ITERATIVE METHOD FOR SOLVING TWO-DIMENSIONAL NONLINEAR FUZZY INTEGRAL EQUATIONS USING FUZZY BIVARIATE BLOCK-PULSE FUNCTIONS WITH ERROR ESTIMATION

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Abstract. In this paper, we propose an iterative procedure based on two dimensional fuzzy block-pulse functions for solving nonlinear fuzzy Fredholm integral equations of the second kind. The error estimation and numerical stability of the proposed method are given in terms of supplementary Lipschitz condition. Finally, illustrative examples are included in order to demonstrate the accuracy and convergence of the proposed method.

1. Introduction

Block pulse functions are a set of orthogonal functions with piecewise constant value and usually applied as a useful tool in the analysis, synthesis identification and other problems of control and systems sciences [26]. The block pulse functions has been frequently used in several papers to approximate solution of differential equations, integral and integro-differential equations in the crisp case.
One of the most interesting research matters in fuzzy sets and systems is to study fuzzy integral equations. The study of fuzzy integral equations from theoretical and practical aspects has been developed by some authors. The investigation of the existence and uniqueness of the solution for fuzzy integral equations has been carried out in [6, 7, 8, 28, 29, 32]. The Banach fixed point theorem is the main tool in studying the existence and uniqueness of the solution for fuzzy integral equations which can appear in numerical procedures for solving fuzzy integral equations, based on the iterative techniques. The iterative numerical methods for solving fuzzy integral equations can be found in [8, 9, 10, 20, 21, 22]. The Nyström technique, Adomian decomposition method, fuzzy Bernstein polynomials and fuzzy Haar wavelet were applied to solve the fuzzy integral equations of the second kind in [1, 4, 15, 35]. Bica and Popescu, in [10], applied the method of successive approximations for solving the fuzzy Hammerstein integral equation. Ezzati and Ziari in [16], proved the convergence of the method of successive approximations for solving nonlinear fuzzy Fredholm integral equations of the second kind, and proposed an iterative procedure based on the trapezoidal quadrature. Recently, Baghmisheh and Ezzati in [5], approximated the fuzzy function by the hybrid Taylor and block-pulse functions

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and estimated the error approximation. Also, an iterative procedure is constructed based on the hybrid Taylor and block-pulse functions for solving nonlinear Fredholm fuzzy integral equations by them. Recently, in [36], Ziari and Bica obtained the new error estimation of the iterative method based on the trapezoidal formula for solving nonlinear fuzzy Hammerstein-Fredholm integral equations of the second kind in terms of uniform and partial modulus of continuity. Moreover, they extended the notion of numerical stability of the solution with respect to the first iteration in the context of using the modulus of continuity. The study of numerical solution of two-dimensional nonlinear fuzzy Fredholm integral equations of the second kind using iterative methods based on successive approximations and two dimensional quadrature rules started by Sadat Rasoul and Ezzati, in [30]. Also, Ezzati and Ziari in [17] proposed a non-iterative numerical method for two-dimensional fuzzy Fredholm integral equations based on Bernstein polynomials. Sadat Rasoul and Ezzati in [31] presented an iterative method of successive approximations to approximate solution of linear and nonlinear two-dimensional Hammerstein fuzzy integral equations by defining and developing an optimal quadrature formula for classes of two-dimensional fuzzy-number-valued functions of Lipschitz type. In [11], Bica and Popescu constructed the fuzzy trapezoidal cubature rule for the case of Lipschitzian functions. As an application, they proposed an iterative numerical method in order to approximate the solution of nonlinear fuzzy Fredholm integral equations in two variables, the fuzzy cubature rule being used in the construction of the numerical method. Recently, Ezzati and Sadatrasoul in [18] applied the bivariate fuzzy Bernstein polynomials to solve two-dimensional fuzzy integral equations. Also, recently, in [12], Bica and Ziari proposed an iterative numerical method for solving fuzzy Volterra linear integral equations in two dimensions. In this paper, we approximate the integral of fuzzy function in a two dimensional case by the bivariate fuzzy block-pulse functions and estimate its error approximation. Moreover, an iterative process is constructed based on two dimensional fuzzy block-pulse functions for solving two dimensional nonlinear Fredholm fuzzy integral equations,

$$F(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t) \odot G(F(s,t))dsdt. \tag{1}$$

In addition the convergence of the presented successive approximations method is proved, and the accuracy of the proposed method is shown through illustrative examples. Finally, the concluding remarks are presented.

2. Preliminaries

Definition 2.1. [14] A fuzzy number is a function $u : R \to [0,1]$ having the properties:

(1) u is normal , that is $\exists x_0 \in R$ such that $u(x_0) = 1$,
(2) u is fuzzy convex set
($i.e.$ $u(\lambda x + (1-\lambda)y) \geq \min u(x), u(y)$, for any $x, y \in R$ and $\lambda \in [0,1]$),
(3) u is upper semicontinuous on R,
(4) the $\overline{\{\, x \in R : u(x) > 0\}}$ is compact set.

