

ON THE BICOMPLETION OF INTUITIONISTIC FUZZY QUASI-METRIC SPACES

F. CASTRO-COMPANY AND P. TIRADO

ABSTRACT. Based on previous results that study the completion of fuzzy metric spaces, we show that every intuitionistic fuzzy quasi-metric space, using the notion of fuzzy metric space in the sense of Kramosil and Michalek to obtain a generalization to the quasi-metric setting, has a bicompletion which is unique up to isometry.

1. Introduction

The core concepts we will use to study on the bicompletion of fuzzy constructions are based on the Kramosil and Michalek notion of a fuzzy metric space [11] (a KM-fuzzy metric space from now on) and on the definition of an intuitionistic fuzzy metric space [1, 14].

Indeed the KM-fuzzy metric notion has a close relationship with probabilistic spaces. As a matter of fact, Kramosil and Michalek observed that the class of fuzzy metric spaces in their sense, is “equivalent” to the class of Menger spaces having a continuous t-norm. Adapting Sherwood’s construction, who proved in [19] that every Menger space belonging to this class has a completion which is unique up to isometry, to the fuzzy metric context, it can be deduced that every KM-fuzzy metric space has a completion which is unique up to isometry.

In addition, the notion of a KM-fuzzy metric space was generalized to the quasi-metric setting in [4, 8], where several properties of these structures were discussed. Recently, there were given in [15, 16, 17], applications of fixed point theorems, in the realm of fuzzy quasi-metric spaces, to deduce the existence and uniqueness of solution for the recurrence equations associated with some types of algorithms.

Both paths lead to the notion of the intuitionistic fuzzy quasi-metric space. In [16] the authors generalize the notion of intuitionistic fuzzy metric space by Alaca et al [1] to the quasi-metric setting and obtain an intuitionistic fuzzy quasi-metric version of the Banach contraction principle which is applied to deduce the existence of solution for the recurrence equation which is typically associated with the complexity analysis of Quicksort.

In this context, the completion of fuzzy quasi-metric spaces appears as a natural and attractive question. After Section 2, where some pertinent definitions in order to introduce these constructs will be exposed, in Section 3 we will recall that every

Received: August 2015; Revised: December 2015; Accepted: February 2016

Key words and phrases: Intuitionistic fuzzy quasi-metric, Bicomplete, Isometry, Bicompletion.

KM-fuzzy quasi-metric space has a (KM-fuzzy quasi-metric) bicompletion which is unique up to isometry [3]. This fact allows us to restate the completion of a KM-fuzzy metric space as a particular case.

Here we shall extend our results announced in [2] by applying these constructions in Section 4 to study the bicompletion of intuitionistic fuzzy quasi-metric spaces.

2. Preliminary Concepts

Our basic references for quasi-uniform and quasi-metric spaces are [6, 12].

Definition 2.1. A quasi-metric on a set X is a nonnegative real valued function defined on $X \times X$ such that for all $x, y, z \in X$:

- (i) $d(x, y) = d(y, x) = 0 \iff x = y$;
- (ii) $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 2.2. A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X .

It is known that each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X , then the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$, is also a quasi-metric on X , called the conjugate of d , and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X .

A quasi-metric space (X, d) is said to be bicomplete if (X, d^s) is a complete metric space (all Cauchy sequences converge). In this case we say that d is a bicomplete quasi-metric on X .

A topological space (X, τ) is called quasi-metrizable if there is a quasi-metric d on X such that $\tau = \tau_d$.

Definition 2.3. [18] A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if $*$ satisfies:

- (i) $*$ is associative and commutative;
- (ii) $*$ is continuous;
- (iii) $a * 1 = a$ for every $a \in [0, 1]$;
- (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

Definition 2.4. [13] A continuous t-conorm is a binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ such that:

- (i) \diamond is commutative and associative;
- (ii) \diamond is continuous;
- (iii) $a \diamond 0 = a$ for all $a \in [0, 1]$;
- (iv) $a \diamond b \leq c \diamond d$ when $a \leq c$ and $b \leq d$ ($a, b, c, d \in [0, 1]$).

