

## HESITANT FUZZY LINGUISTIC ARITHMETIC AGGREGATION OPERATORS IN MULTIPLE ATTRIBUTE DECISION MAKING

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**ABSTRACT.** In this paper, we investigate the multiple attribute decision making (MADM) problem based on the arithmetic and geometric aggregation operators with hesitant fuzzy linguistic information. Then, motivated by the idea of traditional arithmetic operation, we have developed some aggregation operators for aggregating hesitant fuzzy linguistic information: hesitant fuzzy linguistic weighted average (HFLWA) operator, hesitant fuzzy linguistic ordered weighted average (HFLOWA) operator and hesitant fuzzy linguistic hybrid average (HFLHA) operator. Furthermore, we propose the concept of the dual hesitant fuzzy linguistic set and develop some aggregation operators with dual hesitant fuzzy linguistic information. Then, we have utilized these operators to develop some approaches to solve the hesitant fuzzy linguistic multiple attribute decision making problems. Finally, a practical example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

### 1. Introduction

Atanassov[1-3] introduced the concept of intuitionistic fuzzy set (IFS), which is a generalization of the concept of fuzzy set[59]. Each element in the IFS is expressed by an ordered pair, and each ordered pair is characterized by a membership degree and a non-membership degree. The sum of the membership degree and the non-membership degree of each ordered pair is less than or equal to 1. The intuitionistic fuzzy set has received more and more attention since its appearance[4,5,6,7,9,12,14,15,25,31,32,33,36,37,39,47,52,58]. Furthermore, Torra and Narukawa[28] and Torra[27] proposed the hesitant fuzzy set which permits the membership having a set of possible values and discussed the relationship between hesitant fuzzy set and intuitionistic fuzzy set, and showed that the envelope of hesitant fuzzy set is an intuitionistic fuzzy set. Xia and Xu[45] gave an intensive study on hesitant fuzzy information aggregation techniques and their application in decision making. Xu and Xia[49] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. Xu and Xia[50] defined the distance and correlation measures for hesitant fuzzy information and then discuss their properties in detail. Xu et al. [51]

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developed several series of aggregation operators for hesitant fuzzy information with the aid of quasi-arithmetic means. Motivated by the idea of prioritized aggregation operators[50], Wei[35] developed some prioritized aggregation operators for aggregating hesitant fuzzy information, and then apply them to develop some models for hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority level. Wei et al.[42] proposed two hesitant fuzzy Choquet integral aggregation operators: hesitant fuzzy choquet ordered averaging (HFCOA) operator and hesitant fuzzy choquet ordered geometric (HFCOG) operator and applied these operators to multiple attribute decision making with hesitant fuzzy information. Wang et al.[30] proposed the generalized hesitant fuzzy hybrid weighted distance (GHFHWD) measure, which is based on the generalized hesitant fuzzy weighted distance (GHFWD) measure and the generalized hesitant fuzzy ordered weighted distance (GHFOWD) measure[56] and studied some desirable properties of the GHFHWD measure. Zhu et al.[62] explored the geometric Bonferroni mean (GBM) considering both the BM and the geometric mean (GM) under hesitant fuzzy environment. They further defined the hesitant fuzzy geometric Bonferroni mean (HFGBM) and the hesitant fuzzy Choquet geometric Bonferroni mean (HFCGBM). Then they gave the definition of hesitant fuzzy geometric Bonferroni element (HFGBE), which is considered as the basic calculation unit in the HFGBM and reflects the conjunction between two aggregated arguments. Wei et al.[44] developed some approaches to hesitant fuzzy multiple attribute decision making with incomplete weight information. Wei and Zhang[38] developed a multiple criteria hesitant fuzzy decision making with Shapley value-based VIKOR method. Wang et a.[29] proposed some dual hesitant fuzzy aggregation operators in Multiple Attribute Decision Making. Wei and Zhang[41] proposed some induced hesitant interval-valued fuzzy einstein aggregation operators and applied these operators to multiple attribute decision making. Wei et al.[43] developed some hesitant triangular fuzzy information aggregation and applied these operators to multiple attribute decision making.

From above analysis, we can see that hesitant fuzzy set is a very useful tool to deal with uncertainty. More and more multiple attribute decision making theories and methods under hesitant fuzzy environment have been developed. Current methods are under the assumption that hesitant fuzzy set permits the membership having a set of possible exact and crisp values. However, under many conditions, for the real multiple attribute group decision making problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human think, thus, exact and crisp values are inadequate or insufficient to model real-life decision problems. Indeed, human judgments including preference information may be stated which permits the membership having a set of possible hesitant fuzzy linguistic values. So, in this paper we shall propose the concept of the hesitant fuzzy linguistic set based on hesitant fuzzy set to overcome this limitation. To do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to hesitant fuzzy linguistic set and some operational laws of hesitant fuzzy linguistic set. In Section 3 we have developed some hesitant

fuzzy linguistic aggregation operators: hesitant fuzzy linguistic weighted average (HFLWA) operator, hesitant fuzzy linguistic ordered weighted average (HFLOWA) operator, hesitant fuzzy linguistic hybrid average (HFLHA) operator and studied some desirable properties of the proposed operators. In Section 4, furthermore, we propose the concept of the dual hesitant fuzzy linguistic set and develop some aggregation operators with dual hesitant fuzzy linguistic information. In Section 5, we have applied these operators to develop some models for multiple attribute decision making (MADM) problems with hesitant fuzzy linguistic information. In Section 6, a practical example is given to verify the developed approach and to demonstrate its practicality and effectiveness. In Section 7, we conclude the paper and give some remarks.

## 2. Hesitant Fuzzy Linguistic Set

### 2.1. Hesitant Fuzzy Set.

In the following, we briefly describe some basic concepts and basic operational laws related to intuitionistic fuzzy set and hesitant fuzzy sets. Atanassov [1-3] extended the fuzzy set to the intuitionistic fuzzy set (IFS), shown as follows.

