TIME-VARYING FUZZY SETS BASED ON A GAUSSIAN MEMBERSHIP FUNCTIONS FOR DEVELOPING FUZZY CONTROLLER

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Abstract. The paper presents a novel type of fuzzy sets, called time-Varying Fuzzy Sets (VFS). These fuzzy sets are based on the Gaussian membership functions, they are depended on the error and they are characterized by the displacement of the kernels to both right and left side of the universe of discourse, the two extremes kernels of the universe are fixed for all time. In this work we focus only on the midpoint movement of the universe, all points of supports (kernels) are shifted by the same distance and in the same direction excepted the two extremes points of supports are always fixed for all computation time. To show the effectiveness of this approach we used these VFS to develop a PDC (Parallel Distributed Compensation) fuzzy controller for a nonlinear and certain system in continuous time described by the T-S fuzzy model, the parameters of the functions defining the midpoint movements are optimized by a PSO (Particle Swarm Optimization) approach.

1. Introduction

Fuzzy controller′s design depends mainly on the IF-THEN rules based on fuzzy sets and membership functions, which contain the linguistic elements, that characterize the functioning of the industrial process. In fact one cannot exactly evaluate the length of an element of fuzzy sets. For example pressure′s linguistic variables are 'Low', 'Medium' and 'High', these linguistics values of fuzzy sets do not have a well-defined numerical range at all the time and they also depend on the process. In general applications, we approximate the linguistics values of fuzzy sets by a proper numerical range, where the membership functions are fixed constant during the computation time, called Fixed Fuzzy Sets (FFS), conventionally this type of fuzzy sets is known as type-1 fuzzy sets [29, 42, 39, 28, 30, 37, 6]. The type-2 fuzzy sets is a set in which we also have uncertainty on the membership function [40, 24, 41].
A number of researchers are interested in fuzzy controllers based on the FFS [19, 1, 10, 18], where TS model [29, 42, 39, 28, 30, 37, 6, 24, 41, 40, 25, 20, 38] and PDC (Parallel Distributed Compensation) fuzzy controller [25, 20, 38] are used. The interest in TS system is due to the fact that the stability and performance characteristics of the system can be analyzed using Lyapunov function approach,

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which is easy to implement and can be expressed as a convex optimization problem in LMI formalism [8, 2, 17, 32]. In PDC scheme a linear control is designed for each local linear system, the overall controller is then a fuzzy blending of all local linear controllers, which is usually nonlinear.
In the context of the Self Organizing Fuzzy Control (S.O.F.C.) [21, 26, 12, 4, 11, 22, 23, 27, 3, 44, 43], we introduce in this work a novel fuzzy set based on a Gaussian membership function, we propose that the range of the linguistics values of these fuzzy sets vary during the computation time, called time-Varying Fuzzy Sets (VFS) [44, 43]. This VFS is based on the displacements of all support points in both left and right directions of the universe of discourse via a temporal functions depending on the error, without altering the universe of discourse. We can imagine the displacements of the membership functions in VFS which are varied in time as a spring movement. In this note we focus only on the midpoint movement of the FFS universe which includes the classical three membership functions (Small, Medium and Big). The displacements of the midpoint in both left and right side are directed by the premise variable and the parameters of the functions defining the midpoint movements are optimized by a PSO approach [15, 9, 14].
By applying this VFS for designing a PDC fuzzy controller for nonlinear system, we use a decay rate controller and relaxed stability conditions [16, 7, 35, 31, 13, 5, 33, 36, 34]. An application for the control of the inverted pendulum is presented to show the robustness of the PDC fuzzy controller based on the VFS.
The paper is organized as follows, the T-S fuzzy model and stability using Lyapunov approach and PDC fuzzy controller are recalled in Fundamentals basics section. Section 3 discusses the time-varying fuzzy sets. Section 4 presents the proposed algorithm design. A simulation example is provided in Section 5. Finally, a conclusion is given in Section 6.

2. Fundamental Basics

2.1. **The TS Fuzzy Model.** Consider a nonlinear system described by the T-S fuzzy model [39, 5]: Plant rule i : IF $z_1(t)$ is M_{i1} and ... and z_p(t) is M_{ip} THEN

$$\begin{cases} \dot{x}(t) & = A_i.x(t) + B_i.u(t) \\ y(t) & = C_i.x(t) \;\; i = 1...r \end{cases} \tag{1}$$

Where :
M_{ij} is the fuzzy set, and r is the number of IF-THEN rules, z(t)=[z_1(t) , z_2(t), ..., z_p(t)] are the premise variable $A_i \in R^{nxn}$, $B_i \in R^{nxm}$ and $C_i \in R^{nx1}$ are system matrices where $m \leqslant n$, $x \in R^n$ is the state , $u \in R^m$ is the control constrained as:

$$\| u(t) \|_2 < \varphi \tag{2}$$

The output y(t) constrained as follow :

$$\| y(t) \|_2 < \rho \tag{3}$$

The considered fuzzy model can be written as :

$$\dot{x}(t) = \frac{\sum_{i=1}^{n} w_i(A_i.x(t) + B_i.u(t))}{\sum_{i=1}^{n} w_i(t)} \tag{4}$$

Where: w_i is defined as :

$$w_i(z(t)) = \prod_{j=1}^{r} M_{ij}(z_j(t)) \tag{5}$$

M_{ij} is membership function of the j^{th} fuzzy set in the i^{th} rule. Let us define

$$h_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))} \tag{6}$$

$$\begin{cases} \sum_{i=1}^{r} h_i(z(t)) & =1 \\ h_i(z(t)) & \geqslant 0 \quad i = 1, ..., r \end{cases} \tag{7}$$

for every input x(t) and u(t), the global output is obtained by the following :

$$\begin{cases} \dot{x}(t) & = \sum_{i=1}^{r} h_i(z(t)).\{A_i.x(t) + B_i.u(t)\} \\ y(t) & = \sum_{i=1}^{r} h_i(z(t)).\{C_i.x(t)\} \quad i = 1...r \end{cases} \tag{8}$$

Where matrices A_i, B_i are constant of appropriate size and satisfy the following assumption: Each pair (A_i, B_i) is stabilizable.

2.2. **Parallel Distributed Compensation Controller.** To stabilize the system represented by (8) we use a PDC controller defined by [38, 20, 25] :
Control rule i :
If $z_1(t)$ is M_{i1} and ... and $z_p(t) is\ M_{ip}$, then

$$u(t) = -K_i.x(t) \quad i = 1, ..., r \tag{9}$$

, where : K_i : is the controller stabilizing the i^{th} subsystem. The global control will be given by :

$$\begin{aligned} u(t) &= -\frac{\sum_{i=1}^{r} w_i(z(t)).K_i.x(t)}{\sum_{i=1}^{r} w_i(z(t))} \\ &= -\sum_{i=1}^{r} h_i(z(t)).K_i.x(t) \quad i = 1, ..., r \end{aligned} \tag{10}$$

2.3. **Quadratic Stability Via Lyapunov Approach.** To guaranty the synthesizable fuzzy controller stability we use theorems giving the sufficient conditions of Lyapunov quadratic stability those exploits LMI formalism [16, 35, 5, 33, 36, 34], see Appendix.

