

## STABILITY OF THE JENSEN'S FUNCTIONAL EQUATION IN MULTI-FUZZY NORMED SPACES

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ABSTRACT. In this paper, we define the notion of (dual) multi-fuzzy normed spaces and describe some properties of them. We then investigate Ulam-Hyers stability of Jensen's functional equation for mappings from linear spaces into multi-fuzzy normed spaces. We establish an asymptotic behavior of the Jensen equation in the framework of multi-fuzzy normed spaces.

### 1. Introduction

The notion of multi-normed spaces was initiated by H. G. Dales and M. E. Polyakov [6]. This concept is somewhat similar to the operator sequence space and has some connections with operator spaces and Banach lattices. An incentive for the study of multi-normed spaces and many examples are given in [6]. Some results of multi-normed spaces are stable under fuzzy normed spaces [3, 4]. In this paper, using some ideas from [6, 18] we first introduce the concept of (dual) multi-fuzzy normed spaces and then we investigate the Ulam-Hyers stability of Jensen's functional equation for mappings from linear spaces into multi-fuzzy normed spaces.

In 1984, A. K. Katsaras [11] defined the notion of fuzzy norm on a linear space to construct a fuzzy vector topological structure. Thereafter, some mathematicians introduced and discussed several notions of fuzzy norms from various points of view [7, 13, 24]. In particular, in 2003 T. Bag and S. K. Samanta [3], following Cheng and Mordeson [5] gave an idea of a fuzzy norm in such a way that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They established a decomposition theorem of a fuzzy norm into a family of "crisp norms" and also described some nice properties of the fuzzy norm in [4].

In 1940, a question that was given by S. M. Ulam [22] concerning the stability of group homomorphisms gave rise to the stability problem of functional equations. D. H. Hyers [9] was the first to come out with a partial affirmative answer to solve the question posed by Ulam on Banach spaces. Hyers's theorem was extended by T. Aoki [2] for additive mappings in 1950 and by Th. M. Rassias [20] for linear mappings by taking into consideration an unbounded Cauchy difference in 1978. Rassias's paper [20] has significantly influenced in the expansion of what we now call Ulam-Hyers-Rassias stability of functional equations. Recently, several fuzzy stability problems of various functional equations and in particular, Jensen

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equation and its generalizations have been studied by some mathematicians; we refer the interested reader to [1, 8, 10, 14, 15, 17, 18, 21].

## 2. Preliminaries

Let  $(E, \|\cdot\|)$  be a complex normed space and let  $k \in \mathbb{N}$ . We denote by  $E^k$ , the linear space  $E \oplus \dots \oplus E$  consisting of  $k$ -tuples  $(x_1, \dots, x_k)$ , where  $x_1, \dots, x_k \in E$ . The linear operations on  $E^k$  are defined coordinatewise. The zero element of either  $E$  or  $E^k$  is denoted by 0. We denote by  $\mathbb{N}_k$  the set  $\{1, \dots, k\}$  and denote by  $\mathfrak{S}_k$  the group of permutations on  $k$  symbols. For  $\sigma \in \mathfrak{S}_k$ ,  $x = (x_1, \dots, x_k) \in E^k$  and  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$  define  $A_\sigma(x) = (x_{\sigma(1)}, \dots, x_{\sigma(k)})$  and  $M_\alpha(x) = (\alpha_1 x_1, \dots, \alpha_k x_k)$ . Let  $n \in \mathbb{N}$ , we set  $x^{[n]} = (x_1, \dots, x_k, \dots, x_1, \dots, x_k) \in E^{nk}$ , where  $x^{[n]}$  consists of  $n$  copies of each block  $(x_1, \dots, x_k)$ .

Take  $k \in \mathbb{N}$  and let  $S$  be a subset of  $\mathbb{N}_k$ . For  $(x_1, \dots, x_k) \in E^k$ , we set  $Q_S(x_1, \dots, x_k) = (y_1, \dots, y_k)$ , where  $y_i = x_i$  ( $i \notin S$ ) and  $y_i = 0$  ( $i \in S$ ). Thus  $Q_S$  is the projection onto the complement of  $S$ . Indeed, we observe that  $Q_S^2 = Q_S^* = Q_S$ , where  $Q_S^*(x_1, \dots, x_k) = (\overline{y_1}, \dots, \overline{y_k})$  (we mean by  $\overline{z}$ , the complex conjugate of a complex number  $z$ ).

**Definition 2.1.** [6] Let  $(E, \|\cdot\|)$  be a complex (real) normed space. A multi-norm on  $\{E^k, k \in \mathbb{N}\}$  is a sequence  $\{\|\cdot\|_k\}_{k \in \mathbb{N}}$  of norms on  $E^k$  ( $k \in \mathbb{N}$ ) such that  $\|x\|_1 = \|x\|$ , for each  $x \in E$  and the following axioms (A1)-(A4) are satisfied for each  $k \in \mathbb{N}$  with  $k \geq 2$ :

(A1) for each  $\sigma \in \mathfrak{S}_k$  and  $x \in E^k$  we have

$$\|A_\sigma(x)\|_k = \|x\|_k;$$

(A2) for each  $\alpha_1, \dots, \alpha_k \in \mathbb{C}$  ( $\mathbb{R}$ ) and  $x \in E^k$  we have

$$\|M_\alpha(x)\|_k \leq \left(\max_{1 \leq i \leq k} |\alpha_i|\right) \|x\|_k;$$

(A3) for each  $x_1, \dots, x_{k-1} \in E$  we have

$$\|(x_1, \dots, x_{k-1}, 0)\|_k = \|(x_1, \dots, x_{k-1})\|_{k-1};$$

(A4) for each  $x_1, \dots, x_{k-1} \in E$  we have

$$\|(x_1, \dots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \dots, x_{k-1})\|_{k-1}.$$

In this case, we say that  $\{(E^k, \|\cdot\|_k), k \in \mathbb{N}\}$  is a multi-normed space. Now if the axiom (A4) is replaced by the following axiom, then  $\{(E^k, \|\cdot\|_k), k \in \mathbb{N}\}$  is called a dual multi-normed space.

(B4) for each  $x_1, \dots, x_{k-1} \in E$ ,  $\|(x_1, \dots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \dots, 2x_{k-1})\|_{k-1}$ .

**Remark 2.2.** [6] Suppose that  $\{(E^k, \|\cdot\|_k), k \in \mathbb{N}\}$  is a (dual) multi-normed space, and take  $k \in \mathbb{N}$ . The following property is an almost immediate consequence of the axioms (A1), (A2) and (A3).

$$\max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \dots, x_k)\|_k \leq \sum_{i=1}^k \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\| \quad (x_1, \dots, x_k \in E). \quad (1)$$

Applying (1) one concludes that if  $(E, \|\cdot\|)$  is a Banach space, then  $(E^k, \|\cdot\|_k)$  is a Banach space for each  $k \in \mathbb{N}$ ; in this case,  $\{(E^k, \|\cdot\|_k), k \in \mathbb{N}\}$  is called a (dual) multi-Banach space.