Definition 2.5. [5] For a given t-norm $*$ and $x, y \in [0, 1]$, the t-conorm $*'$ defined as:

$$x *' y = 1 - ((1 - x) * (1 - y))$$

is called the dual t-conorm of $*$. Similarly, for any given t-conorm \diamond , the t-norm \diamond' defined as:

$$x \diamond' y = 1 - ((1 - x) \diamond (1 - y))$$

is called the dual t-norm of \diamond .

Definition 2.6. [8] A KM-fuzzy quasi-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

- (KM1) $M(x, y, 0) = 0$;
- (KM2) $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$;
- (KM3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
- (KM4) $M(x, y, -) : [0, \infty) \rightarrow [0, 1]$ is left continuous;

Definition 2.7. [11] A KM-fuzzy metric on a set X is a KM-fuzzy quasi-metric $(M, *)$ on X such that for each $x, y \in X$:

- (KM5) $M(x, y, t) = M(y, x, t)$ for all $t > 0$.

Definition 2.8. [4, 8] A KM-fuzzy quasi-metric space is a triple $(X, M, *)$ such that X is a nonempty set and $(M, *)$ is a KM-fuzzy quasi-metric on X .

Definition 2.9. [16] An intuitionistic fuzzy quasi-metric on a set X is a 4-tuple $(M, N, *, \diamond)$ such that $*$ is a continuous t-norm, \diamond is a continuous t-conorm and M, N are fuzzy sets in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

- (1) $M(x, y, t) + N(x, y, t) \leq 1$ for all $t \geq 0$;
- (2) $M(x, y, 0) = 0$;
- (3) $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$;
- (4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
- (5) $M(x, y, -) : [0, \infty) \rightarrow [0, 1]$ is left continuous;
- (6) $N(x, y, 0) = 1$;
- (7) $x = y$ if and only if $N(x, y, t) = N(y, x, t) = 0$ for all $t > 0$;
- (8) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$ for all $t, s \geq 0$;
- (9) $N(x, y, -) : [0, \infty) \rightarrow [0, 1]$ is left continuous.

Definition 2.10. [16] An intuitionistic fuzzy quasi-metric space is a 5-tuple $(X, M, N, *, \diamond)$ such that $(M, N, *, \diamond)$ is an intuitionistic fuzzy quasi-metric on a set X .

Remark 2.11. It is clear that if $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy quasi-metric space, then $(X, M, *)$ is a KM-fuzzy quasi-metric space.

Similarly to the fuzzy metric case, each fuzzy quasi-metric $(M, *)$ on a set X induces a T_0 topology τ_M on X which has as a base the family of open balls $\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ (see [8, 9]).

If $(M, N, *, \diamond)$ is an intuitionistic fuzzy quasi-metric on X , then one has that $(M^{-1}, N^{-1}, *, \diamond)$ is also an intuitionistic fuzzy quasi-metric on X , where M^{-1} is the fuzzy set in $X \times X \times [0, \infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$ and N^{-1} is the fuzzy set in $X \times X \times [0, \infty)$ defined by $N^{-1}(x, y, t) = N(y, x, t)$.

Moreover, if we define M^i as the fuzzy set in $X \times X \times [0, \infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$ and denote by N^s the fuzzy set in $X \times X \times [0, \infty)$ given by $N^s(x, y, t) = \max\{N(x, y, t), N^{-1}(x, y, t)\}$ then $(M^i, N^s, *, \diamond)$ is an intuitionistic fuzzy metric in the sense of [1].

