SELECTIVE GROUPOIDS AND FRAMEWORKS INDUCED BY FUZZY SUBSETS

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ABSTRACT. In this paper, we show that every selective groupoid induced by a fuzzy subset is a pogroupoid, and we discuss several properties in quasi ordered sets by introducing the notion of a framework.

1. **Introduction**

J. Neggers [7] has defined a pogroupoid and he obtained a functorial connection between posets and pogroupoids and associated structure mappings. J. Neggers and H. S. Kim [8] demonstrated that a pogroupoid $(X, \cdot)$ is modular* if and only if its associated poset $(X, \leq)$ is $(C_2+\underline{1})$-free, a condition which corresponds naturally to the notion of sublattice (in the sense of Kelly-Rival [2, 4]) isomorphic to N_5, and that this is equivalent to the associativity of the pogroupoid. J. Neggers and H. S. Kim [10] showed that the Jacobi form is 0 precisely when the pogroupoid $(X, \cdot)$ is a semigroup, i.e., modular*, precisely when the associated poset $(X, \leq)$ is $(C_2+\underline{1})$-free. Moreover, they showed that a *pg*-algebra KS over a field K is a Lie algebra with respect to the commutator product if and only if its associated poset $(X, \leq)$ is $(C_2+\underline{1})$-free.

The concept of a fuzzy set was introduced by L. A. Zadeh [15]. A *fuzzy subset* of a set X is a function $\mu : X \to [0, 1]$. The applications of fuzzy concepts to posets and groupoids have been investigated by several authors including [1, 5, 11, 13, 14].

J. Neggers and H. S. Kim [12] showed that, for a given pogroupoid $(X, \cdot)$, the associated poset $(X, \leq)$ is $(C_2+\underline{1})$-free if and only if the relation $\rhd_\mu$ is transitive for any fuzzy subset μ of X. Also they determined the set $C(X, \cdot)$ of fuzzy subsets μ such that $\mu(x \cdot y) = \mu(y \cdot x)$ for all $x, y \in X$. Furthermore, they introduced two polynomial invariants associated with fuzzy subsets of finite posets which appear to be of interest also. H. S. Kim and J. Neggers [6] introduced the notion $Bin(X)$ of all binary systems(groupoids, algebras) defined on a set X, and showed that it becomes a semigroup under suitable operation.

As a goal in this paper we seek to make further connections between the various ways that poset may be identified. Thus, the original study of pogroupoids was one attempt to give a poset an algebraic structure whose properties revealed those of posets not otherwise clear in a way analogous to the Hasse diagram of a poset

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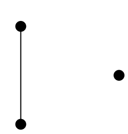
for example. Similarly, once a Hasse diagram has been constructed as a particular digraph, one may then obtain a different groupoid associated with a digraph, which is then restricted to the particular type corresponding to the Hasse diagram. The final move made is to make a connection between fuzzy subsets and selective groupoids which extends the modeling of posets to this level as well. Naturally, in doing so, one obtains new perspectives and new results as well, as we have shown in the text that follows. Although posets in the abstract, as special ordered systems, have already been proven to have many applications, such as scheduling theory and associated algorithms, it stands to reason that with new classes of descriptions available, these may be used to provide alternative ways of seeing such concepts as "schedules" for example, as well as to increase the probability of using the insights obtained in new settings for applied questions and possible answers for these. Whether and to what extent this will ultimately be the case will be proven out in the future ultimately. References as give above already support the argument as we have made them.

In this paper, we extend the existing results connecting posets with pogroupoids both in general and in specific situations, e.g., the case where the pogroupoids is $(C_2 + \underline{1})$-free. In particular, a selective groupoid $(X, *)$ has the property that $x * y \in \{x, y\}$ and is able to model posets $(X, \leq)$ by the condition $x \leq y$ provided $x * y = y = y * x$. Given the conditions defining a pogroupoid it is clear that such type of selective groupoid is itself a special type of selective groupoid and thus that existing theory may be cast into other results involving selective groupoid. And additional new connection involves special kinds of fuzzy subsets whose induced selective groupoids are of the special type indicated above. Finally, these observations are joined with the effects of homomorphisms on groupoids on the special conditions already mentioned. A useful structure in this theory, introduced in this paper, is a framework (X, I, F) for a fuzzy subset $\mu : X \to [0, 1]$ of X, whose properties are more closely considered.