**Definition 2.3.** [4] Let  $X$  be a real vector space. A function  $N : X \times \mathbb{R} \rightarrow [0, 1]$  is a fuzzy norm on  $X$  if for all  $x, y \in X$  and  $s, t \in \mathbb{R}$ , the following conditions hold:

- (N1)  $N(x, t) = 0$  for  $t \leq 0$ ;
- (N2)  $x = 0$  if and only if  $N(x, t) = 1$  for all  $t > 0$ ;
- (N3)  $N(cx, t) = N(x, \frac{t}{|c|})$  if  $c \neq 0$ ;
- (N4)  $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$ ;
- (N5)  $N(x, \cdot)$  is a non-decreasing function of  $\mathbb{R}$  and  $\lim_{t \rightarrow \infty} N(x, t) = 1$ ;
- (N6) for  $x \neq 0$ ,  $N(x, \cdot)$  is continuous on  $\mathbb{R}$ .

The pair  $(X, N)$  is called a fuzzy normed space.

**Definition 2.4.** Let  $(X, N)$  be a fuzzy-normed space. Then

(i) A sequence  $\{x_n\}$  in  $X$  is said to be convergent if there exists  $x \in X$  such that  $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$  for all  $t > 0$  and we write  $N\text{-}\lim_{n \rightarrow \infty} x_n = x$ .

(ii) A sequence  $\{x_n\}$  in  $X$  is called Cauchy if for each  $0 < \varepsilon < 1$  and each  $\delta > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $N(x_m - x_n, \delta) > 1 - \varepsilon$  for all  $n, m \geq n_0$ .

The fuzzy-normed space  $(X, N)$  is complete if every Cauchy sequence in  $X$  converges in  $X$ . In this case fuzzy normed space is called a fuzzy Banach space.

For more details on this issue see [8, 17, 23] and the bibliography quoted there.

### 3. Multi-fuzzy Normed Spaces

We begin this section with the notion of (dual) multi-fuzzy normed space.

**Definition 3.1.** Let  $(E, N)$  be a fuzzy normed space. A multi-fuzzy norm on  $\{E^k, k \in \mathbb{N}\}$  is a sequence  $\{N_k\}$  such that  $N_k$  is a fuzzy norm on  $E^k$  ( $k \in \mathbb{N}$ ),  $N_1(x, t) = N(x, t)$  for each  $x \in E$  and  $t \in \mathbb{R}$  and the following axioms are satisfied for each  $k \in \mathbb{N}$  with  $k \geq 2$ :

(MF1) for each  $\sigma \in \mathfrak{S}_k$ ,  $x \in E^k$  and  $t \in \mathbb{R}$ ,

$$N_k(A_\sigma(x), t) = N_k(x, t);$$

(MF2) for each  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ ,  $x \in E^k$  and  $t \in \mathbb{R}$ ,

$$N_k(M_\alpha(x), t) \geq N_k(\max_{i \in \mathbb{N}_k} |\alpha_i| |x|, t);$$

(MF3) for each  $x_1, \dots, x_k \in E$  and  $t \in \mathbb{R}$ ,

$$N_{k+1}((x_1, \dots, x_k, 0), t) = N_k((x_1, \dots, x_k), t);$$

(MF4) for each  $x_1, \dots, x_k \in E$  and  $t \in \mathbb{R}$ ,

$$N_{k+1}((x_1, \dots, x_k, x_k), t) = N_k((x_1, \dots, x_k), t).$$

In such a case  $\{(E^k, N_k), k \in \mathbb{N}\}$  is called a multi-fuzzy normed space. Moreover, if axiom (MF4) is replaced by the following axiom, then  $\{N_k\}$  is called a dual

multi-fuzzy norm and  $\{(E^k, N_k), k \in \mathbb{N}\}$  is called a dual multi-fuzzy normed space. (DF4) for each  $x_1, \dots, x_k \in E$  and  $t \in \mathbb{R}$ ,

$$N_{k+1}((x_1, \dots, x_k, x_k), t) = N_k((x_1, \dots, 2x_k), t).$$

We present the definition just in the case where the index set is  $\mathbb{N}$ , however there is also an obvious definition of (dual) multi-fuzzy normed space of level  $n$  ( $n \in \mathbb{N}$ ), that is the index set is  $\mathbb{N}_n$ .

The following examples guarantee the existence of notable source of examples of (dual) multi-fuzzy normed spaces.

**Example 3.2.** Let  $(E, N)$  be a fuzzy normed space. For each  $k \in \mathbb{N}$ , set

$$N_k((x_1, \dots, x_k), t) = \min\{N(x_i, t), i = 1, \dots, k\}, \quad (x_1, \dots, x_k \in E \text{ and } t \in \mathbb{R}).$$

It is easily verified that,  $\{(E^k, N_k), k \in \mathbb{N}\}$  is a multi-fuzzy normed space.

**Example 3.3.** Let  $\{(E^k, N_k^\alpha), k \in \mathbb{N}\}_\alpha$  be a family of (dual) multi-fuzzy normed spaces. For each  $k \in \mathbb{N}$ ,  $x_1, \dots, x_k \in E$  and  $t \in \mathbb{R}$  define

$$N_k((x_1, \dots, x_k), t) = \inf_\alpha N_k^\alpha((x_1, \dots, x_k), t).$$

Then  $\{(E^k, N_k), k \in \mathbb{N}\}$  is a (dual) multi-fuzzy normed space, too.

**Example 3.4.** Let  $\{(E^k, \|\cdot\|_k), k \in \mathbb{N}\}$  be a (dual) multi-normed space. For each  $x \in E^k$ ,  $t \in \mathbb{R}$  and  $\alpha, \beta \geq 0$ , define  $N_k^1, N_k^2$  and  $N_k^3$  by setting

$$N_k^1(x, t) = \begin{cases} \frac{\alpha t}{\alpha t + \beta \|x\|_k}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

$$N_k^2(x, t) = \begin{cases} 0, & t \leq \|x\|_k, \\ 1, & t > \|x\|_k. \end{cases}$$

$$N_k^3(x, t) = \begin{cases} \frac{t^2 - \|x\|_k^2}{t^2 + \|x\|_k^2}, & t > \|x\|_k, \\ 0, & t \leq \|x\|_k. \end{cases}$$

It is rutin to check that  $N_k^1, N_k^2$  and  $N_k^3$  are (dual) multi-fuzzy normed spaces (see also [8, 19]).

According to Example 3.4 and also using some examples in [6], one can give examples that show axioms for multi- and dual multi-fuzzy normed spaces are independent of each other.

• *Independence of the axiom (MF1)*

(1) Assume that  $(E, \|\cdot\|)$  is a normed space.

(I) For each  $k \in \mathbb{N}$ , set  $\|(x_1, \dots, x_k)\|_k = \max\{\|x_1\|, \frac{\|x_2\|}{2}, \dots, \frac{\|x_k\|}{k}\}$ , where  $x_1, \dots, x_k$  are in  $E$ . Let  $x \in E$  with  $\|x\| = 1$ , then for some  $t \in \mathbb{R}^+$ ,  $N_2^i((2x, 3x), t) \neq N_2^i((3x, 2x), t)$  ( $i = 1, 2, 3$ ). Clearly  $\{(E^k, N_k^i), k \in \mathbb{N}\}$  ( $i = 1, 2, 3$ ) is a family of fuzzy normed spaces in which satisfies axioms (MF2), (MF3) and (MF4), however it does not satisfy axiom (MF1).

(II) Define  $\|x\|_1 = \|x\|$  ( $x \in E$ ) and  $\|(x_1, x_2)\|_2 = \max\{\|x_1\|, 2\|x_2\|\}$ , where  $x_1, x_2 \in E$ . Let  $x \in E$  with  $\|x\| = 1$ , then for some  $t \in \mathbb{R}^+$ ,  $N_2^i((2x, 3x), t) \neq N_2^i((3x, 2x), t)$  ( $i = 1, 2, 3$ ). It is relatively easy to see that  $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$

( $i = 1, 2, 3$ ) is a family of fuzzy normed spaces in which satisfies axioms (MF2), (MF3) and (DF4), however it does not satisfy axiom (MF1).

