GRADED DIUNIFORMITIES

R. EKMEKÇİ AND R. ERTÜRK

Abstract. Graded ditopological texture spaces have been presented and discussed in categorical aspects by Lawrence M. Brown and Alexander Šostak in [7]. In this paper, the authors generalize the structure of diuniformity in ditopological texture spaces defined in [13] to the graded ditopological texture spaces and investigate graded ditopologies generated by graded diuniformities. The autors also compare the properties of diuniformities and graded diuniformities.

1. **Introduction**

Uniform spaces are significant tools for investigation of topological spaces in many respects. Many concepts and properties such as uniform continuity, completeness and uniform convergence are defined by using uniform structure. So, setting and investigating uniform structure in a topological structure is reasonable and necessary for the deeper understanding of the topological structure.

The concept of fuzzy topological space was defined in 1968 by C.Chang as ordinary subset of the family of all fuzzy subsets of a given set in [8]. As a more suitable approach to the idea of fuzzyness, in 1985, Šostak and Kubiak independently redefined fuzzy topology where a fuzzy subset has a degree of openness rather than being open or not [16, 11] (for historical developments and basic ideas of the theory of fuzzy topology see [17]).

In classical topolgy the notion of open set is usually taken as primitive with that of closed set being auxiliary. However, since the closed sets are easily obtained as the complements of open sets they often play an important, sometimes dominating role in topological arguments. A similar situation holds for topologies on lattices where an order reversing involution plays the role of set complement. It is the case, however, that there may be no order reversing involution available, or that the presence of such an involution is otherwise irrelevant to the topic under consideration. To deal with such cases it is natural to consider a topological structure consisting of *a priori* unrelated families of open sets and of closed sets. This was the approach adapted from the beginning for the topological structures on textures, originally introduced as a point-based representation for fuzzy sets [2, 3]. These topological structures were given the name of a dichotomous topology, or ditopology for short. They consist of a family τ of open sets and a generally unrelated family κ of closed

Received: November 2015; Revised: May 2106; Accepted: November 2016
Key words and phrases: Texture, Graded ditopology, Graded diuniformity, Fuzzy topology.

sets. Hence, both the open and the closed sets are regarded as primitive concepts for a ditopology.

A ditopology (τ, κ) on the discrete texture $(X, \mathcal{P}(X))$ gives rise to a bitopological space (X, τ, κ^c). This link with bitopological spaces has had a powerful influence on the development of the theory of ditopological texture spaces, but it should be emphasized that a ditopology and a bitopology are conceptually different. Indeed, a bitopology consists of two separate topological structures (complete with their open and closed sets) whose interrelations we wish to study, whereas a ditopology represents a single topological structure.

Ditopological texture spaces were introduced by L. M. Brown as a natural extention of the work of the second autor on the representation of lattice-valued topologies by bitopologies in [10]. The concept of ditopology is more general than general topology, bitopology and fuzzy topology in Chang's sense. An adequate introduction to the theory of texture spaces and ditopological texture spaces may be obtained from [2, 3, 4, 5, 6]. Diuniform texture spaces were introduced by S. Özçağ and L. M. Brown in [13] and then several papers have been published on this subject such as [14, 15].

Recently, L. M. Brown and A. Šostak have presented the concept "graded ditopology" on textures as an extention of the concept of ditopology to the case where openness and closedness are given in terms of a priori unrelated grading functions [7]. The concept of graded ditopology is more general than ditopology and fuzzy topology in Šostak's sence. Two sorts of neighborhood structure on graded ditopological texture spaces are presented and investigated by the authors in [9].

The aim of this work is to generalize the structure of diuniformity in ditopological texture spaces defined in [13] to the graded ditopological texture spaces and investigate graded ditopologies generated by graded diuniformities. We also compare the properties of diuniformities and graded diuniformities and finally study categorical perspective of this new structure.

2. Preliminaries

We recall various concepts and properties from [3, 4, 5, 6] under the following subtitle.

2.1. **Ditopological Texture Spaces:** Let S be a set. A texturing $\mathcal{S}$ on S is a subset of $\mathcal{P}(S)$ which is a point separating (i.e. for all $s, t \in S$, $s \neq t$ there exists a set $A \in \mathcal{S}$ such that $s \in A$, $t \notin A$ or $s \notin A$, $t \in A$), complete, completely distributive lattice with respect to inclusion which contains S, $\emptyset$ and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. The pair $(S, \mathcal{S})$ is then called a texture or a texture space.

In general, a texturing of S need not be closed under set complementation, but it may be that there exist a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. In this case σ is called a complementation on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is said to be a complemented texture.

For any texture $(S, \mathcal{S})$, many properties are conveniently defined in terms of the $p - sets$

$$P_s = \bigcap \{A \in \mathcal{S} \mid s \in A\}$$

and the $q - sets$

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}.$$

A texture $(S, \mathcal{S})$ is called a plain texture if it satisfies any of the following equivalent conditions:

(1) $P_s \nsubseteq Q_s$ for all $s \in S$
(2) $A = \bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ for all $A_i \in \mathcal{S}, i \in I$

Recall that $M \in \mathcal{S}$ is called a molecule in $\mathcal{S}$ if $M \neq \emptyset$ and $M \subseteq A \cup B$, $A, B \in \mathcal{S}$ implies $M \subseteq A$ or $M \subseteq B$. The sets $P_s, s \in S$ are molecules, and the texture $(S, \mathcal{S})$ is called "simple" if all molecules of $\mathcal{S}$ are in the form $\{P_s \mid s \in S\}$. For a set $A \in \mathcal{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap\big\{\bigcup\{A_i \mid i \in I\} \mid \{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\big\}.$$

Theorem 2.1. [4] *In any texture space $(S, \mathcal{S})$, the following statements hold:*

(1) $s \notin A \Rightarrow A \subseteq Q_s \Rightarrow s \notin A^\flat$ for all $s \in S$, $A \in \mathcal{S}$.
(2) $A^\flat = \{s \mid A \nsubseteq Q_s\}$ for all $A \in \mathcal{S}$.
(3) For $A_j \in \mathcal{S}$, $j \in J$ we have $(\bigvee_{j\in J} A_j)^\flat = \bigcup_{j\in J} A_j^\flat$.
(4) A is the smallest element of $\mathcal{S}$ containing $A^\flat$ for all $A \in \mathcal{S}$.
(5) For $A, B \in \mathcal{S}$, if $A \nsubseteq B$ then there exists $s \in S$ with $A \nsubseteq Q_s$ and $P_s \nsubseteq B$.
(6) $A = \bigcap\{Q_s \mid P_s \nsubseteq A\}$ for all $A \in \mathcal{S}$.
(7) $A = \bigvee\{P_s \mid A \nsubseteq Q_s\}$ for all $A \in \mathcal{S}$.

Example 2.2. (1) If $\mathcal{P}(X)$ is the powerset of a set X, then $(X, \mathcal{P}(X))$ is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$. The mapping $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is a complementation on the texture $(X, \mathcal{P}(X))$.
(2) Setting $\mathbb{I} = [0, 1]$, $\mathcal{J} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ gives the unit interval texture $(\mathbb{I}, \mathcal{J})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$. And the mapping $\iota : \mathcal{J} \to \mathcal{J}$, $\iota[0, r] = [0, 1 - r)$, $\iota[0, r) = [0, 1 - r]$ is a complementation on this texture.
(3) The texture $(L, \mathcal{L}, \lambda)$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$, $\lambda((0, r]) = (0, 1 - r]$. For $r \in L$, $P_r = (0, r] = Q_r$.
(4) $\mathcal{S} = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, S\}$ is a simple texturing of $S = \{a, b, c\}$. $P_a = \{a, b\}$, $P_b = \{b\}$, $P_c = \{b, c\}$. It is not possible to define a complementation on $(S, \mathcal{S})$.
(5) If $(S, \mathcal{S}), (V, \mathcal{V})$ are textures, the product texturing $\mathcal{S} \otimes \mathcal{V}$ of $S \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{V}$, and $(S \times V, \mathcal{S} \otimes \mathcal{V})$ is called the product of $(S, \mathcal{S})$ and $(V, \mathcal{V})$. For $s \in S, v \in V$, $P_{(s,v)} = P_s \times P_v$ and $Q_{(s,v)} = (Q_s \times V) \cup (S \times Q_v)$. The p-sets and q-sets of the product $(S, \mathcal{P}(S)) \times (V, \mathcal{V})$ will be denoted by $\overline{P}_{(s,v)}$ and $\overline{Q}_{(s,v)}$ respectively.

Proposition 2.3. [18] *For the product textures $\mathcal{P}(S) \otimes \mathcal{V}$ and $\mathcal{P}(V) \otimes \mathcal{S}$, the following properties are satisfied.*

(1) $\overline{P}_{(s,v)} \nsubseteq \overline{Q}_{(s,v')} \Leftrightarrow P_v \nsubseteq Q_{v'}$
(2) $\overline{P}_{(v,s)} \nsubseteq \overline{Q}_{(v,s')} \Leftrightarrow P_s \nsubseteq Q_{s'}$

A dichotomous topology, or ditopology for short, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of open sets τ satisfies

(T_1) $S, \emptyset \in \tau$
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set of closed sets κ satisfies

(CT_1) $S, \emptyset \in \kappa$
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no a priori relation between the open and closed sets.

Definition 2.4. [6] Let (τ, κ) be a ditopology on $(S, \mathcal{S})$.

(1) Let $s \in S^\flat$. Then a set $N \in \mathcal{S}$ is called a neighborhood of s if there exists $G \in \tau$ satisfying $P_s \subseteq G \subseteq N \not\subseteq Q_s$.
(2) Let $s \in S$. Then a set $M \in \mathcal{S}$ is called a coneighborhood of s if there exists $K \in \kappa$ satisfying $P_s \not\subseteq M \subseteq K \subseteq Q_s$.

If the set of nhds (conhds) of s is denoted by $\eta(s)$ $(\mu(s))$ respectively, then (η, μ) is called dinhd system of (τ, κ).

2.2. **Direlational Uniformities and the Uniform Ditopology.** [4, 12, 13] Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be textures. $\overline{P}_{(s,v)}$, $\overline{Q}_{(s,v)}$ will denote the p-sets and q-sets for the texture $(S \times V, \mathcal{P}(S) \otimes \mathcal{V})$ and $\overline{P}_{(v,s)}$, $\overline{Q}_{(v,s)}$ will denote the p-sets and q-sets for the texture $(V \times S, \mathcal{P}(V) \otimes \mathcal{S})$.

Definition 2.5. [4] Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be textures. Then

(1) $r \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies
  R1 $r \not\subseteq \overline{Q}(s, v)$, $P_{s'} \not\subseteq Q_s \Rightarrow r \not\subseteq \overline{Q}(s', v)$.
  R2 $r \not\subseteq \overline{Q}(s, v) \Rightarrow \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}(s', v)$.
(2) $R \in \mathcal{P}(S) \otimes \mathcal{V}$ is called a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies
  CR1 $\overline{P}(s, v) \not\subseteq R$, $P_s \not\subseteq Q_{s'} \Rightarrow \overline{P}(s', v) \not\subseteq R$.
  CR2 $\overline{P}(s, v) \not\subseteq R \Rightarrow \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}(s', v) \not\subseteq R$.
(3) A pair (r, R), where r is a relation and R a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ is called a direlation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$.

The direlations can be ordered as follows: for direlations (p, P), (q, Q) on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ it is written $(p, P) \sqsubseteq (q, Q)$ if and only if $p \subseteq q$ and $Q \subseteq P$.

For a texture $(S, \mathcal{S})$, $i = i_S = \bigvee\{\overline{P}_{(s,s)} \mid s \in S\}$ is a relation and $I = I_S = \bigcap\{\overline{Q}_{(s,s)} \mid s \in S\}$ is a co-relation on $(S, \mathcal{S})$ to $(S, \mathcal{S})$. That is, (i, I) is a direlation and we call it the identity direlation on $(S, \mathcal{S})$.

Let (r, R) be a direlation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. The inverses of r and R are defined by $r^{\leftarrow} = \bigcap\{\overline{Q}_{(v,s)} \mid r \not\subseteq \overline{Q}_{(s,v)}\}$ and $R^{\leftarrow} = \bigvee\{\overline{P}_{(v,s)} \mid \overline{P}_{(s,v)} \not\subseteq R\}$ where $R^{\leftarrow}$ is a relation and $r^{\leftarrow}$ is a co-relation on $(V, \mathcal{V})$ to $(S, \mathcal{S})$. The direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ is called the inverse of (r, R).

For $A \in \mathcal{S}$, $r^{\to}A = \bigcap\{Q_v \mid \forall s, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow A \subseteq Q_s\}$ is called the A-section of r and $R^{\to}A = \bigvee\{P_v \mid \forall s, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow P_s \subseteq A\}$ is called the A-section of R.

For $B \in \mathcal{V}$, $r^{\leftarrow}B = \bigvee\{P_s \mid \forall v, r \not\subseteq \overline{Q}_{(s,v)} \Rightarrow P_v \subseteq B\}$ is called the B-presection of r and $R^{\leftarrow}B = \bigcap\{Q_s \mid \forall v, \overline{P}_{(s,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$ is called the B-presection of R.