The set of all fuzzy numbers is denoted by R_F.
The LU-representation of a fuzzy number is specified by its level sets $[u]^r = \{x \in R : u(x) \geq r\}$ as follows:

$$[u]^r = [u_-^r, u_+^r], \quad \forall r \in (0,1],$$

where u_-^r, u_+^r can be considered as functions $u_-^r, u_+^r : [0,1] \to \mathbb{R}$, such that u_- is increasing and u_+ is decreasing (according to [24], page 32). Moreover, $[u]^0 = \overline{\{x \in R : u(x) > 0\}}$. So, the level sets of a fuzzy number $[u]^r$, $r \in [0,1]$, are compact intervals of the real axis. For $u, v \in R_F$, $k \in \mathbb{R}$, the addition and the scalar multiplication based on levelsetwise are defined as follows

(1) $[u \oplus v]^r = [u_-^r + v_-^r, u_+^r + v_+^r]$, $\forall r \in [0,1]$,

(2) $[k \odot u]^r = \begin{cases} [ku_-^r, ku_+^r], & \text{if } k \geq 0, \\ \\ [ku_+^r, ku_-^r], & \text{if } k < 0. \end{cases}$

Theorem 2.2. *(Stacking Theorem,* [25]. *A fuzzy number u satisfies the following conditions:*

(1) *its r-cuts are nonempty closed intervals, for all $r \in [0,1]$;*
(2) *if $0 \leq r_1 \leq r_2 \leq 1$, then $[u]^{r_2} \subset [u]^{r_1}$;*
(3) *for any nondecreasing sequence (r_n) in $[0,1]$ converging to $r \in (0,1]$ we have:*

$$\bigcap_{n=1}^{\infty} [u]^{r_n} = [u]^r;$$

(4) *for any nonincreasing sequence (r_n) in $[0,1]$ converging to zero we have:*

$$cl\left(\bigcup_{n=1}^{\infty} [u]^{r_n}\right) = [u]^0;$$

Theorem 2.3. *(Characterization Theorem,* [25]. *If $\{[u]^r : r \in [0,1]\}$ is a family of subsets of $\mathbb{R}$
and assumptions (1)-(4) of the above theorem are hold then there exists a unique fuzzy number u
such that $\{[u]^r : r \in [0,1]\}$ are its r-cuts.*

Definition 2.4. (See [10, 34]). For arbitrary fuzzy numbers u, $v \in R_F$, the quantity

$$D(u,v) = \sup_{r \in [0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$$

is the distance between u and v.

Lemma 2.5. [34] *The following properties are hold:*

(1) $D(u \oplus w, v \oplus w) = D(u,v)\ \forall\ u, v, w \in R_F$,
(2) $D(k \odot u, k \odot v) = |k|\, D(u,v)\ \forall\ u, v \in R_F\ \forall k \in R$,
(3) $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e)\ \forall\ u, v, w, e \in R_F$,
(4) $D(u \oplus v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0}),\ \forall\ u, v \in R_F$,

In [34], it is proved that (R_F, D) is a complete metric space.

Lemma 2.6. *(see [3]). For any $k_1, k_2 \in \mathbb{R}$ with $k_1 \cdot k_2 \geq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have $D(k_1 \cdot u, k_2 \cdot u) = |k_1 - k_2| D\left(u, \widetilde{0}\right)$.*

Remark 2.7. The properties (4) in Lemma 1 introduce the definition of a function $\|.\| : R_F \to R^+$ by $\|u\| = D(u, \tilde{0})$, which has the properties of the usual norms. In [5] the properties of this function are presented as follows:
(i) $\|u\| \geq 0, \ \forall\, u \in R_F$, and $\|u\| = 0$ iff $u = \tilde{0}$,
(ii) $\|\lambda.u\| = |\lambda|\|u\|$ and $\|u \oplus v\| \leq \|u\| + \|v\|, \ \forall\, u, v \in R_F, \ \forall\, \lambda \in R$,
(iii) $|\|u\| - \|v\|| \leq D(u, v)$ and $D(u, v) \leq \|u\| + \|v\| \ \forall\, u, v \in R_F$.

Definition 2.8. [19] A fuzzy real number valued function $f : [a, b] \to R_F$ is said to be continuous in $x_0 \in [a, b]$, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $D(f(x), f(x_0)) < \varepsilon$, whenever $x \in [a, b]$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a, b]$, and denote the space of all such functions by $C_F[a, b]$.