A KM-fuzzy metric space $(X, M, *)$ is complete [7, 20] provided that each Cauchy sequence in X is convergent with respect to τ_M , where a sequence $(x_n)_n$ in X is said to be a Cauchy sequence if for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

According to [8, 9], a KM-fuzzy quasi-metric space $(X, M, *)$ is called bicomplete if $(X, M^i, *)$ is a complete fuzzy metric space. We say that $(M, *)$ is a bicomplete fuzzy quasi-metric on X .

In order to construct a suitable topology on an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$, Romaguera and Tirado considered in [16] the natural “balls” $B(x, \varepsilon, t)$ defined, similarly to [14] and [1], by:

$$B(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon, N(x, y, t) < \varepsilon\}$$

for all $x \in X$, $0 < \varepsilon < 1$, and $t > 0$.

Then, they proved that $B(x, \varepsilon, t) = B_M(x, \varepsilon, t)$ (compare [10] for the metric case), and thus the topology induced by $(M, N, *, \diamond)$ coincides with the topology τ_M induced by $(M, *)$.

In [14], Park introduced the notion of a complete intuitionistic fuzzy metric space. It is proved in [10] that an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is complete if and only if $(X, M, *)$ is complete.

For the quasi-metric case we have the following definition and lemma according to [16] which will be useful in section 4.

Definition 2.12. (a) A sequence $(x_n)_n$ in an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is called Cauchy if for each $\varepsilon \in (0, 1)$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, and $N(x_n, x_m, t) < \varepsilon$, for all $n, m \geq n_0$.

(b) An intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is called bicomplete if $(X, M^i, N^s, *, \diamond)$ is a complete intuitionistic fuzzy metric space.

Lemma 2.13. (a) A sequence in an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the fuzzy metric space $(X, M^i, *)$.

(b) An intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is bicomplete if and only if the fuzzy quasi-metric space $(X, M, *)$ is bicomplete.

3. The Completion of a Fuzzy Metric Space and the bicompletion of a Fuzzy Quasi-Metric Space

In this section we will recall some known and crucial results on the completion of fuzzy metric spaces and the bicompletion of fuzzy quasi-metric spaces in the sense of Kramosil-Michalek which should be useful to provide a better understanding of the constructions made in the rest of the paper.

It was shown in [3] that the bicompletion of KM-fuzzy quasi-metric spaces can be achieved using the suprema of subsets of $[0, 1]$ and lower limits of sequences in $[0, 1]$ as it is sketched below:

Let $(X, M, *)$ be a KM-fuzzy quasi-metric space.

Denote by S the collection of all Cauchy sequences in $(X, M^i, *)$.

Definition 3.1. Define a relation \sim on S by

$$(x_n)_n \sim (y_n)_n \iff \sup_{0 < s < t} \lim_{n \geq k} M^i(x_n, y_n, s) = 1 \text{ for all } t > 0,$$

where by $\lim_{n \geq k} M^i(x_n, y_n, s)$ we denote, as usual, the lower limit of the sequence $(M^i(x_n, y_n, s))_n$, i.e.,

$$\lim_{n \geq k} M^i(x_n, y_n, s) = \sup_k \inf_{n \geq k} M^i(x_n, y_n, s).$$

Lemma 3.2. [3, Lemma 1] \sim is an equivalence relation on S .

Now denote by \tilde{X} the quotient S/\sim , and by $[(x_n)_n]$ the class of the element $(x_n)_n$ of S .

Lemma 3.3. [3, Lemma 3] For each $(x_n)_n, (y_n)_n \in S$ and each $(a_n)_n \in [(x_n)_n]$, $(b_n)_n \in [(y_n)_n]$, one has

$$\sup_{0 < s < t} \lim_{n \geq k} M((x_n)_n, (y_n)_n, t) = \sup_{0 < s < t} \lim_{n \geq k} M((a_n)_n, (b_n)_n, t),$$

for all $t > 0$.