The family of direlations on a texture space $(S, \mathcal{S})$ will be denoted by $\mathfrak{DR}_S$ or if there is no confision just by $\mathfrak{DR}$.

For a direlation (d, D), $d^{\to}P_t$ and $D^{\to}Q_t$ will be denoted by $d[t]$ and $D[t]$ respectively.

Lemma 2.6. [4, 18] *Let r, r_1, r_2 be relations, R, R_1, R_2 co-relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ with $r_1 \subseteq r_2$, $R_1 \subseteq R_2$, $A_1, A_2 \in \mathcal{S}$, $A_1 \subseteq A_2$, $B_1, B_2 \in \mathcal{V}$, $B_1 \subseteq B_2$, $A_j \in \mathcal{S}$, $j \in J$, $B_k \in \mathcal{V}$, $k \in K$.*

(1) $r \not\subseteq \overline{Q}_{(s,v)} \Leftrightarrow \overline{P}_{(v,s)} \not\subseteq r^{\leftarrow}$ and $\overline{P}_{(s,v)} \not\subseteq R \Leftrightarrow R^{\leftarrow} \not\subseteq \overline{Q}_{(v,s)}$ for all $s \in S$, $v \in V$.

(2) $r_1^{\to}A_1 \subseteq r_2^{\to}A_2$, $R_1^{\to}A_1 \subseteq R_2^{\to}A_2$, $r_2^{\leftarrow}B_1 \subseteq r_1^{\leftarrow}B_2$, $R_2^{\leftarrow}B_1 \subseteq R_1^{\leftarrow}B_2$

(3) $r^{\to}(\bigvee_{j\in J} A_j) = \bigvee_{j\in J} r^{\to}A_j$, $R^{\to}(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} R^{\to}A_j$, $r^{\leftarrow}(\bigcap_{k\in K} B_k) = \bigcap_{k\in K} r^{\leftarrow}B_k$, $R^{\leftarrow}(\bigvee_{k\in K} B_k) = \bigvee_{k\in K} R^{\leftarrow}B_k$.

(4) $r \not\subseteq \overline{Q}_{(s,v)} \Leftrightarrow r^{\to}P_s \not\subseteq Q_v$ and $\overline{P}_{(s,v)} \not\subseteq R \Leftrightarrow P_v \not\subseteq R^{\to}Q_s$.

Definition 2.7. [4] Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ and $(Y, \mathcal{Y})$ be textures.

(1) If p is a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ and q is a relation on $(V, \mathcal{V})$ to $(Y, \mathcal{Y})$ then their composition is the relation $q \circ p$ on $(S, \mathcal{S})$ to $(Y, \mathcal{Y})$ defined by

$$q \circ p = \bigvee\{\overline{P}_{(s,y)} \mid \exists v \in V \textit{ with } p \not\subseteq \overline{Q}_{(s,v)} \textit{ and } q \not\subseteq \overline{Q}_{(v,y)}\}.$$

(2) If P is a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ and Q is a co-relation on $(V, \mathcal{V})$ to $(Y, \mathcal{Y})$ then their composition is the co-relation $Q \circ P$ on $(S, \mathcal{S})$ to $(Y, \mathcal{Y})$ defined by

$$Q \circ P = \bigcap\{\overline{Q}_{(s,y)} \mid \exists v \in V \textit{ with } \overline{P}_{(s,v)} \not\subseteq P \textit{ and } \overline{P}_{(v,y)} \not\subseteq Q\}.$$

(3) The composition of direlations (p, P) and (q, Q) is the direlation $(q, Q) \circ (p, P)$ defined by $(q, Q) \circ (p, P) = (q \circ p, Q \circ P)$.

Also it is shown in [4] that the composition of direlations is associative and $[(q, Q) \circ (p, P)]^{\leftarrow} = (p, P)^{\leftarrow} \circ (q, Q)^{\leftarrow}$.

Definition 2.8. [4] Let (f, F) be a direlation from $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then (f, F) is called a difunction from $(S, \mathcal{S})$ to $(V, \mathcal{V})$ if it satisfies the following two conditions:

(DF1) For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists v \in V$ with $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s',v)} \not\subseteq F$.

(DF2) For $v, v' \in V$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v')} \not\subseteq F \Rightarrow P_{v'} \not\subseteq Q_v$.

It is clear that (i_S, I_S) is a difunction on $(S, \mathcal{S})$ and we call it the identity difunction on $(S, \mathcal{S})$.

Definition 2.9. [12] Let (p, P) and (q, Q) be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then

$$p \sqcap q = \bigvee\{\overline{P}_{(s,v)} \mid \exists t \in S \textit{ with } P_s \not\subseteq Q_t \textit{ and } p, q \not\subseteq \overline{Q}_{(t,v)}\},$$

$$P \sqcup Q = \bigcap \{\overline{Q}_{(s,v)} \mid \exists t \in S \text{ with } P_t \not\subseteq Q_s \text{ and } \overline{P}_{(t,v)} \not\subseteq P, Q\},$$

$$(p, P) \sqcap (q, Q) = (p \sqcap q, P \sqcup Q).$$

Proposition 2.10. [12] *Let (p, P) and (q, Q) be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then*

(1) $p \sqcap q$ is a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. It is the greatest lower bound of p and q in the set of all relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by inclusion.
(2) $P \sqcup Q$ is a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. It is the least upper bound of P and Q in the set of all co-relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by inclusion.
(3) The direlation $(p, P) \sqcap (q, Q)$ is the greatest lower bound of (p, P) and (q, Q) in the set of all direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by the relation $\sqsubseteq$.
(4) $(p \sqcap q)^{\leftarrow} = p^{\leftarrow} \sqcup q^{\leftarrow}$ and $(P \sqcup Q)^{\leftarrow} = P^{\leftarrow} \sqcap Q^{\leftarrow}$.
(5) For $A \in \mathcal{S}$, $(p \sqcap q)^{\rightarrow}(A) \subseteq p^{\rightarrow}(A) \cap q^{\rightarrow}(A)$ and $P^{\rightarrow}(A) \cup Q^{\rightarrow}(A) \subseteq (P \sqcup Q)^{\rightarrow}(A)$.
(6) For $B \in \mathcal{V}$, $p^{\leftarrow}(B) \cup q^{\leftarrow}(B) \subseteq (p \sqcap q)^{\leftarrow}(B)$ and $(P \sqcup Q)^{\leftarrow}(B) \subseteq P^{\leftarrow}(B) \cap Q^{\leftarrow}(B)$.
(7) Let (p_1, P_1), (p_2, P_2) be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ and (q_1, Q_1), (q_2, Q_2) be direlations on $(V, \mathcal{V})$ to $(Y, \mathcal{Y})$. Then $((q_1, Q_1) \sqcap (q_2, Q_2)) \circ ((p_1, P_1) \sqcap (p_2, P_2)) \sqsubseteq ((q_1, Q_1) \circ (p_1, P_1)) \sqcap ((q_2, Q_2) \circ (p_2, P_2))$.

Definition 2.11. [13] Let $(S, \mathcal{S})$ be a texture and $\mathcal{U}$ a nonempty family of direlations on $(S, \mathcal{S})$, i.e. $\emptyset \neq \mathcal{U} \subseteq \mathfrak{DR}_S$. If $\mathcal{U}$ satisfies the conditions

(U_1) $(i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathcal{U}$,
(U_2) $(d, D) \in \mathcal{U}$, $(e, E) \in \mathfrak{DR}$ *and* $(d, D) \sqsubseteq (e, E)$ *implies* $(e, E) \in \mathcal{U}$,
(U_3) (d, D), $(e, E) \in \mathcal{U}$ *implies* $(d, D) \sqcap (e, E) \in \mathcal{U}$,
(U_4) Given for all $(d, D) \in \mathcal{U}$ there exists $(e, E) \in \mathcal{U}$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$,
(U_5) Given for all $(d, D) \in \mathcal{U}$ there exists $(c, C) \in \mathcal{U}$ satisfying $(c, C)^{\leftarrow} \sqsubseteq (d, D)$,

then $\mathcal{U}$ is called a direlational uniformity on $(S, \mathcal{S})$ and the triple $(S, \mathcal{S}, \mathcal{U})$ is known as a direlational uniform texture space. We'll use "diuniformity" and "diuniform texture spaces" instead of the terms "direlational uniformity" and "direlational uniform texture space" respectively.

Example 2.12. [13] Let $(\mathbb{I}, \mathcal{J})$ be the unit interval texture. For $\epsilon > 0$ define $d_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s < r + \epsilon\}$, $D_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s \leq r - \epsilon\}$. Then the family $\mathcal{U}_{\mathbb{I}} = \{(d, D) \mid (d, D) \in \mathfrak{DR}$ and there exist $\epsilon > 0$ with $(d_\epsilon, D_\epsilon) \sqsubseteq (d, D)\}$ is a diuniformity on $(\mathbb{I}, \mathcal{J})$.

Proposition 2.13. [13] *Let $(S, \mathcal{S}, \mathcal{U})$ be a diuniform texture space. Then the family $(\eta_{\mathcal{U}}(s), \mu_{\mathcal{U}}(s))$, $s \in S^\flat$, defined by*

$$\eta_{\mathcal{U}}(s) = \{N \in \mathcal{S} \mid N \not\subseteq Q_s,\ P_s \not\subseteq Q_t \Rightarrow\ \exists (d, D) \in \mathcal{U},\ d[t] \subseteq N\}$$

$$\mu_{\mathcal{U}}(s) = \{M \in \mathcal{S} \mid P_s \not\subseteq M,\ P_t \not\subseteq Q_s \Rightarrow\ \exists (d, D) \in \mathcal{U},\ M \subseteq D[t]\}$$

is the dineighborhood system for a ditopology on $(S, \mathcal{S})$.

Definition 2.14. [13] Let $(S, \mathcal{S}, \mathcal{U})$ be a diuniform texture space and $\eta_{\mathcal{U}}(s), \mu_{\mathcal{U}}(s)$ defined as above. The ditopology with dineighborhood system $\{(\eta_{\mathcal{U}}(s), \mu_{\mathcal{U}}(s) \mid s \in S^\flat\}$ is called the uniform ditopology induced by $\mathcal{U}$ and we denote it by $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$.

2.3. **Graded Ditopological Texture Spaces.** [7] Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and consider $\mathcal{T}, \mathcal{K} : \mathcal{S} \to \mathcal{V}$ satisfying

(GT_1) $\mathcal{T}(S) = \mathcal{T}(\emptyset) = V$
(GT_2) $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)$ $\forall A_1, A_2 \in \mathcal{S}$
(GT_3) $\bigcap_{j \in J} \mathcal{T}(A_j) \subseteq \mathcal{T}(\bigvee_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$

and

(GCT_1) $\mathcal{K}(S) = \mathcal{K}(\emptyset) = V$
(GCT_2) $\mathcal{K}(A_1) \cap \mathcal{K}(A_2) \subseteq \mathcal{K}(A_1 \cup A_2)$ $\forall A_1, A_2 \in \mathcal{S}$
(GCT_3) $\bigcap_{j \in J} \mathcal{K}(A_j) \subseteq \mathcal{K}(\bigcap_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$

Then $\mathcal{T}$ is called a $(V, \mathcal{V})$-graded topology, $\mathcal{K}$ a $(V, \mathcal{V})$-graded cotopology and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. The tuple $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is called a graded ditopological texture space. For $v \in V$ we define

$$\mathcal{T}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{T}(A)\},\ \mathcal{K}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{K}(A)\}.$$

Then $(\mathcal{T}^v, \mathcal{K}^v)$ is a ditopology on $(S, \mathcal{S})$ for each $v \in V$. That is, if $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is any graded ditopological texture space then there exists a ditopology $(\mathcal{T}^v, \mathcal{K}^v)$ on $(S, \mathcal{S})$ for each $v \in V$.

If $(S, \mathcal{S}, \sigma)$ is a complemented texture space and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$, then $(\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$ is also a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. Besides $(\mathcal{T}, \mathcal{K})$ is called complemented if $(\mathcal{T}, \mathcal{K}) = (\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$.

Example 2.15. [7] Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathcal{V})$ the discrete texture on a singleton. Take $(V, \mathcal{V}) = (1, \mathcal{P}(1))$ (The notation 1 denotes the set $\{0\}$) and define $\tau^g : \mathcal{S} \to \mathcal{P}(1)$ by $\tau^g(A) = 1 \Leftrightarrow A \in \tau$. Then τ^g is a $(V, \mathcal{V})$-graded topology on $(S, \mathcal{S})$. Likewise, κ^g defined by $\kappa^g(A) = 1 \Leftrightarrow A \in \kappa$ is a $(V, \mathcal{V})$-graded cotopology on $(S, \mathcal{S})$ and (τ^g, κ^g) is called the graded ditopology on $(S, \mathcal{S})$ corresponding to ditopology (τ, κ).

