(w) \text{ and } (\beta_f)_2([w]) = f_2(w).$$

$\forall [w] \in Q_f$ and $\forall x \in X$.
Both the maps are well defined, which is shown as under:

Let $w_1, w_2 \in X^*$ such that $[w_1] = [w_2]$. Then $w_1 \approx w_2$. Now,

$$
\begin{aligned}
w_1 \approx w_2 \quad &\Rightarrow \quad f_1(w_1w) = f_1(w_2w) \text{ and } f_2(w_1w) = f_2(w_2w), \forall w \in X^* \\
&\Rightarrow \quad f_1(w_1) = f_1(w_2) \text{ and } f_2(w_1) = f_2(w_2) \\
&\Rightarrow \quad (\beta_f)_1([w_1]) = (\beta_f)_1([w_2]) \text{ and } (\beta_f)_2([w_1]) = (\beta_f)_2([w_2]), \\
&\Rightarrow \quad ((\beta_f)_1, (\beta_f)_2)([w_1]) = ((\beta_f)_1, (\beta_f)_2)([w_2]) \\
&\Rightarrow \quad (\beta_f)([w_1]) = (\beta_f)([w_2]),
\end{aligned}
$$

and thus β_f is well-defined.
Again, let $w_1, w_2 \in X^*$ such that $[w_1] = [w_2]$. Then $w_1 \approx w_2$. Now,

$$
\begin{aligned}
w_1 \approx w_2 \quad &\Rightarrow \quad f_1(w_1w) = f_1(w_2w) \text{ and } f_2(w_1w) = f_2(w_2w), \forall w \in X^* \\
&\Rightarrow \quad f_1(w_1xw) = f_1(w_2xw) \text{ and } f_2(w_1xw) = f_2(w_2xw), \forall x \in X \\
&\Rightarrow \quad w_1x \approx w_2x, \forall x \in X \\
&\Rightarrow \quad [w_1x] = [w_2x], \forall x \in X \\
&\Rightarrow \quad \delta_f([w_1], x) = \delta_f([w_2], x),
\end{aligned}
$$

and so δ_f is well-defined. Thus $N_f = (Q_f, X, \delta_f, [e], \beta_f)$ is a deterministic automaton with IF-outputs. Also, by induction, it is easy to verify that δ_f can be extended to $\delta_f^* : Q_f \times X^* \to Q_f$ such that $\delta_f^*([w], w') = [ww']$.
Finally,

$$
\begin{aligned}
E(N_f)(w) \quad &= \quad f_{[e]}(w) \\
&= \quad ((f_{[e]})_1(w), (f_{[e]})_2(w)) \\
&= \quad ((\beta_f)_1(\delta_f^*([e], w)), (\beta_f)_2(\delta_f^*([e], w))) \\
&= \quad ((\beta_f)_1([w]), (\beta_f)_2([w])) \\
&= \quad (f_1(w), f_2(w)) \\
&= \quad f(w).
\end{aligned}
$$

Thus $E(N_f) = f$. Hence N_f realizes f. $\qquad\square$

Proposition 4.2. *The realization N_f of IF-language $f : X^* \to I \times I$ is minimal.*
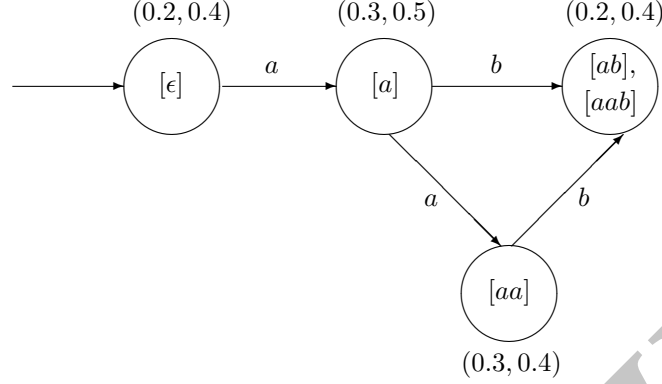
FIGURE 10. Minimal Deterministic Automaton with
IF-outputs for Example 4.3

Proof. Let $[w] \in Q_f$. Then $r(w) = \delta^*([e], w) = [ew] = [w]$. Thus r is onto, and so N_f is reachable. Also, let $w_1, w_2 \in X^*$ such that $\sigma([w_1]) = \sigma([w_2])$.
Now, $\sigma([w_1]) = \sigma([w_2])$