2. Preliminaries

A groupoid $(X, \cdot)$ is said to be a *pogroupoid* [7] if (i) $x \cdot y \in \{x, y\}$; (ii) $x \cdot (y \cdot x) = y \cdot x$; (iii) $(x \cdot y) \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in X$. For a given pogroupoid $(X, \cdot)$, its associated poset $(X, \leq)$ is defined by the condition $x \leq y$ iff $y \cdot x = y = x \cdot y$. On the other hand, for a given poset $(X, \leq)$ its associated pogroupoid $(X, \cdot)$ is defined by $y \cdot x = y$ if $x \leq y$, $y \cdot x = x$ otherwise. This means that there is a natural isomorphism between the category of pogroupoids and the category of posets. We call a pogroupoid *modular** if it is a semigroup [8]. Given a poset $(X, \leq)$ it is *Q-free* if there is no full subposet $(P, \leq)$ of $(X, \leq)$ which is order isomorphic to the poset $(Q, \leq)$. If C_n denotes a chain of length n and if $\underline{n}$ denotes an antichain of cardinal number n, while $+$ denotes the disjoint union of posets, then the poset $(C_2 + \underline{1})$ (or $C_2 + C_1$) has Hasse-diagram:



Theorem 2.1. [12] *Let* $(X, \cdot)$ *be a pogroupoid and let its associated poset* $(X, \leq)$ *be* $(C_2 + \underline{1})$*-free. Let* $\mu : X \to [0,1]$ *be a fuzzy subset of* X*. Define a relation* $x \rhd_\mu y$ *by*

$$x \rhd_\mu y \iff \mu(x \cdot y) < \mu(y \cdot x), \tag{1}$$

for all $x, y \in X$*. Then* $(X, \rhd_\mu)$ *is a poset.*

Definition 2.2. A *d-algebra* [9] is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y \in X$.

A BCK-algebra is a d-algebra $(X; *, 0)$ satisfying the following additional axioms:

(IV) $(x * (x * y)) * y = 0$
(V) $((x * y) * (x * z)) * (z * y) = 0$,

for all $x, y, z \in X$.

Example 2.3. [3, 9] Let R be the set of all real numbers. Define a binary operation "$*$" on R by $x * y := x(x - y)$, $\forall x, y \in R$. Then $x * x = 0$ and $0 * x = 0$. If $x * y = y * x = 0$, then $x(x - y) = y(y - x) = 0$, and hence $x = y$. Hence $(R; *, 0)$ is a d-algebra, but not BCK-algebra, since $(2 * 0) * 2 \neq 0$.

Given two groupoids $(X, *)$ and $(X, \bullet)$, we define a new binary operation $\Box$ by $x \Box y := (x * y) \bullet (y * x)$ for all $x, y \in X$. Then we obtain a new groupoid $(X, \Box)$, i.e., $(X, \Box) = (X, *) \Box (X, \bullet)$. We denote the collection of all binary systems(groupoid, algebras) defined on X by $Bin(X)$ [6].

Theorem 2.4. [6] $(Bin(X), \Box)$ *is a semigroup and the left zero semigroup is an identity.*

3. Selective Groupoids Induced by Fuzzy Subsets

Let $\mu : X \to [0,1]$ be a fuzzy subset. Define a binary operation " $\odot$ " on X by

$$x \odot y := \begin{cases} x & \text{if } \mu(x) > \mu(y), \\ y & \text{if } \mu(x) \leq \mu(y). \end{cases} \tag{2}$$

for all $x, y \in X$. We call such a groupoid $(X, \odot)$ a *selective groupoid induced by a fuzzy subset* μ, and we denote it by $(X, \odot)_\mu$.