Therefore graded ditopological texture spaces are more general than ditopological texture spaces.

The graded dineighborhood systems of the graded ditopological texture spaces were defined in [9]. To avoid a long preliminaries we will give the following equivalent proposition instead of the definition.

Proposition 2.16. [9] *Let $(\mathcal{T}, \mathcal{K})$ be a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$ and $N : S^\flat \to \mathcal{V}^{\mathcal{S}}$, $M : S \to \mathcal{V}^{\mathcal{S}}$ mappings where $N(s) = N_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S^\flat$ and $M(s) = M_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S$. Then (N, M) is a graded dinhd system of the graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ iff*

$$N_s(A) = \begin{cases} sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}, & A \not\subseteq Q_s \\ \emptyset, & A \subseteq Q_s \end{cases} \tag{1}$$

for each $s \in S^\flat$, $A \in \mathcal{S}$ and

$$M_s(A) = \begin{cases} sup\{\mathcal{K}(B) : P_s \not\subseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\}, & P_s \not\subseteq A \\ \emptyset, & P_s \subseteq A \end{cases} \quad (2)$$

for each $s \in S$, $A \in \mathcal{S}$.

Theorem 2.17. [9] *Let* $(\mathcal{T}, \mathcal{K})$ *be a* $(V, \mathcal{V})$*-graded ditopology on a texture space* $(S, \mathcal{S})$*. If* (N, M) *is the graded dinhd system of the graded ditopological texture space* $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$*, then the following properties hold for all* $A, A_1, A_2 \in \mathcal{S}$*:*

(**1**) *For each* $s \in S^\flat$*;*

(**N1**) $N_s(A) \neq \emptyset \Rightarrow A \not\subseteq Q_s$

(**N2**) $N_s(\emptyset) = \emptyset$ *and* $N_s(S) = V$

(**N3**) $A_1 \subseteq A_2 \Rightarrow N_s(A_1) \subseteq N_s(A_2)$

(**N4**) $A_1 \cap A_2 \not\subseteq Q_s \Rightarrow N_s(A_1) \wedge N_s(A_2) \subseteq N_s(A_1 \cap A_2)$

(**N5**) $N_s(A) \subseteq sup\{\bigwedge_{s' \in B^\flat} N_{s'}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}$

(**2**) *For each* $s \in S$*;*

(**M1**) $M_s(A) \neq \emptyset \Rightarrow P_s \not\subseteq A$

(**M2**) $M_s(S) = \emptyset$ *and* $M_s(\emptyset) = V$

(**M3**) $A_1 \subseteq A_2 \Rightarrow M_s(A_2) \subseteq M_s(A_1)$

(**M4**) $M_s(A_1) \wedge M_s(A_2) \subseteq M_s(A_1 \cup A_2)$

(**M5**) $M_s(A) \subseteq sup\{\bigwedge_{s' \in (S \setminus B)} M_{s'}(B) : P_s \not\subseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\}$

Theorem 2.18. [9] *If the mappings* $N : S^\flat \to \mathcal{V}^{\mathcal{S}}$*,* $M : S \to \mathcal{V}^{\mathcal{S}}$ *satisfy the conditions* $N1-N4$ *and* $M1-M4$ *in Theorem 2.17. respectively then the mappings* $\mathcal{T}_N, \mathcal{K}_M : \mathcal{S} \to \mathcal{V}$*, defined by*

$$\mathcal{T}_N(A) = \bigcap_{s \in A^\flat} N_s(A) \quad (3)$$

$$\mathcal{K}_M(A) = \bigcap_{s \in S \setminus A} M_s(A) \quad (4)$$

where $A \in \mathcal{S}$*, form a* $(V, \mathcal{V})$*-graded ditopology on texture* $(S, \mathcal{S})$*.*

3. The Least Upper Bound of Direlations

The greatest lower bound of two direlations is defined in [12]. We'll need the least upper bound of any family of direlations in the next section, so we'll define it and give some properties of it in this section. We begin to define with the extention of the greatest lower bound of two direlations to the greatest lower bound of any family of direlations. The extention of Proposition 2.10. to "any family" case will be given as Proposition 3.2. with similar proof of Proposition 2.10.

Definition 3.1. Let $(p_i, P_i)_{i \in I}$ be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then

$$\bigsqcap_{i \in I} p_i = \bigvee\{\overline{P}_{(s,v)} \mid \exists t \in S \text{ with } P_s \not\subseteq Q_t \text{ and } \forall i \in I, p_i \not\subseteq \overline{Q}_{(t,v)}\} \quad (5)$$

$$\bigsqcup_{i \in I} P_i = \bigcap\{\overline{Q}_{(s,v)} \mid \exists t \in S \text{ with } P_t \not\subseteq Q_s \text{ and } \forall i \in I, \overline{P}_{(t,v)} \not\subseteq P_i\} \quad (6)$$

$$\prod_{i\in I}(p_i, P_i) = (\prod_{i\in I} p_i, \bigsqcup_{i\in I} P_i). \tag{7}$$

Proposition 3.2. *Let $(p_i, P_i)_{i\in I}$ be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then*

(1) $\prod_{i\in I} p_i$ is a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. It is the greatest lower bound of $\{p_i\}_{i\in I}$ in the set of all relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by inclusion.

(2) $\bigsqcup_{i\in I} P_i$ is a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. It is the least upper bound of $\{P_i\}_{i\in I}$ in the set of all co-relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by inclusion.

(3) The direlation $\prod_{i\in I}(p_i, P_i)_{i\in I}$ is the greatest lower bound of $(p_i, P_i)_{i\in I}$ in the set of all direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by the relation $\sqsubseteq$.

(4) $(\prod_{i\in I} p_i)^{\leftarrow} = \bigsqcup_{i\in I} p_i^{\leftarrow}$ and $(\bigsqcup_{i\in I} P_i)^{\leftarrow} = \prod_{i\in I} P_i^{\leftarrow}$.

(5) For $A \in \mathcal{S}$, $(\prod_{i\in I} p_i)(A) \subseteq \bigcap_{i\in I} p_i(A)$ and $\bigvee_{i\in I} P_i(A) \subseteq (\bigsqcup_{i\in I} P_i)(A)$.

(6) For $B \in \mathcal{V}$, $\bigvee_{i\in I} p_i^{\leftarrow}(B) \subseteq (\prod_{i\in I} p_i)^{\leftarrow}(B)$ and $(\bigsqcup_{i\in I} P_i)^{\leftarrow}(B) \subseteq \bigcap_{i\in I} P_i^{\leftarrow}(B)$.

(7) Let $(p_i, P_i)_{i\in I}$ be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ and $(q_i, Q_i)_{i\in I}$ be direlations on $(V, \mathcal{V})$ to $(Y, \mathcal{Y})$. Then $(\prod_{i\in I}(q_i, Q_i)) \circ (\prod_{i\in I}(p_i, P_i)) \sqsubseteq \prod_{i\in I}((q_i, Q_i) \circ (p_i, P_i))$.

Proof. (1) At first, to show that $\prod_{i\in I} p_i$ is a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ we will show that $\prod_{i\in I} p_i$ satisfies the conditions (R1-R2) in Definition 2.5.

R1: Let $\prod_{i\in I} p_i \not\subseteq \overline{Q}(s, v)$ and $P_{s'} \not\subseteq Q_s$. Since $\prod_{i\in I} p_i \not\subseteq \overline{Q}(s, v)$, there exists $t \in S$, $v' \in V$ such that $P_s \not\subseteq Q_t$, "$\forall i \in I,\ p_i \not\subseteq \overline{Q}(t, v')$", and $\overline{P}(s, v') \not\subseteq \overline{Q}(s, v)$. Now, considering $P_{s'} \not\subseteq Q_s$ we have $P_s \subseteq P_{s'}$ and so "$\exists t \in S\ :\ P_{s'} \not\subseteq Q_t$ and $\forall i \in I,\ p_i \not\subseteq \overline{Q}(t, v')$". Hence, we get

$$\overline{P}(s', v') \in \{\overline{P}(s, v) \mid \exists t \in S \text{ with } P_s \not\subseteq Q_t \text{ and } \forall i \in I,\ p_i \not\subseteq \overline{Q}(t, v)\}. \tag{8}$$

On the other hand, $\overline{P}(s, v') \not\subseteq \overline{Q}(s, v) \Rightarrow P_{v'} \not\subseteq Q_v \Rightarrow \overline{P}(s', v') \not\subseteq \overline{Q}(s', v)$. Therefore, considering (8) we have $\prod_{i\in I} p_i \not\subseteq \overline{Q}(s', v)$.

R2: Let $\prod_{i\in I} p_i \not\subseteq \overline{Q}(s, v)$. Then there exists $t \in S$ such that $P_s \not\subseteq Q_t$ and "$\forall i \in I,\ p_i \not\subseteq \overline{Q}(t, v')$"; $\overline{P}(s, v') \not\subseteq \overline{Q}(s, v)$. From $\overline{P}(s, v') \not\subseteq \overline{Q}(s, v)$ we have $P_{v'} \not\subseteq Q_v$. Considering this with "$P_s \not\subseteq Q_t$" we get $\overline{P}(s, v') \not\subseteq \overline{Q}(t, v)$ and so $\prod_{i\in I} p_i \not\subseteq \overline{Q}(t, v)$. Therefore we obtain that $\exists t \in S\ :\ P_s \not\subseteq Q_t,\ \prod_{i\in I} p_i \not\subseteq \overline{Q}(t, v)$.

Suppose that $\prod_{i\in I} p_i \not\subseteq p_j$ for some $j \in I$. Then there exist $s \in S$, $v \in V$ such that $\prod_{i\in I} p_i \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v)} \not\subseteq p_j$. From (5) and Proposition 2.3. (1), there exist $v' \in V$, $t \in S$ such that $P_{v'} \not\subseteq Q_v$, $P_s \not\subseteq Q_t$ and $p_i \not\subseteq \overline{Q}_{(t,v')}$ for all $i \in I$. Since p_j is a relation, from (R1) we get $p_j \not\subseteq \overline{Q}_{(s,v')}$. Also we have $Q_v \subseteq Q_{v'}$ from $P_{v'} \not\subseteq Q_v$. So, we obtain $p_j \not\subseteq \overline{Q}_{(s,v)}$ and it follows that $\overline{P}_{(s,v)} \subseteq p_j$. But this result contradicts with $\overline{P}_{(s,v)} \not\subseteq p_j$. Therefore we get $\prod_{i\in I} p_i \subseteq p_j$ for all $j \in I$, that is $\prod_{i\in I} p_i$ is a lower bound of $\{p_i\}_{i\in I}$.

Let r be a relation with $r \subseteq p_i$ for each $i \in I$. Suppose that $r \not\subseteq \prod_{i\in I} p_i$. Then there exist $s \in S$, $v \in V$ such that $r \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{P}_{(s,v)} \not\subseteq \prod_{i\in I} p_i$. Since r is a relation, using (R2), there exists $s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',v)}$. So we get $p_i \not\subseteq \overline{Q}_{(s',v)}$ since $r \subseteq p_i$ for each $i \in I$. Hence we obtain $\overline{P}_{(s,v)} \subseteq \prod_{i\in I} p_i$ by (5) but this result contradicts with $\overline{P}_{(s,v)} \not\subseteq \prod_{i\in I} p_i$. Therefore $\prod_{i\in I} p_i$ is the greatest

lower bound of $\{p_i\}_{i\in I}$.
(2) Similar to (1).
(3) It is clear from (1) and (2).
(4) Suppose that $(\bigsqcap_{i\in I} p_i)^{\leftarrow} \not\subseteq \bigsqcup_{i\in I} p_i^{\leftarrow}$. Then there exist $s \in S$, $v \in V$ such that $(\bigsqcap_{i\in I} p_i)^{\leftarrow} \not\subseteq \overline{Q}_{(v,s)}$ and $\overline{P}_{(v,s)} \not\subseteq \bigsqcup_{i\in I} p_i^{\leftarrow}$. From (6), there exist $s' \in S$ with $\overline{P}_{(v,s)} \not\subseteq \overline{Q}_{(v,s')}$ and $t \in V$ such that $P_t \not\subseteq Q_v$, $\overline{P}_{(t,s')} \not\subseteq p_i^{\leftarrow}$ for each $i \in I$. Using Lemma 2.6., we get $p_i \not\subseteq \overline{Q}_{(s',t)}$. So, from $P_s \not\subseteq Q_{s'}$ and (5) we have $\overline{P}_{(s,t)} \subseteq \bigsqcap_{i\in I} p_i$. Hence, considering $P_v \subseteq P_t$ obtained the contradiction $\overline{P}_{(s,v)} \subseteq \bigsqcap_{i\in I} p_i$.