3. The time-Varying Fuzzy Sets (VFS)

3.1. **Definition and Presentation of the VFS.** On the fuzzy sets form there is no confusion in defining the two extremes numerical ranges of their corresponding linguistics values. Around the midpoint of the universe of discourse there is always a wide margin for intersection of linguistics values. Indeed, it is difficult to determine exactly their numerical ranges and also to maintain their fixed constant for all computation time. Hence, we propose that the ranges of the fuzzy sets varies in time on the universe of discourse $[\underline{Z}\ ,\overline{Z}\ ]$, this approach called time-Varying Fuzzy Sets (VFS) [44, 43].

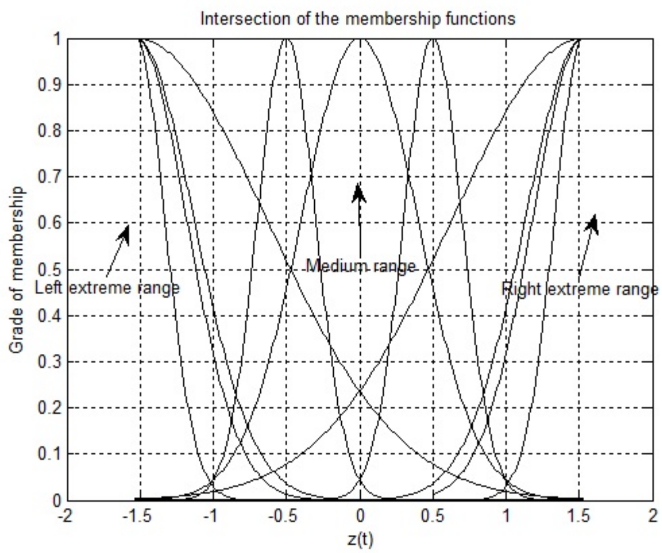

FIGURE 1. Exemple of Intersection of the Membership Functions

Let us define the e(t) error :

$$e(t) = x(t) - x_d(t) \tag{11}$$

Where :
x(t): current system state, x_d(t): desired system state.

If the error e(t) is big then you need a high effort, and you must decrease this effort if the error is small when approaching the desired state by an adequate acceleration. For example, consider the fuzzy set (Low, High), if the range of one linguistic value decreases by one step, the other range increases by the same quantity.
To carry out this objective, we propose to adjust the midpoint of the universe of discourse defined by α, shown in figures 2 and 3. If the midpoint is shifted in one

side by a well computed distance, all other points which define the membership functions are shifted in the same direction and by the same distance. The displacements of this midpoint are characterized by a continuous function depending on the error where [43]:

$$\alpha(t) = f(e(t)) \;/\; \alpha(t) \in [\underline{\alpha}\,, \overline{\alpha}] \subset \,]\underline{Z}\,, \overline{Z}[\tag{12}$$

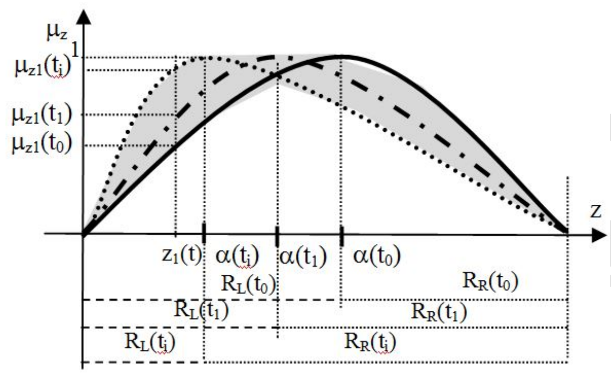

FIGURE 2. The Gaussian Membership Functions with the Time-varying Midpoint α(t) (2D) (μ: Membership Grade, Z: Premise Variable, R_R(t) : Right Range, R_L(t) : Left Range, α(t) : Midpoint)

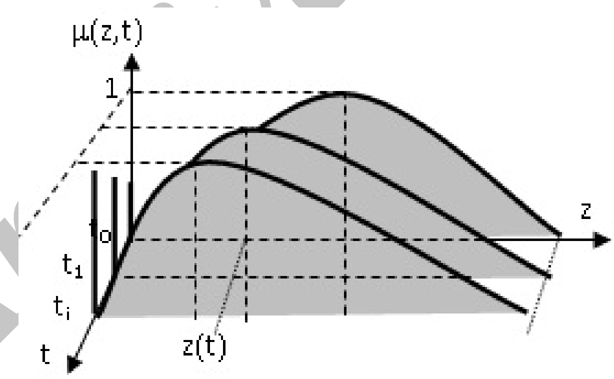

FIGURE 3. The Gaussian Membership Functions with Time-varying Midpoint (3D)

Through figures 2 and 3, if the midpoint is shifted towards the left ($\alpha(t_0)$, $\alpha(t_1)$, ..., $\alpha(t_i)$), (the decrease of the left range) it causes a higher membership grades on the left side ($\mu_{z1}(\mathrm{t}_0) < \mu_{z1}(\mathrm{t}_1) < ... < \mu_{z1}(\mathrm{t}_i)$) and the increase of these membership grades generates an increase of the control law, see equation (10).

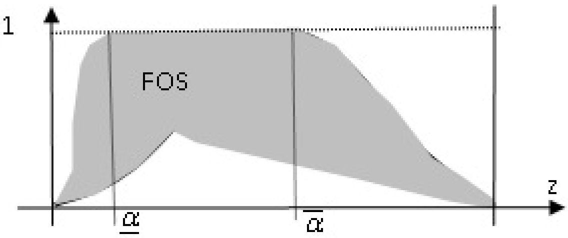

FIGURE 4. Footprint of Shifting ⋆ FOS ⋆

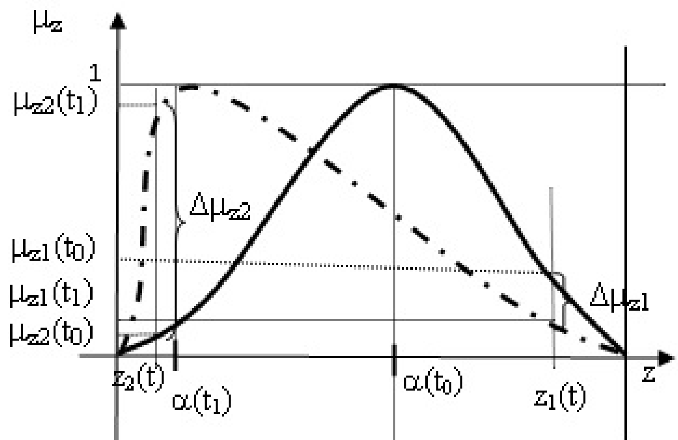

FIGURE 5. The Variation of the Gaussian Membership Grades Generated by the Time-varying Midpoint

Figure 4 represents the area of the membership functions covered by the shifting of the time-varying midpoint, called Footprint of Shifting "FOS".
In the case where α(t) is displaced to the left side as shown in figure 5, the membership grades of the premise variable z_2(t) is considerably increased, while those of the premise variable z_1(t) are slightly decreased.