$\Rightarrow$ $(f_{[w_1]})_1 = (f_{[w_2]})_1$ *and* $(f_{[w_1]})_2 = (f_{[w_2]})_2, \forall w \in X^*$

$\Rightarrow$ $(f_{[w_1]})_1(w) = (f_{[w_2]})_1(w)$ *and* $(f_{[w_1]})_2(w) = (f_{[w_2]})_2(w), \forall w \in X^*$

$\Rightarrow$ $(\beta_f)_1(\delta_f^*([w_1], w)) = (\beta_f)_1(\delta_f^*([w_2], w))$ *and*
        $(\beta_f)_2(\delta_f^*([w_1], w)) = (\beta_f)_2(\delta_f^*([w_2], w)), \forall w \in X^*$

$\Rightarrow$ $(\beta_f)_1([w_1 w]) = (\beta_f)_1([w_2 w])$ *and* $(\beta_f)_2([w_1 w]) = (\beta_f)_2([w_2 w]), \forall w \in X^*$

$\Rightarrow$ $f_1(w_1 w) = f_1(w_2 w)$ *and* $f_2(w_1 w) = f_2(w_2 w), \forall w \in X^*$

$\Rightarrow$ $(f_1(w_1 w), f_2(w_1 w)) = (f_1(w_2 w), f_2(w_2 w)), \forall w \in X^*$

$\Rightarrow$ $f(w_1 w) = f(w_2 w), \forall w \in X^*$,

and so $w_1 \approx w_2$ or $[w_1] = [w_2]$, whereby σ is one-one. Hence N_f is minimal realization of f. $\qquad\square$

Example 4.3. Consider an IF-language $f : X^* \to I \times I$ such that

$$f(w) = \begin{cases} (0.2, 0.4), & \text{if } w = e, ab, aab \\ (0.3, 0.5), & \text{if } w = a \\ (0.3, 0.4), & \text{if } w = aa \\ 0, & \text{otherwise} \end{cases}$$

over monoid generated by $X = \{a, b\}$. Then the minimal deterministic automaton with IF-outputs, which realizes f is given in Figure 10. Also, it can easily verified that the reachability map $r : X^* \to Q_f$ is onto, i.e., N_f is reachable and the observability map σ is one-one. Thus realization N_f of IF-language f is minimal.

The following is a functorial relationship between the categories **L** and **D**.

FIGURE 11. Commuting Diagram of Proposition 4.4

Proposition 4.4. *Let $N : \mathbf{L} \to \mathbf{D}$ be a map which sends each $f \in \mathbf{L}$ to N_f, and to each $\mathbf{L}$-morphism $a : f \to f'$ to $(a, b) : N_f \to N_{f'}$, where $b : Q_f \to Q_{f'}$ is the map such that $b([w]) = [a^*(w)]$. Then N is a functor.*

Proof. We only show that for given $\mathbf{L}$-morphism $a : f \to f'$, $(a, b) : N_f \to N_{f'}$, where $b : Q_f \to Q_{f'}$ is the map such that $b([w]) = [a^*(w)]$ is a $\mathbf{D}$-morphism. First, we show that the map $b : Q_f \to Q_{f'}$ is well-defined, which is shown as under:
Let $w_1, w_2 \in X^*$ such that $[w_1] = [w_2]$. Then $w_1 \approx w_2$. Now,

$$
\begin{aligned}
w_1 \approx w_2 \;\; \Rightarrow \;\; & f_1(w_1w) \;=\; f_1(w_2w) \; and \; f_2(w_1w) \;=\; f_2(w_2w), \forall w \in X^* \\
\Rightarrow \;\; & (f_1' \circ a^*)(w_1w) \;=\; (f_1' \circ a^*)(w_2w) \; and \; (f_2' \circ a^*)(w_1w) \\
& = \; (f_2' \circ a^*)(w_2w), \forall w \in X^*, \; (As \; 'a' \; is \; an \; \mathbf{L} - morphism) \\
\Rightarrow \;\; & f_1'(a^*(w_1)a^*(w)) \;=\; f_1'(a^*(w_2)a^*(w)) \; and \; f_2'(a^*(w_1)a^*(w)) \\
& = \; f_2'(a^*(w_2)a^*(w)), \; \forall a^*(w) \in X'^* \\
\Rightarrow \;\; & a^*(w_1) \approx a^*(w_2) \\
\Rightarrow \;\; & [a^*(w_1)] \;=\; [a^*(w_2)] \\
\Rightarrow \;\; & b([w_1]) \;=\; b([w_2]).
\end{aligned}
$$

Thus the map b is well-defined.