**Definition 2.1.** [1-3]. An IFS  $A$  in  $X$  is given by

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle | x \in X. \}, \quad (1)$$

where  $\mu_A : X \rightarrow [0, 1]$  and  $\nu_A : X \rightarrow [0, 1]$ , with the condition  $0 \leq \mu_A(x) + \nu_A(x) \leq 1, \forall x \in X$ . The numbers  $\mu_A(x)$  and  $\nu_A(x)$  represent, respectively, the membership degree and non- membership degree of the element to the set  $A$ .

However, when giving the membership degree of an element, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. For such cases, Torra [22] proposed another generation of FS.

**Definition 2.2.** [27]. Given a fixed set  $X$ , then a hesitant fuzzy set (HFS) on  $X$  is in terms of a function that when applied to  $X$  returns a subset of  $[0, 1]$ , the HFS can be expressed by mathematical symbol:

$$E = (\langle x, h_E(x) \rangle | x \in X.), \quad (2)$$

where  $h_E(x)$  is a set of some values in  $[0, 1]$ , denoting the possible membership degree of the element  $x \in X$  to the set  $E$ .

For convenience, Xia and Xu[45] called  $h = h_E(x)$  a hesitant fuzzy element (HFE) and  $H$  the set of all HFEs.

**Definition 2.3.** [45]. For a HFE  $h$ ,  $s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$  is called the score function of  $h$ , where  $\#h$  is the number of the elements in  $h$ . For two HFEs  $h_1$  and  $h_2$ , if  $s(h_1) > s(h_2)$ , then;  $h_1 > h_2$  if  $s(h_1) = s(h_2)$ , then  $h_1 = h_2$ .

Based on the relationship between the HFEs and IFVs, Xia and Xu[45] define some new operations on the HFEs  $h$ ,  $h_1$  and  $h_2$ :

- (1)  $h^\lambda = \cup_{\gamma \in h} \{\gamma^\lambda\}$ ;
- (2)  $\lambda h = \cup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}$

$$(3) h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\};$$

$$(4) h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}.$$

**2.2. Linguistic Term Set.** Let  $S = \{s_i | i = 1, 2, \dots, t\}$  be a linguistic term set with odd cardinality. Any label,  $s_i$  represents a possible value for a linguistic variable, and it should satisfy the following characteristics [8, 10, 11, 13, 16, 17, 21, 23, 24, 34, 40, 60]: ① The set is ordered:  $s_i > s_j$ , if  $i > j$ ; ② There is the negation operator:  $neg(s_i) = s_j$  such that  $i + j = t + 1$ ; ③ Max operator:  $\max(s_i, s_j) = s_i$ , if  $s_i \geq s_j$ ; ④ Min operator:  $\min(s_i, s_j) = s_i$ , if  $s_i \leq s_j$ . For example,  $S$  can be defined as

$$S = \{s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{medium}, \\ s_5 = \text{good}, s_6 = \text{very good}, s_7 = \text{extremely good}\}$$

To preserve all the given information, we extend the discrete term set  $S$  to a continuous term set  $\bar{S} = \{s_a | s_1 \leq s_a \leq s_q, a \in [1, q]\}$ , where  $q$  is a sufficiently large positive integer. If  $s_a \in S$ , then we call  $s_a$  the original linguistic term, otherwise, we call  $s_a$  the virtual linguistic term. In general, the decision maker uses the original linguistic term to evaluate attributes and alternatives, and the virtual linguistic terms can only appear in calculation [46, 61].

Consider any two linguistic variables  $s_\alpha$  and  $s_\beta$ ,  $s_\alpha, s_\beta \in \bar{S}$ ,  $\lambda \in [0, 1]$  we define their operational laws as follows [46]:

$$(1) s_\alpha \oplus s_\beta = s_{\alpha+\beta};$$

$$(2) s_\alpha \otimes s_\beta = s_{\alpha\beta};$$

$$(3) \lambda s_\alpha = s_{\lambda\alpha};$$

$$(4) (s_\alpha)^\lambda = s_{\alpha^\lambda}.$$

### 2.3. Hesitant Fuzzy Linguistic Set.

From above analysis, we can see that hesitant fuzzy set is a very useful tool to deal with uncertainty. More and more multiple attribute decision making theories and methods under hesitant fuzzy environment have been developed. Current methods are under the assumption that hesitant fuzzy set permits the membership having a set of possible exact and crisp values. However, under many conditions, for the real multiple attribute group decision making problems, the decision information about alternatives is usually uncertain or fuzzy due to the increasing complexity of the socio-economic environment and the vagueness of inherent subjective nature of human think, thus, exact and crisp values are inadequate or insufficient to model real-life decision problems. Indeed, human judgments including preference information may be stated which permits the membership having a set of possible hesitant fuzzy linguistic values. In the following, we shall propose some basic concepts and basic operational laws related to hesitant fuzzy linguistic set.

**Definition 2.4.** Given a fixed set  $X$ , then a hesitant fuzzy linguistic set (HFLS) on  $X$  is in terms of a function that when applied to  $X$  returns a subset of  $[0, 1]$ . To be easily understood, the HFLS can be expressed by mathematical symbol as follows:

$$A = (\langle x, s_{\theta(x)}, h_A(x) \rangle | x \in X.), \quad (3)$$

where  $h_A(x)$  is a set of some values in  $[0, 1]$ , denoting the possible membership degree of the element  $x \in X$  to the linguistic set  $s_{\theta(x)}$ . For convenience, we called  $a = \langle s_{\theta(x)}, h_A(x) \rangle$  a hesitant fuzzy linguistic element (HFLE) and the set of all HFLEs.