3.2. **Relation Between the Control Law and the Membership Grades.** The linguistics ranges are inversely proportional to membership grades, figures 2 and 5, which are directly proportional to the control law defined by PDC fuzzy controller, see proof.

Proof. This section presents a proof of the relation between the control effort and the grade of the membership functions. From equation (6) we have :

$$h_i = \frac{w_i}{\sum_{i=1}^{r} wi} = \frac{wi}{w_1 + w_2 + .. + w_i + .. + w_r} \geq 0 \tag{13}$$

$$W = w_1 + w_2 + .. + w_{i-1} + w_{i+1} .. + w_r > 0 \tag{14}$$

$$for \quad w_i^{'} \quad we \quad have \quad h_i^{'} = \frac{w_i^{'}}{W + w_i^{'}} \tag{15}$$

$$\begin{aligned} then \quad h_i^{'} - h_i \quad &= \frac{w_i^{'}}{W + w_i^{'}} - \frac{w_i}{W + w_i} \\ &= \frac{W}{(W + w_i).(W + w_i^{'})}.(w_i^{'} - w_i) \end{aligned} \tag{16}$$

$$So \quad if \quad w_i^{'} - w_i > 0 \quad then \quad h_i^{'} - h_i > 0 \tag{17}$$

So, the increase of W_i generates an increase in h_i, and from equation (10) the increase of h_i generates an increase in the control effort u(t). □

Corollary 3.1. *In PDC fuzzy controller if the membership grade is increased then the control effort is increased.*

4. Algorithm Design for VFS

To design an algorithm for the VFS appraoch we propose five steps defined as follow :

(1) Setting of α(t) midpoint into the algorithm of fuzzy system ;
(2) Defining the functions giving the α(t) displacements;
(3) Defining the direction criterion of the α(t) displacements ;
(4) Determining the effect of the α(t) function on the stability ;
(5) Identification of the parameters of α(t) functions.

In the following, we explain each step of this algorithm using a Gaussian membership functions to develop a PDC fuzzy controller for a non linear system in continuous time described by TS fuzzy model.

4.1. **Setting of α(t) Midpoint Into the Algorithm of Fuzzy System.** Let us consider Gaussian membership functions in figure 6 that caracterize the three membership functions : Small, Medium and Big.

The grade of membership fucntion is given by :

$$M(t) = \exp\left(\beta \frac{[z(t) - \alpha]^2}{[\sigma]^2}\right) \tag{18}$$

where : β : negative coefficient, σ : variance, α : center (midpoint).

In FFS appraoch these three parameters (β, σ, α) are fixed constant for all computation time. Now, the following example, figure 7, illustrate the VFS approach. The membership functions representation whose α(t) is time-varying midpoint. For example, when α(t) is shifted to the left side (α(t) is shifted from 0 to -0.75 for Medium membership function), both extreme points of support {(-1.5,0), (1.5,0)} and extremes core points {(-1.5,1), (1.5,1)} of the universe of discourse are fixed constant for all time. This generates a new membership function that could be either Gaussian or Rayleigh functions.

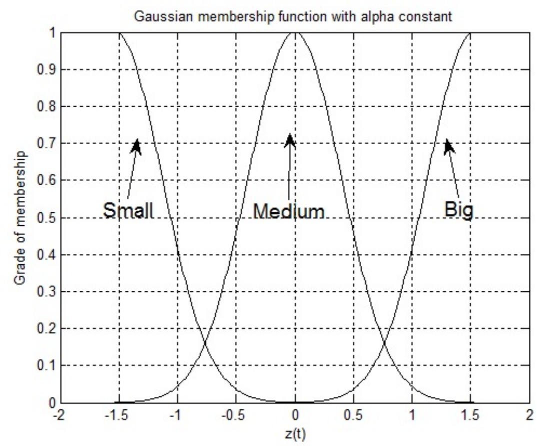

FIGURE 6. Membership Functions with α Constant

Note, here, that the Rayleigh function is product by the compression to the left side and the extension to right side of a Gaussian function, this looks like the movement of a spring. These obtained membership functions presented in figure 7 are

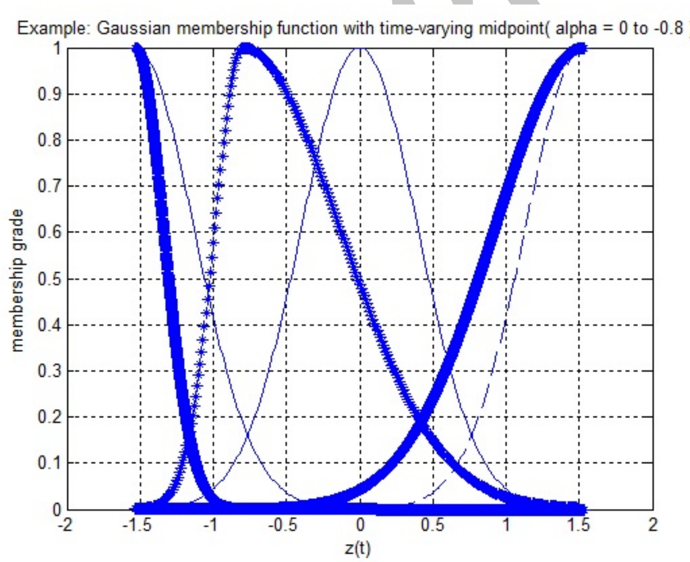

FIGURE 7. Membership Functions with α Time-varying

characterized by (β_i D_i $[\sigma_i \quad \alpha_i]$).
where :
D_i : The universe for each membership function

σ_i : The variance for each membership function
α_i : The center for each membership function
β_i : The coefficient for each membership function.

So, the parameters (β_i, σ_i, α_i, D_i), which characterize the new desired membership function, are inevitably time-varying parameters.
To calculate the grades of the premise variable $M_i(z_i(t))$ of each membership function we propose the following equations:

(1) **Small membership function :**

$$M_S(t) = \exp\left(\beta_S \frac{[z_i(t) - \alpha(t)]^2}{[\sigma_S(t)]^2}\right) \tag{19}$$

(2) **Big membership function :**

$$M_B(t) = \exp\left(\beta_B \frac{[z_i(t) - \alpha(t)]^2}{[\sigma_B(t)]^2}\right) \tag{20}$$

(3) **Medium membership function :**

$$M_M(t) = \begin{cases} \exp\left(\beta_M \frac{[z_i(t) - \alpha(t)]^2}{[\sigma_{MR}(t)]^2}\right) & if \quad z_i(t) \geqslant \alpha(t) \\ \exp\left(\beta_M \frac{[z_i(t) - \alpha(t)]^2}{[\sigma_{ML}(t)]^2}\right) & if \quad z_i(t) < \alpha(t) \end{cases} \tag{21}$$

We define the time-varying variance of each membership function separately as follows :

$$\begin{aligned} \sigma_S(t) &= \sigma_{S0} \frac{\alpha(t) - \underline{z}_i}{\alpha(t-1) - \underline{z}_i} > 0 \\ \sigma_B(t) &= \sigma_{B0} \frac{\alpha(t) - \overline{z}_i}{\alpha(t-1) - \overline{z}_i} > 0 \\ \sigma_{MR}(t) &= \sigma_{M0} \frac{\alpha(t) - \overline{z}_i}{\alpha(t-1) - \overline{z}_i} > 0 \\ \sigma_{ML}(t) &= \sigma_{M0} \frac{\alpha(t) - \underline{z}_i}{\alpha(t-1) - \underline{z}_i} > 0 \end{aligned} \tag{22}$$

Where:

$$\alpha(t) \in [\underline{\alpha},\ \overline{\alpha}] \subset]\underline{z}_i,\ \overline{z}_i[\tag{23}$$

with :

β_S , β_M and β_B are well defined negative constants coefficients.