Now, suppose that $\bigsqcup_{i\in I} p_i^{\leftarrow} \not\subseteq (\bigsqcap_{i\in I} p_i)^{\leftarrow}$. Then there exist $s \in S$, $v \in V$ such that $\bigsqcup_{i\in I} p_i^{\leftarrow} \not\subseteq \overline{Q}_{(v,s)}$ and $\overline{P}_{(v,s)} \not\subseteq (\bigsqcap_{i\in I} p_i)^{\leftarrow}$. From Lemma 2.6. we get $\bigsqcap_{i\in I} p_i \not\subseteq \overline{Q}_{(s,v)}$ and considering (5) there exist $v' \in V$, $t \in S$ such that $\overline{P}_{(s,v')} \not\subseteq \overline{Q}_{(s,v)}$, $P_s \not\subseteq Q_t$, $p_i \not\subseteq \overline{Q}_{(t,v')}$ for each $i \in I$. Considering Lemma 2.6., we get $\overline{P}_{(v',t)} \not\subseteq p_i^{\leftarrow}$ for each $i \in I$ and so by using (6) we have $\bigsqcup_{i\in I} p_i^{\leftarrow} \subseteq \overline{Q}_{(v,t)}$. Since $Q_t \subseteq Q_s$, it follows that $\bigsqcup_{i\in I} p_i^{\leftarrow} \subseteq \overline{Q}_{(v,s)}$ which contradicts with $\bigsqcup_{i\in I} p_i^{\leftarrow} \not\subseteq \overline{Q}_{(v,s)}$. Hence we get $\bigsqcup_{i\in I} p_i^{\leftarrow} = (\bigsqcap_{i\in I} p_i)^{\leftarrow}$. Similarly it can be shown that $(\bigsqcup_{i\in I} P_i)^{\leftarrow} = \bigsqcap_{i\in I} P_i^{\leftarrow}$.

(5) In the contrary, let it be $(\bigsqcap_{i\in I} p_i)(A) \not\subseteq \bigcap_{i\in I} p_i(A)$. Then we have $(\bigsqcap_{i\in I} p_i)(A) \not\subseteq Q_v$ and $P_v \not\subseteq \bigcap_{i\in I} p_i(A)$ for some $v \in V$. Since $(\bigsqcap_{i\in I} p_i)(A) \not\subseteq Q_v$ there exists a $s \in S$ such that $(\bigsqcap_{i\in I} p_i) \not\subseteq \overline{Q}_{(s,v)}$ and $A \not\subseteq Q_s$. So there exist $v' \in V$, $t \in S$ such that $\overline{P}_{(s,v')} \not\subseteq \overline{Q}_{(s,v)}$, $P_s \not\subseteq Q_t$ and $p_i \not\subseteq \overline{Q}_{(t,v)}$ for each $i \in I$. Now we get $p_i \not\subseteq \overline{Q}_{(s,v)}$ for each $i \in I$ by (R1). Further we have at least a $j \in I$ with $P_v \not\subseteq p_j(A)$. So there exists $u \in V$ with $P_v \not\subseteq Q_u$ so that $p_j \not\subseteq \overline{Q}_{(s',u)} \Rightarrow A \subseteq Q'_s$ for each $s' \in S$. Since $p_j \not\subseteq \overline{Q}_{(s,v)}$ and $\overline{Q}_{(s,u)} \subseteq \overline{Q}_{(s,v)}$ we get $p_j \not\subseteq \overline{Q}_{(s,u)}$. Hence, considering "$p_j \not\subseteq \overline{Q}_{(s',u)} \Rightarrow A \subseteq Q'_s$ for each $s' \in S$" we obtain $A \subseteq Q_s$ which contradicts with $A \not\subseteq Q_s$.

Similarly, it can be shown that $\bigvee_{i\in I} P_i(A) \subseteq (\bigsqcup_{i\in I} P_i)(A)$.

(6) It is clear from (4) and (5).

(7) To show that $(\bigsqcap_{i\in I}(q_i, Q_i)) \circ (\bigsqcap_{i\in I}(p_i, P_i)) \sqsubseteq \bigsqcap_{i\in I}((q_i, Q_i) \circ (p_i, P_i))$ and equivalently $(\bigsqcap_{i\in I} q_i \circ \bigsqcap_{i\in I} p_i, \bigsqcup_{i\in I} Q_i \circ \bigsqcup_{i\in I} P_i) \sqsubseteq (\bigsqcap_{i\in I}(q_i \circ p_i), \bigsqcup_{i\in I}(Q_i \circ P_i))$ we must show that $(\bigsqcap_{i\in I} q_i) \circ (\bigsqcap_{i\in I} p_i) \subseteq \bigsqcap_{i\in I}(q_i \circ p_i)$ and $\bigsqcup_{i\in I}(Q_i \circ P_i) \subseteq (\bigsqcup_{i\in I} Q_i \circ \bigsqcup_{i\in I} P_i)$. Firstly, suppose that $(\bigsqcap_{i\in I} q_i) \circ (\bigsqcap_{i\in I} p_i) \not\subseteq \bigsqcap_{i\in I}(q_i \circ p_i)$. Then there exist $s \in S$, $y \in Y$ such that $(\bigsqcap_{i\in I} q_i) \circ (\bigsqcap_{i\in I} p_i) \not\subseteq \overline{Q}_{(s,y)}$ and $\overline{P}_{(s,y)} \not\subseteq \bigsqcap_{i\in I}(q_i \circ p_i)$ and so, there exist $v \in V$ such that $\bigsqcap_{i\in I} p_i \not\subseteq \overline{Q}_{(s,v)}$ and $\bigsqcap_{i\in I} q_i \not\subseteq \overline{Q}_{(v,y)}$.

Now, considering $\bigsqcap_{i\in I} p_i \not\subseteq \overline{Q}_{(s,v)}$ and (5), there exist $v' \in V$, $t \in S$ with $P_s \not\subseteq Q_t$ such that $\overline{P}_{(s,v')} \not\subseteq \overline{Q}_{(s,v)}$ and $p_i \not\subseteq \overline{Q}_{(t,v')}$ for each $i \in I$. Similarly, from $\bigsqcap_{i\in I} q_i \not\subseteq \overline{Q}_{(v,y)}$ and (6), there exist $y' \in Y$, $v'' \in V$ with $P_v \not\subseteq Q_{v''}$ such that $\overline{P}_{(v,y')} \not\subseteq \overline{Q}_{(v,y)}$ and $q_i \not\subseteq \overline{Q}_{(v'',y')}$ for each $i \in I$. Since $\overline{P}_{(s,v')} \not\subseteq \overline{Q}_{(s,v)}$ we have $P_{v'} \not\subseteq Q_v$ and so $P_v \subseteq P_{v'}$. So, considering $P_v \not\subseteq Q_{v''}$ and $P_v \subseteq P_{v'}$ we get $P_{v'} \not\subseteq Q_{v''}$. Since $q_i \not\subseteq \overline{Q}_{(v'',y')}$ for each $i \in I$ and $P_{v'} \not\subseteq Q_{v''}$, by (R1), we obtain that $q_i \not\subseteq \overline{Q}_{(v',y')}$ for each $i \in I$ and since "$p_i \not\subseteq \overline{Q}_{(t,v')}$ for each $i \in I$" we get $q_i \circ p_i \not\subseteq \overline{Q}_{(t,y')}$ that is $\overline{P}_{(t,y')} \subseteq q_i \circ p_i$ for each $i \in I$. On the other hand, since $\overline{P}_{(v,y')} \not\subseteq \overline{Q}_{(v,y)}$ we have $P_{y'} \not\subseteq Q_y$ and so $\overline{P}_{(t,y')} \not\subseteq \overline{Q}_{(t,y)}$. Hence, we get $q_i \circ p_i \not\subseteq \overline{Q}_{(t,y)}$ for each $i \in I$. Since $P_s \not\subseteq Q_t$ and (R1) we get $\overline{P}_{(s,y)} \subseteq \bigsqcap_{i\in I}(q_i \circ p_i)$ which contradicts with $\overline{P}_{(s,y)} \not\subseteq \bigsqcap_{i\in I}(q_i \circ p_i)$.

It can be also shown that $\bigsqcup_{i\in I}(Q_i \circ P_i) \subseteq (\bigsqcup_{i\in I} Q_i) \circ (\bigsqcup_{i\in I} P_i)$ in the same way. □

Definition 3.3. Let $(p_i, P_i)_{i\in I}$ be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then

$$\bigsqcup_{i\in I} p_i = \bigsqcap\{q \mid \forall i \in I,\ p_i \subseteq q,\ q \text{ is a relation from } (S,\mathcal{S}) \text{ to } (V,\mathcal{V})\},$$

$$\bigsqcap_{i\in I} P_i = \bigsqcup\{Q \mid \forall i \in I,\ Q \subseteq P_i,\ Q \text{ is a corelation from } (S,\mathcal{S}) \text{ to } (V,\mathcal{V})\},$$

$$\bigsqcup_{i\in I}(p_i, P_i)_{i\in I} = (\bigsqcup_{i\in I} p_i, \bigsqcap_{i\in I} P_i).$$

Proposition 3.4. *Let $(p_i, P_i)_{i\in I}$ be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. Then*

(1) $\bigsqcup_{i\in I} p_i$ is a relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. It is the least upper bound of $\{p_i\}_{i\in I}$ in the set of all relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by inclusion.

(2) $\bigsqcap_{i\in I} P_i$ is a co-relation on $(S, \mathcal{S})$ to $(V, \mathcal{V})$. It is the greatest lower bound of $\{P_i\}_{i\in I}$ in the set of all co-relations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by inclusion.

(3) The direlation $\bigsqcup_{i\in I}(p_i, P_i)_{i\in I}$ is the least upper bound of $(p_i, P_i)_{i\in I}$ in the set of all direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$, ordered by the relation $\sqsubseteq$.

(4) $(\bigsqcup_{i\in I} p_i)^{\leftarrow} = \bigsqcap_{i\in I} p_i^{\leftarrow}$ and $(\bigsqcap_{i\in I} P_i)^{\leftarrow} = \bigsqcup_{i\in I} P_i^{\leftarrow}$.

(5) Let $(p_i, P_i)_{i\in I}$ be direlations on $(S, \mathcal{S})$ to $(V, \mathcal{V})$ and $(q_i, Q_i)_{i\in I}$ be direlations on $(V, \mathcal{V})$ to $(Y, \mathcal{Y})$. Then $(\bigsqcup_{i\in I}(q_i, Q_i)) \circ (\bigsqcup_{i\in I}(p_i, P_i)) \sqsubseteq \bigsqcup_{i\in I}((q_i, Q_i) \circ (p_i, P_i))$.

Proof. (1), (2) and (3) are straightforward from Definition 3.3.

(4) $(\bigsqcup_{i\in I} p_i)^{\leftarrow} = (\bigsqcap\{q \mid \forall i \in I,\ p_i \subseteq q\})^{\leftarrow} = \bigsqcup\{q^{\leftarrow} \mid \forall i \in I,\ q^{\leftarrow} \subseteq p_i^{\leftarrow}\} = \bigsqcap_{i\in I} p_i^{\leftarrow}$.

$(\bigsqcap_{i\in I} P_i)^{\leftarrow} = (\bigsqcup\{Q \mid \forall i \in I,\ Q \subseteq P_i\})^{\leftarrow} = \bigsqcap\{Q^{\leftarrow} \mid \forall i \in I,\ P_i^{\leftarrow} \subseteq Q^{\leftarrow}\} = \bigsqcup_{i\in I} P_i^{\leftarrow}$.

(5) From Definition 3.3. and Proposition 3.2. we get: $(\bigsqcup_{i\in I} q_i) \circ (\bigsqcup_{i\in I} p_i) = (\bigsqcap\{q \mid \forall i \in I,\ q_i \subseteq q\}) \circ (\bigsqcap\{p \mid \forall i \in I,\ p_i \subseteq p\}) \sqsubseteq \bigsqcap\{(q \circ p) \mid \forall i \in I,\ q_i \subseteq q \text{ and } p_i \subseteq p\} \sqsubseteq \bigsqcap\{(q \circ p) \mid \forall i \in I,\ (q_i \circ p_i) \subseteq (q \circ p)\} = \bigsqcup_{i\in I}(q_i \circ p_i)$. Similarly it can be shown that $\bigsqcap_{i\in I}(Q_i \circ P_i) \sqsubseteq (\bigsqcap_{i\in I} Q_i) \circ (\bigsqcap_{i\in I} P_i)$. □

4. Graded Diuniformity and Uniform Graded Ditopology

Definition 4.1. Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and $\mathfrak{DR}$ denote the family of all direlations on $(S, \mathcal{S})$. A mapping $\mathfrak{U} : \mathfrak{DR} \to \mathcal{V}$ is called a $(V, \mathcal{V})$-graded diuniformity on $(S, \mathcal{S})$ if it satisfies:

(GU1) $\mathfrak{U}(d, D) \neq \emptyset \Rightarrow (i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathfrak{DR}$

(GU2) $(d, D) \sqsubseteq (e, E) \Rightarrow \mathfrak{U}(d, D) \subseteq \mathfrak{U}(e, E)$ for all $(d, D), (e, E) \in \mathfrak{DR}$

(GU3) $\mathfrak{U}(d, D) \wedge \mathfrak{U}(e, E) \subseteq \mathfrak{U}((d, D) \sqcap (e, E))$ for all $(d, D), (e, E) \in \mathfrak{DR}$

(GU4) $\forall (d,D) \in \mathfrak{DR}\ \exists (e,E) \in \mathfrak{DR} : \mathfrak{U}(d,D) \subseteq \mathfrak{U}(e,E)$ and $(e,E) \circ (e,E) \sqsubseteq (d,D)$
(GU5) $\forall (d,D) \in \mathfrak{DR}\ \exists (c,C) \in \mathfrak{DR} : \mathfrak{U}(d,D) \subseteq \mathfrak{U}(c,C)$ and $(c,C)^{\leftarrow} \sqsubseteq (d,D)$
(GU6) $\bigvee \{\mathfrak{U}(d,D) \mid (d,D) \in \mathfrak{DR}\} = V$.