**Definition 2.5.** For a HFLE  $a = \langle s_{\theta(x)}, h_A(x) \rangle$ ,  $s(a) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma \cdot s_{\theta(x)}$  is called the score function of  $a$ , where  $\#h$  is the number of the elements in  $h$ . For two HFLEs  $a_1$  and  $a_2$ , if  $s(a_1) > s(a_2)$ , then  $a_1 > a_2$ ; if  $s(a_1) = s(a_2)$ , then  $a_1 = a_2$ .

Based on the relationship between the HFLEs, we shall define some new operations on the HFLEs  $a = \langle s_{\theta(a)}, h(a) \rangle$ ,  $a_1 = \langle s_{\theta(a_1)}, h(a_1) \rangle$  and  $a_2 = \langle s_{\theta(a_2)}, h(a_2) \rangle$ :

- (1)  $a^\lambda = \langle s_{\theta(a)^\lambda}, \cup_{\gamma(a) \in h(a)} \{\gamma(a)^\lambda\} \rangle$ ;
- (2)  $\lambda a = \langle s_{\lambda \theta(a)}, \cup_{\gamma(a) \in h(a)} \{1 - (1 - \gamma(a))^\lambda\} \rangle$ ;
- (3)  $a_1 \oplus a_2 = \langle s_{\theta(a_1) + \theta(a_2)}, \cup_{\gamma(a_1) \in h(a_1), \gamma(a_2) \in h(a_2)} \{\gamma(a_1) + \gamma(a_2) - \gamma(a_1)\gamma(a_2)\} \rangle$ ;
- (4)  $a_1 \otimes a_2 = \langle s_{\theta(a_1) \times \theta(a_2)}, \cup_{\gamma(a_1) \in h(a_1), \gamma(a_2) \in h(a_2)} \{\gamma(a_1)\gamma(a_2)\} \rangle$ .

### 3. Hesitant Fuzzy Linguistic Arithmetic Aggregation Operators

Based on the traditional arithmetic and geometric aggregating operators[19,20, 48, 54, 56] and motivated by the operational law of hesitant fuzzy linguistic sets, in the following, we shall develop some hesitant fuzzy linguistic arithmetic aggregation operator as follows.

**Definition 3.1.** Let  $a_j = (s_{\theta(a_j)}, h(a_j)) (j = 1, 2, \dots, n)$  be a collection of HFLEs, then we define the hesitant fuzzy linguistic weighted average (HFLWA) operator as follows:

$$\text{HFLWA}_\omega(a_1, a_2, \dots, a_n) = \bigoplus_{j=1}^n (\omega_j a_j), \tag{4}$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  be the weight vector of  $a_j (j = 1, 2, \dots, n)$ , and  $\omega_j > 0$ ,  $\sum_{j=1}^n \omega_j = 1$ .

Based on operations of the hesitant fuzzy linguistic values described, we can derive Theorem 3.2.

**Theorem 3.2.** Let  $a_j = (s_{\theta(a_j)}, h(a_j)) (j = 1, 2, \dots, n)$  be a collection of HFLEs, then their aggregated value by using the HFLWA operator is also a HFLE, and

$$\begin{aligned} \text{HFLWA}_\omega(a_1, a_2, \dots, a_n) &= \bigoplus_{j=1}^n (\omega_j a_j) \\ &= \left\langle \sum_{j=1}^n \omega_j s_{\theta(a_j)}, \left( \cup_{\gamma(a_1) \in h(a_1), \gamma(a_2) \in h(a_2), \dots, \gamma(a_n) \in h(a_n)} \left\{ 1 - \prod_{j=1}^n (1 - \gamma(a_j))^{\omega_j} \right\} \right) \right\rangle \end{aligned} \tag{5}$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  be the weight vector of  $a_j (j = 1, 2, \dots, n)$ , and  $\omega_j > 0$ ,  $\sum_{j=1}^n \omega_j = 1$

**Definition 3.3.** Let  $a_j = (s_{\theta(a_j)}, h(a_j)) (j = 1, 2, \dots, n)$  be a collection of HFLEs, the hesitant fuzzy linguistic ordered weighted average (HFLOWA) operator is defined as follows:

$$\text{HFLOWA}_w(a_1, a_2, \dots, a_n) = \bigoplus_{j=1}^n (w_j a_{\sigma(j)}) \quad (6)$$

where  $(\sigma(1), \sigma(2), \dots, \sigma(n))$  is a permutation of  $(1, 2, \dots, n)$ , such that  $a_{\sigma(j-1)} \geq a_{\sigma(j)}$  for all  $j = 2, \dots, n$ , and  $w = (w_1, w_2, \dots, w_n)^T$  is the aggregation-associated weight vector such that  $w_j \in [0, 1]$  and  $\sum_{j=1}^n w_j = 1$ .

Based on operations of the hesitant fuzzy linguistic values described, we can derive Theorem 3.4.

**Theorem 3.4.** Let  $a_j = (s_{\theta(a_j)}, h(a_j)) (j = 1, 2, \dots, n)$  be a collection of HFLEs, then their aggregated value by using the HFLOWA operator is also a HFLE, and

$$\begin{aligned} \text{HFLOWA}_w(a_1, a_2, \dots, a_n) &= \bigoplus_{j=1}^n (w_j a_{\sigma(j)}) \\ &= \left\langle \sum_{j=1}^n w_j s_{\theta(a_{\sigma(j)})}, \left( \bigcup_{\gamma(a_{\sigma(1)}) \in h(a_{\sigma(1)}), \gamma(a_{\sigma(2)}) \in h(a_{\sigma(2)}), \dots, \gamma(a_{\sigma(n)}) \in h(a_{\sigma(n)})} \right. \right. \\ &\quad \left. \left. \left\{ 1 - \prod_{j=1}^n (1 - \gamma(a_{\sigma(j)}))^{w_j} \right\} \right) \right\rangle \end{aligned} \quad (7)$$

where  $(\sigma(1), \sigma(2), \dots, \sigma(n))$  is a permutation of  $(1, 2, \dots, n)$ , such that  $a_{\sigma(j-1)} \geq a_{\sigma(j)}$  for all  $j = 2, \dots, n$ , and  $w = (w_1, w_2, \dots, w_n)^T$  is the aggregation-associated weight vector such that  $w_j \in [0, 1]$  and  $\sum_{j=1}^n w_j = 1$

From Definitions 3.1 and 3.3, we know that the HFLWA operator weights the hesitant fuzzy linguistic argument itself, while the HFLOWA operator weights the ordered positions of the hesitant fuzzy linguistic arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the HFLWA and HFLOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose a hesitant fuzzy linguistic hybrid average (HFLHA) operator.