σ_{S0} , σ_{M0} and σ_{B0} are positive constants.

$\sigma_S(t)$ and $\sigma_B(t)$: respectively represent the time-varying variance of the SMALL and BIG membership functions, figure 6.

$\sigma_{MR}(t)$ and $\sigma_{ML}(t)$: respectively represent the time-varying variance of both right and left side of the Medium membership function, figure 6. Generally, all membership functions do not have a Gaussian function, they are not symmetri and they have a look at a Rayleigh function.

$\overline{Z}_i$ and $\underline{Z}_i$ represent the minimum and the maximum of $z_i(t)$ for $x(t) \in [\underline{x},\ \overline{x}]$

$M_S(t)$, $M_B(t)$ and $M_M(t)$ are the membership grades of $z_i(t)$ with $\alpha(t)$.

Remark 4.1. For any shifting of the MEDIUM membership function in both right or left side σ_{ML} and σ_{MR} are decreased, but if the shifting is in left side σ_S is decreased and σ_B is increased.

4.2. **Defining the Functions Giving the α(t) Displacements.** Let $\alpha_R(t)$ be the right displacement of the midpoint on the universe, ensured by a function depending on the error as :

$$\alpha_R(t) = f_R(e(t)) = \theta_{1R}(t).\left(1 - e^{-\theta_{2R}(t).|e(t)|}\right) \tag{24}$$

where :

$\theta_{1R}(t)$, $\theta_{2R}(t)$ are the maximum and the growth rate of the α_R(t), respectively.

α_L(t) is the left displacement of the midpoint on the universe, ensured by:

$$\alpha_L(t) = f_L(e(t)) = -\theta_{1L}(t).\left(1 - e^{-\theta_{2L}(t).|e(t)|}\right) \tag{25}$$

where :

$\theta_{1L}(t), \theta_{2L}(t)$: are the minimum and the decay rate of the α_L(t), respectively. this is shown in figure 8.

4.3. **Defining the Direction Criterion of the α(t) Displacements.** The direction criterion depends on the relation between the membership grades and the control law based on the absolute error distance e(t). In this note, we use a PDC fuzzy controller based on the relationship (10). The displacement α(t) midpoint to both left or right side is directed by the position of the premise variable Z(t). If the premise variable is set to the left side of the midpoint then α(t) must approach to the minimum $\underline{\alpha}$ by the function defined in relationship (25). On the other hand, if the premise variable is set to the right side of the midpoint then α(t) must approach to the maximum $\overline{\alpha}$ by the function defined in relationship (24). The relationship (23) is must always be checked.

We propose that the displacements α(t) will follow the premise variable Z(t) by an acceleration determined by the output Y_m(t) of a Mamdani fuzzy model reference whose inputs are error and change in error as shown in figures 9 and 10. The switch function between both right and left displacement is illustrated in figure 8 and is given by this sub-program :

$$\begin{array}{lrl} if\ \alpha(t) < z(t+1) & then & \alpha(t+1) = \alpha_R(t+1) \\ & else & \alpha(t+1) = \alpha_L(t+1) \\ end\ if & & \end{array} \tag{26}$$

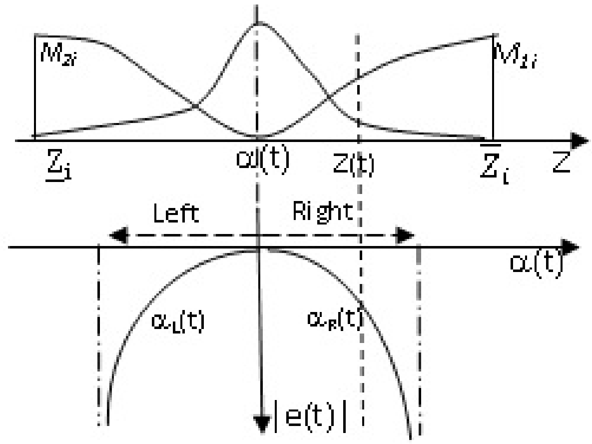

FIGURE 8. Direction of the Displacement of the Time-varying Midpoint

4.4. **Determining the Effect of α(t) Functions on the Stability.** The subsystems [A_i B_i; C_i 0] of T-S [39, 6] do not change as the value of α(t), and also the criteria of the stability (Stability theorems [5, 33, 36, 34]) do not change). To ensure stability used in this note, see Appendix, it is necessary to ensure that the equation (23) is always checked.

4.5. **Identification of α(t) Parameters.** The overall configuration of the closed loop system with VFS is shown in figure 9, in the following we describe each subsystem.

with :

Y_m(t): The output of the Mamdani model reference,
e(t): error, e_a(t): Adaptive error,
r(t): Reference, y(t): System output.
u(t): Control law, $\triangle$e(t): Change in error,
α_L(t): The left displacement function of the midpoint,
α_R (t): The right displacement function of the midpoint,
α(t): The general displacement function of the midpoint,
z_1(t), ..., z_n(t) : Premise variable vector ,
α_1(t), ... , α_i(t) : The vector of the midpoint function.

(1) **Representation of the closed loop system with VFS :** Figure 9 presents the S.O.F.C. of the VFS approach, which is composed of a self-organizing part that includes a classical Model Reference Adaptive Control and a traditional PDC fuzzy controller.
(2) **The reference model :** The reference model is defined by the output Y_m(t) of a Mamdani fuzzy system, which is based on the triangular membership fucntion (for example). As indicated above it represent the acceleration of the midpoint displacements presented in figure 10.