Lemma 3.4. [8, Proposition 4.5] Let $(X, M, *)$ be a fuzzy quasi-metric space and $(Y, N, *)$ a bicomplete fuzzy quasi-metric space. If there is a τ_{M^i} -dense subset A of X and an isometry $f : (A, M, *) \rightarrow (Y, N, *)$, then there exists a unique isometry $F : (X, M, *) \rightarrow (Y, N, *)$ such that $F|_A = f$.

Definition 3.5. For each $[(x_n)_n], [(y_n)_n] \in \tilde{X}$, define

$$\tilde{M}([(x_n)_n], [(y_n)_n], 0) = 0,$$

and

$$\tilde{M}([(x_n)_n], [(y_n)_n], t) = \sup_{0 < s < t} \lim_{n \geq k} M(x_n, y_n, t),$$

for $t > 0$.

Then \tilde{M} is a function from $\tilde{X} \times \tilde{X} \times [0, \infty)$ to $[0, 1]$.

Definition 3.6. Define $i : X \rightarrow \tilde{X}$ such that, for each $x \in X$, $i(x)$ is the class of the constant sequence x, x, \dots

And from the above constructions we obtain:

Theorem 3.7. [3, Theorem 1] Let $(X, M, *)$ be a fuzzy quasi-metric space. Then:

- $(\tilde{M}, *)$ is a fuzzy quasi-metric on \tilde{X} .
- $i(X)$ is dense in $(\tilde{X}, \tilde{M}^i, *)$.
- $(X, M, *)$ is isometric to $(i(X), \tilde{M}, *)$.

(d) $(\widetilde{M}, *)$ is bicomplete.

(e) If $(Y, N, *)$ is a bicomplete fuzzy quasi-metric space such that $(X, M, *)$ is isometric to a τ_{N^i} -dense subspace of Y , then $(Y, N, *)$ and $(\widetilde{X}, \widetilde{M}, *)$ are isometric.

Remark 3.8. The preceding theorem implies that every fuzzy quasi-metric space $(X, M, *)$ has a bicompletion which is unique up to isometry. We refer to $(\widetilde{X}, \widetilde{M}, *)$ as the bicompletion of $(X, M, *)$.

4. The Bicompletion of an Intuitionistic Fuzzy Quasi-Metric Space

Now we shall show, with the help of the construction of the bicompletion of a KM-fuzzy quasi-metric space recalled above, that every intuitionistic fuzzy quasi-metric space in the sense of Kramosil and Michalek has a bicompletion which is unique up to isometry.

To this end we shall use the following two natural definitions.

Definition 4.1. A mapping f from an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ to an intuitionistic fuzzy quasi-metric space $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ is called an isometry if for each $x, y \in X$ and each $t > 0$,

$$M_Y(f(x), f(y), t) = M(x, y, t) \quad \text{and} \quad N_Y(f(x), f(y), t) = N(x, y, t).$$

It is clear that every isometry is a one-to-one mapping.

Two intuitionistic fuzzy quasi-metric spaces $(X, M, N, *, \diamond)$ and $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ are called isometric if there is an isometry from X onto Y .

Definition 4.2. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy quasi-metric space. A bicompletion of $(X, M, N, *, \diamond)$ is a bicomplete fuzzy quasi-metric space $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ such that $(X, M, N, *, \diamond)$ is isometric to a $\tau_{(M_Y)^i}$ -dense subspace of Y .

Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy quasi-metric space. Consider the fuzzy quasi-metric space $(X, M, *)$, and let $(\widetilde{X}_M, \widetilde{M}, *)$ be its bicompletion as constructed in Theorem 3.7.

Remark 4.3. Recall that, in particular, from Theorem 3.7 the mapping $i : X \rightarrow \widetilde{X}_M$ such that, for each $x \in X$, $i(x)$ is the class of the constant sequence x, x, \dots , is an isometry between $(X, M, *)$ and the dense subspace $(i(X), \widetilde{M}, *)$ of $(\widetilde{X}_M, \widetilde{M}, *)$.

In the following we will refer to i as i_M .