• *Independence of the axiom (MF2)*

(2) Let  $E = \mathbb{R}$  and  $\|x\|_1 = |x|$  for each  $x \in E$ .

(III) Define  $\|(x_1, \dots, x_k)\|_k = \max\{\max_{i \in \mathbb{N}_k} |x_i|, \max_{i, j \in \mathbb{N}_k} |x_i - x_j|\}$ , where  $k \in \mathbb{N}$  and  $x_1, \dots, x_k \in E$  ([6, Example 2.4]). Then  $\{(E^k, N_k^i), k \in \mathbb{N}\}$  ( $i = 1, 2$ ) is a family of fuzzy normed spaces in which satisfies axioms (MF1), (MF3) and (MF4) but (MF2) fails.

(IV) Define  $\|(x, y)\|_2 = \frac{1}{2}(|x + y| + |x| + |y|)$ , where  $x, y \in E$ . It is immediately checked that  $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$  ( $i = 1, 2$ ) is a family of fuzzy normed spaces in which satisfies axioms (MF1), (MF3) and (DF4) but (MF2) fails.

• *Independence of the axiom (MF3)*

(3) (V) Suppose  $E = \mathbb{R}$ . Set  $\|x\|_1 = |x|$  and  $\|(x, y)\|_2 = \frac{1}{2}(|x| + |y|)$  ( $x, y \in E$ ) ([6, Example 2.5]). Clearly,  $N_2^i((1, 0), t) \neq N_1^i(1, t)$  ( $i = 1, 2, 3$ ) for some  $t \in \mathbb{R}^+$ . Therefore  $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$  ( $i = 1, 2, 3$ ) is a family of fuzzy normed spaces in which satisfies axioms (MF1), (MF2) and (MF4) but (MF3) does not hold.

(VI) Assume that  $E = \mathbb{R}^2$ ,  $\|(x, y)\|_1 = |x - y| + |x| + |y|$  and  $\|(x, y), (z, w)\|_2 = |x| + |y| + |z| + |w| + 2 \max\{|x - y|, |z - w|\}$  ( $x, y, z, w \in \mathbb{R}$ ). It is readily verified that  $\{(E^k, N_k^i), k \in \mathbb{N}_2\}$  ( $i = 1, 2, 3$ ) is a family of fuzzy normed spaces in which satisfies axioms (MF1), (MF2) and (DF4) but (MF3) is not true.

• *Independence of the axiom (MF4) and (DF4)*

(4) Let  $(E, \|\cdot\|)$  be a normed space and  $p > 1$ . For each  $k \in \mathbb{N}$ , define  $\|(x_1, \dots, x_k)\|_k = (\sum_{i=1}^k \|x_i\|^p)^{\frac{1}{p}}$  ([6, Example 2.6]). Then  $\{(E^k, N_k^i), k \in \mathbb{N}\}$  ( $i = 1, 2, 3$ ) is a family of fuzzy normed spaces in which satisfies axioms (MF1), (MF2) and (MF3) but it does not satisfy (MF4) and (DF4).

The following lemma is an immediate consequence of the definition of multi-fuzzy normed space.

**Lemma 3.5.** *Let  $\{(E^k, N_k), k \in \mathbb{N}\}$  be a (dual) multi-fuzzy normed space,  $k, n \in \mathbb{N}$ ,  $x_1, \dots, x_k, x_{k+1}, \dots, x_{k+n} \in E$  and  $\eta_1, \dots, \eta_k$  be real numbers of absolute value 1, then we have*

(i)  $N_k((\eta_1 x_1, \dots, \eta_k x_k), t) = N_k((x_1, \dots, x_k), t)$ .

(ii)  $N_k((x_1, \dots, x_k), t) \geq N_{k+1}((x_1, \dots, x_k, x_{k+1}), t)$ .

(iii)  $N_{k+n}((x_1, \dots, x_k, x_{k+1}, \dots, x_{k+n}), t) \geq$

$$\min\{N_k((x_1, \dots, x_k), \alpha t), N_n((x_{k+1}, \dots, x_{k+n}), \beta t)\},$$

where  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 1$ .

(iv)  $\min_{i \in \mathbb{N}_k} N(x_i, t) \geq N_k((x_1, \dots, x_k), t) \geq \min_{i \in \mathbb{N}_k} N(kx_i, t)$ .

As an important result of the preceding lemma we get the following corollary.

**Corollary 3.6.** *Let  $\{(E^k, N_k), k \in \mathbb{N}\}$  be a (dual) multi-fuzzy normed space, and  $(E, N_1)$  be a fuzzy Banach space. Then for each  $k \in \mathbb{N}$ ,  $(E^k, N_k)$  is a fuzzy Banach space, too.*

*Proof.* Fix an arbitrary  $k \in \mathbb{N}$ . Let  $\{X_n\}$  be a Cauchy sequence in  $E^k$ , where  $X_n = (x_{n,1}, \dots, x_{n,k})$ . Also let  $0 < \varepsilon < 1$  and  $\delta > 0$  be given, then there exists  $n_0 \in \mathbb{N}$  such that  $N_k(X_m - X_n, \delta) > 1 - \varepsilon$  for all  $n, m \geq n_0$ . As a result of Lemma 3.5, we observe that for  $i = 1, \dots, k$ ,  $N_1(x_{m,i} - x_{n,i}, \delta) > 1 - \varepsilon$  for all  $n, m \geq n_0$ . Hence  $\{x_{n,i}\}$ ,  $i = 1, \dots, k$ , is a Cauchy sequence in fuzzy Banach space  $(E, N_1)$  and therefore it converges to some  $x_i \in E$ . Thus there exists  $n_i$ ,  $i = 1, \dots, k$  for which  $N_1(x_{n,i} - x_i, \delta) > 1 - \varepsilon$  for all  $n \geq n_i$ . Take  $n' = \max\{n_1, \dots, n_k\}$ . Applying again Lemma 3.5, we obtain that

$$\begin{aligned} N_k((x_{n,1} - x_1, \dots, x_{n,k} - x_k), \delta) &\geq \min_{i \in \mathbb{N}_k} N_1(x_{n,i} - x_i, \frac{\delta}{k}) \\ &\geq 1 - \varepsilon, \end{aligned}$$

for all  $n \geq n'$ . This proves  $N\text{-}\lim_{n \rightarrow \infty} X_n = X$ , where  $X = (x_1, \dots, x_k)$ . Therefore  $(E^k, N_k)$  is a fuzzy Banach space.  $\square$

In the light of the previous corollary, the following definition is reasonable.

**Definition 3.7.** Suppose that  $\{(E^k, N_k), k \in \mathbb{N}\}$  is a (dual) multi-fuzzy normed space, for which  $(E, N_1)$  is a fuzzy Banach space, then we say  $\{(E^k, N_k), k \in \mathbb{N}\}$  is a (dual) multi-fuzzy Banach space.

Using the similar techniques applied in [6, Lemmata 2.16, 2.19] we get the following two propositions.