Again, in order to prove that $(a, b) : N_f \to N_{f'}$ is a $\mathbf{D}$-morphism, we have to show that the diagrams in Figure 11 commute. To show the commutativity of square (A) in Figure 11, let $([w], x) \in Q_f \times X$. Then $(\delta_{f'} \circ (b \times a))([w], x) = \delta_{f'}(b([w]), a(x)) = \delta_{f'}([a^*(w)], a(x)) = [a^*(w)a(x)] = [a^*(wx)]$. Also, $(b \circ \delta_f)([w], x) = b(\delta_f([w], x)) = b([wx]) = [a^*(wx)]$. Hence square (A) in Figure 11 commutes. Commutativity of Diagram (B) in Figure 11 follows from the fact that $(b \circ \tau_f)(0) = b(\tau_f(0)) = b([e]) = [a^*(e)] = [e'] = \tau_{f'}(0)$. Lastly, to show the commutativity of the Diagram (C) in Figure 11, let $[w] \in Q_f$. Then for $i = 1, 2$; $((\beta_{f'})_i \circ b)([w]) = (\beta_{f'})_i(b([w])) = (\beta_{f'})_i([a^*(w)]) = f_i'(a^*(w)) = f_i(w) = (\beta_f)_i([w])$ (c.f., Definition 3.16 (b)), i.e., $(\beta_{f'})_1 \circ b = (\beta_f)_1$ and $(\beta_{f'})_2 \circ b = (\beta_f)_2$. Thus the Diagram (C) in Figure 11 commutes. Hence $\beta_{f'} \circ b = \beta_f$. $\qquad\square$

4.2. **Minimal Realization Based on Derivatives of IF-language.** This subsection is towards the construction of minimal realization of a given IF-language based on derivatives of the given IF-language. We begin with the following concept of derivatives.

Definition 4.5. Let $f : X^* \to I \times I$ be an IF-language. The **derivative** of f with respect to $u \in X^*$ is a map $f^u : X^* \to I \times I$ such that $f^u = (f_1^u, f_2^u)$, where $f_1^u(v) = f_1(uv)$ and $f_2^u(v) = f_2(uv), \forall v \in X^*$.

For given an IF-language $f : X^* \to I \times I$, we put $Q^f = \{f^u : u \in X^*\}$.

Proposition 4.6. *Given an IF-language $f : X^* \to I \times I$, there exists $N^f \in$ **DIFA** which realizes f.*

Proof. Let $f : X^* \to I \times I$ be an IF-language. Define the maps δ^f and β^f as :

$$\delta^f : Q^f \times X \to Q^f \text{ such that } \delta^f(g, x) = g^x, \forall g \in Q^f \text{ and } \forall x \in X.$$
$$\beta^f : Q^f \to I \times I \text{ such that } \beta^f(g) = g(e), \text{ i.e.,}$$
$$\beta_1^f(g_1) = g_1(e), \text{ and } \beta_2^f(g_2) = g_2(e), \forall g \in Q^f.$$

The maps δ^f and β^f are well defined, which are shown as follow:
Let $g, h \in Q^f$ such that $g = h$. Then $\exists u, v \in X^*$ such that $g = f^u$, $h = f^v$ and $f^u = f^v$. Now,

$$
\begin{aligned}
g = h \quad &\Rightarrow \quad f^u = f^v \\
&\Rightarrow \quad f_1^u(w) = f_1^v(w), f_2^u(w) = f_2^v(w), \forall w \in X^* \\
&\Rightarrow \quad f_1^u(xw) = f_1^v(xw), f_2^u(xw) = f_2^v(xw), \forall w \in X^* and \forall x \in X \\
&\Rightarrow \quad g_1(xw) = h_1(xw), g_2(xw) = h_2(xw), \forall w \in X^* and \forall x \in X \\
&\Rightarrow \quad g_1^x(w) = h_1^x(w), g_2^x(w) = h_2^x(w), \forall w \in X^* and \forall x \in X \\
&\Rightarrow \quad g^x = h^x, \forall x \in X \\
&\Rightarrow \quad \delta^f(g, x) = \delta^f(h, x), \forall x \in X,
\end{aligned}
$$

and so δ^f is well defined. Also, let $h, k \in Q^f$ such that $h = k$. Then $\exists u, v \in X^*$ such that $h = f^u$ and $k = f^v$. Now,