In this case the tuple $(S, \mathcal{S}, \mathfrak{U}, V, \mathcal{V})$ is called a graded (direlational) diuniform texture space. From now on, we call graded direlational diuniform texture space just by graded diuniform texture space.

Proposition 4.2. *Let $(S, \mathcal{S}, \mathfrak{U}, V, \mathcal{V})$ be a graded diuniform texture space. For each $s \in S^\flat$ the mapping $N_s^{\mathfrak{U}} : \mathcal{S} \to \mathcal{V}$ defined by*

$$N_s^{\mathfrak{U}}(A) = \begin{cases} \bigcap_{P_s \not\subseteq Q_t} \bigvee_{d[t] \subseteq A} \mathfrak{U}(d,D), & A \not\subseteq Q_s \\ \emptyset, & A \subseteq Q_s \end{cases}$$

for all $A \in \mathcal{S}$, holds the properties $(N1) - (N4)$ of Theorem 2.17. For each $s \in S$ the mapping $M_s^{\mathfrak{U}} : \mathcal{S} \to \mathcal{V}$ defined by

$$M_s^{\mathfrak{U}}(A) = \begin{cases} \bigcap_{P_t \not\subseteq Q_s} \bigvee_{A \subseteq D[t]} \mathfrak{U}(d,D), & P_s \not\subseteq A \\ \emptyset, & P_s \subseteq A \end{cases}$$

for all $A \in \mathcal{S}$, holds the properties $(M1) - (M4)$ of Theorem 2.17.

Proof. $(N1)$ and $(N2)$ are clear. $(N3)$: Let $A_1, A_2 \in \mathcal{S}$, $A_1 \subseteq A_2$. If $A_1 = \emptyset$ then $N_s^{\mathfrak{U}}(A_1) = \emptyset \subseteq N_s^{\mathfrak{U}}(A_2)$. If $A_1 \neq \emptyset$ then we have

$$N_s^{\mathfrak{U}}(A_1) = \bigcap_{P_s \not\subseteq Q_t} \bigvee_{d[t] \subseteq A_1} \mathfrak{U}(d,D) \subseteq \bigcap_{P_s \not\subseteq Q_t} \bigvee_{d[t] \subseteq A_2} \mathfrak{U}(d,D) = N_s^{\mathfrak{U}}(A_2).$$

$(N4)$: Let $A_1, A_2 \in \mathcal{S}$, $A_1 \cap A_2 \neq \emptyset$. So, using $(GU3)$ we get

$$N_s^{\mathfrak{U}}(A_1) \wedge N_s^{\mathfrak{U}}(A_2) = \Big(\bigcap_{P_s \not\subseteq Q_t} \bigvee_{d[t] \subseteq A_1} \mathfrak{U}(d,D)\Big) \wedge \Big(\bigcap_{P_s \not\subseteq Q_t} \bigvee_{e[t] \subseteq A_2} \mathfrak{U}(e,E)\Big)$$

$$= \bigcap_{P_s \not\subseteq Q_t} \Big(\bigvee_{d[t] \subseteq A_1} \mathfrak{U}(d,D) \wedge \bigvee_{e[t] \subseteq A_2} \mathfrak{U}(e,E)\Big) = \bigcap_{P_s \not\subseteq Q_t} \Big(\bigvee_{d[t] \subseteq A_1,\ e[t] \subseteq A_2} (\mathfrak{U}(d,D) \wedge \mathfrak{U}(e,E))\Big)$$

$$\subseteq \bigcap_{P_s \not\subseteq Q_t} \Big(\bigvee_{d[t] \subseteq A_1,\ e[t] \subseteq A_2} \mathfrak{U}((d,D) \sqcap (e,E))\Big) \subseteq \bigcap_{P_s \not\subseteq Q_t} \Big(\bigvee_{k[t] \subseteq A_1 \cap A_2} \mathfrak{U}(k,K) = N_s^{\mathfrak{U}}(A_1 \cap A_2)$$

since $(d \sqcap e)[t] = (d \sqcap e)^{\to} P_t \subseteq d^{\to} P_t \cap e^{\to} P_t \subseteq d[t] \cap e[t] \subseteq A_1 \cap A_2$ and $(d,D) \sqcap (e,E) = (d \sqcap e, D \sqcup E) \in \mathfrak{DR}$.

Similarly it can be shown that $M_s^{\mathfrak{U}}$ holds the properties $(M1) - (M4)$ of Theorem 2.17. □

Corollary 4.3. *Let $(S, \mathcal{S}, \mathfrak{U}, V, \mathcal{V})$ be a graded diuniform texture space. Then the mappings $\mathcal{T}_{\mathfrak{U}}, \mathcal{K}_{\mathfrak{U}} : \mathcal{S} \to \mathcal{V}$ defined by*

$$\mathcal{T}_{\mathfrak{U}}(A) = \bigcap_{s \in A^\flat} N_s^{\mathfrak{U}}(A) = \bigcap_{s \in A^\flat} \bigcap_{P_s \not\subseteq Q_t} \bigvee_{d[t] \subseteq A} \mathfrak{U}(d,D), \tag{9}$$

$$\mathcal{K}_{\mathfrak{U}}(A) = \bigcap_{s \in S \setminus A} M_s^{\mathfrak{U}}(A) = \bigcap_{s \in S \setminus A} \bigcap_{P_t \not\subseteq Q_s} \bigvee_{A \subseteq D[t]} \mathfrak{U}(d,D) \tag{10}$$

where $A \in \mathcal{S}$, *form a* $(V, \mathcal{V})$*-graded ditopology* $(\mathcal{T}_{\mathfrak{U}}, \mathcal{K}_{\mathfrak{U}})$ *on* $(S, \mathcal{S})$.

Proof. It is clear from Theorem 2.18. □

Corollary 4.4. *The mappings* $\mathcal{T}_{\mathfrak{U}}, \mathcal{K}_{\mathfrak{U}} : \mathcal{S} \to \mathcal{V}$ *defined in Corollary 4.3. may also be written as*

$$\mathcal{T}_{\mathfrak{U}}(A) = \bigcap_{t \in A^\flat} \bigvee_{d[t] \subseteq A} \mathfrak{U}(d, D), \quad \mathcal{K}_{\mathfrak{U}}(A) = \bigcap_{t \in S \setminus A} \bigvee_{A \subseteq D[t]} \mathfrak{U}(d, D) \tag{11}$$

where $A \in \mathcal{S}$.

Proof. If we define the sets $Z_1 = \{t \in S \mid A \not\subseteq Q_s,\ P_s \not\subseteq Q_t \ for\ some\ s \in S\}$, $Z_2 = \{t \in S \mid P_s \not\subseteq A,\ P_t \not\subseteq Q_s \ for\ some\ s \in S\}$ then we have $Z_1 = A^\flat$ and $Z_2 = S \setminus A$ by Theorem 2.1 (5). So, for each $A \in \mathcal{S}$,

$$\mathcal{T}_{\mathfrak{U}}(A) = \bigcap_{s \in A^\flat} N_s^{\mathfrak{U}}(A) = \bigcap_{s \in A^\flat} \bigcap_{P_s \not\subseteq Q_t} \bigvee_{d[t] \subseteq A} \mathfrak{U}(d, D) = \bigcap_{t \in Z_1} \bigvee_{d[t] \subseteq A} \mathfrak{U}(d, D) = \bigcap_{t \in A^\flat} \bigvee_{d[t] \subseteq A} \mathfrak{U}(d, D),$$

$$\mathcal{K}_{\mathfrak{U}}(A) = \bigcap_{s \in S \setminus A} M_s^{\mathfrak{U}}(A) = \bigcap_{s \in S \setminus A} \bigcap_{P_t \not\subseteq Q_s} \bigvee_{A \subseteq D[t]} \mathfrak{U}(d, D) = \bigcap_{t \in Z_2} \bigvee_{A \subseteq D[t]} \mathfrak{U}(d, D) = \bigcap_{t \in S \setminus A} \bigvee_{A \subseteq D[t]} \mathfrak{U}(d, D)$$

is obtained. □

Definition 4.5. A graded ditopolgy generated by a graded diuniformity as in Corollary 4.3. is called a uniform graded ditopology.

Example 4.6. (1) Let $(S, \mathcal{S}, \mathfrak{U}, V, \mathcal{V})$ be a graded diuniform texture space. Then the set $\mathfrak{U}^v = \{(d, D) \in \mathfrak{D}\mathfrak{R} \mid P_v \subseteq \mathfrak{U}(d, D)\} \neq \emptyset$ is a diuniformity on $(S, \mathcal{S})$ for each $v \in V^\flat$.

(2) If $\mathcal{U}$ is a diuniformity on $(S, \mathcal{S})$ then the mapping $\mathfrak{U}_{\mathcal{U}} : \mathfrak{D}\mathfrak{R} \to \mathcal{P}(1)$ defined by

$$\mathfrak{U}_{\mathcal{U}}(d, D) = \begin{cases} 1, & (d, D) \in \mathcal{U} \\ \emptyset, & (d, D) \notin \mathcal{U} \end{cases}$$

is a $(1, \mathcal{P}(1))$-graded diuniformity on $(S, \mathcal{S})$.

Thus, graded diuniformities which we introduced in Definition 4.1. are more general than diuniformities on texture spaces.

Definition 4.7. Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and $\mathcal{U}_v$ diuniformity on $(S, \mathcal{S})$ for each $v \in V$. The family $\{\mathcal{U}_v\}_{v \in V}$ is called $\mathcal{V}$-compatible if $\mathcal{U}_v = \bigcap\{\mathcal{U}_{v'} \mid P_v \not\subseteq Q_{v'}\}$ for each $v \in V$.

Proposition 4.8. *Let* $(S, \mathcal{S})$, $(V, \mathcal{V})$ *be textures. If* $\{\mathcal{U}_v\}_{v \in V}$ *is a* $\mathcal{V}$*-compatible family of diuniformities on* $(S, \mathcal{S})$ *then*

$$\bigvee\{P_v \mid (d, D) \in \mathcal{U}_v\} = \bigcap\{Q_v \mid (d, D) \notin \mathcal{U}_v\} \tag{12}$$

for each $(d, D) \in \mathfrak{D}\mathfrak{R}$.

Proof. Suppose that $\bigvee\{P_v \mid (d,D) \in \mathcal{U}_v\} \not\subseteq \bigcap\{Q_v \mid (d,D) \notin \mathcal{U}_v\}$. Then there exists $v \in V$ with $(d,D) \in \mathcal{U}_v$ such that $P_v \not\subseteq \bigcap\{Q_v \mid (d,D) \notin \mathcal{U}_v\}$. So we get that $P_v \not\subseteq Q_t$ and $(d,D) \notin \mathcal{U}_t$ for a $t \in V$. Since $\{\mathcal{U}_v\}_{v\in V}$ is $\mathcal{V}$-compatible, we obtain that $\mathcal{U}_v \subseteq \mathcal{U}_t$ and this implies the contradiction $(d,D) \notin \mathcal{U}_v$.

Now we suppose that $\bigcap\{Q_v \mid (d,D) \notin \mathcal{U}_v\} \not\subseteq \bigvee\{P_v \mid (d,D) \in \mathcal{U}_v\}$. Then there exists $t \in V$ such that $\bigcap\{Q_v \mid (d,D) \notin \mathcal{U}_v\} \not\subseteq Q_t$ and $P_t \not\subseteq \bigvee\{P_v \mid (d,D) \in \mathcal{U}_v\}$. So we get the contradiction $(d,D) \in \mathcal{U}_t$ and $(d,D) \notin \mathcal{U}_t$. Hence we have the equality $\bigvee\{P_v \mid (d,D) \in \mathcal{U}_v\} = \bigcap\{Q_v \mid (d,D) \notin \mathcal{U}_v\}$. □

Theorem 4.9. *Let $(S,\mathcal{S})$, $(V,\mathcal{V})$ be textures and $\{\mathcal{U}_v\}_{v\in V}$ be a $\mathcal{V}$-compatible family of diuniformities on $(S,\mathcal{S})$. Then the mapping $\mathfrak{U} : \mathfrak{DR} \to \mathcal{V}$ defined by*

$$\mathfrak{U}(d,D) = \bigvee\{P_v \mid (d,D) \in \mathcal{U}_v\}, \quad (d,D) \in \mathfrak{DR} \tag{13}$$

is a $(V,\mathcal{V})$-graded diuniformity on $(S,\mathcal{S})$.