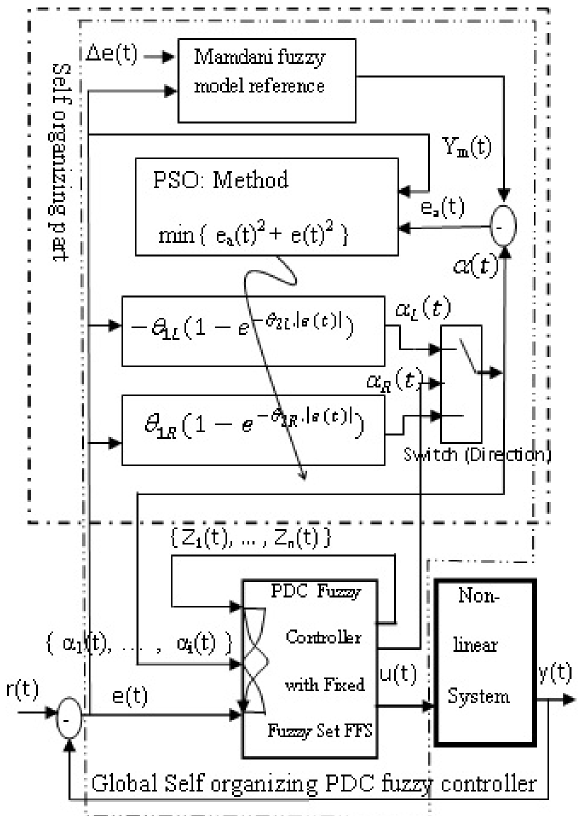

Figure 9. Closed Loop System with Time-varying Fuzzy Sets

(3) **The rule-base of $\alpha(\mathbf{t})$ displacements :** Based on figure 10, as an example, we can define in Table 1 the rule-base of the system.

e(t) / $\triangle$e(t)	S	M	B
$\triangle$S	L	Mo	G
$\triangle$M	L	Mo	G
$\triangle$B	Mo	G	G

Table 1. RULE BASE (S:Small, M:Medium, B:Big, L:Litle, Mo:Moderate, G:Great.)

(4) **The approach of the Particle Swarm Optimization PSO :** The PSO method is a member of wide category of swarm intelligence methods for solving the optimization problems. It is a population based search algorithm where each individual is referred to as particle and represents a candidate solution. Each particle in PSO flies through the search space with an adaptable velocity that is dynamically modified according to its own

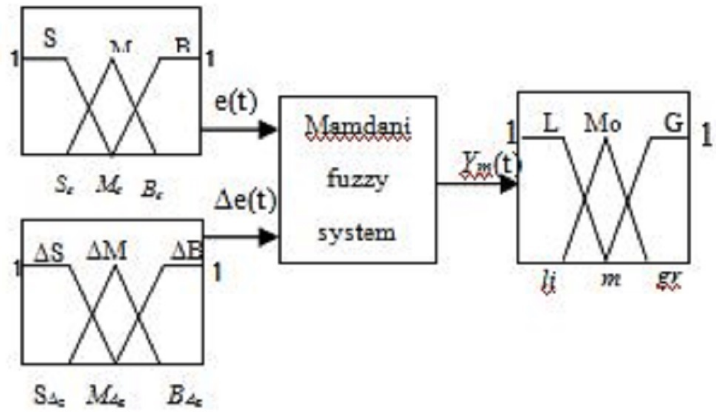

FIGURE 10. Reference model representation

flying experience and also to the flying experience of the other particles. In PSO all particles strive to improve themselves by imitating traits of their successful peers. Further, each particle has a memory and hence it is capable of remembering the best position in the search space ever visited by it. The position corresponding to the best fitness of one particle is known as pbest and the overall best out of all the particles in the population is called gbest [9, 14, 15].

The velocity and the position of each particle can be calculated using the current velocity and the distances from the $\text{pbest}_{j,g}$ to gbest_g as shown in the following formulas :

$$v_{j,g}^{(ite+1)} = I_w * v_{j,g}^{(ite)} + c_1 * r_1 * (pbest_{j,g} - x_{j,g}^{(ite)}) + c_2 * r_2 * (gbest_g - x_{j,g}^{(ite)}) \tag{27}$$

$$x_{j,g}^{(ite+1)} = x_{j,g}^{(ite)} + v_{j,g}^{(ite+1)} \tag{28}$$

with :
j=1,2,...,n , g=1,2,...,m, Iw: inertia weight factor,
n: number of particles in the swarm,
m: number of components for the vector v_j and x_j,
ite: number of iterations (generations),
$v_{j,g}^{(ite)}$: the g^{th} component of the velocity of the particle j at iteration t,
c_1, c_2: cognitive and social acceleration factors, respectively,
r_1, r_2: random numbers uniformly distributed in the range [0, 1],
$x_{j,g}^{(ite)}$: the g^{th} component of the position of particle at iteration t,
pbest_j: pbest of particle j ; gbest_g: gbest of group.

The j^{th} particle in the swarm is represented by a d-dimensional vector $x_j=(x_{j,1}\ , x_{j,2}\ , ..., x_{j,d})$ and its rate of position change (velocity) is denoted by another d-dimensional vector $v_j=(v_{j,1}\ , v_{j,2}\ , ..., v_{j,d})$. The best previous position of the j^{th} particle is represented as pbest$_j$=(pbest$_{j,1}$, pbest$_{j,2}$, ..., pbest$_{j,d}$). The index of best particle among all of the particles in the swarm is represented by the gbest$_g$. In PSO, each particle moves in the search space with a velocity according to its own previous best solution and its group′s previous best solution. The velocity update in a PSO consists of three parts; namely momentum, cognitive and social parts equation (27). The balance among these parts determines the performance of a PSO algorithm. The parameters c_1 and c_2 determine the relative pull of pbest and gbest and the parameters r_1 and r_2 help in stochastically varying these pulls. In the above equations, superscripts denote the iteration number. Figue 11 shows the velocity and the position in two-dimensional parameter space.

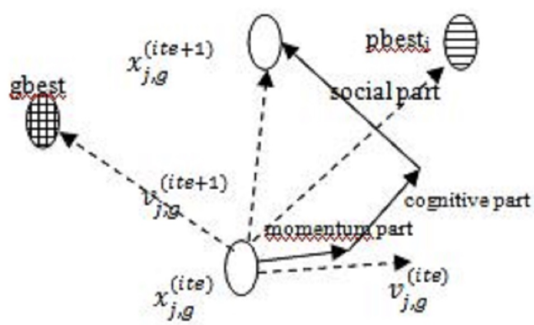

Figure 11. Description of Velocity and Position Updates in PSO for 2-dimensional Parameter Space

Based on the closed loop system with VFS illustrated in figure 9, we have :

$$e_a(t) = \alpha(t) - Y_m(t) \tag{29}$$

The general form of $\alpha(t)$ midpoint function is given by :

$$\alpha(t) = \theta_1(t)(1 - e^{-\theta_2(t).e(t)}) \tag{30}$$

The objective for using PSO approach is to find the parameters $\theta_1(t)$ and $\theta_2(t)$ as the objective function J is optimized, which is defined by :

$$\begin{aligned} J &= min\left(e_a^2(t) + e^2(t)\right) \\ &= min\left[(\alpha(t) - Y_m(t))^2 + e^2(t)\right] \\ &= min\left[\left(\theta_1(t)(1 - e^{-\theta_2(t).e(t)}) - Y_m(t)\right)^2\right. \\ &\quad \left. + (x(t) - x_d(t))^2\right] \end{aligned} \tag{31}$$

The final optimization problem is given by :

$$\min_{(\theta_{ij}(t): \ / \ i=1,2 \ ; \ j=R,L)} \left(e_a^2(t)+e^2(t)\right) \tag{32}$$

subject

$$\begin{cases} \underline{Z} < \alpha(t) < \overline{Z} \\ \theta_{ij}(t) \geqslant 0, \ \ / \ \ i=1,\ 2 \ ; \ j=R,\ L \\ \sigma_l(t) > 0, \ \ / \ \ l=S,\ M,\ B \end{cases} \tag{33}$$

Remark 4.2. The purpose of the inclusion of these constraints is to guarantee the stability of the closed loop system, which depends on the universe of $\alpha(t) \in [\underline{\alpha},\ \overline{\alpha}] \subset]\underline{z}_i,\ \overline{z}_i[$, or $\theta_1(t)$ and $\theta_2(t)$ should be limited and can not escape.