**Proposition 3.8.** Let  $\{(E^k, N_k), k \in \mathbb{N}\}$  be a multi-fuzzy normed space,  $k \in \mathbb{N}$  with  $k \geq 2$ ,  $x_1, \dots, x_k \in E$ ,  $\alpha, \beta$  be nonnegative real numbers with  $\alpha + \beta = 1$  and  $x = \alpha x_{k-1} + \beta x_k$ . Then for each  $t \in \mathbb{R}$ ,  $N_k((x_1, \dots, x_{k-2}, x, x), t) \geq N_k((x_1, \dots, x_{k-2}, x_{k-1}, x_k), t)$ .

**Proposition 3.9.** Let  $\{(E^k, N_k), k \in \mathbb{N}\}$  be a dual multi-fuzzy normed space,  $k \in \mathbb{N}$  with  $k \geq 2$ ,  $x_1, \dots, x_k \in E$  and  $t \in \mathbb{R}$ . Then for each  $t \in \mathbb{R}$ ,  $N_{k-1}((x_1, \dots, x_{k-1} + x_k), t) \geq N_k((x_1, \dots, x_{k-1}, x_k), t)$ .

The following proposition is proved in framework of multi-normed spaces ([6, Proposition 2.7]). Slightly modification in the proof shows this proposition holds in the category of multi-fuzzy normed spaces.

**Proposition 3.10.** Let  $(E, N)$  be a fuzzy normed space,  $\{N_k\}$  be a sequence such that  $N_k$  is a fuzzy norm on  $E^k$  for each  $k \in \mathbb{N}$  and  $N_1(x, t) = N(x, t)$  for each  $x \in E$  and  $t \in \mathbb{R}$ . Also assume that axioms (MF1), (MF2) and (MF4) are satisfied for each  $k \in \mathbb{N}$ . Then  $\{(E^k, N_k), k \in \mathbb{N}\}$  is a multi-fuzzy normed space.

*Proof.* By Definition 3.1, it is enough to show that axiom (MF3) holds. Let  $k \in \mathbb{N}$  and  $x = (x_1, \dots, x_k)$  be an arbitrary element of  $E^k$ . Obviously if  $x = 0$ , then  $N_{k+1}((x_1, \dots, x_k, 0), t) = N_k((x_1, \dots, x_k), t)$  by (MF4) (or (N2)) and so in this case (MF3) holds. Now assume that  $x$  is nonzero and  $n \in \mathbb{N}$  is arbitrary. For  $i = 1, \dots, n+1$ , let  $B_i = \{(i-1)k+1, \dots, ik\}$  be the subset of  $\mathbb{N}_{k(n+1)}$ , and  $Q_{B_i}$  be the projection onto the complement of  $B_i$ . From (N4) it follows that

$$N_{k(n+1)}\left(\sum_{i=1}^{n+1} Q_{B_i}(x^{[n+1]}), (n+1)t\right) \geq \min\{N_{k(n+1)}(Q_{B_i}(x^{[n+1]}), t), i = 1, \dots, n+1\}. \quad (2)$$

Since  $\sum_{i=1}^{n+1} Q_{B_i}(x^{[n+1]}) = nx^{[n+1]}$ , so the left hand side of (2) is equal to  $N_{k(n+1)}(nx^{[n+1]}, (n+1)t) = N_k((x_1, \dots, x_k), \frac{n+1}{n}t)$  by (N3), (MF1) and (MF4). On the other hand, the right hand side of (2) is equal to  $N_{k+1}((x_1, \dots, x_k, 0), t)$  by (MF1) and (MF4). Consequently  $N_k((x_1, \dots, x_k), \frac{n+1}{n}t) \geq N_{k+1}((x_1, \dots, x_k, 0), t)$ . Since  $n$  is arbitrary, then let  $n \rightarrow \infty$ , by (N6) we get

$$N_k((x_1, \dots, x_k), t) \geq N_{k+1}((x_1, \dots, x_k, 0), t). \quad (3)$$

For the reverse direction applying (MF2) and (MF4), we deduce that

$$\begin{aligned} N_{k+1}((x_1, \dots, x_k, 0), t) &= N_{k+1}(M_\alpha(x_1, \dots, x_k, x_k), t) \\ &\geq N_{k+1}((x_1, \dots, x_k, x_k), t) \\ &= N_k((x_1, \dots, x_k), t), \end{aligned} \quad (4)$$

where  $\alpha = (1, \dots, 1, 0) \in \mathbb{C}^{k+1}$ . Now from (3) and (4) we achieve our goal.  $\square$

In the following, we are going to show that the above result holds for dual multi-fuzzy norms, too.

**Proposition 3.11.** *Let  $(E, N)$  be a fuzzy normed space,  $\{N_k\}$  be a sequence such that  $N_k$  is a fuzzy norm on  $E^k$  for each  $k \in \mathbb{N}$  and  $N_1(x, t) = N(x, t)$  for each  $x \in E$  and  $t \in \mathbb{R}$ . Also assume that axioms (MF1), (MF2) and (DF4) are satisfied for each  $k \in \mathbb{N}$ . Then  $\{(E^k, N_k), k \in \mathbb{N}\}$  is a dual multi-fuzzy normed space.*

*Proof.* Suppose that  $x = (x_1, \dots, x_k)$  is in  $E^k$  ( $k \in \mathbb{N}$ ),  $n$  is an arbitrary element of  $\mathbb{N}$  and  $t \in \mathbb{R}$ . For  $i = 1, \dots, 2^n$ , let  $B_i$  be the subset  $\{(i-1)k+1, \dots, ik\}$  of  $\mathbb{N}_{(2^n)k}$ , and  $Q_{B_i}$  be the projection onto the complement of  $B_i$ . Then we obtain

$$\begin{aligned} N_k(x, t) &= N_k(2^n x, 2^n t) \\ &= N_{2^n k}((2^n - 1)x^{[2^n]}, 2^n(2^n - 1)t) \quad (DF4) \\ &= N_{2^n k}(\sum_{i=1}^{2^n} Q_{B_i}(x^{[2^n]}), 2^n(2^n - 1)t) \\ &\geq \min\{N_{2^n k}(Q_{B_i}(x^{[2^n]}), (2^n - 1)t), i = 1, \dots, 2^n\} \\ &= N_{k+1}((2^n - 1)(x_1, \dots, x_k, 0), (2^n - 1)t) \end{aligned}$$

and so

$$N_k(x, t) \geq N_{k+1}((x_1, \dots, x_k, 0), t). \quad (5)$$

For the reverse direction assume that  $x = (x_1, \dots, x_k, 0)$ . Without loss of generality we may assume that (using (MF1))  $x^{[2^n]} = (x_1, \dots, x_1, \dots, x_k, \dots, x_k, 0, \dots, 0)$ , where the number of repetitions of each item is  $2^n$ . For  $i = 1, \dots, k$ , let  $C_i = \{(i-1)2^n + 1, \dots, i2^n\}$  and  $Q_{C_i}$  be the projection onto the complement of  $C_i$ . Also for  $i = 1, \dots, k-2$ , put

$$X_1^i = (x_1, \dots, x_1, \dots, x_{i-1}, \dots, x_{i-1}, 0, \dots, 0, x_{i+1}, \dots, x_{i+1}, \dots, x_k, \dots, x_k, 0, \dots, 0),$$

$$X_2^i = (x_1, \dots, x_1, \dots, x_{i-1}, \dots, x_{i-1}, x_{i+1}, \dots, x_{i+1}, \dots, x_k, \dots, x_k, 0, \dots, 0),$$

where the number of repetitions of each item  $x_t$ ,  $t = 1, \dots, i-1, i+1, \dots, k$  is  $2^n$  and also zero has repeated  $2 \times 2^n$  times,

$$X_3^i = (x_1, \dots, x_1, \dots, x_{i-1}, \dots, x_{i-1}, x_{i+1}, \dots, x_{i+1}, \dots, x_k, \dots, x_k, 0, \dots, 0),$$

$$X_4^i = (x_1, \dots, x_1, \dots, x_{i-1}, \dots, x_{i-1}, x_{i+1}, \dots, x_{i+1}, \dots, x_k, \dots, x_k, x_i, \dots, x_i),$$

$$X_5 = (x_1, \dots, x_1, \dots, x_k, \dots, x_k),$$

where the number of repetitions of each item in  $X_3^i, X_4^i$  and  $X_5$  is  $2^n$ .