**Definition 3.5.** A hesitant fuzzy linguistic hybrid average (HFLHA) operator is defined as follows:

$$\text{HFLHA}_{w,\omega}(a_1, a_2, \dots, a_n) = \bigoplus_{j=1}^n (w_j \dot{a}_{\sigma(j)}) \quad (8)$$

where  $w = (w_1, w_2, \dots, w_n)$  is the associated weighting vector, with  $w_j \in [0, 1]$ ,  $\sum_{j=1}^n w_j = 1$ , and  $\dot{h}_{\sigma(j)}$  is the  $j$ -th largest element of the hesitant fuzzy linguistic arguments  $\dot{a}_{\sigma(j)}$  ( $\dot{a}_{\sigma(j)} = n\omega_j a_j$ ,  $j = 1, 2, \dots, n$ ),  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$  is the weighting vector of hesitant fuzzy linguistic arguments  $a_i (i = 1, 2, \dots, n)$ , with  $\omega_i \in [0, 1]$ ,  $\sum_{i=1}^n \omega_i = 1$ , and  $n$  is the balancing coefficient. Especially, if  $w =$

$(1/n, 1/n, \dots, 1/n)^T$ , then HFLHA is reduced to the hesitant fuzzy linguistic weighted average (HFLWA) operator; if  $\omega = (1/n, 1/n, \dots, 1/n)$ , then HFLHA is reduced to the hesitant fuzzy linguistic ordered weighted average (HFLOWA) operator.

Based on operations of the hesitant fuzzy linguistic values described, we can derive Theorem 3.6.

**Theorem 3.6.** *Let  $a_j = (s_{\theta(a_j)}, h(a_j)) (j = 1, 2, \dots, n)$  be a collection of HFLEs, then their aggregated value by using the HFLHA operator is also a HFLE, and*

$$\begin{aligned} \text{HFLHA}_{w,\omega}(a_1, a_2, \dots, a_n) &= \bigoplus_{j=1}^n (w_j \dot{a}_{\sigma(j)}) \\ &= \left\langle \sum_{j=1}^n w_j s_{\theta(\dot{a}_{\sigma(j)})}, \left( \cup_{\gamma(\dot{a}_{\sigma(1)}) \in h(\dot{a}_{\sigma(1)}), \gamma(\dot{a}_{\sigma(2)}) \in h(\dot{a}_{\sigma(2)}), \dots, \gamma(\dot{a}_{\sigma(n)}) \in h(\dot{a}_{\sigma(n)})} \right. \right. \\ &\quad \left. \left. \left\{ 1 - \prod_{j=1}^n (1 - \gamma(\dot{a}_{\sigma(j)}))^{w_j} \right\} \right) \right\rangle \end{aligned} \quad (9)$$

where  $w = (w_1, w_2, \dots, w_n)$  is the associated weighting vector, with  $w_j \in [0, 1]$ ,  $\sum_{j=1}^n w_j = 1$ , and  $\dot{h}_{\sigma(j)}$  is the  $j$ -th largest element of the hesitant fuzzy linguistic arguments  $\dot{a}_{\sigma(j)}$  ( $\dot{a}_{\sigma(j)} = n\omega_j a_j, j = 1, 2, \dots, n$ ),  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$  is the weighting vector of hesitant fuzzy linguistic arguments  $a_i (i = 1, 2, \dots, n)$ , with  $\omega_i \in [0, 1]$ ,  $\sum_{i=1}^n \omega_i = 1$ , and  $n$  is the balancing coefficient.

#### 4. Dual Hesitant Fuzzy Linguistic Set

In the following, we shall propose some basic concepts and basic operational laws related to dual hesitant fuzzy linguistic set.

**Definition 4.1.** Given a fixed set  $X$ , then a dual hesitant fuzzy linguistic set (DHFLS) on  $X$  is in terms of a function that when applied to  $X$  returns a subset of  $[0, 1]$ . To be easily understood, the DHFLS can be expressed by mathematical symbol as follows:

$$A = (\langle x, s_{\theta(x)}, (h_A(x), g(x)) \rangle | x \in X.) \quad (10)$$

which  $h(x)$  and  $g(x)$  are two sets of some values in  $[0, 1]$ , denoting the possible membership degree and non-membership degrees of the element  $x \in X$  to the linguistic set  $s_{\theta(x)}$  respectively, with the conditions:

$$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+, \eta^+ \leq 1$$

where

$$\gamma \in h(x), \eta \in g(x), \gamma^+ \in h^+(x) = \cup_{\gamma \in h(x)} \max\{\gamma\}, \eta^+ \in g^+(x) = \cup_{\eta \in g(x)} \max\{\eta\}$$

for all  $x \in X$  to the linguistic set  $s_{\theta(x)}$ .

For convenience, we called  $a = \langle s_{\theta(x)}, (h_A(x), g(x)) \rangle$  a dual hesitant fuzzy linguistic element (DHFLE) and  $A$  the set of all DHFLEs.