5. **Simulation Example**

5.1. **Simulation System.**

(1) **System description :** To illustrate the idea of this note, we consider the problem of an inverted pendulum on a cart [39]:

$$\begin{cases} x_1(t) & = x_2(t) \\ x_2(t) & = \dfrac{g.sin(x_1(t)) - a.m.x_2^2(t).\frac{sin(2.x_1(t))}{2} - h(t)}{4.l/3 - a.m.l.cos^2(x_1(t))} \\ h(t) & = a.cos(x_1(t)).u(t) \end{cases} \tag{34}$$

We approximate the system by the following two-rule fuzzy model :

$$\begin{cases} A_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{4lm-aml} & 0 \end{bmatrix} & B_1 = \begin{bmatrix} 0 \\ \frac{a}{4.l.m/3-a.m.l} \end{bmatrix} \\ A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2.g}{\Pi(4.l/3-a.m.l\gamma^2)} & 0 \end{bmatrix} & B_2 = \begin{bmatrix} 0 \\ \frac{-a.\gamma}{4.l/3-a.m.l.\gamma^2} \end{bmatrix} \\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} & C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{cases} \tag{35}$$

(2) **Initialization parameters :**
$-\Pi/2 < x_1(t) < \Pi/2$
x(0)= $\begin{bmatrix} \Pi/3 \\ 0 \end{bmatrix}$

(3) **Reference model parameters :** We can use the rule-base of the Mamdani reference model defined in Table 1 as follow :
S_e=01%.$|x_1(0)|$=0.5^o , M_e=03%.$|x_1(0)|$=1.5^o
B_e=06%.$|x_1(0)|$=3.0^o, $S_{\triangle e}$ = 01%.S_e,
$M_{\triangle e}$ = 03%.M_e, $B_{\triangle e}$ = 06%.B_e

The movements acceleration are dependent of the universe, we can take the maximum of the range :

$li = 0,\ mo = 0.785,\ gr = 1.57$

(4) **Membership fucntion parameters :** The membership functions are shown in Figure 6, with:
$\sigma_{S0} = \sigma_{B0} = \sigma_{M0} = 0.9$
$\beta_S = \beta_B = \beta_M = -0.5$

(5) **PSO parameters :**
n=49: number of particles in swarm , ite=100: number of iterations
$c_1 = c_2 = 2;\quad r_1 = r_2 = random[0, 1]$
$I_w = w_{max}.\frac{(w_{max}-w_{min})}{iter_{max}}.iter$: inertia weight factor [9]
$\theta_{1L}(0) = \theta_{1R}(0) = 1,\ \theta_{2L}(0) = \theta_{2R}(0) = 0$
$v_1 = 1,\ v_2 = 0$: initial parameters

(6) **Simulation results :** Figures 12 to 21 represent the obtained responses :

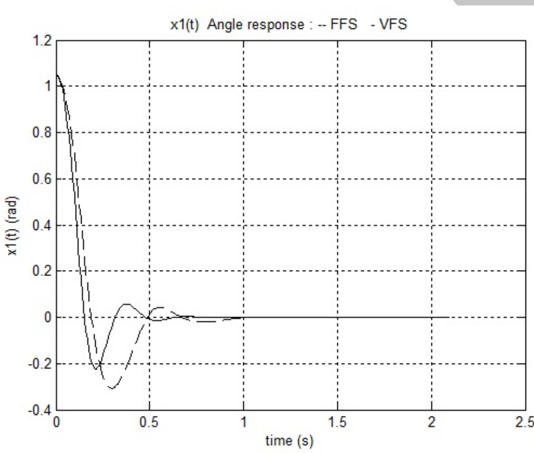

FIGURE 12. $x_1(t)$ Response (- VFS, - - FFS)

5.2. **Comments and Comparison.** The fuzzy system with the VFS approach gives a good stability and the dynamic specifications are better than the system with the FFS. Our results are improved compared to the results found in [39, 26] particularly the settling time in [26, 3]. In figure. 14 the VFS requests relatively high effort, that reflects accurately that the VFS approach has given the necessary power and enough time to the controller for stabilizing the system. We can observe that the effect of α(t) function is in the transient and converge to the final stability midpoint, see figure. 21. From the figure 21 and in the transient, the $\alpha(t)$ follows the acceleration movement Y_m defined by the Mamdani fuzzy model reference which defined by the rule-base.

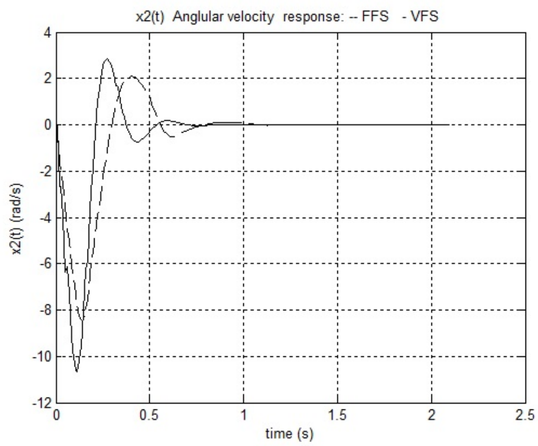

FIGURE 13. $x_2(t)$ Response (- VFS, - - FFS)

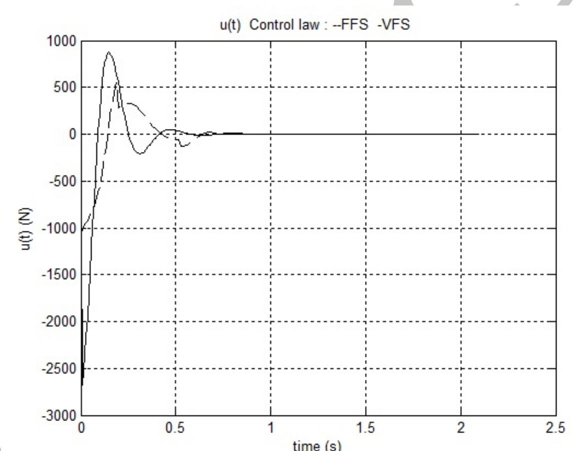

FIGURE 14. Control Effort Response (- VFS, - - FFS)

In the end, we define the Footprint of Shifting (FOS), which represents the area covered by the variation of fuzzy sets, which itself is an interval-valued fuzzy set, figure 22. The initial $\alpha(0)$ is set to the midpoint of the universe. Through the response of the figure 21, we can compute the universe of discourse of α(t):

$$FOS = \overline{\alpha} + \mid \underline{\alpha} \mid = 0 + 1.57 = 1.57 \in] - \Pi/2, \Pi/2[\tag{36}$$

The FOS, which represents the area covered by the shifting of the membership functions during simulation, is illustate in figure 22.
Also through the response of the figure 21, we can observe that α(t) is bounded in the universe of discourse. Hence the stability is checked by this VFS approach.