Proof. To show that $\mathfrak{U}(d,D)$ is a $(V,\mathcal{V})$-graded diuniformity on $(S,\mathcal{S})$ we will show that the properties of Definition 4.1. are satisfied.

GU1: Let $(d,D) \in \mathfrak{DR}$. $\mathfrak{U}(d,D) \neq \emptyset \Rightarrow \exists v \in V$ so that $(d,D) \in \mathcal{U}_v \Rightarrow (i,I) \sqsubseteq (d,D)$.

GU2: Let (d,D), $(e,E) \in \mathfrak{DR}$, $(d,D) \sqsubseteq (e,E)$. If $\mathfrak{U}(d,D) = \emptyset$ then (GU2) holds. So, let $\mathfrak{U}(d,D) \neq \emptyset$. Then we have $(d,D) \in \mathcal{U}_v$ for some $v \in V$. We get $\mathfrak{U}(d,D) = \bigvee\{P_v \mid (d,D) \in \mathcal{U}_v\} \subseteq \bigvee\{P_v \mid (e,E) \in \mathcal{U}_v\} = \mathfrak{U}(e,E)$ since "$(d,D) \in \mathcal{U}_v \Rightarrow (e,E) \in \mathcal{U}_v$" for each $v \in V$.

GU3: Let (d,D), $(e,E) \in \mathfrak{DR}$. If $\mathfrak{U}(d,D) = \emptyset$ or $\mathfrak{U}(e,E) = \emptyset$ then (GU3) is hold. So, let $\mathfrak{U}(d,D) \neq \emptyset$ and $\mathfrak{U}(e,E) \neq \emptyset$. Then we have $(d,D) \in \mathcal{U}_v$ and $(e,E) \in \mathcal{U}_u$ for some $v,u \in V$. Since "$(d,D),(e,E) \in \mathcal{U}_v \Rightarrow (d,D) \sqcap (e,E) \in \mathcal{U}_v$" for all $v \in V$ from Definition 2.11 (U_3), we have the fact "$(d,D) \sqcap (e,E) \notin \mathcal{U}_v \Rightarrow (d,D) \notin \mathcal{U}_v$ or $(e,E) \notin \mathcal{U}_v$" for all $v \in V$. Using this fact we obtain

$$\begin{aligned}\mathfrak{U}(d,D) \cap \mathfrak{U}(e,E) &= \bigcap\{Q_v \mid (d,D) \notin \mathcal{U}_v\} \cap \bigcap\{Q_v \mid (e,E) \notin \mathcal{U}_v\} \\ &= \bigcap\{Q_v \mid (d,D) \notin \mathcal{U}_v \text{ or } (e,E) \notin \mathcal{U}_v\} \\ &\subseteq \bigcap\{Q_v \mid (d,D) \sqcap (e,E) \notin \mathcal{U}_v\} = \mathfrak{U}((d,D) \sqcap (e,E)).\end{aligned}$$

GU4: Let $(d,D) \in \mathfrak{DR}$. Since $\mathcal{U}_v$ is a diuniformity, we have "$(d,D) \in \mathcal{U}_v \Rightarrow \exists (e,E)_v = (e_v,E_v) \in \mathcal{U}_v \ : \ (e,E)_v \circ (e,E)_v \sqsubseteq (d,D)$" for each $v \in V$. If we set $(e,E) = \bigsqcup_{v\in V}(e,E)_v$, then $(e,E) \in \mathfrak{DR}$ and using the fact "$(d,D) \in \mathcal{U}_v \Rightarrow (e,E) \in \mathcal{U}_v$" we have $\mathcal{U}_v(d,D) \subseteq \mathcal{U}_v(e,E)$. Moreover, considering Proposition 3.4. (5), we get $(e,E) \circ (e,E) = \bigsqcup_{v\in V}(e,E)_v \circ \bigsqcup_{v\in V}(e,E)_v \sqsubseteq \bigsqcup_{v\in V}((e,E)_v \circ (e,E)_v) \sqsubseteq (d,D)$.

GU5: Let $(d,D) \in \mathfrak{DR}$. Since $\mathcal{U}_v$ is a diuniformity, we have "$(d,D) \in \mathcal{U}_v \Rightarrow \exists (c,C)_v = (c_v,C_v) \in \mathcal{U}_v \ : \ (c,C)_v^{\leftarrow} \sqsubseteq (d,D)$" for each $v \in V$. If we set $(c,C) = \bigsqcup_{v\in V}(c,C)_v$, then $(c,C) \in \mathfrak{DR}$ and considering Proposition 3.4. (4),

$$(c,C)^{\leftarrow} = (\bigsqcup_{v\in V} c_v, \bigsqcap_{v\in V} C_v)^{\leftarrow} = (\bigsqcup_{v\in V} C_v^{\leftarrow}, \bigsqcap_{v\in V} c_v^{\leftarrow}) = \bigsqcup_{v\in V}(c,C)_v^{\leftarrow} \sqsubseteq (d,D).$$

GU6: Since $\mathcal{U}_v \neq \emptyset$ for each $v \in V$ we have $\bigvee\{\mathfrak{U}(d,D) \mid (d,D) \in \mathfrak{DR}\} = \bigvee\{\bigvee\{P_v \mid (d,D) \in \mathcal{U}_v\} \mid (d,D) \in \mathfrak{DR}\} = V$. □

One can obtain diuniformities from a graded diuniformitiy as in Example 4.6. and Theorem 4.9. also shows that a family of diuniformities under some conditions form a graded diuniformity. In this context, the relationship between the uniform ditopologies generated by the family of diuniformities and the uniform graded ditopology generated by the graded diuniformity is given in the next proposition.

Proposition 4.10. *Let $(S,\mathcal{S})$, $(V,\mathcal{V})$ be textures and $\{\mathcal{U}_v\}_{v\in V}$ be a $\mathcal{V}$-compatible family of diuniformities on $(S,\mathcal{S})$. Then $(\tau_{\mathcal{U}_v},\kappa_{\mathcal{U}_v}) \subseteq (\mathcal{T}^v_{\mathfrak{U}},\mathcal{K}^v_{\mathfrak{U}})$ and in case of the texture $\mathcal{V}$ is plain $(\tau_{\mathcal{U}_v},\kappa_{\mathcal{U}_v}) = (\mathcal{T}^v_{\mathfrak{U}},\mathcal{K}^v_{\mathfrak{U}})$ for each $v\in V$ where $\mathfrak{U}$ is the $(V,\mathcal{V})$-graded diuniformity on $(S,\mathcal{S})$ generated by the family $\{\mathcal{U}_v\}_{v\in V}$ by (13).*

Proof. At first, we will see that $\tau_{\mathcal{U}_v} \subseteq \mathcal{T}^v_{\mathfrak{U}}$.

$$A \in \tau_{\mathcal{U}_v} \Rightarrow \forall s \in A^\flat \; A \in \eta_{\mathcal{U}_v}(s)$$
$$\Rightarrow \text{"} A \not\subseteq Q_s,\; P_s \not\subseteq Q_t \Rightarrow \exists (d,D) \in \mathcal{U}_v \;:\; d[t] \subseteq A\text{"}$$
$$\Rightarrow P_v \subseteq \bigcap_{s\in A^\flat} \bigcap_{P_s \not\subseteq Q_t} \bigvee_{d[t]\subseteq A} \mathfrak{U}(d,D) = \mathcal{T}_{\mathfrak{U}}(A) \Rightarrow A \in \mathcal{T}^v_{\mathfrak{U}}$$

Now, if $\mathcal{V}$ is plain then we have "$P_v \subseteq \bigvee_{d[t]\subseteq A} \mathfrak{U}(d,D) = \bigcup_{d[t]\subseteq A} \mathfrak{U}(d,D) \Rightarrow \exists (d,D) \in \mathfrak{DR} \;:\; d[t] \subseteq A,\; P_v \subseteq \mathfrak{U}(d,D)$" and so $\mathcal{T}^v_{\mathfrak{U}} \subseteq \tau_{\mathcal{U}_v}$.

Using similar method, it can be seen that $\kappa_{\mathcal{U}_v} \subseteq \mathcal{K}^v_{\mathfrak{U}}$ and in case of $\mathcal{V}$ is plain $\kappa_{\mathcal{U}_v} = \mathcal{K}^v_{\mathfrak{U}}$ for each $v \in V$. □

5. Graded Uniform Bicontinuity and the Category dfGDiU

We begin this section with continuity concepts and their some basic properties in ditopological texture spaces, diuniform texture spaces and graded ditopological texture spaces. We also need the concept of *inverse of a direlation under a difunction* defined in [13]. Our reference for category theory is [1].

Definition 5.1. [5] Let $(S_k,\mathcal{S}_k,\tau_k,\kappa_k)$, $k=1,2$ be ditopological texture spaces and $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ a difunction. (f,F) is called continuous if

$$\forall A \in \tau_2, \quad F^{\leftarrow} A \in \tau_1$$

and cocontinuous if

$$\forall A \in \kappa_2, \quad f^{\leftarrow} A \in \kappa_1.$$

The difunction (f,F) is called bicontinuous if it is both continuous and cocontinuous.

Theorem 5.2. [5] *Ditopological texture spaces and bicontinuous difunctions form a category denoted by* **dfDiTop**.

Proposition 5.3. [13] *Let $(S,\mathcal{S})$, $(V,\mathcal{V})$ be texture spaces, (d,D) a relation on $(V,\mathcal{V})$ and $(f,F):(S,\mathcal{S})\to(V,\mathcal{V})$ a difunction.*

(1) For the sets

$$(f,F)^{-1}(d) = \bigvee\{\overline{P}_{(s_1,s_2)} \mid \exists P_{s_1} \not\subseteq Q_{s_1'} \;:\; \overline{P}_{(s_1',v_1)} \not\subseteq F,\; f \not\subseteq \overline{Q}_{(s_2,v_2)} \Rightarrow \overline{P}_{(v_1,v_2)} \subseteq d\}$$

and

$$(f,F)^{-1}(D)=\bigcap\{\overline{Q}_{(s_1,s_2)} \mid \exists P_{s_1'} \not\subseteq Q_{s_1} \ : \ f \not\subseteq \overline{Q}_{(s_1',v_1)},\ \overline{P}_{(s_2,v_2)} \not\subseteq F \Rightarrow D \subseteq \overline{Q}_{(v_1,v_2)}\},$$

$$(f,F)^{-1}(d,D)=((f,F)^{-1}(d),(f,F)^{-1}(D))$$

is a direlation on $(S,\mathcal{S})$.

(2) $(f,F)^{-1}(i_V,I_V)=(i_S,I_S)$
(3) $(i_S,I_S)^{-1}(d,D)=(d,D)$ *for all* $(d,D)\in\mathfrak{DR}_S$.

Definition 5.4. [13] Let $(S_k,\mathcal{S}_k,\mathcal{U}_k)$, $k=1,2$ be diuniform texture spaces and $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ a difunction. (f,F) is called $\mathcal{U}_1-\mathcal{U}_2$ uniformly bicontinuous if $(f,F)^{-1}(d,D)\in\mathcal{U}_1$ for each $(d,D)\in\mathcal{U}_2$.

Theorem 5.5. [15] *The class of diuniform texture spaces and uniformly bicontinuous difunctions between them form a category denoted by* **dfDiU**. *Considering Definition 2.14., the functor* $\mathfrak{F}':\mathbf{dfDiU}\to\mathbf{dfDiTop}$ *is defined by*

$$\mathfrak{F}'((f,F):(S_1,\mathcal{S}_1,\mathcal{U}_1)\to(S_2,\mathcal{S}_2,\mathcal{U}_2))=((f,F):(S_1,\mathcal{S}_1,\tau_{\mathcal{U}_1},\kappa_{\mathcal{U}_1})\to(S_2,\mathcal{S}_2,\tau_{\mathcal{U}_2},\kappa_{\mathcal{U}_2})).$$

Definition 5.6. [7] Let $(S_k,\mathcal{S}_k,\mathcal{T}_k,\mathcal{K}_k,V_k,\mathcal{V}_k)$, $k=1,2$ be graded ditopological texture spaces, $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$, $(h,H):(V_1,\mathcal{V}_1)\to(V_2,\mathcal{V}_2)$ difunctions. For the pair $((f,F),(h,H))$, (f,F) is called continuous with respect to (h,H) if

$$\forall A\in\mathcal{S}_2,\quad H^{\leftarrow}\mathcal{T}_2(A)\subseteq\mathcal{T}_1(F^{\leftarrow}A)$$

and cocontinuous with respect to (h,H) if

$$\forall A\in\mathcal{S}_2,\quad h^{\leftarrow}\mathcal{K}_2(A)\subseteq\mathcal{K}_1(f^{\leftarrow}A).$$

The difunction (f,F) is called bicontinuous with respect to (h,H) if it is both continuous and cocontinuous with respect to (h,H).