**Definition 4.2.** For three DHFLEs  $a = \langle s_{\theta(x)}, (h_A(x), g(x)) \rangle$ ,  $a_1 = \langle s_{\theta(x_1)}, (h_A(x_1), g(x_1)) \rangle$  and  $a_2 = \langle s_{\theta(x_2)}, (h_A(x_2), g(x_2)) \rangle$ ,

$$s(a) = \frac{(1 + \frac{1}{\#h} \sum_{\gamma \in h} \gamma - \frac{1}{\#g} \sum_{\eta \in g} \eta) s_{\theta(x)}}{2}$$

is called the score function of  $a$ , and  $p(a) = (\frac{1}{\#h} \sum_{\gamma \in h} \gamma + \frac{1}{\#g} \sum_{\eta \in g} \eta) s_{\theta(x)}$  the accuracy function of  $a$ , where  $\#h$  and  $\#g$  are the numbers of the elements in  $h$  and  $g$  respectively, then

- If  $s(a_1) > s(a_2)$ , then  $a_1$  is superior to  $a_2$ , denoted by  $a_1 \succ a_2$ ;
- If  $s(a_1) = s(a_2)$ , then
  - (1) If  $p(a_1) = p(a_2)$ , then  $a_1$  is equivalent to  $a_2$ , denoted by  $a_1 \sim a_2$ ;
  - (2) If  $p(a_1) > p(a_2)$ , then  $a_1$  is superior to  $a_2$ , denoted by  $a_1 \succ a_2$ .

Based on the relationship between the DHFLEs, we shall define some new operations on the DHFLEs  $a = \langle s_{\theta(x)}, (h_A(x), g(x)) \rangle$ ,  $a_1 = \langle s_{\theta(x_1)}, (h_A(x_1), g(x_1)) \rangle$  and  $a_2 = \langle s_{\theta(x_2)}, (h_A(x_2), g(x_2)) \rangle$ ,

- (1)  $a^\lambda = \langle s_{\theta(a)^\lambda}, \cup_{\gamma \in h, \eta \in g} (\{\gamma^\lambda\}, \{1 - (1 - \eta)^\lambda\}) \rangle, \lambda > 0$ ;
- (2)  $\lambda a = \langle s_{\lambda \theta(a)}, \cup_{\gamma \in h, \eta \in g} (\{1 - (1 - \gamma)^\lambda\}, \{\mu^\lambda\}) \rangle, \lambda > 0$ ;
- (3)  $a_1 \oplus a_2 = \langle s_{\theta(a_1) + \theta(a_2)}, \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} (\{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}, \{\eta_1 \eta_2\}) \rangle$ ;
- (4)  $a_1 \otimes a_2 = \langle s_{\theta(a_1) \times \theta(a_2)}, \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} (\{\gamma_1 \gamma_2\}, \{\eta_1 + \eta_2 - \eta_1 \eta_2\}) \rangle$ .

#### 4.1. Dual Hesitant Fuzzy Linguistic Arithmetic Aggregation Operators.

Motivated by the operational law of dual hesitant fuzzy linguistic set, in the following, we shall develop some dual hesitant fuzzy linguistic arithmetic aggregation operator based on the operations of DHFLEs.

**Definition 4.3.** Let  $a_j = \langle s_{\theta(a_j)}, (h_j, g_j) \rangle (j = 1, 2, \dots, n)$  be a collection of DHFLEs, then we define the dual hesitant fuzzy linguistic weighted average (DHFLWA) operator as follows:

$$\begin{aligned} \text{DHFLWA}_\omega(a_1, a_2, \dots, a_n) &= \bigoplus_{j=1}^n (\omega_j a_j) \\ &= \langle \sum_{j=1}^n \omega_j s_{\theta(a_j)}, \cup_{\gamma_j \in h_j, \eta_j \in g_j} (\{1 - \prod_{j=1}^n (1 - \gamma_j)^{\omega_j}\}, \{\prod_{j=1}^n (\eta_j)^{\omega_j}\}) \rangle \end{aligned} \quad (11)$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  be the weight vector of  $a_j (j = 1, 2, \dots, n)$ , and  $\omega_j > 0$ ,  $\sum_{j=1}^n \omega_j = 1$ .

**Definition 4.4.** Let  $a_j = \langle s_{\theta(a_j)}, (h_j, g_j) \rangle (j = 1, 2, \dots, n)$  be a collection of DHFLEs, then we define the dual hesitant fuzzy linguistic ordered weighted average (DHFLOWA) operator as follows:

$$\begin{aligned} \text{DHFLOWA}_w(a_1, a_2, \dots, a_n) &= \bigoplus_{j=1}^n (w_j a_{\sigma(j)}) \\ &= \langle \sum_{j=1}^n w_j s_{\theta(a_{\sigma(j)})}, \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} (\{1 - \prod_{j=1}^n (1 - \gamma_{\sigma(j)})^{w_j}\}, \{\prod_{j=1}^n (\eta_{\sigma(j)})^{w_j}\}) \rangle \end{aligned} \quad (12)$$



where  $(\sigma(1), \sigma(2), \dots, \sigma(n))$  is a permutation of  $(1, 2, \dots, n)$ , such that  $a_{\sigma(j-1)} \geq a_{\sigma(j)}$  for all  $j = 2, \dots, n$ , and  $w = (w_1, w_2, \dots, w_n)^T$  is the aggregation-associated weight vector such that  $w_j \in [0, 1]$  and  $\sum_{j=1}^n w_j = 1$ .

From Definitions 4.3 and 4.4, we know that the DHFLWA operator weights the dual hesitant fuzzy linguistic argument itself, while the DHFLOWA operator weights the ordered positions of the dual hesitant fuzzy linguistic arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the DHFLWA and DHFLOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the dual hesitant fuzzy linguistic hybrid average (DHFLHA) operator.