Lemma 4.4. [16] $(X, 1 - N, \diamond')$, where \diamond' is the dual t -norm of \diamond , is a fuzzy quasi-metric space.

Now we construct a fuzzy set $\widetilde{1 - N}$ in $\widetilde{X}_M \times \widetilde{X}_M \times [0, \infty)$ by

$$\widetilde{1 - N}([(x_n)_n], [(y_n)_n], 0) = 0,$$

and

$$\widetilde{1 - N}([(x_n)_n], [(y_n)_n], t) = \sup_{0 < s < t} \underline{\lim} (1 - N)(x_n, y_n, s),$$

whenever $t > 0$.

Remark 4.5. From the fact that $M + N \leq 1$, it obviously follows that every Cauchy sequence in the fuzzy metric space $(X, M^i, *)$ is a Cauchy sequence in the fuzzy metric space $(X, (1 - N)^i, \diamond')$.

We shall prove the following theorem that shows that each intuitionistic fuzzy quasi-metric space has a bicompletion which is unique up to isometry.

Theorem 4.6. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy quasi-metric space. Then:*

- (a) $(\widetilde{M}, 1 - (1 - N), *, \diamond)$ is an intuitionistic fuzzy quasi-metric on \widetilde{X}_M .
- (b) $i_M(X)$ is dense in $(\widetilde{X}_M, \widetilde{M}^i, *)$.
- (c) $(X, M, N, *, \diamond)$ is isometric to $(i_M(X), \widetilde{M}, 1 - (1 - N), *, \diamond)$.
- (d) $(\widetilde{M}, 1 - (1 - N), *, \diamond)$ is bicomplete.
- (e) If $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ is a bicomplete intuitionistic fuzzy quasi-metric space such that $(X, M, N, *, \diamond)$ is isometric to a $\tau_{(M_Y)^i}$ -dense subspace of Y , then $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ and $(\widetilde{X}_M, \widetilde{M}, 1 - (1 - N), *, \diamond)$ are isometric.

Proof. (a) Since $(X, M, *)$ is a fuzzy quasi-metric space, we know that $(\widetilde{M}, *)$ is a fuzzy quasi-metric on \widetilde{X} according to Theorem 3.7 (a).

Next we prove that $(\widetilde{X}_M, 1 - N, \diamond')$ is a fuzzy quasi-metric space.

Indeed $(1 - N, \diamond')$ satisfies property (KM1) of Definition 2.6.

Now let $(x_n)_n, (y_n)_n \in S$ such that $1 - N([(x_n)_n], [(y_n)_n], t) = 1$ for all $t > 0$. If $(z_n)_n \in [(y_n)_n]$, it follows from Lemma 3.3 that

$$\sup_{0 < s < t} \liminf (1 - N)((z_n)_n, (y_n)_n, t) = 1$$

for all $t > 0$, i.e., $(z_n)_n \in [(y_n)_n]$. The same argument shows that $(z_n)_n \in [(x_n)_n]$ whenever $(z_n)_n \in [(y_n)_n]$. We conclude that $1 - N([(x_n)_n], [(y_n)_n], t) = 1$ for all $t > 0$, if and only if $[(x_n)_n] = [(y_n)_n]$ (KM2).

For (KM3) to hold we need to prove that $(1 - N)(([x_n]_n), [(y_n]_n), t + s) \geq (1 - N)(([x_n]_n), [(z_n]_n), t) \diamond' (1 - N)(([z_n]_n), [(y_n]_n), s)$.

Indeed, since $(1 - N)(x_n, y_n, t + s) \geq (1 - N)(x_n, z_n, t) \diamond' (1 - N)(z_n, y_n, s)$ then

$$\begin{aligned} \sup_{0 < r < t+s} \liminf (1 - N)(x_n, y_n, t + s) &\geq \sup_{0 < r < t} \liminf (1 - N)(x_n, z_n, t) \diamond' \\ &\sup_{0 < r < s} \liminf (1 - N)(z_n, y_n, s) \end{aligned}$$

and hence the desired conclusion.