Theorem 3.1. *Every selective groupoid induced by a fuzzy subset is a pogroupoid.*

Proof. Let $(X, \odot)_\mu$ be a selective groupoid induced by a fuzzy subset $\mu : X \to [0,1]$. Clearly, we have $x \odot y \in \{x, y\}$ for any $x, y \in X$. Consider $x \odot (y \odot x)$ and $y \odot x$. If $\mu(y) > \mu(x)$, then $x \odot y = y$ and $y \odot x = y$, and hence $x \odot (y \odot x) = x \odot y = y = y \odot x$. If $\mu(y) \leq \mu(x)$, then $x \odot y = x$ and $y \odot x = x$, and hence $x \odot (y \odot x) = x \odot x = x = y \odot x$.

Consider $(x \odot y) \odot z$ and $(x \odot y) \odot (y \odot z)$. We need to check 12 cases for proving that $(x \odot y) \odot z = (x \odot y) \odot (y \odot z)$. (1). If $\mu(z) < \mu(y) < \mu(x)$, then $(x \odot y) \odot (y \odot z) = x \odot y = x$ and $(x \odot y) \odot z = x \odot z = x$. (2). If $\mu(y) < \mu(z) < \mu(x)$, then $(x \odot y) \odot (y \odot z) = x \odot z = x$ and $(x \odot y) \odot z = x \odot z = x$. (3). If $\mu(z) < \mu(x) < \mu(y)$, then $(x \odot y) \odot (y \odot z) = y \odot y = y$ and $(x \odot y) \odot z = y \odot z = y$. (4). If $\mu(x) < \mu(z) < \mu(y)$, then $(x \odot y) \odot (y \odot z) = y \odot y = y$ and $(x \odot y) \odot z = y \odot z = y$. (5). If $\mu(y) < \mu(x) < \mu(z)$, then $(x \odot y) \odot (y \odot z) = x \odot z = z$ and $(x \odot y) \odot z = x \odot z = z$. (6). If $\mu(x) < \mu(y) < \mu(z)$, then $(x \odot y) \odot (y \odot z) = y \odot z = z$ and $(x \odot y) \odot z = y \odot z = z$. (7). If $\mu(z) < \mu(x) = \mu(y)$, then $(x \odot y) \odot (y \odot z) = y \odot y = y$ and $(x \odot y) \odot z = y \odot z = y$. (8). If $\mu(z) = \mu(y) < \mu(x)$, then $(x \odot y) \odot (y \odot z) = x \odot z = x$ and $(x \odot y) \odot z = x \odot z = x$. (9). If $\mu(y) < \mu(x) = \mu(z)$, then $(x \odot y) \odot (y \odot z) = x \odot z = z$ and $(x \odot y) \odot z = x \odot z = z$. (10). If $\mu(x) = \mu(z) < \mu(y)$, then $(x \odot y) \odot (y \odot z) = y \odot y = y$ and $(x \odot y) \odot z = y \odot z = y$. (11). If $\mu(x) < \mu(y) = \mu(z)$, then $(x \odot y) \odot (y \odot z) = y \odot z = z$ and $(x \odot y) \odot z = y \odot z = z$. (12). If $\mu(x) = \mu(y) < \mu(z)$, then $(x \odot y) \odot (y \odot z) = y \odot z = z$ and $(x \odot y) \odot z = y \odot z = z$. This proves that $(x \odot y) \odot z = (x \odot y) \odot (y \odot z)$ holds. Hence $(X, \odot)_\mu$ is a pogroupoid. □

As we have seen [8], if we define a relation " $\leq_\mu$ " on X by

$$x \leq_\mu y \iff y \odot x = y = x \odot y \tag{3}$$

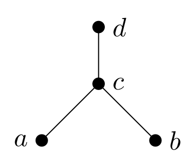
where $(X, \odot)_\mu$ is a pogroupoid induced by μ. Then $(X, \leq_\mu)$ is a poset, called an *induced poset by a fuzzy subset* μ.

Note that, using the notion of the pogroupoid, we can construct a partially ordered set on a set X by a fuzzy subset $\mu : X \to [0, 1]$.

Example 3.2. Let $X := \{a, b, c, d\}$ be a set. Define a map $\mu : X \to [0, 1]$ by $\mu(a) = \mu(b) < \mu(c) < \mu(d)$. Then we can construct a selective groupoid $(X, \odot)_\mu$ induced by μ as below.