**Definition 4.5.** A dual hesitant fuzzy linguistic hybrid average (DHFLHA) operator is defined as follows:

$$\begin{aligned} \text{DHFLHA}_{w,\omega}(a_1, a_2, \dots, a_n) &= \bigoplus_{j=1}^n (w_j \dot{a}_{\sigma(j)}) \\ &= \left\langle \sum_{j=1}^n w_j s_{\theta(\dot{a}_{\sigma(j)})}, \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in \dot{g}_{\sigma(j)}} \left\{ \left\{ 1 - \prod_{j=1}^n (1 - \gamma_{\sigma(j)})^{w_j} \right\}, \left\{ \prod_{j=1}^n (\dot{\eta}_{\sigma(j)})^{w_j} \right\} \right\} \right\rangle \end{aligned} \quad (13)$$

where  $w = (w_1, w_2, \dots, w_n)$  is the associated weighting vector, with  $w_j \in [0, 1]$ ,  $\sum_{j=1}^n w_j = 1$ , and  $\dot{a}_{\sigma(j)}$  is the  $j$ -th largest element of the dual hesitant fuzzy linguistic arguments  $\dot{a}_j (\dot{a}_j = n\omega_j a_j, j = 1, 2, \dots, n)$ ,  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$  is the weighting vector of dual hesitant fuzzy linguistic arguments  $a_i (i = 1, 2, \dots, n)$ , with  $\omega_i \in [0, 1]$ ,  $\sum_{i=1}^n \omega_i = 1$ , and  $n$  is the balancing coefficient. Especially, if  $w = (1/n, 1/n, \dots, 1/n)^T$ , then DHFLHA is reduced to the dual hesitant fuzzy linguistic weighted average (DHFLWA) operator; if  $\omega = (1/n, 1/n, \dots, 1/n)$ , then DHFLHA is reduced to the dual hesitant fuzzy linguistic ordered weighted average (DHFLOWA) operator.

#### 4.2. Dual Hesitant Fuzzy Linguistic Geometric Aggregation Operators.

Based on the dual hesitant fuzzy linguistic arithmetic aggregation operators and the geometric mean, here we define some dual hesitant fuzzy linguistic geometric aggregation operators:

**Definition 4.6.** Let  $a_j = \langle s_{\theta(a_j)}, (h_j, g_j) \rangle (j = 1, 2, \dots, n)$  be a collection of DHFLEs, then we define the dual hesitant fuzzy linguistic weighted geometric (DHFLWG) operator as follows:

$$\begin{aligned} \text{DHFLWG}_{\omega}(a_1, a_2, \dots, a_n) &= \bigotimes_{j=1}^n (a_j)^{\omega_j} \\ &= \left\langle \prod_{j=1}^n (s_{\theta(a_j)})^{\omega_j}, \cup_{\gamma_j \in h_j, \eta_j \in g_j} \left( \left\{ \prod_{j=1}^n (\gamma_j)^{\omega_j} \right\}, \left\{ 1 - \prod_{j=1}^n (1 - \eta_j)^{\omega_j} \right\} \right) \right\rangle \end{aligned} \quad (14)$$

where  $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$  be the weight vector of  $a_j (j = 1, 2, \dots, n)$ , and  $\omega_j > 0$ ,  $\sum_{j=1}^n \omega_j = 1$ .

**Definition 4.7.** Let  $a_j = \langle s_{\theta(a_j)}, (h_j, g_j) \rangle (j = 1, 2, \dots, n)$  be a collection of DHFLEs, the dual hesitant fuzzy linguistic ordered weighted geometric (DHFLOWG) operator is defined as follows:

$$\begin{aligned} \text{DHFLOWG}_w(a_1, a_2, \dots, a_n) &= \bigotimes_{j=1}^n (a_{\sigma(j)})^{w_j} \\ &= \left\langle \prod_{j=1}^n (s_{\theta(a_{\sigma(j)})})^{w_j}, \cup_{\gamma_{\sigma(j)} \in h_{\sigma(j)}, \eta_{\sigma(j)} \in g_{\sigma(j)}} (\{\prod_{j=1}^n (\gamma_{\sigma(j)})^{w_j}\}, \{1 - \prod_{j=1}^n (1 - \eta_{\sigma(j)})^{w_j}\}) \right\rangle \end{aligned} \quad (15)$$

where  $(\sigma(1), \sigma(2), \dots, \sigma(n))$  is a permutation of  $(1, 2, \dots, n)$ , such that  $a_{\sigma(j-1)} \geq a_{\sigma(j)}$  for all  $j = 2, \dots, n$ , and  $w = (w_1, w_2, \dots, w_n)^T$  is the aggregation-associated weight vector such that  $w_j \in [0, 1]$  and  $\sum_{j=1}^n w_j = 1$ .

From Definitions 4.6 and 4.7, we know that the DHFLWG operator weights the dual hesitant fuzzy linguistic argument itself, while the DHFLOWG operator weights the ordered positions of the dual hesitant fuzzy linguistic arguments instead of weighting the arguments themselves. Therefore, weights represent different aspects in both the FHFLWG and DHFLOWG operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose a dual hesitant fuzzy linguistic hybrid geometric (DHFLHG) operator.