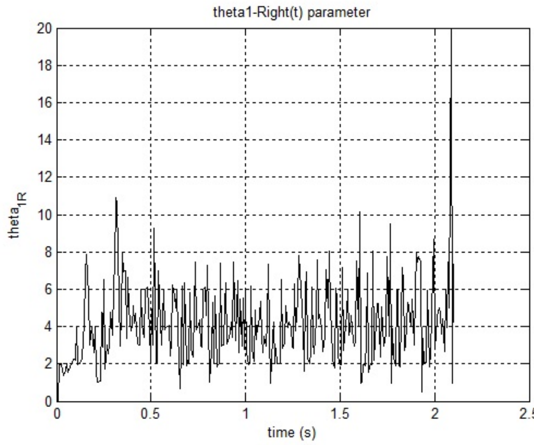

FIGURE 15. $\theta_{1R}(t)$ Response

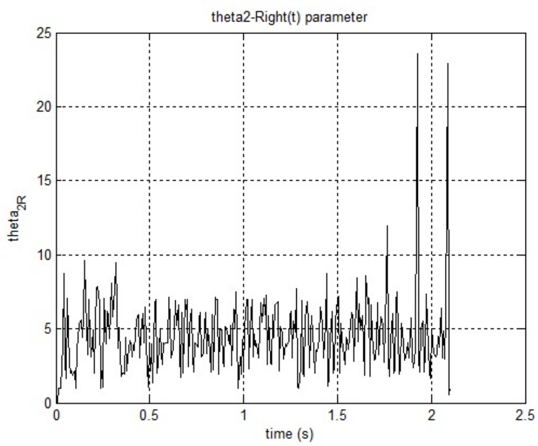

FIGURE 16. $\theta_{2R}(t)$ Response

6. Conclusion

The basic principle of the VFS approach is based on the absolute error that is proportional to the effort, in the case of PDC controller. If error is high VFS system responds with great effort and if the error is small the system requires a small effort.The role of the $\alpha(t)$ function (midpoint in this work) is to ensure an adequate effort depending on the error and what time to apply this effort.
The following points summarize some characteristics of the VFS approach which are verified in this work :

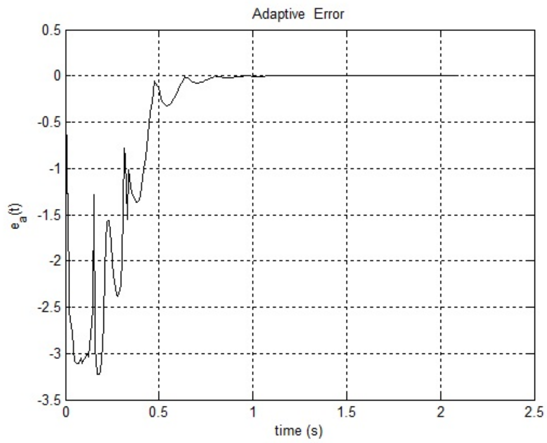

FIGURE 17. Adaptive Error e_a(t)

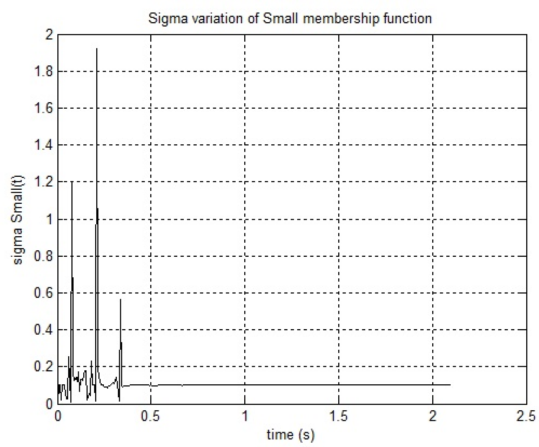

FIGURE 18. Time-varying $\sigma_S(t)$ Response (Small Membership Function)

⟶ The VFS approach is built on the rule base that defines accelerator of the movement functions of the midpoint, when it is very important to give a high or low control effort.
⟶ The VFS approach is built on the criterion of the direction of shifting to accelerate or decelerate the movement.
⟶ The types of functions provided to represent the midpoint movement must be bounded.
⟶ VFS is an approach that encompasses a fuzzy system with more rules.

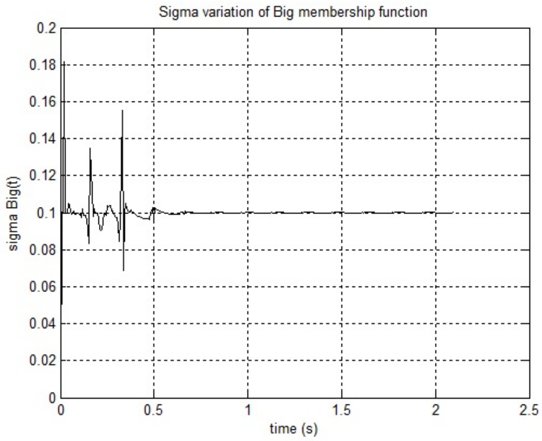

FIGURE 19. Time-varying $\sigma_B(t)$ Response (Big Membership Function)

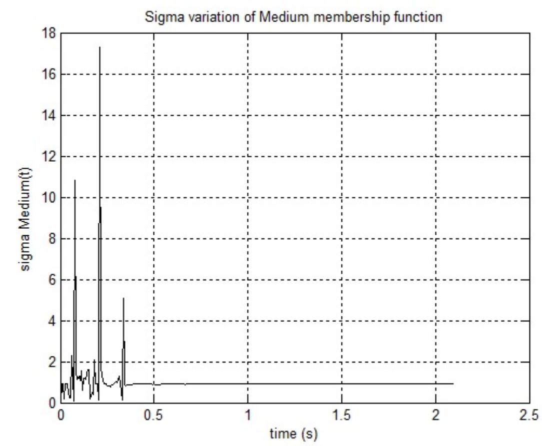

FIGURE 20. Time-varying $\sigma_M(t)$ Response (Medium Membership Function)

$\longrightarrow$ Through the VFS approach the ranges of the linguistics values of the fuzzy sets change in time according to the variation of the linguistics values of the error.

$\longrightarrow$ The effect on the $\alpha(t)$ (midpoint particular case)is in the transient after it stabilized at the final stability midpoint, which depends on the proposed reference model (rule base).

$\longrightarrow$ This approach has an inconvenient that present in the chattering phenomenon of control law. We can reduce these oscillations by taking in consideration some preventive measures : well defined criteria of direction, well defined universe, well defined rule-base,... .