$\odot$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	c	d
d	d	d	d	d

Using formula (3) we obtain the poset $(X, \leq_\mu)$ induced by a fuzzy subset μ as below.

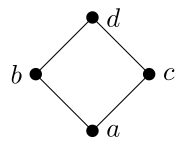


$(X, \leq_\mu)$

If we define another fuzzy subset $\nu : X \to [0,1]$ by $\nu(a) < \nu(b) = \nu(c) < \nu(d)$, then we obtain the selective groupoid $(X, \odot)_\nu$ induced by ν is as follows.

$\odot$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	b	c	d
d	d	d	d	d

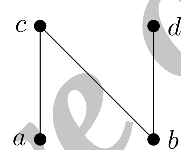
Using formula (3) we obtain the poset $(X, \leq_\nu)$ induced by a fuzzy subset ν as below.



$(X, \leq_\nu)$

In fact, $b \odot a = a \odot b = b$ implies $a \leq_\nu b$, $c \odot a = c \odot a = c$ implies $a \leq_\nu c$. Moreover, $d \odot \alpha = \alpha \odot d = d$ for all $\alpha \in X$ shows that $\alpha \leq_\nu d$.

Example 3.3. Let $(X, \leq)$ be a poset with the following diagram:

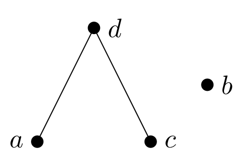


$(X, \leq)$

Then its associated pogroupoid $(X, \cdot)$ is as follows:

$\cdot$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	c	c	c	d
d	a	d	c	d

Define a map $\mu : X \to [0,1]$ by $\mu(a) = \mu(b) < \mu(c) < \mu(d)$ as in Example 3.2. Then $\mu(d \cdot c) = \mu(c) < \mu(d) = \mu(c \cdot d)$, i.e, $d \rhd_\mu c$, and $\mu(d \cdot a) = \mu(a) < \mu(d) = \mu(a \cdot d)$, i.e, $d \rhd_\mu a$. Moreover, $\mu(a \cdot c) = \mu(c \cdot a), \mu(a \cdot b) = \mu(b \cdot a), \mu(b \cdot c) = \mu(c \cdot b)$ and $\mu(d \cdot b) = \mu(b \cdot d)$. Using the fact we obtain a poset $(X, \rhd_\mu)$ induced by μ as follows.



$(X, \rhd_\mu)$

Since the poset $(X, \leq)$ is not $(C_2 + \underline{1})$-free, we can not apply Theorem 2.1. But in this special case, i.e., for some particularly defined fuzzy subset μ, we could be able to obtain the poset $(X, \rhd_\mu)$.

4. Structures of Frameworks by Fuzzy Subsets

A d-algebra (or BCK-algebra) $(X, *, 0)$ is said to be *interval* if $X = [0, 1]$. Given an interval d-algebra $(X, *, 0)$, we define a binary operation "$\circledast$" by $(u_1, u_2) \circledast (v_1, v_2) := (u_1 * v_1, u_2 * v_2)$ on $X \times X (= X^2)$. It is easy to see that $(X^2, \circledast, (0, 0))$ is also a d-algebra, called a *product interval d-algebra*.

Given an interval d-algebra $(X, *, 0)$, a map $F : X^2 \to [0, 1]$ is said to be *d-order preserving* on $(X, *, 0)$ if $(u_1, u_2) \circledast (v_1, v_2) := (0, 0)$ implies $F(u_1, v_1) * F(u_2, v_2) = 0$. Note that every constant function is always d-order preserving, since $x * x = 0$ for all $x \in [0, 1]$.

Example 4.1. Let $(X, *, 0)$ be an interval d-algebra. If we define a map $F : X^2 \to [0, 1]$ by $F(x, y) := x$, then it is easy to see that F is d-order preserving.