Similarly $(1 - N)(([x_n]_n), [(y_n]_n), -) : [0, \infty] \rightarrow [0, 1]$ (KM4) is left continuous as $(1 - N)(x_n, y_n, -)$ is left continuous as both the supremum and the inferior limit maintain the continuity.

Consequently $(\widetilde{1 - N}, \diamond')$ is a fuzzy quasi-metric on \widetilde{X} and then $(\widetilde{M}, 1 - (1 - N), *, \diamond)$ is an intuitionistic fuzzy quasi-metric space.

(b) As shown in Theorem 3.7 (b) and due to the way that $i_M(X)$ is constructed in Remark 4.3 we immediately have that $i_M(X)$ is dense in $(\widetilde{X}_M, \widetilde{M}^i, *)$.

(c) The mapping i_M satisfies for each $x, y \in X$ and $t > 0$,

$$\widetilde{M}(i_M(x), i_M(y), t) = M(x, y, t),$$

and, also using Definitions 3.5 and 4.1,

$$(1 - (\widetilde{1 - N}))(i_M(x), i_M(y), t) = (1 - (1 - N))(x, y, t) = N(x, y, t).$$

Thus i_M is an isometry between $(X, M, N, *, \diamond)$ and the subspace $i_M(X)$ of $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$, which is dense with respect to τ_{M^i} .

(d) Since $M \leq 1 - N$, we deduce that

$$\sup_{0 < s < t} \lim M(x_n, y_n, s) \leq \sup_{0 < s < t} \lim (1 - N)(x_n, y_n, s)$$

for each $t > 0$ and each pair $(x_n)_n, (y_n)_n$, of Cauchy sequences in $(X, M^i, *)$. Thus

$$\widetilde{M} \leq \widetilde{1 - N}.$$

Since, according to Theorem 3.7, $(\widetilde{X}_M, \widetilde{M}, *)$ is a bicomplete fuzzy quasi-metric space, we conclude using Lemma 2.13 (b) that $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ is a bicomplete intuitionistic fuzzy quasi-metric space.

(e) Let $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ be any bicompletion of $(X, M, N, *, \diamond)$. Then, there is an isometry j from $(X, M, N, *, \diamond)$ to $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ as defined in 4.1. On the other hand, since the fuzzy quasi-metric space $(Y, M_Y, *_Y)$ is a bicompletion of $(X, M, *)$, following Lemma 3.4, there is a unique isometry F from $(\widetilde{X}_M, \widetilde{M}, *)$ onto $(Y, M_Y, *_Y)$ such that $F(i_M) = j$. Taking into account Remark 4.5 we deduce from standard arguments (see for instance the proof of [8, Proposition 4.5]) that

$$N_Y(F(\tilde{x}), F(\tilde{y}), t) = (1 - (\widetilde{1 - N}))(\tilde{x}, \tilde{y}, t),$$

whenever $\tilde{x}, \tilde{y} \in \widetilde{X}_M$ and $t > 0$.

Then F is an isometry from $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ onto $(Y, M_Y, N_Y, *_Y, \diamond_Y)$. \square

Remark 4.7. The preceding theorem implies that every intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ has a bicompletion which is unique up to isometry. We refer to $(\widetilde{X}_M, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ as the bicompletion of $(X, M, N, *, \diamond)$.