Take  $A = N_{2^n(k+1)}(Q_{C_{k-1}}(x^{[2^n]}) + Q_{C_k}(x^{[2^n]}), 2^n t)$ . Then

$$\begin{aligned} A &= N_{2^n(k+1)}((2x_1, \dots, 2x_1, \dots, 2x_{k-2}, \dots, 2x_{k-2}, \\ &\quad x_{k-1}, \dots, x_{k-1}, x_k, \dots, x_k, 0, \dots, 0), 2^n t) \\ &\geq N_{2^n(k+1)}(2M_\alpha(x_1, \dots, x_1, \dots, x_{k-1}, \dots, x_{k-1}, x_k, \dots, x_k, x_k, \dots, x_k), 2^n t) \\ &\geq N_{k+1}(2^{n+1}(x_1, \dots, x_{k-1}, x_k, x_k), 2^n t) \\ &\geq N_k((x_1, \dots, x_k), \frac{2^n}{2^{n+2}} t), \end{aligned}$$

where  $\alpha = (\underbrace{1, \dots, 1}_{2^k \text{-tuples}}, \underbrace{0, \dots, 0}_{2^n \text{-tuples}})$ . Also, we observe that

$$\begin{aligned} &N_{2^n(k+1)}(\sum_{i=1}^k Q_{C_i}(x^{[2^n]}), 2^n(k-1)t) \\ &\geq \min\{A, N_{2^n(k+1)}(Q_{C_i}(x^{[2^n]}), 2^n t) : i = 1, \dots, k-2\} \\ &= \min\{A, N_{2^n(k+1)}(X_1^i, 2^n t) : i = 1, \dots, k-2\} \\ &= \min\{A, N_{2^n(k+1)}(X_2^i, 2^n t) : i = 1, \dots, k-2\} \quad (MF1) \\ &= \min\{A, N_{2^n k}(X_3^i, 2^n t) : i = 1, \dots, k-2\} \quad (DF4) \\ &= \min\{A, N_{2^n k}(M_\beta X_4^i, 2^n t) : i = 1, \dots, k-2\} \\ &\geq \min\{A, N_{2^n k}(X_5, 2^n t) : i = 1, \dots, k-2\} \quad (MF1), (MF2) \\ &= \min\{A, N_{2^n k}(2^n(x_1, \dots, x_k), 2^n t)\} \\ &\geq N_k((x_1, \dots, x_k), \frac{2^n}{2^{n+2}} t), \end{aligned}$$

where  $\beta = (\underbrace{1, \dots, 1}_{2^{n(k-1)} \text{-tuples}}, \underbrace{0, \dots, 0}_{2^n \text{-tuples}})$ .

Furthermore, let  $x'^{[2^n]} = (x_1, \dots, x_k, 0, \dots, x_1, \dots, x_k, 0)$ . It is easily verified that

$\sum_{i=1}^k Q_{C_i}(x^{[2^n]}) = (k-1)x^{[2^n]}$ . Take  $n$  sufficiently large in which  $k \leq 2^n$ , then we deduce

$$\begin{aligned} N_{k+1}(x, \frac{2^n}{2^n-1} t) &= N_{k+1}(2^n x, \frac{2^{2n}}{2^n-1} t) \\ &\geq N_{2^n(k+1)}(x'^{[2^n]}, 2^n \frac{2^n-1}{2^n-1} t) \\ &= N_{2^n(k+1)}(x'^{[2^n]}, 2^n \frac{k-1}{k-1} t) \\ &= N_{2^n(k+1)}(\sum_{i=1}^k Q_{C_i}(x^{[2^n]}), 2^n(k-1)t) \\ &\geq N_k((x_1, \dots, x_k), \frac{2^n}{2^{n+2}} t). \end{aligned}$$

Thus we find that  $N_{k+1}(x, \frac{2^n}{2^n-1} t) \geq N_k((x_1, \dots, x_k), \frac{2^n}{2^{n+2}} t)$ . Let  $n \rightarrow \infty$ , it obtains that

$$N_{k+1}((x_1, \dots, x_k, 0), t) \geq N_k((x_1, \dots, x_k), t). \quad (6)$$



Now the result follows by (5) and (6).  $\square$

#### 4. Ulam-Hyers Stability of the Jensen's Functional Equation

This section is devoted to establishing Ulam-Hyers stability of the Jensen's functional equation for mappings from linear spaces into multi-fuzzy normed spaces. To this end, we extend some results of [18] in the framework of multi-fuzzy normed spaces.

**Theorem 4.1.** *Let  $E$  be a linear space,  $\{(F^k, N_k), k \in \mathbb{N}\}$  be a multi-fuzzy Banach space,  $(F, N')$  be a fuzzy normed space and  $f : E \rightarrow F$  be a mapping satisfying  $f(0) = 0$ . Suppose that  $\alpha$  is a nonzero fixed vector in  $F$  such that*

$$N_k\left(\left(f\left(\frac{x_1 + y_1}{2}\right) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f\left(\frac{x_k + y_k}{2}\right) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}\right), t\right) \geq N'(\alpha, t), \quad (7)$$

for all  $x_1, \dots, x_k, y_1, \dots, y_k \in E$  and for all  $t > 0$ . Then there exists a unique additive mapping  $T : E \rightarrow F$  such that

$$N_k\left(\left(f(x_1) - T(x_1), \dots, f(x_k) - T(x_k)\right), 2t\right) \geq N'(\alpha, t), \quad (8)$$

where  $x_1, \dots, x_k \in E$  and  $t > 0$ .

*Proof.* Substituting  $y_i = 0$  for  $i = 1, \dots, k$  and replacing  $x_1, \dots, x_k$  by  $2^n x_1, \dots, 2^n x_k$  in (7), we get

$$N_k\left(\left(\frac{f(2^{n-1}x_1)}{2^{n-1}} - \frac{f(2^n x_1)}{2^n}, \dots, \frac{f(2^{n-1}x_k)}{2^{n-1}} - \frac{f(2^n x_k)}{2^n}\right), \frac{t}{2^{n-1}}\right) \geq N'(\alpha, t). \quad (9)$$

Regarding (9), we conclude that

$$N_k\left(\left(\frac{f(2^m x_1)}{2^m} - \frac{f(2^n x_1)}{2^n}, \dots, \frac{f(2^m x_k)}{2^m} - \frac{f(2^n x_k)}{2^n}\right), t\left(\sum_{i=m}^{n-1} 2^{-i}\right)\right) \geq N'(\alpha, t), \quad (10)$$

for nonnegative integer numbers  $m, n$ .