Proposition 5.7. [7] *For the above notations, the followings are equivalent:*

(1) (f,F) *is bicontinuous with respect to* (h,H).
(2) (f,F) *is* $(\mathcal{T}_1^{v_1},\mathcal{K}_1^{v_1})-(\mathcal{T}_2^{v_2},\mathcal{K}_2^{v_2})$ *bicontinuous for all* $v_1\in V_1$, $v_2\in V_2$ *satisfying* $P_{v_1}\subseteq H^{\leftarrow}P_{v_2}$.
(3) (f,F) *is* $(\mathcal{T}_1^{v_1},\mathcal{K}_1^{v_1})-(\mathcal{T}_2^{v_2},\mathcal{K}_2^{v_2})$ *bicontinuous for all* $v_1\in V_1$, $v_2\in V_2$ *satisfying* $H^{\leftarrow}P_{v_2}\not\subseteq Q_{v_1}$.

Theorem 5.8. [7] *The class of graded ditopological texture spaces and relatively bicontinuous difunction pairs between them form a category denoted by* **dfGDiTop**. *Considering Example 2.15., the functor* $\mathfrak{G}':\mathbf{dfDiTop}\to\mathbf{dfGDiTop}$ *defined by*

$$\mathfrak{G}'((f,F):(S_1,\mathcal{S}_1,\tau_1,\kappa_1)\to(S_2,\mathcal{S}_2,\tau_2,\kappa_2))$$
$$=(((f,F),(i,I)):(S_1,\mathcal{S}_1,\tau_1^g,\kappa_1^g,1,\mathcal{P}(1))\to(((f,F),(i,I)):(S_1,\mathcal{S}_1,\tau_2^g,\kappa_2^g,1,\mathcal{P}(1)))$$

is an embedding.

Lemma 5.9. [12] *(6.13. Prop.) Let* $(S_k,\mathcal{S}_k)$, $k=1,2$ *be texture spaces,* $(f,F):(S_1,\mathcal{S}_1)\to(S_2,\mathcal{S}_2)$ *a difunction and* $(d,D)\in\mathfrak{DR}_{S_2}$. *If* $\overline{P}_{(s_1,s_2)}\not\subseteq F$ *and* $d[s_2]\subseteq A$ *for* $s_1\in S_1$, $s_2\in S_2$, $A\in\mathcal{S}_2$ *then* $(f,F)^{-1}(d)[s_1]\subseteq F^{\leftarrow}A$.

Definition 5.10. Let $(S_k, \mathcal{S}_k, \mathfrak{U}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ be graded diuniform texture spaces and $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ difunctions. If $H^{\leftarrow}(\mathfrak{U}_2(d, D)) \subseteq \mathfrak{U}_1((f, F)^{-1}(d, D))$ for each $(d, D) \in \mathfrak{DR}_{S_2}$ then (f, F) is called $\mathfrak{U}_1$ - $\mathfrak{U}_2$ uniformly bicontinuous with respect to (h, H).

Example 5.11. Let $(S, \mathcal{S}, \mathfrak{U}, V, \mathcal{V})$ be graded diuniform texture spaces and $(i_S, I_S) : (S, \mathcal{S}) \to (S, \mathcal{S})$, $(i_V, I_V) : (V, \mathcal{V}) \to (V, \mathcal{V})$ identity difunctions. For each $(d, D) \in \mathfrak{DR}_S$ we have $I_V^{\leftarrow}(\mathfrak{U}(d, D)) = \mathfrak{U}(d, D) = \mathfrak{U}((i_S, I_S)^{-1}(d, D))$. Hence (i_S, I_S) is uniformly bicontinuous with respect to (i_V, I_V).

Proposition 5.12. *Relatively uniformly bicontinuity is preserved under composition of difunctions.*

Proof. Let $(S_j, \mathcal{S}_j, \mathfrak{U}_j, V_j, \mathcal{V}_j)$, $j = 1, 2, 3$ be graded diuniform texture spaces, $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$, $(g, G) : (S_2, \mathcal{S}_2) \to (S_3, \mathcal{S}_3)$, $(k, K) : (V_2, \mathcal{V}_2) \to (V_3, \mathcal{V}_3)$ difunctions where (f, F) is uniformly bicontinouos with respect to (h, H) and (g, G) is uniformly bicontinouos with respect to (k, K). For each $(d, D) \in \mathfrak{DR}_{S_3}$ we have

$$(K \circ H)^{\leftarrow}(\mathfrak{U}_3(d, D)) = H^{\leftarrow}(K^{\leftarrow}\mathfrak{U}_3(d, D)) \subseteq H^{\leftarrow}(\mathfrak{U}_2(g, G)^{-1}(d, D))$$
$$\subseteq \mathfrak{U}_1((f, F)^{-1}((g, G)^{-1}(d, D))) = \mathfrak{U}_1(((g, G) \circ (f, F))^{-1}(d, D)) = \mathfrak{U}_1((g \circ f, G \circ F)^{-1}(d, D))$$

So, $(g \circ f, G \circ F)$ is uniformly bicontinuous with respect to $(k \circ h, K \circ H)$. □

Corollary 5.13. *Graded diuniform texture spaces and relatively uniformly bicontionuous difunction pairs between them form a category that we will denote by* **dfGDiU**.

Proof. It is clear from Example 5.11. and Proposition 5.12. □

Theorem 5.14. *For the above notations, the functor* $\mathfrak{G}$: **dfDiU** $\to$ **dfGDiU** *defined by*

$$\mathfrak{G}((f, F) : (S_1, \mathcal{S}_1, \mathcal{U}_1) \to (S_2, \mathcal{S}_2, \mathcal{U}_2)$$
$$= ((f, F), (i_1, I_1)) : (S_1, \mathcal{S}_1, \mathfrak{U}_{\mathcal{U}_1}, 1, \mathcal{P}(1)) \to (S_2, \mathcal{S}_2, \mathfrak{U}_{\mathcal{U}_2}, 1, \mathcal{P}(1))$$

is an embedding of the category **dfDiU** *as a full subcategory* $\mathbf{dfGDiU}_{(1,\mathcal{P}(1))}$ *of the category* **dfGDiU**.

Proof. If a difunction $(f, F) : (S_1, \mathcal{S}_1, \mathcal{U}_1) \to (S_2, \mathcal{S}_2, \mathcal{U}_2)$ is uniformly bicontinuous then it is clearly $\mathfrak{U}_{\mathcal{U}_1} - \mathfrak{U}_{\mathcal{U}_2}$ uniformly bicontinuous with respect to (i_1, I_1). So $\mathfrak{G}$ is a functor. $\mathfrak{G}$ is also a full embedding from Example 4.6. (2), Definition 5.4. and Definition 5.10. □

Theorem 5.15. *Let* $(S_k, \mathcal{S}_k, \mathfrak{U}_k, V_k, \mathcal{V}_k)$, $k = 1, 2$ *be graded diuniform texture spaces and* $(f, F) : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$, $(h, H) : (V_1, \mathcal{V}_1) \to (V_2, \mathcal{V}_2)$ *difunctions. If* (f, F) *is* $\mathfrak{U}_1$ - $\mathfrak{U}_2$ *uniform bicontinuous with respect to* (h, H) *then it is* $(\mathcal{T}_{\mathfrak{U}_1}, \mathcal{K}_{\mathfrak{U}_1})$ - $(\mathcal{T}_{\mathfrak{U}_2}, \mathcal{K}_{\mathfrak{U}_2})$ *bicontinuous with respect to* (h, H).

Proof. Let (f, F) be $\mathfrak{U}_1$ - $\mathfrak{U}_2$ uniform bicontinuous with respect to (h, H). We will show that $A \in \mathcal{T}_{\mathfrak{U}_2}^{v_2} \Rightarrow F^{\leftarrow}A \in \mathcal{T}_{\mathfrak{U}_1}^{v_1}$ for all $v_1 \in V_1$, $v_2 \in V_2$ satisfying $P_{v_1} \subseteq H^{\leftarrow}P_{v_2}$.

So, let $P_{v_1} \subseteq H^{\leftarrow} P_{v_2}$ and $P_{v_2} \subseteq \mathcal{T}_{\mathfrak{U}_2}(A)$. Using Corollary 4.4. and Lemma 2.6.(3) we get

$$P_{v_1} \subseteq H^{\leftarrow} P_{v_2} \subseteq H^{\leftarrow}(\mathcal{T}_{\mathfrak{U}_2}(A)) = h^{\leftarrow}(\bigcap_{t\in A^\flat} \bigvee_{d[t]\subseteq A} \mathfrak{U}_2(d,D)) = \bigcap_{t\in A^\flat} h^{\leftarrow}(\bigvee_{d[t]\subseteq A} \mathfrak{U}_2(d,D))$$

$$= \bigcap_{t\in A^\flat} H^{\leftarrow}(\bigvee_{d[t]\subseteq A} \mathfrak{U}_2(d,D)) = \bigcap_{t\in A^\flat} \bigvee_{d[t]\subseteq A} H^{\leftarrow}(\mathfrak{U}_2(d,D)) \subseteq \bigcap_{t\in A^\flat} \bigvee_{d[t]\subseteq A} \mathfrak{U}_1((f,F)^{-1}(d,D)).$$

Therefore

$$\forall t \in A^\flat \Rightarrow \exists (d,D) \in \mathfrak{DR}_{S_2} \ : \ d[t] \subseteq A \text{ and } P_{v_1} \subseteq \mathfrak{U}_1((f,F)^{-1}(d,D)) \qquad (14)$$

is obtained.

Now, to show that $P_{v_1} \subseteq \mathcal{T}_{\mathfrak{U}_1}(F^{\leftarrow}A)$ we recall Corollary 4.4. Let $F^{\leftarrow}A \nsubseteq Q_{s_1}$. Then there exists a $s_2 \in S_2$ such that $\overline{P}_{(s_1,s_2)} \nsubseteq F$ and $A \nsubseteq Q_{s_2}$. Since $s_2 \in A^\flat$, considering (14) there exists a $(d,D) \in \mathfrak{DR}_{S_2}$ such that $d[s_2] \subseteq A$ and $P_{v_1} \subseteq \mathfrak{U}_1((f,F)^{-1}(d,D))$. Besides, we have $(e,E) = (f,F)^{-1}(d,D) \in \mathfrak{DR}_{S_1}$ by Proposition 5.3. and $e[s_1] \subseteq F^{\leftarrow}A$ by Lemma 5.9. Hence we obtain $P_{v_1} \subseteq \mathcal{T}_{\mathfrak{U}_1}(F^{\leftarrow}A)$ and so that (f,F) is $(\mathcal{T}_{\mathfrak{U}_1}, \mathcal{K}_{\mathfrak{U}_1})$ - $(\mathcal{T}_{\mathfrak{U}_2}, \mathcal{K}_{\mathfrak{U}_2})$ continuous with respect to (h,H).

The cocontinuity part of the proof is similar. □

Corollary 5.16. *For the above notations,* $\mathfrak{F} : \mathbf{dfGDiU} \to \mathbf{dfGDiTop}$ *defined by*

$$\mathfrak{F}(((f,F),(h,H)) : (S_1, \mathcal{S}_1, \mathfrak{U}_1, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, \mathfrak{U}_2, V_2, \mathcal{V}_2))$$
$$= ((f,F),(h,H)) : (S_1, \mathcal{S}_1, \mathcal{T}_{\mathfrak{U}_1}, \mathcal{K}_{\mathfrak{U}_1}, V_1, \mathcal{V}_1) \to (S_2, \mathcal{S}_2, \mathcal{T}_{\mathfrak{U}_2}, \mathcal{K}_{\mathfrak{U}_2}, V_2, \mathcal{V}_2)$$

is a faithful and full functor.