Acknowledgements. Pedro Tirado acknowledges the support of the Ministry of Economy and Competitiveness of Spain, grant MTM2012-37894-C02-01

REFERENCES

- [1] C. Alaca, D. Turkoglu and C. Yildiz, *Fixed points in intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals, **29** (2006), 1073–1078.
- [2] F. Castro-Company and P. Tirado, *The bicompletion of intuitionistic fuzzy quasi-metric spaces*, 10th International Conference of Numerical Analysis and Applied Mathematics (ICNAAM), (2016), 844–847.
- [3] F. Castro-Company, S. Romaguera and P. Tirado, *The bicompletion of fuzzy quasi-metric spaces*, Fuzzy Sets and Systems, **166** (2011), 56–64.
- [4] Y. J. Cho, M. Grabiec and V. Radu, *On non symmetric topological and probabilistic structures*, Nova Sci. Publ. Inc., New York, 2006.
- [5] D. Dubois and P. Prade, *Fundamentals of fuzzy sets, the handbooks of fuzzy sets series*, FSHS, 7, Kluwer Academic, 2000.
- [6] P. Fletcher and W. F. Lindgren, *Quasi-uniform spaces*, Marcel Dekker, New York, 1982.
- [7] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64** (1994), 395–399.
- [8] V. Gregori and S. Romaguera, *Fuzzy quasi-metric spaces*, Appl. Gen. Topology, **5** (2004), 129–136.
- [9] V. Gregori, S. Romaguera and A. Sapena, *A characterization of bicompletable fuzzy quasi-metric spaces*, Fuzzy Sets and Systems, **152** (2005), 395–402.
- [10] V. Gregori, S. Romaguera and P. Veeramani, *A note on intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals, **28** (2006), 902–905.
- [11] I. Kramosil and J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11** (1975), 326–334.
- [12] H. P. A. Künzi, *Nonsymmetric topology*, In: Proceedings of the Colloquium on Topology, Szekszárd, Colloq. Math. Soc. János Bolyai Math. Studies, Hungary, **4** (1995), 303–338.
- [13] K. Menger, *Statistical metrics*, In: Proceedings of the National Academy of Sciences of the United States of America, **28** (1942), 535–537.
- [14] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals, **22** (2004), 1039–1046.
- [15] S. Romaguera, A. Sapena and P. Tirado, *The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words*, Topology Appl., **154** (2007), 2196–2203.
- [16] S. Romaguera and P. Tirado, *Contraction maps on ifqm-spaces with application to recurrence equations of Quicksort*, Electronic Notes in Theoret. Comput. Sci., **225** (2009), 269–279.
- [17] R. Saadati, S. M. Vaezpour and Y. J. Cho, *Quicksort algorithm: Application of a fixed point theorem in intuitionistic fuzzy quasi-metric spaces at a domain of words*, J. Comput. Appl. Math., **228** (2009), 219–225.
- [18] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math., **10** (1960), 314–334.
- [19] H. Sherwood, *On the completion of probabilistic metric spaces*, Z. Wahrsch. verw. Geb., **6** (1966), 62–64.
- [20] R. Vasuki and P. Veeramani, *Fixed point theorems and Cauchy sequences in fuzzy metric spaces*, Fuzzy Sets and Systems, **135** (2003), 415–417.

FRANCISCO CASTRO-COMPANY*, GILMATION S.L., CALLE 232, 66 LA CAÑADA, PATERNA, 46182, SPAIN

E-mail address: fracasco@mat.upv.es

PEDRO TIRADO, INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA, UNIVERSIDAD POLITÉCNICA DE VALENCIA, 46022, VALENCIA, SPAIN

E-mail address: pedtipe@mat.upv.es

*CORRESPONDING AUTHOR

ON THE BICOMPLETION OF INTUITIONISTIC FUZZY QUASI-METRIC SPACES

F. CASTRO-COMPANY AND P. TIRADO

کامل سازی مضاعف فضاهای شبکه متریک فازی شهودی

چکیده. براساس نتایج قبلی از مطالعه کامل سازی فضاهای متریک فازی، با بکار بردن مفهوم فضای متریک فازی به مفهوم Michalek و Kramosil برای بدست آوردن تعمیمی از شبکه متریک، نشان می دهیم که هر فضای شبه متریک فازی شهودی دارای یک کبمل سازی است که با تقریب ایزومتري یکتا است.

Archive of SID