Fix nonzero  $x \in E$ , put  $x_1 = \dots = x_k = x$  in (10) we thus find that

$$N\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, t\left(\sum_{i=m}^{n-1} 2^{-i}\right)\right) \geq N'(\alpha, t).$$

Let  $\varepsilon > 0$  and  $\delta > 0$  be given. Since  $\lim_{t \rightarrow \infty} N'(\alpha, t) = 1$ , there is some  $t_0 > 0$  such that

$N'(\alpha, t_0) > 1 - \varepsilon$ . Since  $\sum_{i=1}^{\infty} t_0 2^{-i} < \infty$ , there is some  $n_0 \in \mathbb{N}$  such that  $\sum_{i=m}^{n-1} t_0 2^{-i} < \delta$

for all  $m \geq n_0 + 1$ . Therefore we derive

$$N\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, \delta\right) \geq N\left(\frac{f(2^m x)}{2^m} - \frac{f(2^n x)}{2^n}, t_0\left(\sum_{i=m}^{n-1} 2^{-i}\right)\right) \geq N'(\alpha, t_0) > 1 - \varepsilon.$$

It follows that  $\left\{\frac{f(2^n x)}{2^n}\right\}$  is Cauchy and so is convergent in the complete fuzzy normed space  $F$ . Set

$$T(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}.$$

In view of Lemma 3.5, we deduce

$$N\text{-}\lim_{n \rightarrow \infty} \left( \frac{f(2^n x_1)}{2^n}, \dots, \frac{f(2^n x_k)}{2^n} \right) = (T(x_1), \dots, T(x_k)).$$

Moreover, if we put  $m = 0$  in (10), we observe that

$$N_k\left(\left(f(x_1) - \frac{f(2^n x_1)}{2^n}, \dots, f(x_k) - \frac{f(2^n x_k)}{2^n}\right), t\right) \geq N'(\alpha, t). \quad (11)$$

Therefore

$$\begin{aligned} N_k((f(x_1) - T(x_1), \dots, f(x_k) - T(x_k)), 2t) &\geq \\ \min\{N_k\left(\left(f(x_1) - \frac{f(2^n x_1)}{2^n}, \dots, f(x_k) - \frac{f(2^n x_k)}{2^n}\right), t\right), \\ N_k\left(\left(\frac{f(2^n x_1)}{2^n} - T(x_1), \dots, \frac{f(2^n x_k)}{2^n} - T(x_k)\right), t\right)\}. \end{aligned}$$

The second term on the right hand side of the above inequality tends to 1 as  $n \rightarrow \infty$  and the first term, by (11) is greater than or equal to  $N'(\alpha, t)$ . It gives us relation (8). Next, we will show that  $T$  is additive. Let  $x, y \in E$ . Put  $x_1 = \dots = x_k = 2^n x$ ,  $y_1 = \dots = y_k = 2^n y$  and replace  $t$  by  $2^n t$  in (7) to obtain

$$N_k\left(\frac{f\left(\frac{2^n(x+y)}{2}\right)}{2^n} - \frac{1}{2} \frac{f(2^n x)}{2^n} - \frac{1}{2} \frac{f(2^n y)}{2^n}, t\right) \geq N'(\alpha, 2^n t). \quad (12)$$

On the other hand,

$$\begin{aligned} N\left(T\left(\frac{x+y}{2}\right) - \frac{T(x)}{2} - \frac{T(y)}{2}, 4t\right) &\geq \min\left\{N\left(T\left(\frac{x+y}{2}\right) - \frac{f(2^n\left(\frac{x+y}{2}\right))}{2^n}, t\right), \right. \\ N\left(\frac{T(x)}{2} - \frac{1}{2} \frac{f(2^n x)}{2^n}, t\right), N\left(\frac{T(y)}{2} - \frac{1}{2} \frac{f(2^n y)}{2^n}, t\right), &N\left(\frac{f(2^n\left(\frac{x+y}{2}\right))}{2^n} - \frac{1}{2} \frac{f(2^n x)}{2^n} - \frac{1}{2} \frac{f(2^n y)}{2^n}, t\right)\}, \end{aligned}$$

for each  $x, y \in E$  and  $t > 0$ . The first three terms on the right hand side of the above inequality tend to 1 as  $n \rightarrow \infty$  and the fourth term, by (12) is greater than or equal to  $N'(\alpha, 2^n t)$ , which tends to 1 as  $n \rightarrow \infty$ . Therefore

$$N\left(T\left(\frac{x+y}{2}\right) - \frac{T(x)}{2} - \frac{T(y)}{2}, 4t\right) = 1,$$

for each  $x, y \in E$  and  $t > 0$ . It enforces that  $T$  satisfies the Jensen equation and by virtue of the fact that  $T(0) = 0$ ,  $T$  is additive. To end the proof, let  $T'$  be another additive mapping satisfying (8) and  $x$  be nonzero element of  $E$ , then

$$\begin{aligned} N(T'(x) - T(x), 2t) &= N\left(\frac{T'(2^n x)}{2^n} - \frac{T(2^n x)}{2^n}, 2t\right) \\ &\geq \min\{N(T'(2^n x) - f(2^n x), 2^n t), N(T(2^n x) - f(2^n x), 2^n t)\} \\ &\geq N'(\alpha, 2^{n-1}t). \end{aligned}$$

By taking the limit as  $n$  tends to infinity we get  $T = T'$ . This proves the uniqueness assertion.  $\square$

**Example 4.2.** Let  $E$  and  $F$  be fuzzy normed space  $(\mathbb{R}, N)$ , where

$$N(x, t) = \begin{cases} 1, & t > |x|, \\ 0, & t \leq |x|. \end{cases}$$

Consider  $N_k((x_1, \dots, x_k), t) = \min\{N(x_i, t) : i = 1, \dots, k\}$  ( $(x_1, \dots, x_k) \in F^k$ ,  $t \in \mathbb{R}$ ). Then  $\{(F^k, N_k), k \in \mathbb{N}\}$  is a multi-fuzzy Banach space. Also consider  $f(x) = \operatorname{sgn}(x)$ , ( $x \in \mathbb{R}$ ), take  $\alpha = 4$  and  $N' = N$ . Now, all conditions of Theorem 4.1 are satisfied and so there is a unique additive mapping  $T$  fulfills condition (8).

Next, we show that the condition  $f(0) = 0$  in Theorem 4.1 is necessary.

**Example 4.3.** In Theorem 4.1, let  $E$  and  $F$  be fuzzy normed space  $(\mathbb{R}, N)$ , where  $N(x, t) = \frac{t}{t+|x|}$  ( $x \in E$  or  $F$ ,  $t > 0$ ) and  $N_k((x_1, \dots, x_k), t) = \min\{N(x_i, t) : i = 1, \dots, k\}$  ( $(x_1, \dots, x_k) \in F^k$ ,  $t \in \mathbb{R}$ ). Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 10$ . Also take  $\alpha = 1$  and  $N' = N$ . Evidently, condition (7) holds. If on the contrary, there exists an additive mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  satisfying (8), then take  $x = n$  ( $n \in \mathbb{N}$  large enough), we get  $N(10 - nT(1), 2t) \geq N(1, t)$  for all  $t > 0$ . Hence

$$\frac{2t}{2t + |10 - nT(1)|} \geq \frac{t}{t + 1}, \quad (13)$$

for all  $t > 0$ . In turn this proves

$$\left| \frac{10}{n} - T(1) \right| \leq \frac{2}{n}. \quad (14)$$

In the light of the relation (14) we conclude that  $T(1) = 0$ , which leads to a contradiction with the condition (13).