Proof. At first note that from Corollary 4.3. and Theorem 5.15. it follows that $\mathfrak{F}$ is a functor. Moreover, from the definition of $\mathfrak{F}$, it is a faithful and full functor. □

From Theorem 5.5, 5.8, 5.14. and Corollary 5.16. we obtain the following diagram.

$$\begin{array}{ccc} \mathbf{dfDiU} & \xrightarrow{\mathfrak{F}'} & \mathbf{dfDiTop} \\ \downarrow \mathfrak{G} & & \downarrow \mathfrak{G}' \\ \mathbf{dfGDiU} & \xrightarrow{\mathfrak{F}} & \mathbf{dfGDiTop} \end{array}$$

Proposition 5.17. *For the above notations, the followings are equivalent:*

(1) (f,F) *is uniformly bicontinuous with respect to* (h,H)*.*
(2) (f,F) *is* $\mathfrak{U}_1^{v_1} - \mathfrak{U}_2^{v_2}$ *uniformly bicontinuous for all* $v_1 \in V_1^\flat$, $v_2 \in V_2^\flat$ *satisfying* $P_{v_1} \subseteq H^{\leftarrow} P_{v_2}$*.*
(3) (f,F) *is* $\mathfrak{U}_1^{v_1} - \mathfrak{U}_2^{v_2}$ *uniformly bicontinuous for all* $v_1 \in V_1^\flat$, $v_2 \in V_2^\flat$ *satisfying* $H^{\leftarrow} P_{v_2} \nsubseteq Q_{v_1}$*.*

Proof. (1) $\Rightarrow$ (2) : Let (f,F) be uniformly bicontinuous with respect to (h,H), $P_{v_1} \subseteq H^{\leftarrow} P_{v_2}$ and $(d,D) \in {\mathfrak{U}_2}^{v_2}$. Then we have $P_{v_2} \subseteq \mathfrak{U}_2(d,D)$ and so, $P_{v_1} \subseteq H^{\leftarrow} P_{v_2} \subseteq H^{\leftarrow}\mathfrak{U}_2(d,D)$ by Lemma 2.6. (2). Since (f,F) is uniformly bicontinuous with respect to (h,H), we get $P_{v_1} \subseteq H^{\leftarrow}\mathfrak{U}_2(d,D) \subseteq \mathfrak{U}_1((f,F)^{-1}(d,D))$ and hence

$(f,F)^{-1}(d,D) \in \mathfrak{U}_1{}^{v_1}$.
$(2) \Rightarrow (3)$: It is obvious since "$H^{\leftarrow} P_{v_2} \nsubseteq Q_{v_1} \Rightarrow P_{v_1} \subseteq H^{\leftarrow} P_{v_2}$".
$(3) \Rightarrow (1)$: Let (3) be satisfied and suppose that (f,F) is not uniformly bicontinuous with respect to (h,H). Then there exists $(d,D) \in \mathfrak{DR}_{S_2}$ such that $H^{\leftarrow}\mathfrak{U}_2(d,D) \nsubseteq \mathfrak{U}_1((f,F)^{-1}(d,D))$. So $H^{\leftarrow}\mathfrak{U}_2(d,D) \nsubseteq Q_{v_1}$ and $P_{v_1} \nsubseteq \mathfrak{U}_1((f,F)^{-1}(d,D))$ for a $v_1 \in V_1^\flat$. Since $H^{\leftarrow}\mathfrak{U}_2(d,D) \nsubseteq Q_{v_1}$ there exists $v_2 \in V_2^\flat$ such that $\overline{P}_{(v_1,v_2)} \nsubseteq H$ and $\mathfrak{U}_2(d,D) \nsubseteq Q_{v_2}$. We have $H^{\leftarrow} \nsubseteq \overline{Q}_{(v_2,v_1)}$ by Lemma 2.6. (1) and so $(H^{\leftarrow})^{\rightarrow} P_{v_2} \nsubseteq Q_{v_1}$ by Lemma 2.6. (4). So, $H^{\leftarrow} P_{v_2} = (H^{\leftarrow})^{\rightarrow} P_{v_2} \nsubseteq Q_{v_1}$ and since (3), (f,F) is $\mathfrak{U}_1^{v_1} - \mathfrak{U}_2^{v_2}$ uniformly bicontinuous.

On the other hand, since $\mathfrak{U}_2(d,D) \nsubseteq Q_{v_2}$ we get $P_{v_2} \subseteq \mathfrak{U}_2(d,D)$ and so $(d,D) \in \mathfrak{U}_2^{v_2}$. Since (f,F) is $\mathfrak{U}_1^{v_1} - \mathfrak{U}_2^{v_2}$ uniformly bicontinuous, we have $(f,F)^{-1}(d,D) \in \mathfrak{U}_1^{v_1}$ and so $P_{v_1} \subseteq \mathfrak{U}_1((f,F)^{-1}(d,D))$ which contradicts with $P_{v_1} \nsubseteq \mathfrak{U}_1((f,F)^{-1}(d,D))$. □

Theorem 5.18. *For a graded diuniform texture space $(S,\mathcal{S},\mathfrak{U},V,\mathcal{V})$, $(\tau_{\mathfrak{U}^v},\kappa_{\mathfrak{U}^v}) \subseteq (\mathcal{T}_{\mathfrak{U}}^v,\mathcal{K}_{\mathfrak{U}}^v)$ for each $v \in V^\flat$ and in case of the texture $\mathcal{V}$ is plain $(\tau_{\mathfrak{U}^v},\kappa_{\mathfrak{U}^v}) = (\mathcal{T}_{\mathfrak{U}}^v,\mathcal{K}_{\mathfrak{U}}^v)$ for each $v \in V = V^\flat$.*

Proof. Let $A \in \mathcal{S}$. $A \in \tau_{\mathfrak{U}^v} \iff \forall s \in A^\flat,\ A \in \eta_{\mathfrak{U}^v}(s) \overset{Prop.\ 2.13.}{\iff}$ "$A \nsubseteq Q_s,\ P_s \nsubseteq Q_t \Rightarrow \exists (d,D) \in \mathfrak{U}^v\ :\ d[t] \subseteq A$" $\iff$ "$A \nsubseteq Q_s,\ P_s \nsubseteq Q_t \Rightarrow \exists (d,D) \in \mathfrak{DR}\ :\ d[t] \subseteq A$ and $P_v \subseteq \mathfrak{U}(d,D)$" $\overset{(9)}{\Longrightarrow} A \in \mathcal{T}_{\mathfrak{U}}^v$ and so, we have $\tau_{\mathfrak{U}^v} \subseteq \mathcal{T}_{\mathfrak{U}}^v$. If $\mathcal{V}$ is plain, since $\bigvee_{d[t]\subseteq A}\mathfrak{U}(d,D) = \bigcup_{d[t]\subseteq A}\mathfrak{U}(d,D)$ we get $A \in \mathcal{T}_{\mathfrak{U}}^v \overset{(9)}{\Longrightarrow}$ "$A \nsubseteq Q_s,\ P_s \nsubseteq Q_t \Rightarrow \exists (d,D) \in \mathfrak{DR}\ :\ d[t] \subseteq A$ and $P_v \subseteq \mathfrak{U}(d,D)$". Hence $\tau_{\mathfrak{U}^v} = \mathcal{T}_{\mathfrak{U}}^v$.

On the other hand, $A \in \kappa_{\mathfrak{U}^v} \iff \forall s \in S \setminus A,\ A \in \mu_{\mathfrak{U}^v}(s) \overset{Prop.\ 2.13.}{\iff}$ "$P_s \nsubseteq A,\ P_t \nsubseteq Q_s \Rightarrow \exists (d,D) \in \mathfrak{U}^v\ :\ A \subseteq D[t]$" $\iff$ "$P_s \nsubseteq A,\ P_t \nsubseteq Q_s \Rightarrow \exists (d,D) \in \mathfrak{DR}\ :\ A \subseteq D[t]$ and $P_v \subseteq \mathfrak{U}(d,D)$" $\overset{(10)}{\Longrightarrow} A \in \mathcal{K}_{\mathfrak{U}}^v$ and so, we have $\kappa_{\mathfrak{U}^v} \subseteq \mathcal{K}_{\mathfrak{U}}^v$. If $\mathcal{V}$ is plain, since $\bigvee_{d[t]\subseteq A}\mathfrak{U}(d,D) = \bigcup_{d[t]\subseteq A}\mathfrak{U}(d,D)$ we get $A \in \mathcal{K}_{\mathfrak{U}}^v \overset{(10)}{\Longrightarrow}$ "$P_s \nsubseteq A,\ P_t \nsubseteq Q_s \Rightarrow \exists (d,D) \in \mathfrak{DR}\ :\ A \subseteq D[t]$ and $P_v \subseteq \mathfrak{U}(d,D)$". Hence $\kappa_{\mathfrak{U}^v} = \mathcal{K}_{\mathfrak{U}}^v$. □

6. Conclusion

Uniform properties such as uniform continuity and uniform convergence are defined in uniform spaces. So, uniform spaces are useful for an investigation of topological spaces. In this work, graded diuniformities are introduced and its relations with diuniformities and graded ditopologies are investigated. Moreover, the category of this new structure **dfGDiU** is formed and its relations with some other categories are given.

Graded diuniformities are a generalization of diuniformities to the graded case. Hence, each diuniformity is an example of a graded diuniformity. However it's not that easy to find a graded diuniformity which is not a diuniformity. We will continue to study to find such further examples. On the other hand, a family of diuniformities generates a graded diuniformity under some conditions (see Theorem 4.9.).

As expected, each graded diuniformity induces a graded ditopology called as uniform graded ditopology (see Corollary 4.3., 4.4.). Thus, a functor can be defined from **dfGDiU** to **dfGDiTop** (see Corollary 5.16.). In this paper, basic categorical properties of graded diuniformities are discussed without the relations with many other categories (e.g. with the category of texture spaces). So, in a later work, we intend to study further categorical properties, relations and problems, such as the problem recommended by one of the referees: Is **dfGDiU** topological over the category of sets or others?

Obviously, the structure of graded diuniformity can be helpful to define and investigate the other uniform concepts in graded ditopological texture spaces.

Acknowledgements. The authors would like to thank the referees for their valuable comments which helped to improve this paper.

References

[1] J. Adámek, H. Herrlich and G. E. Strecer, *Abstract and concrete categories*, New York, Chichester, Brisbane, Toronto, Singapore: John Wiley & Sons, Inc., 1990.
[2] L. M. Brown and M. Diker, *Ditopological texture spaces and intuitionistic sets*, Fuzzy Sets and Systems, **98** (1998), 217–224.
[3] L. M. Brown and R. Ertürk, *Fuzzy sets as texture spaces, I. Representation theorems*, Fuzzy Sets and Systems, **110(2)** (2000), 227–236.
[4] L. M. Brown, R. Ertürk and Ş. Dost, *Ditopological texture spaces and fuzzy topology, I. Basic concepts*, Fuzzy Sets and Systems, **147(2)** (2004), 171–199.
[5] L. M. Brown, R. Ertürk and Ş. Dost, *Ditopological texture spaces and fuzzy topology, II. Topological considerations*, Fuzzy Sets and Systems, **147(2)** (2004), 201–231.
[6] L. M. Brown, R. Ertürk and Ş. Dost, *Ditopological texture spaces and fuzzy topology, III. Separation Axioms*, Fuzzy Sets and Systems, **157(14)** (2006), 1886–1912.
[7] L. M. Brown and A. Šostak, *Categories of fuzzy topology in the context of graded ditopologies on textures*, Iranian Journal of Fuzzy Systems, **11(6)** (2014), 1–20.
[8] C. L. Chang, *Fuzzy topological spaces*, Math. Anal. Appl., **24** (1968), 182–190.
[9] R. Ekmekçi and R. Ertürk, *Neighborhood structures of graded ditopological texture spaces*, Filomat, **29(7)** (2015), 1445–1459.
[10] R. Ertürk, *Separation axioms in fuzzy topology characterized by bitopologies*, Fuzzy Sets and Systems, **58** (1993), 206–209.
[11] T. Kubiak, *On fuzzy topologies*, PhD Thesis, A. Mickiewicz University Poznan, Poland (1985).
[12] S. Özçağ, *Uniform texture spaces*, PhD Thesis, Hacettepe University, Ankara, Turkey (2004).
[13] S. Özçağ and L. M. Brown, *Di-uniform texture spaces*, Appiled General Topology, **4(1)** (2003), 157–192.
[14] S. Özçağ, L. M. Brown and K. Biljana, *Di-uniformities and Hutton uniformities*, Fuzzy Sets and Systems, **195** (2012), 58–74.
[15] S. Özçağ and Ş. Dost, *A categorical view of di-uniform texture spaces*, Bol. Soc. Mat. Mexicana, **3(15)** (2009), 63–80.
[16] A. Šostak, *On a fuzzy topological structure*, Rend. Circ. Matem. Palermo, Ser. II, **11** (1985), 89–103.
[17] A. Šostak, *Two decates of fuzzy topology: basic ideas, notions and results*, Russian Math. Surveys, **44(6)** (1989), 125–186.
[18] G. Yıldız, *Ditopological spaces on texture spaces*, MSc Thesis, Hacettepe University, Ankara, Turkey (2005).

Ramazan Ekmekçi*, Department of Mathematics, Çanakkale Onsekiz Mart University, Çanakkale, TURKEY
E-mail address: ekmekciramazan@yahoo.com

Riza Ertürk, Department of Mathematics, Hacettepe University, Ankara, TURKEY
E-mail address: rerturk@hacettepe.edu.tr

*Corresponding author

GRADED DIUNIFORMITIES

R. EKMEKÇI AND R. ERTÜRK

یکریختی های دو گانه مدرج

چکیده. فضاهای ترکیب توپولوژیکی دوگانه مدرجی که ارائه گردیده اند در [7] توسط Lawrence ، Brown ، Alexander Sostak از منظر رسته ای مورد بررسی قرار گرفته اند. در این مقاله ، مؤلفین ساختار یکریختی دوگانه در فضاهای ترکیب توپولوژیکی دو گانه را که در [13] تعریف شده است به فضاهای ترکیب توپولوژیکی دو گانه مدرج تعمیم می دهند و توپولوژی های دو گانه مدرج تولید شده توسط یکریختی های دوگانه مدرج را مورد مطالعه قرار می دهند. آنها همچنین خواص یکریختی های دوگانه و یکریختی های مدرج را مورد مقایسه قرار می دهند.