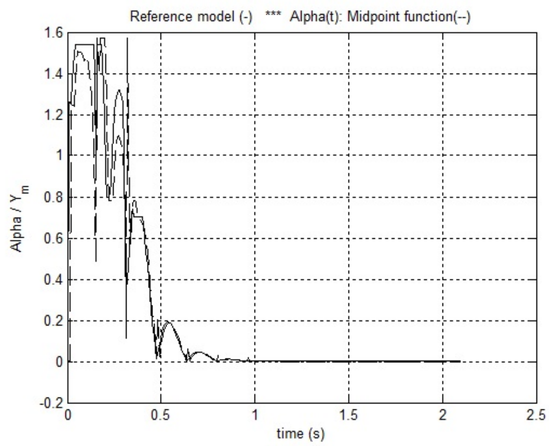

FIGURE 21. α(t) Response (- -) and Y_m Reference (—)

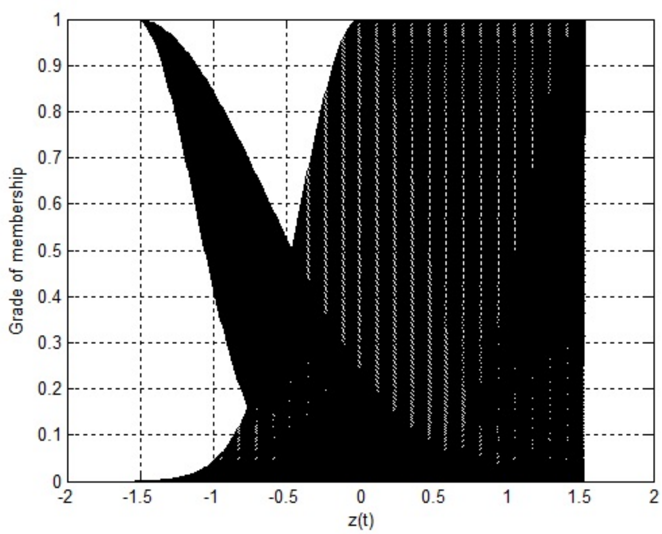

FIGURE 22. FOS Generated by VFS Based on the Gaussian Membership Functions

$\longrightarrow$ Finally, the following table presents a comparison between three fuzzy sets:
As perspective, in this paper we have studied a particular case of VFS, where the movement of all points (support points and core points), which define the membership functions, are shifted in the same direction and by the same distance.

Type of fuzzy set	Vertical interval FOU	Horizontal interval FOS
Type-1 fuzzy set	FOU $= 0$, $\forall t$	FOS $= 0$, $\forall t$
Type-2 fuzzy set	FOU $\neq 0$, $\forall t$	FOS $= 0$, $\forall t$
VFS	FOU $\neq 0$, $\forall t$	FOS $\neq 0$, $\forall t$

TABLE 2. Comparison Between Three Fuzzy Sets

In general case each point, which define the VFS, can be displaced in different direction and by different distance. The closed loop system with VFS could be complicate. Also we can applied this VFS in the case of type-2 fuzzy sets.

7. Appendix

Theorem 7.1. *Decay rate controller design using relaxed stability conditions: The condition that $\dot{V}(x) + 2.\beta.V(x(t)) \leq 0$ for all trajectories is equivalent to* [7, 35, 31, 13] *:*

$$\exists P > 0, \exists Q > 0 \quad G_{ii}^T P + PG_{ii} + (s-1)Q + 2\beta P < 0, \beta > 0 \tag{37}$$

$$\left(\frac{G_{ij} + G_{ji}}{2}\right)^T P + P\frac{G_{ij} + G_{ji}}{2} - Q + 2\beta P \leqslant 0 \tag{38}$$

$i < j \quad s.t. \quad h_i \cap h_j \neq \phi \quad where : 1 < s < r$

We can find the K_i controller by this optimization problem :
$max_{(X,Z,Y_1,\dots,Y_r)}\beta$

$$\begin{cases} -XA_i^T - A_iX + B_iY_i + Y_i^T B_i^T - (s-1)Z - 2\beta X > 0 \\ \quad i = 1, 2, .., r \\ -XA_i^T - A_iX - XA_j^T - A_jX + B_iY_j + \\ \quad Y_j^T B_i^T + B_jY_i + Y_i^T B_j^T + 2Z - 4\beta X > 0 \\ X > 0 \quad , \quad Y \geqslant 0 \quad i < j \quad s.t. \quad h_i \cap h_j \neq \phi \\ where \quad K_i = Y_i X^{-1} \ , \ X = P^{-1} \ , \ Z = XQX \end{cases} \tag{39}$$

Theorem 7.2. *Assume that the initial condition x(o) is known then* [7, 35, 31, 13]*:*

- *The constraint on control input $\|u(t)\|_2 < \varphi$ for $t \geqslant 0$ can be represented by :*

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & X \end{bmatrix} \geqslant 0 \tag{40}$$

$$\begin{bmatrix} X & Y_i^T \\ Y_i^T & \varphi^2.I \end{bmatrix} \geqslant 0 \tag{41}$$

- *The constraint on output* $\|y(t)\|_2 < \rho$ *for* $t \geqslant 0$ *can be represented by :*

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & X \end{bmatrix} \geqslant 0 \tag{42}$$

$$\begin{bmatrix} X & XC_i^T \\ XC_i^T & \rho^2.I \end{bmatrix} \geqslant 0 \tag{43}$$

where :

$$K_i = Y_i X^{-1} \quad X = P^{-1} \tag{44}$$

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TIME-VARYING FUZZY SETS BASED ON A GAUSSIAN MEMBERSHIP FUNCTIONS FOR DEVELOPING FUZZY CONTROLLER

S. ZIANI

مجموعه های فازی تابع زمان بر اساس توابع عضویت گوسی برای گسترش یک کنترل کننده فازی

چکیده. این مقاله نوع جدیدی از مجموعه های فازی را معرفی می کند که مجموعه های فازی تابع زمان (VFS) نامیده می شوند. اساس این مجموعه های فازی توابع عضویت گوسی می باشند، آنها وابسته به خطا هستند و با جابجایی هسته ها به چپ و راست عالم سخن توصیف می شوند، دو هسته اکسترمم عالم برای همیشه ثابت شده اند. در این کار ما تنها بر حرکت نقطه میانی عالم تمرکز می کنیم، تمام نقاط تکیه گاهها (هسته ها) با یک فاصله و در یک جهت انتقال داده شده اند. به جز دو نقطه اکسترمم مجزا که همواره در تمام مدت محاسبه ثابت می مانند. برای اینکه تأثیر این رویکرد را نشان دهیم VFS ها را به کار می بریم تا یک PDS (جبران توزیع شده موازی) کنترل کننده فازی برای یک سیستم مطمئن و غیر خطی در زمان پیوسته توصیف شده توسط مدل فازی $T-S$ را گسترش دهیم. پارامترهای توابعی که نقطه میانی حرکات را تعریف می کنند توسط یک روش PSO (بهینه سازی ازدحام ذره) بهینه می شوند.