$$
\begin{aligned}
h = k \quad &\Rightarrow \quad f^u(w) = f^v(w), \forall w \in X^* \\
&\Rightarrow \quad f_1^u(w) = f_1^v(w) \text{ and } f_2^u(w) = f_2^v(w), \forall w \in X^* \\
&\Rightarrow \quad f_1^u(e) = f_1^v(e) \text{ and } f_2^u(e) = f_2^v(e) \\
&\Rightarrow \quad h_1(e) = k_1(e) \text{ and } h_2(e) = k_2(e) \\
&\Rightarrow \quad \beta_1^f(h_1) = \beta_1^f(k_1) \text{ and } \beta_2^f(h_1) = \beta_2^f(k_1) \\
&\Rightarrow \quad \beta^f(h) = \beta^f(k),
\end{aligned}
$$

and so β^f is well defined. Thus $N^f = (Q^f, X, \delta^f, f^e, \beta^f) \in$ **DIFA**. Also, by induction it is easy to verify that δ^f can be extended to $(\delta^*)^f : Q_f \times X^* \to Q_f$ such that $(\delta^*)^f(g, w) = g^w, \forall g \in Q^f$ and $\forall w \in X^*$. Finally, it remains to show that

$E(N^f) = f$. For this, let $w \in X^*$. Then

$$
\begin{aligned}
E(N^f)(w) &= f^e(w) \\
&= (f_1^e(w), f_2^e(w)) \\
&= (\beta_1^f((\delta^*)^f(f_1^e, w)), \beta_2^f((\delta^*)^f(f_2^e, w))) \\
&= (\beta_1^f(f_1^{ew}), \beta_2^f(f_2^{ew})) \\
&= (\beta_1^f(f_1^w), \beta_2^f(f_2^w)) \\
&= (f_1^w(e), f_2^w(e)) \\
&= (f_1(we), f_2(we)) \\
&= (f_1(w),\ f_2(w)) \\
&= f(w).
\end{aligned}
$$

Hence $E(N^f) = f$, whereby N^f realizes f. $\qquad\square$

Proposition 4.7. *The realization $N^f = (Q^f, X, \delta^f, f^e, \beta^f)$ of IF-language $f : X^* \to I \times I$ is minimal.*

Proof. First we show that the observability map σ is one-one. Let $w_1, w_2 \in X^*$ such that $\sigma(f^{w_1}) = \sigma(f^{w_2})$. Then

$$
\begin{aligned}
\sigma(f^{w_1}) = \sigma(f^{w_2}) &\Rightarrow \sigma(f^{w_1})(w) = \sigma(f^{w_2})(w), \forall w \in X^* \\
&\Rightarrow \beta_1^f((\delta^*)^f(f_1^{w_1}, w)) = \beta_1^f((\delta^*)^f(f_1^{w_2}, w))\ and \\
&\quad\ \beta_2^f((\delta^*)^f(f_2^{w_1}, w)) = \beta_2^f((\delta^*)^f(f_2^{w_2}, w)) \\
&\Rightarrow \beta_1^f(f_1^{w_1 w}) = \beta_1^f(f_1^{w_2 w})\ and\ \beta_2^f(f_2^{w_1 w}) = \beta_2^f(f_2^{w_2 w}) \\
&\Rightarrow f_1^{w_1 w}(e) = f_1^{w_2 w}(e)\ and\ f_2^{w_1 w}(e) = f_2^{w_2 w}(e) \\
&\Rightarrow f_1^{w_1}(w) = f_1^{w_2}(w)\ and\ f_2^{w_1}(w) = f_2^{w_2}(w) \\
&\Rightarrow f^{w_1} = f^{w_2}.
\end{aligned}
$$

Thus σ is one-one. Finally, to show that the reachability map r is onto, let $g \in Q^f$. Then there exists $w \in X^*$ such that $g = f^w$. Now, $r(w) = (\delta^*)^f(f^e, w) = f^{ew} = f^w = g$. Thus for each $g = f^w \in Q^f$ there exists $w \in Q$ such that $r(w) = g$, whereby r is onto. Hence the realization N^f of IF-language f is minimal. $\qquad\square$

Now, we provide another functorial relationship between the categories **L** and **D**.

Proposition 4.8. *Let $N : \mathbf{L} \to \mathbf{D}$ be a map which sends each $f \in \mathbf{L}$ to N^f, and to each **L**-morphism $a : f \to f'$ to $(a, b) : N^f \to N^{f'}$, where $b : Q^f \to Q^{f'}$ is the map such that $b(f^w) = f^{a^*(w)}$. Then N is a functor.*

Proof. Similar to that of Proposition 4.4. $\qquad\square$

We close this subsection by providing an isomorphism between deterministic automata with outputs N_f and N^f.