**Definition 4.8.** A dual hesitant fuzzy linguistic hybrid geometric (DHFLHG) operator is defined as follows:

$$\begin{aligned} \text{DHFLHG}_{w,\omega}(a_1, a_2, \dots, a_n) &= \bigotimes_{j=1}^n (\hat{a}_{\sigma(j)})^{w_j} \\ &= \left\langle \prod_{j=1}^n (s_{\theta(\hat{a}_{\sigma(j)})})^{w_j}, \cup_{\dot{\gamma}_{\sigma(j)} \in \dot{h}_{\sigma(j)}, \dot{\eta}_{\sigma(j)} \in \dot{g}_{\sigma(j)}} (\{\prod_{j=1}^n (\dot{\gamma}_{\sigma(j)})^{w_j}\}, \{1 - \prod_{j=1}^n (1 - \dot{\eta}_{\sigma(j)})^{w_j}\}) \right\rangle \end{aligned} \quad (16)$$

where  $w = (w_1, w_2, \dots, w_n)$  is the associated weighting vector, with  $w_j \in [0, 1]$ ,  $\sum_{j=1}^n w_j = 1$ , and  $\hat{a}_{\sigma(j)}$  is the  $j$ -th largest element of the dual hesitant linguistic fuzzy arguments  $\hat{a}_j (\hat{a}_j = (a_j)^{n\omega_j}, j = 1, 2, \dots, n)$ ,  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$  is the weighting vector of dual hesitant fuzzy linguistic arguments  $a_j (j = 1, 2, \dots, n)$ , with  $\omega_j \in [0, 1]$ ,  $\sum_{j=1}^n \omega_j = 1$ , and  $n$  is the balancing coefficient. Especially, if  $w = (1/n, 1/n, \dots, 1/n)^T$ , then DHFLHG is reduced to the dual hesitant fuzzy linguistic weighted geometric (DHFLWG) operator; if  $\omega = (1/n, 1/n, \dots, 1/n)$ , then DHFLHG is reduced to the dual hesitant fuzzy linguistic ordered weighted geometric (DHFLOWG) operator.

## 5. An Approach to Multiple Attribute Decision Making with Hesitant Fuzzy Linguistic Information

In this section, we shall utilize the hesitant linguistic aggregation operators to multiple attribute decision making with hesitant fuzzy linguistic information.

The following assumptions or notations are used to represent the MADM problems for potential evaluation of emerging technology commercialization with hesitant fuzzy linguistic information. Let  $A = \{A_1, A_2, \dots, A_m\}$  be a discrete set of alternatives, and  $G = \{G_1, G_2, \dots, G_n\}$  be the state of nature. If the decision makers provide several values for the alternative  $A_i$  under the state of nature  $G_j$  with respect to  $s_{\theta_{ij}}$  with anonymity, these values can be considered as a hesitant fuzzy linguistic element  $\langle s_{\theta_{ij}}, h_{ij} \rangle$ . In the case where two decision makers provide the same value, then the value emerges only once in  $h_{ij}$ . Suppose that the decision matrix  $H = (\tilde{h}_{ij})_{m \times n} = (\langle s_{\theta_{ij}}, h_{ij} \rangle)_{m \times n}$  is the hesitant fuzzy linguistic decision matrix, where  $\langle s_{\theta_{ij}}, h_{ij} \rangle (i = 1, 2, \dots, m, j = 1, 2, \dots, n)$  are in the form of HFLEs.

In the following, we apply the HFLWA operator to the MADM problems for potential evaluation of emerging technology commercialization with hesitant fuzzy linguistic information.

**Step 1:** We utilize the decision information given in matrix  $H$ , and the HFLWA operator to derive the overall preference values  $\tilde{h}_i (i = 1, 2, \dots, m)$  of the alternative  $A_i$ , i.e.,

$$\begin{aligned} \tilde{h}_i &= (\langle s_{\theta_i}, h_i \rangle) = \text{HFLWA}_\omega(\tilde{h}_{i1}, \tilde{h}_{i2}, \dots, \tilde{h}_{in}) \\ &= \text{HFLWA}_\omega(\langle s_{\theta_{i1}}, h_{i1} \rangle, \langle s_{\theta_{i2}}, h_{i2} \rangle, \dots, \langle s_{\theta_{in}}, h_{in} \rangle) = \bigoplus_{j=1}^n (\omega_j \tilde{h}_{ij}) \\ &= \langle \sum_{j=1}^n \omega_j s_{\theta_{ij}}, (\cup_{\gamma_{i1} \in h_{i1}, \gamma_{i2} \in h_{i2}, \dots, \gamma_{in} \in h_{in}} \{1 - \prod_{j=1}^n (1 - \gamma_{ij})^{\omega_j}\}) \rangle \end{aligned} \tag{17}$$

**Step 2:** Calculate the scores  $S(\tilde{h}_i) (i = 1, 2, \dots, m)$  of the overall hesitant fuzzy linguistic preference values  $\tilde{h}_i (i = 1, 2, \dots, m)$  to rank all the alternatives  $A_i (i = 1, 2, \dots, m)$  and then to select the best one(s).

**Step 3:** Rank all the alternatives  $A_i (i = 1, 2, \dots, m)$  and select the best one(s) in accordance with  $S(\tilde{h}_i) (i = 1, 2, \dots, m)$ .

**Step 4:** End.

## 6. Numerical Example

Thus, in this section we shall present a numerical example to show potential evaluation of emerging technology commercialization with hesitant fuzzy linguistic information in order to illustrate the method proposed in this paper. There is a panel with five possible emerging technology enterprises  $A_i (i = 1, 2, 3, 4, 5)$  to select. The experts selects four attribute to evaluate the five possible emerging technology enterprises: ①  $G_1$  is the technical advancement; ②  $G_2$  is the potential market risk; ③  $G_3$  is the industrialization infrastructure, human resources and financial conditions; ④  $G_4$  is the employment creation and the development of science and technology. In order to avoid influence each other, the decision makers are required to evaluate the five possible emerging technology enterprises  $A_i (i = 1, 2, 3, 4, 5)$  under the above four attributes in anonymity and the hesitant fuzzy linguistic decision matrix  $H = (\tilde{h}_{ij})_{5 \times 4} = (\langle s_{\theta_{ij}}, h_{ij} \rangle)_{5 \times 4}$  is presented in Table 1, where  $\langle s_{\theta_{ij}}, h_{ij} \rangle (i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$  are in the form of HFLEs. The information about the attribute weights is known as follows:  $\omega = (0.3, 0.2, 0.4, 0.1)$ .