Example 4.2. If we define a binary operation "$*$" by $x * y := \max\{0, x - y\}$ on $X = [0, 1]$, then $(X, *, 0)$ is a d-algebra. Define a map $F : X^2 \to [0, 1]$ by $F(x, y) := \frac{1}{2}(x + y)$. If $(x, y) \circledast (\alpha, \beta) = (0, 0)$, then $x * \alpha = 0 = y * \beta$, i.e., $\max\{0, x - \alpha\} = 0 = \max\{0, y - \beta\}$. This shows that $x \leq \alpha,\ y \leq \beta$. It follows that

$$\begin{aligned} F(x, y) * F(\alpha, \beta) &= \max\{0,\ F(x, y) - F(\alpha, \beta)\} \\ &= \max\{0,\ \frac{1}{2}(x + y) - \frac{1}{2}(\alpha + \beta)\} \\ &= \max\{0,\ \frac{1}{2}(x - \alpha) + \frac{1}{2}(y - \beta)\} \\ &= 0 \end{aligned}$$

Hence F is d-order preserving on $(X, *, 0)$.

Let $\mu : X \to [0, 1]$ be a fuzzy subset of X. A *framework* for μ consists of a triple (X, I, F) where (i) $(X, \cdot)$ is a groupoid; (ii) $(I = [0, 1], *, 0)$ is an interval d-algebra; (iii) F is d-order preserving on $(I, *, 0)$ such that $F(\mu(z \cdot y), \mu(y \cdot x)) = \mu(z \cdot x) = F(\mu(y \cdot x), \mu(z \cdot y))$ for all $x, y, z \in X$.

Given a framework (X, I, F) for the fuzzy subset $\mu : X \to [0, 1]$ of X, we define a binary relation "$\lhd_\mu$" by, for all $x, y \in X$,

$$x \lhd_\mu y \iff \mu(y \cdot x) * \mu(x \cdot y) = 0 \tag{4}$$

Remark 4.3. Even though $(X, \rhd_\mu)$ in Example 3.3 is a poset, the relation "$\rhd_\mu$" defined in Theorem 2.1 may not hold for the transitivity relation in general, since there are many posets of height ≥ 2 which are not $(C_2 + \underline{1})$-free. Using the notion of the framework, we discuss the transitivity relation.

Note that a quasi order relation is reflexive and transitive.

Proposition 4.4. *If (X, I, F) is a framework for the fuzzy subset μ, then $(X, \lhd_\mu)$ is a quasi ordered set.*

Proof. Since $(I, *, 0)$ is an interval d-algebra, if we let $x = y$, then $\mu(y\cdot x)*\mu(x\cdot y) = \mu(x \cdot x) * \mu(x \cdot x) = 0$, proving that $x \lhd_\mu x$ for all $x \in X$.

If $x \lhd_\mu y$ and $y \lhd_\mu z$, then $\mu(y \cdot x) * \mu(x \cdot y) = 0, \mu(z \cdot y) * \mu(y \cdot z) = 0$. It follows that $(\mu(y \cdot x), \mu(z \cdot y)) \circledast (\mu(x \cdot y), \mu(y \cdot z)) = (0, 0)$. Since F is d-order preserving, we obtain $\mu(z \cdot x) * \mu(x \cdot z) = F(\mu(y \cdot x), \mu(z \cdot y)) * F(\mu(x \cdot y), \mu(y \cdot z)) = 0$. This proves that $x \lhd_\mu z$. □

Given groupoids $(X, \cdot)$ and $(Y, \star)$, we consider a groupoid homomorphism $\varphi : (X, \cdot) \to (Y, \star)$. For any fuzzy subset $\mu : (Y, \star) \to [0, 1]$, we define a map $\mu^\varphi : (X, \cdot) \to [0, 1]$ by $\mu^\varphi(x) := \mu(\varphi(x))$ for all $x \in X$.

Given a fuzzy subset $\mu : X \to [0, 1]$, we define a set $\Omega(\mu)$ by

$$\Omega(\mu) := \{(X, \cdot) \in Bin(X) \,|\, \mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in X\} \tag{5}$$

It can be easily seen that that $(\Omega(\mu), \square)$ forms a subsemigroup of the semigroup $(Bin(X), \square)$.