**Remark 4.4.** A significant fact about Ulam-Hyers stability in the setup multi-fuzzy normed spaces which is interesting in own right is that we can obtain some results in multi-normed spaces as particular cases in such spaces. For instance [18, Theorem 1] can be obtained from Theorem 4.1, when we consider Banach space  $(F, \|\cdot\|)$  with the fuzzy norm  $N(x, t) = \frac{t}{t + \|x\|}$  ( $x \in F$ ,  $t > 0$ ),  $(F^k, N_k)$  is defined as Example 3.2 and  $N'(\alpha, t) = \frac{t}{t + \|\alpha\|}$ .

The following example shows the usefulness of our results. Indeed, we give an example of a multi-fuzzy normed space such that its topology is not multi-normable.

**Example 4.5.** Let  $E$  be the space of complex-valued continuous functions on the real line. Then  $E$  is not normable [25]. Define

$$N(f, t) = \begin{cases} 0, & t \leq 0, \\ \sup\{\frac{n}{n+1}, \|f\|_n \leq t\}, & t > 0, \end{cases}$$

where  $\|\cdot\|_n$  denotes the supremum norm on  $[-n, n]$ ,  $n \in \mathbb{N}$ . By applying a similar argument as in the proof of [3, Theorem 2.2], one can show that  $(E, N)$  is a fuzzy normed space (see [16]). Now consider  $\{(E^k, N_k), k \in \mathbb{N}\}$  as Example 3.2. Then it is a multi-fuzzy normed space which is not multi-normable.

In the following we give a result by utilizing the strategy applied in [18, Page 458]. Note that this theorem in comparison with the similar one in multi-normed spaces is applicable in a different area.

**Theorem 4.6.** Let  $\{(E^k, N_k), k \in \mathbb{N}\}$  be a multi-fuzzy normed space,  $\{(F^k, N_k), k \in \mathbb{N}\}$  be a multi-fuzzy Banach space and  $(E, N'), (F, N')$  be fuzzy normed spaces. Suppose that  $\{\beta_k\}$  is a sequence in  $E$ ,  $\alpha$  is a nonzero fixed vector in  $F$  and  $f : E \rightarrow F$  is a mapping satisfying  $f(0) = 0$  and

$$N_k\left(\left(f\left(\frac{x_1+y_1}{2}\right) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f\left(\frac{x_k+y_k}{2}\right) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}\right), \frac{t}{5}\right) \geq N'(\alpha, t), \quad (15)$$

for all  $k \in \mathbb{N}$ ,  $x_1, \dots, x_k, y_1, \dots, y_k \in E$  and all  $t > 0$  with

$$\min\{N_k((x_1, \dots, x_k), t), N_k((y_1, \dots, y_k), t)\} \leq N'(\beta_k, t).$$

Then there exists a unique additive mapping  $T : E \rightarrow F$  such that

$$N_k((f(x_1) - T(x_1), \dots, f(x_k) - T(x_k)), 2t) \geq N'(\alpha, t), \quad (16)$$

for all  $x_1, \dots, x_k \in E$  and all  $t > 0$ .

*Proof.* Fix  $k \in \mathbb{N}$  and  $x = (x_1, \dots, x_k)$  and  $y = (y_1, \dots, y_k)$  in  $E^k$ . Assume that  $\min\{N_k(x, t), N_k(y, t)\} > N'(\beta_k, t)$ . Obviously, if  $x = y = 0$ , then the condition (7) of Theorem 4.1 holds. Now if  $x$  and  $y$  are nonzero, then there exist sufficiently large natural numbers  $k_1$  and  $k_2$ , in which  $N_k(x, \frac{t}{k_1}) \leq N'(\beta_k, t)$  and  $N_k(y, \frac{t}{k_2-1}) \leq N'(\beta_k, t)$ . Put  $z_1 = k_1x$  and  $z_2 = k_2y$ , then  $N_k(z_1 + x, t) \leq N'(\beta_k, t)$  and  $N_k(z_2 - y, t) \leq N'(\beta_k, t)$ . Set

$$z := \begin{cases} z_1 + x, & \text{if } N_k(x, t) \leq N_k(y, t), \\ z_2 - y, & \text{if } N_k(x, t) > N_k(y, t), \end{cases}$$

for all  $t > 0$ . Clearly,  $N_k(z, t) \leq N'(\beta_k, t)$ . Now if  $N_k(x, t) > N_k(y, t)$ , then  $\min\{N_k(x - z, t), N_k(y + z, t)\} \leq N_k(y + z, t) = N_k(z_2, t) \leq N'(\beta_k, t)$ , and if  $N_k(x, t) \leq N_k(y, t)$ , then  $\min\{N_k(x - z, t), N_k(y + z, t)\} \leq N_k(x - z, t) = N_k(z_1, t) \leq N'(\beta_k, t)$ . Taking all these considerations into account we observe that, the following relations hold.

$$\begin{aligned} \min\{N_k(x, t), N_k(z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(y, t), N_k(2z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(y + z, t), N_k(z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(2z, t), N_k(x - z, t)\} &\leq N'(\beta_k, t). \\ \min\{N_k(x - z, t), N_k(y + z, t)\} &\leq N'(\beta_k, t). \end{aligned} \quad (17)$$

If  $x = 0$  and  $y \neq 0$  (or  $x \neq 0$  and  $y = 0$ ), then an easy verification proves that (17) holds, too. From (15) and (17) we get

$$\begin{aligned} &N_k\left(\left(f\left(\frac{x_1+y_1}{2}\right) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f\left(\frac{x_k+y_k}{2}\right) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}\right), t\right) \geq \\ &\min\left\{N_k\left(f\left(\frac{x_1+y_1}{2}\right) - \frac{f(x_1-z_1)+f(y_1+z_1)}{2}, \dots, f\left(\frac{x_k+y_k}{2}\right) - \frac{f(x_k-z_k)+f(y_k+z_k)}{2}, \frac{t}{5}\right), \right. \\ &N_k\left(f\left(\frac{x_1+z_1}{2}\right) - \frac{f(2z_1)+f(x_1-z_1)}{2}, \dots, f\left(\frac{x_k+z_k}{2}\right) - \frac{f(2z_k)+f(x_k-z_k)}{2}, \frac{t}{5}\right), \\ &N_k\left(f\left(\frac{y_1+2z_1}{2}\right) - \frac{f(y_1)+f(2z_1)}{2}, \dots, f\left(\frac{y_k+2z_k}{2}\right) - \frac{f(y_k)+f(2z_k)}{2}, \frac{t}{5}\right), \\ &N_k\left(f\left(\frac{y_1+2z_1}{2}\right) - \frac{f(y_1+z_1)+f(z_1)}{2}, \dots, f\left(\frac{y_k+2z_k}{2}\right) - \frac{f(y_k+z_k)+f(z_k)}{2}, \frac{t}{5}\right), \\ &N_k\left(f\left(\frac{x_1+z_1}{2}\right) - \frac{f(x_1)+f(z_1)}{2}, \dots, f\left(\frac{x_k+z_k}{2}\right) - \frac{f(x_k)+f(z_k)}{2}, \frac{t}{5}\right)\left. \right\} \\ &\geq N'(\alpha, t). \end{aligned}$$

This inequality holds for all  $k \in \mathbb{N}$ , all  $x_1, \dots, x_k, y_1, \dots, y_k \in E$  and all  $t > 0$  (if  $\min\{N_k(x, t), N_k(y, t)\} \leq N'(\beta_k, t)$  we get it immediately by (15)). Now, the result is deduced from Theorem 4.1.  $\square$