	$G_1$	$G_2$	$G_3$	$G_4$
$A_1$	$\langle s_4, (0.3, 0.5) \rangle$	$\langle s_2, (0.6, 0.7, 0.8) \rangle$	$\langle s_1, (0.7, 0.8) \rangle$	$\langle s_3, (0.8, 0.9) \rangle$
$A_2$	$\langle s_1, (0.3, 0.4, 0.5) \rangle$	$\langle s_4, (0.6, 0.9) \rangle$	$\langle s_2, (0.6, 0.7) \rangle$	$\langle s_4, (0.4, 0.5) \rangle$
$A_3$	$\langle s_5, (0.4, 0.6) \rangle$	$\langle s_1, (0.7, 0.8) \rangle$	$\langle s_4, (0.3, 0.5, 0.7) \rangle$	$\langle s_2, (0.6, 0.7) \rangle$
$A_4$	$\langle s_4, (0.7, 0.9) \rangle$	$\langle s_6, (0.3, 0.4) \rangle$	$\langle s_7, (0.5, 0.7) \rangle$	$\langle s_1, (0.3, 0.5) \rangle$
$A_5$	$\langle s_3, (0.2, 0.3) \rangle$	$\langle s_1, (0.6, 0.7) \rangle$	$\langle s_3, (0.5, 0.6) \rangle$	$\langle s_1, (0.7, 0.8, 0.9) \rangle$

TABLE 1. Hesitant Fuzzy Linguistic Decision Matrix

In the following, we utilize the approach developed to show potential evaluation of emerging technology commercialization of five possible emerging technology enterprises with hesitant fuzzy linguistic information.

**Step 1:** We utilize the hesitant fuzzy linguistic decision information given in matrix  $H$ , and the HFLWA operator to obtain the overall preference values  $\tilde{h}_i$  of the emerging technology enterprises  $A_i$  ( $i = 1, 2, 3, 4, 5$ ). Take alternative  $A_1$  for an example, we have

$$\begin{aligned}
\tilde{h}_1 &= (\langle s_{\theta_1}, h_1 \rangle) = \text{HFLWA}_\omega(\tilde{h}_{11}, \tilde{h}_{12}, \tilde{h}_{13}, \tilde{h}_{14}) \\
&= \text{HFLWA}_\omega(\langle s_{\theta_{11}}, h_{11} \rangle, \langle s_{\theta_{12}}, h_{12} \rangle, \langle s_{\theta_{13}}, h_{13} \rangle, \langle s_{\theta_{14}}, h_{14} \rangle) \\
&= \left\langle \sum_{j=1}^4 \omega_j s_{\theta_{1j}}, \left( \cup_{\gamma_{11} \in h_{11}, \gamma_{12} \in h_{12}, \gamma_{13} \in h_{13}, \gamma_{14} \in h_{14}} \left\{ 1 - \prod_{j=1}^4 (1 - \gamma_{1j})^{\omega_j} \right\} \right) \right\rangle \\
&= \text{HFLWA} \{ \langle s_4, (0.3, 0.5) \rangle, \langle s_2, (0.6, 0.7, 0.8) \rangle, \langle s_1, (0.7, 0.8) \rangle, \langle s_3, (0.8, 0.9) \rangle \} \\
&= \{ s_{2.30}, (0.6065\dots, 0.6285, 0.6329, 0.6443, 0.6534, 0.6575, 0.6642, 0.6655, 0.6681, \\
&\quad 0.6804, 0.6842, 0.6867, 0.6879, 0.6904, 0.6976, 0.7053, 0.7088, 0.7111, 0.7145, 0.7178, \\
&\quad 0.7283, 0.7336, 0.7367, 0.7544) \}
\end{aligned}$$

**Step 2:** Calculate the scores  $S(\tilde{h}_i)$  ( $i = 1, 2, 3, 4, 5$ ) of the overall hesitant fuzzy linguistic preference values  $\tilde{h}_i$  ( $i = 1, 2, 3, 4, 5$ ):

$$s(\tilde{h}_1) = s_{1.58}, s(\tilde{h}_2) = s_{1.41}, s(\tilde{h}_3) = s_{2.06}, s(\tilde{h}_4) = s_{3.40}, s(\tilde{h}_5) = s_{1.31}.$$

**Step 3:** Rank all the emerging technology enterprises  $A_i$  ( $i = 1, 2, 3, 4, 5$ ) in accordance with the scores  $S(\tilde{h}_i)$  ( $i = 1, 2, 3, 4, 5$ ) of the overall hesitant fuzzy linguistic preference values:  $A_4 \succ A_3 \succ A_1 \succ A_2 \succ A_5$ , and thus the most desirable emerging technology enterprise is  $A_4$ .

## 7. Conclusions

In this paper, we investigate the multiple attribute decision making (MADM) problem based on the arithmetic and geometric aggregation operators with hesitant fuzzy linguistic information. Then, motivated by the idea of traditional arithmetic operation[54-57], we have developed some aggregation operators for aggregating hesitant fuzzy linguistic information: hesitant fuzzy linguistic weighted average (HFLWA) operator, hesitant fuzzy linguistic ordered weighted average (HFLOWA) operator and hesitant fuzzy linguistic hybrid average (HFLHA) operator. The prominent characteristic of these proposed operators are studied. Furthermore, we propose the concept of the dual hesitant fuzzy linguistic set and develop some aggregation operators with dual hesitant fuzzy linguistic information. Then, we

have utilized these operators to develop some approaches to solve the hesitant fuzzy linguistic multiple attribute decision making problems. Finally, a practical example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

It would be interesting that we shall deepen in the analysis of the new aggregation operators. For example, we could study how the hesitant fuzzy sets become a normal fuzzy set. Moreover, it would be good research directions that there are other methodologies for integrating the OWA with the weighted average instead of the hybrid average including the WOWA[26], the OWAWA[18], the immediate weights[22, 55] and importance weights[53].

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