Proposition 4.9. *Given an IF-language $f : X^* \to I \times I$, the **DIFA**-object $N^f = (Q^f, X, \delta^f, f^e, \beta^f)$ is isomorphic to **DIFA**-object $N_f = (Q_f, X, \delta_f, [e], \beta_f)$.*
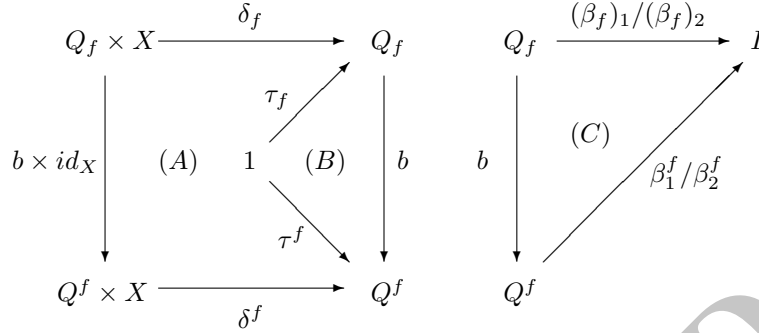
FIGURE 12. Commuting Diagram for Proposition 4.9

Proof. Define a map $\phi = (id_X, b) : N_f \to N^f$ such that $b([w]) = f^w$, $\forall w \in X^*$. Then for given $[w] \in Q_f, \exists\ f^w \in Q^f$ such that $b([w]) = f^w$. Thus ϕ is onto. Also, it is clear that ϕ is one-one. Finally, to show that $\phi = (id_X, b) : N_f \to N^f$ is a homomorphism, we have to show that the diagrams in figure 12 commute. To show the commutativity of square (A) in Figure 12, Let $([w], x) \in Q_f \times X$. Then $(\delta^f \circ (b \times id_X))([w], x) = \delta^f(b([w]), x) = \delta^f(f^w, x) = f^{wx}$. Also, $(b \circ \delta_f)([w], x) = b([wx]) = f^{wx}$. Thus square (A) in Figure 12 commutes. Commutativity of diagram (B) in Figure 12 follows from the fact that $b(\tau_f(0)) = b([e]) = f^e = \tau^f(0)$. Lastly, to show the commutativity of diagram (C) in Figure 12, let $[w] \in Q_f$. Then for $i = 1$ and 2; $(\beta_i^f \circ b)([w]) = \beta_i^f(b([w])) = \beta_i^f(f^w) = f_i^w(e) = f_i(w) = (\beta_f)_i([w])$, i.e., $\beta_1^f \circ b = (\beta_f)_1$ and $\beta_2^f \circ b = (\beta_f)_2$. Thus the Diagram (C) in Figure 12 commutes. Hence the **DIFA**-object $N^f = (Q^f, X, \delta^f, f^e, \beta^f)$ is isomorphic to **DIFA**-object $N_f = (Q_f, X, \delta_f, [e], \beta_f)$. $\square$

5. **Conclusion**

In this paper, we have tried to answer the question that "given an IF-language, can we design a minimal deterministic automaton with IF-outputs, which realize it" in category theoretic setup. Specifically, we introduce and study two minimal deterministic automata with IF-outputs which realize the given IF-language. The construction of one such automaton is based on Myhill-Nerode's theory, while construction of the other is based on derivative of the given IF-language. Interestingly, we have shown that both the realizations are isomorphic. In future, we will try to introduce the concept of realization of IF-languages by monoids.

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ON MINIMAL REALIZATION OF IF-LANGUAGES: A CATEGORICAL APPROACH

V. K. YADAV, V. GAUTAM AND S. P. TIWARI

ادراک مینیمال IF– زبانها : یک رویکرد رسته ای

چکیده. هدف از این کار معرفی ومطالعه ی مفهوم آدم ماشینی قطعی مینیمال با IF– خروجی هاست که IF– زبان را تشخیص می دهد. اینجا یکی از دو روش ارائه شده برای ساختن یک چنین آدم ماشینی بر اساس نظریه Myhill- Nerode می باشد در صورتیکه دیگری بر اساس مشتقاتی از IF– زبان داده شده است. در ضمن، رسته های آدم ماشینی های قطعی با IF– خروجی و IF– زبان ها با یک رابطه ی فانکتوری بین آنها معرفی شده اند.