**Example 4.7.** Consider  $\{(E^k, \|\cdot\|), k \in \mathbb{N}\}$  and  $\{(F^k, \|\cdot\|), k \in \mathbb{N}\}$  are multi-normed space and multi-Banach space, respectively. Equip  $E^k$  and  $F^k$  with the multi-fuzzy norm  $N_k^i$  ( $i = 1$  or  $2$  or  $3$ ) as Example 3.2 and take  $N' = N_1^i$ . Suppose that  $\alpha$  is in  $F$  and  $\{\beta_k\}$ ,  $k \in \mathbb{N}$  is a sequence in  $E$ . Now take  $f : E \rightarrow F$  is a mapping satisfying all conditions of Theorem 4.6. It follows that if

$$\left\| \left( f\left(\frac{x_1 + y_1}{2}\right) - \frac{f(x_1) + f(y_1)}{2}, \dots, f\left(\frac{x_k + y_k}{2}\right) - \frac{f(x_k) + f(y_k)}{2} \right) \right\|_k \leq \frac{\|\alpha\|}{5},$$

for all  $k \in \mathbb{N}$  and for all  $x_1, \dots, x_k, y_1, \dots, y_k \in E$  with

$$\max \left\{ \|(x_1, \dots, x_k)\|_k, \|(y_1, \dots, y_k)\|_k \right\} \geq \|\beta_k\|,$$

then there exists a unique additive mapping  $T : E \rightarrow F$  such that

$$\sup_{k \in \mathbb{N}} \|(f(x_1) - T(x_1), \dots, f(x_k) - T(x_k))\|_k \leq 2\|\alpha\|,$$

for all  $x_1, \dots, x_k \in E$ .

**Example 4.8.** Let  $\{(E^k, \|\cdot\|), k \in \mathbb{N}\}$  and  $\{(F^k, \|\cdot\|), k \in \mathbb{N}\}$  be as Example 4.7. Also let  $N'(x, t) = \frac{t^2 - \|x\|^2}{t^2 + \|x\|^2}$ ,  $x \in E$  or  $F$ , when  $t > \|x\|$ . Suppose that  $\alpha \in F$  and  $\beta_k \in E$  ( $k \in \mathbb{N}$ ) with  $\|\alpha\| = \|\beta_k\| = 1$  and  $f : E \rightarrow F$  is a mapping satisfying all conditions of Theorem 4.6. It follows that if

$$\left\| \left( f\left(\frac{x_1 + y_1}{2}\right) - \frac{f(x_1) + f(y_1)}{2}, \dots, f\left(\frac{x_k + y_k}{2}\right) - \frac{f(x_k) + f(y_k)}{2} \right) \right\|_k \leq \frac{2t}{5(t^2 - 1)},$$

for all  $k \in \mathbb{N}$ , all  $x_1, \dots, x_k, y_1, \dots, y_k \in E$  and all  $t > 1$  with

$$\max \left\{ \|(x_1, \dots, x_k)\|_k, \|(y_1, \dots, y_k)\|_k \right\} \geq \frac{2t}{t^2 - 1},$$

then there exists a unique additive mapping  $T : E \rightarrow F$  such that

$$\sup_{k \in \mathbb{N}} \|(f(x_1) - T(x_1), \dots, f(x_k) - T(x_k))\|_k \leq \frac{4t}{t^2 - 1},$$

for all  $x_1, \dots, x_k \in E$  and all  $t > 1$ .

In the above example, we observe that the stability of Jensen's functional equation depends on another parameter  $t$  such that for each fixed  $t > 1$  it reduces to the same result as Example 4.7.

**Example 4.9.** Under the hypotheses of Example 4.3 except  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(0) = f(\frac{1}{2}) = 0$  and  $f(x) = 1$  elsewhere,  $\alpha = 100$  and  $\beta_k = 1$  ( $k \in \mathbb{N}$ ), then all conditions of Theorem 4.6 hold. Therefore there exists a unique additive mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  fulfilling condition (16).

Now we are ready to prove our main result.

**Theorem 4.10.** Let  $\{(E^k, N_k), k \in \mathbb{N}\}$  be a multi-fuzzy normed space,  $\{(F^k, N_k), k \in \mathbb{N}\}$  be a multi-fuzzy Banach space and  $f : E \rightarrow F$  be a mapping such that  $f(0) = 0$ . Then  $f$  is additive if and only if for all  $k \in \mathbb{N}$

$$N_k\left(\left(f\left(\frac{x_1 + y_1}{2}\right) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f\left(\frac{x_k + y_k}{2}\right) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}\right), \frac{t}{5}\right) \rightarrow 1 \quad (18)$$

as

$$\min\{N_k((x_1, \dots, x_k), t), N_k((y_1, \dots, y_k), t)\} \rightarrow 0.$$

*Proof.* If  $f$  is additive, then evidently (18) holds. Conversely, let  $(E, N')$  and  $(F, N')$  be fuzzy-normed spaces. Fix nonzero vector  $\alpha$  in  $F$ . Employing the condition (18) we can find for each  $n \in \mathbb{N}$  a sequence  $\{\beta_{n_k}\}$  in  $E$  such that

$$N_k\left(\left(f\left(\frac{x_1 + y_1}{2}\right) - \frac{f(x_1)}{2} - \frac{f(y_1)}{2}, \dots, f\left(\frac{x_k + y_k}{2}\right) - \frac{f(x_k)}{2} - \frac{f(y_k)}{2}\right), \frac{t}{5}\right) \geq N'\left(\frac{\alpha}{n}, t\right),$$

for all  $k \in \mathbb{N}$  and all  $x_1, \dots, x_k, y_1, \dots, y_k \in E$ , with

$$\min\{N_k((x_1, \dots, x_k), t), N_k((y_1, \dots, y_k), t)\} \leq N'(\beta_{n_k}, t).$$

In view of Theorem 4.6, for every  $n \in \mathbb{N}$  there exists a unique additive mapping  $T_n$  such that

$$N(T_n(x) - f(x), 2t) \geq N'\left(\frac{\alpha}{n}, t\right), \quad (19)$$

for all  $x \in E$  and all  $t > 0$ . Since  $N(T_1(x) - f(x), 2t) \geq N'(\alpha, t)$  and  $N(T_n(x) - f(x), 2t) \geq N'\left(\frac{\alpha}{n}, t\right) \geq N'(\alpha, t)$  by the uniqueness of  $T_1$  we conclude that  $T_n = T_1$  for each  $n \in \mathbb{N}$ . Now tending with  $n$  to infinity in (19), we deduce that  $f = T_1$  and hence  $f$  is additive.  $\square$

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## STABILITY OF THE JENSEN'S FUNCTIONAL EQUATION IN MULTI-FUZZY NORMED SPACES

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### پایداری معادله تابعی ینسن در فضاهای چند نرم دار فازی

**چکیده.** در این مقاله ، مفهوم فضاهای (دوگان) چند نرم دار فازی را تعریف می کنیم و بعضی از خواص آن ها را توصیف می نماییم. سپس پایداری اولام – اریس معادله تابعی ینسن برای نگاشت هایی از فضاهای خطی به توی فضاهای چند نرم دار فازی را بررسی می کنیم. یک رفتار مجانبی از معادله ینسن در قالب فضاهای چند نرم دار فازی می سازیم.

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