Example 4.5. (a). If $(X, \cdot)$ is the left zero semigroup, i.e., $x\cdot y = x$ for all $x, y \in X$, then $\mu(x \cdot y) = \mu(x) \geq \min\{\mu(x), \mu(y)\}$ for all fuzzy subset μ, i.e., $(X, \cdot) \in \Omega(\mu)$ for all fuzzy subset $\mu \in [0, 1]^X$.

(b). If $X := \mathbf{R}$, the real numbers, then $\mu(x) := \exp^{-x^2}$ has $\mu(X) = (0, 1]$, and thus if $(X, \cdot) \in \Omega(\mu)$ then $\mu(x * y) = \exp^{-(x*y)^2} \geq \min\{\exp^{-x^2}, \exp^{-y^2}\}$. It follows that if $|x*y| \leq \min\{|x|, |y|\}$, then $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$, with many choices possible for product $x * y$. One such choice might be $x * y := \frac{xy}{|(x+1)(y+1)|}$ for example. Thus $2 * \frac{1}{2} = \frac{2}{9} < \frac{1}{2} = \min\{2, \frac{1}{2}\}$.

Proposition 4.6. *If $(Y, \star) \in \Omega(\mu)$ and $\varphi : (X, \cdot) \to (Y, \star)$ is a groupoid homomorphism, then $(X, \cdot) \in \Omega(\mu^\varphi)$.*

Proof. Given $x, y \in X$, since $(Y, \star) \in \Omega(\mu)$, we have

$$\begin{aligned} \mu^\varphi(x \cdot y) &= \mu(\varphi(x \cdot y)) \\ &= \mu(\varphi(x) \star \varphi(y)) \\ &\geq \min\{\mu(\varphi(x)), \mu(\varphi(y))\} \\ &= \min\{\mu^\varphi(x), \mu^\varphi(y)\}. \end{aligned}$$

This shows that $(X, \cdot) \in \Omega(\mu^\varphi)$. □

Proposition 4.7. *Let $\varphi : (X, \cdot) \to (Y, \star)$ be a groupoid homomorphism and let $\mu : (Y, \star) \to [0, 1]$ be a fuzzy subset. If (Y, I, F) is a framework for μ, then (X, I, F) is a framework for μ^φ.*

Proof. Since (Y, I, F) is a framework for μ, we have $F(\mu(z \star y), \mu(y \star x)) = \mu(z \star x)$ for all $x, y, z \in Y$. It follows that

$$\begin{aligned} F(\mu^\varphi(c \cdot b), \mu^\varphi(b \cdot a)) &= F(\mu(\varphi(c \cdot b)), \mu(\varphi(b \cdot a))) \\ &= F(\mu(\varphi(c) \star \varphi(b)), \mu(\varphi(b) \star \varphi(a))) \\ &= \mu(\varphi(c) \star \varphi(a)) \\ &= \mu(\varphi(c \cdot a)) \\ &= \mu^\varphi(c \cdot a), \end{aligned}$$

which shows that (X, I, F) is a framework for μ^φ. □

Proposition 4.8. *Let $\varphi : (X, \cdot) \to (Y, \star)$ be a groupoid homomorphism and let $\mu : (Y, \star) \to [0, 1]$ be a fuzzy subset. If (Y, I, F) is a framework for μ, then $\varphi : (X, \lhd_{\mu^\varphi}) \to (Y, \lhd_\mu)$ is order preserving, i.e., $a \lhd_{\mu^\varphi} b$ implies $\varphi(a) \lhd_\mu \varphi(b)$.*

Proof. If $a \lhd_{\mu^\varphi} b$, then $\mu^\varphi(b \cdot a) * \mu^\varphi(a \cdot b) = 0$. It follows that $0 = \mu(\varphi(b \cdot a)) * \mu(\varphi(a \cdot b)) = \mu(\varphi(b) \star \varphi(a)) * \mu(\varphi(a) \star \varphi(b))$. This shows that $\varphi(a) \lhd_\mu \varphi(b)$. □

The converse of Proposition 4.8 holds for any onto homomorphism of groupoids $\varphi : (X, \cdot) \to (Y, \star)$.

Proposition 4.9. *Let $\varphi : (X, \cdot) \to (Y, \star)$ be an onto homomorphism of groupoids and let $\mu : (Y, \star) \to [0, 1]$ be a fuzzy subset. If (Y, I, F) is a framework for μ, then $\varphi : (X, \lhd_{\mu^\varphi}) \to (Y, \lhd_\mu)$ is order reversing, i.e., $\varphi(a) \lhd_\mu \varphi(b)$ implies $a \lhd_{\mu^\varphi} b$.*

Proof. Let $\varphi(a)$ and $\varphi(b)$ be arbitrary elements of $(Y, \star)$ such that $\varphi(a) \lhd_\mu \varphi(b)$. Then $\mu(\varphi(b \cdot a)) * \mu(\varphi(a \cdot b)) = \mu(\varphi(b) \star \varphi(a)) * \mu(\varphi(a) \star \varphi(b)) = 0$. It follows that $\mu(\varphi(b \cdot a)) * \mu(\varphi(a \cdot b)) = 0$, i.e., $a \lhd_{\mu^\varphi} b$, proving the proposition. □

Given a groupoid homomorphism $\varphi : (X, \cdot) \to (Y, \star)$ and a fuzzy subset $\mu : (X, \cdot) \to [0, 1]$, we define a map $\mu_\varphi : (Y, \star) \to [0, 1]$ by $\mu_\varphi(\varphi(a)) := \sup\{\mu(b) \mid \varphi(a) = \varphi(b)\}$.

Theorem 4.10. *Let $\varphi : (X, \cdot) \to (Y, \star)$ be an onto groupoid homomorphism. If $(X, \cdot) \in \Omega(\mu)$, then $(Y, \star) \in \Omega(\mu_\varphi)$.*

Proof. Given $\alpha, \beta \in Y$, since φ is onto, there exist $x, y \in X$ such that $\alpha = \varphi(x), \beta = \varphi(y)$. It follows that

$$\begin{aligned}
\mu_\varphi(\alpha \star \beta) &= \mu_\varphi(\varphi(x) \star \varphi(y)) \\
&= \mu_\varphi(\varphi(x \cdot y)) \\
&= \sup\{\mu(z) | \varphi(z) = \varphi(x \cdot y)\} \\
&\geq \sup\{\mu(p \cdot q) | \varphi(p \cdot q) = \varphi(x \cdot y)\} \\
&\geq \sup\{\mu(p \cdot q) | \varphi(p) = \varphi(x), \varphi(q) = \varphi(y)\}
\end{aligned}$$

Since $\mu_\varphi(\alpha) = \mu_\varphi(\varphi(x)) = \sup\{\mu(z) | \varphi(z) = \varphi(x)\}$, there exists $p_0 \in X$ such that $\mu(p_0) > \mu_\varphi(\alpha) - \epsilon$ and $\varphi(p_0) = \varphi(x)$ for any $\epsilon > 0$. Similarly, there exists $q_0 \in X$ such that $\mu(q_0) > \mu_\varphi(\beta) - \epsilon$ and $\varphi(q_0) = \varphi(y)$ for any $\epsilon > 0$. It follows that

$$\begin{aligned}
\mu(p_0 \cdot q_0) &\geq \min\{\mu(p_0), \mu(q_0)\} \\
&> \min\{\mu_\varphi(\alpha), \mu_\varphi(\beta)\} - \epsilon
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\mu_\varphi(\alpha \star \beta) &\geq \sup\{\mu(p \cdot q) \,|\, \varphi(p) = \varphi(x), \varphi(q) = \varphi(y)\} \\
&\geq \sup\{\min\{\mu(p), \mu(q)\} \,|\, \varphi(p) = \varphi(x), \varphi(q) = \varphi(y)\} \\
&\geq \sup\{\min\{\mu_\varphi(\alpha), \mu_\varphi(\beta)\} - \epsilon \,|\, \varphi(p) = \varphi(x), \varphi(q) = \varphi(y)\} \\
&= \min\{\mu_\varphi(\alpha), \mu_\varphi(\beta)\} - \epsilon
\end{aligned}$$

for any $\epsilon > 0$. It follows that $\mu_\varphi(\alpha * \beta) \geq \min\{\mu_\varphi(\alpha), \mu_\varphi(\beta)\}$, proving that $(Y, \star) \in \Omega(\mu_\varphi)$. □

Corollary 4.11. *Let $\varphi : (X, \cdot) \to (Y, \star)$ be an onto groupoid homomorphism. If $(Y, \star) \in \Omega(\mu_\varphi)$, then $(X, \cdot) \in \Omega((\mu_\varphi)^\varphi)$.*

Proof. It follows immediately from Proposition 4.6. □

Note that the mapping $(\mu_\varphi)^\varphi$ is a constant function on inverse image $\varphi^{-1}(y), y \in Y$. In fact, assume that $\varphi(a) = \varphi(b) = y$. Then we have $(\mu_\varphi)^\varphi(a) = \mu_\varphi(\varphi(a)) = \sup\{\mu(x) \mid \varphi(x) = \varphi(a)\} = \sup\{\mu(x) \mid \varphi(x) = \varphi(b)\} = (\mu_\varphi)^\varphi(b)$.

Theorem 4.12. *Given a groupoid homomorphism $\varphi : (X, \cdot) \to (Y, \star)$ and a fuzzy subset $\mu : (Y, \star) \to [0, 1]$, we have*

(i). $[(\mu^\varphi)_\varphi]^\varphi = \mu^\varphi$,

(ii). $[(\mu_\varphi)^\varphi]_\varphi = \mu_\varphi$ *if φ is onto.*

Proof. (i). If we let $\xi := \mu^\varphi$, then, for all $a \in X$, we have

$$\begin{aligned}
[(\mu^\varphi)_\varphi]^\varphi(a) &= (\xi_\varphi)(\varphi(a)) \\
&= \sup\{\xi(b) \mid \varphi(b) = \varphi(a)\} \\
&= \sup\{(\mu^\varphi)(b) \mid \varphi(b) = \varphi(a)\} \\
&= \sup\{\mu(\varphi(a))\} \\
&= \mu(\varphi(a)) = \mu^\varphi(a).
\end{aligned}$$

(ii). Assume that φ is onto and assume that $\alpha := (\mu_\varphi)^\varphi$. Given $a \in X$, we have

$$\begin{aligned}
[(\mu_\varphi)^\varphi]_\varphi(\varphi(a)) &= \sup\{\alpha(b) \mid \varphi(b) = \varphi(a)\} \\
&= \sup\{(\mu_\varphi)^\varphi \mid \varphi(b) = \varphi(a)\} \\
&= \sup\{\mu_\varphi(\varphi(b)) \mid \varphi(b) = \varphi(a)\} \\
&= \sup\{\sup\{\mu(z) \mid \varphi(z) = \varphi(b)\} \mid \varphi(b) = \varphi(a)\} \\
&= \sup\{\mu(z) \mid \varphi(z) = \varphi(a)\} \\
&= \mu_\varphi(\varphi(a)).
\end{aligned}$$

This proves that $[(\mu_\varphi)^\varphi]_\varphi = \mu_\varphi$. □

5. **Conclusion**

In this paper we discussed some ideas associated with posets, fuzzy subsets as fuzzy groupoids and pogroupoids. The culminating notion here is probably that of a framework for a quasi ordered set which provides a very flexible structure incorporating the poset notion in several ways, and thus making order concepts accessible to many contexts.

It should be noted that if D is the distribution function of any real random variable on $\mathbf{R}$, the real numbers, then as a fuzzy subset of $\mathbf{R}$ itself, the theory developed above can then be introduced and help to yield new approaches to statistics/probability theory which may ultimately turn out to be useful.

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SELECTIVE GROUPOIDS AND FRAMEWORKS INDUCED BY FUZZY SUBSETS

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گروه واره های انتخابی و چارچوب های القایی توسط زیر مجموعه های فازی

چکیده. در این مقاله، نشان می دهیم که هر گروه وار انتخابی القایی توسط یک زیر مجموعه فازی ، یک گروه وار مرتب جزئی است و با معرفی مفهوم یک چارچوب ، خواص متعددی در مجموعه های شبه مرتب را مورد بحث قرار می دهیم.