Definition 2.9. [8] Let $f : [a, b] \to R_F$. f is fuzzy-Riemann integrable to $I \in R_F$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u, v]\,; \xi\}$ of $[a, b]$ with the norms $\Delta(p) < \delta$, we have

$$D\left(\sum_P (v - u) \odot f(\xi), I\right) < \varepsilon,$$

where $\sum$ denotes the fuzzy summation. In this case it is denoted by

$$I = (FR) \int_a^b f(x)dx.$$

Lemma 2.10. [8, 23] *If $f, \ g : [a, b] \subseteq R \to R_F$ are fuzzy continuous functions, then the function $F : [a, b] \to R_+$ by $F(x) = D(f(x), g(x))$ is continuous on $[a, b]$ and*

$$D\left((FR) \int_a^b f(x)dx, (FR) \int_a^b g(x)dx\right) \leq \int_a^b D(f(x), g(x))dx.$$

Definition 2.11. (see [8]). For $L \geq 0$, a function $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is L-Lipschitz if

$$D(f(x), f(y)) \leq L\,|x - y|$$

for any $x, y \in [a, b]$. A function $F : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is Lipschitz if there exists $L' \geq 0$ such that $D(F(u), F(v)) \leq L' \cdot D(u, v)$, for any $u, v \in \mathbb{R}_{\mathcal{F}}$.

According to [8], any Lipschitz function is continuous.

Definition 2.12. (see [26]). Block-pulse functions on the unit interval $[0, 1)$ is defined as follows:

$$\phi_i(t) = \begin{cases} 1 & t \in [\frac{i-1}{m}, \frac{i}{m}), \\ 0 & otherwise, \end{cases} \tag{2}$$

where $i = 1, 2, ..., m$ with a positive integer value for m. Also, ϕ_i is called i^{th} block-pulse function (BPF).

The BPFs satisfy in the properties of disjointness, orthogonality and completeness [16].
Now, we defined the fuzzy block-pulse function like operator as follows:

Definition 2.13. For $f \in C_F[0,1)$, the fuzzy block-pulse function like operator defined by

$$\Phi_m^{(F)}(f(t)) = \sum_{i=1}^{m} f\left(\frac{i-0.5}{m}\right) \odot \phi_i(t)\ ,\ m \in N\ ,\ t \in [0,1)$$

where $\phi_i(t)$ defined by (2).
It is obvious that $\phi_i(t) \geq 0,\ i = 1,...,m$ for all $t \in [0,1)$, $\{\phi_i\}_{i=1}^m$ are linearly independent, and

$$\sum_{i=1}^{m} \phi_i(t) = 1.$$

The following definitions are related to fuzzy-number-valued functions in two variables.

Definition 2.14. (see [30]). Let $f : [a,b] \times [c,d] \to \mathbb{R}_{\mathcal{F}}$. For $\Delta_n^x : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ a partition of the interval $[a,b]$ and $\Delta_m^y : c = y_0 < y_1 < ... < y_{m-1} < y_m = d$ a partition of the interval $[c,d]$, let us consider the intermediate points $\xi_i \in [x_{i-1}, x_i]\,,\ i = \overline{1,n}$, $\eta_j \in [y_{j-1}, y_j]\,,\ j = \overline{1,m}$, and the functions $\delta : [a,b] \to \mathbb{R}_+$. $\sigma : [c,d] \to \mathbb{R}_+$. The partitions $P_x = \{([x_{i-1}, x_i]; \xi_i),\ i = \overline{1,n}\}$ denoted by $P_x = (\Delta_n^x, \xi)$ and $P_y = \{([y_{j-1}, y_j]; \eta_j),\ j = \overline{1,m}\}$ denoted by $P_y = (\Delta_m^y, \eta)$ are said to be δ-fine iff $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))\,, \forall i = \overline{1,n}$, and σ-fine iff $[y_{j-1}, y_j]$
$\subseteq (\eta_j - \sigma(\eta_j), \eta_j + \sigma(\eta_j))\,, \forall j = \overline{1,m}$, respectively. The function f is said to be fuzzy Henstock integrable if there exists $I(f) \in \mathbb{R}_{\mathcal{F}}$ with the property that for any $\epsilon > 0$ there is a function $\delta : [a,b] \to \mathbb{R}_+$ and a function $\sigma : [c,d] \to \mathbb{R}_+$ such that for any partition δ-fine P_x, and for any partition σ-fine P_y, we have

$$D\left(\sum_{i=1}^{n}\sum_{j=1}^{m} (x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j), I\right) < \epsilon.$$

The fuzzy number I is named the fuzzy Henstock double integral of f and will be denoted by

$$I(f) = (FH)\int_c^d (FH)\left(\int_a^b f(s,t)\,ds\right)dt.$$

Remark 2.15. If the above mentioned functions δ and σ are constant then it obtains the fuzzy-Riemann double integrability. In this case $I(f) \in \mathbb{R}_{\mathcal{F}}$ is called the fuzzy-Riemann double integral of f on $[a,b] \times [c,d]$, being denoted by

$$I(f) = (FR)\int_c^d (FR)\left(\int_a^b f(s,t)\,ds\right)dt$$

or simply, $\int_c^d \left(\int_a^b f(s,t)\,ds\right) dt$. Consequently, the fuzzy-Riemann double integrability is a particular case of the fuzzy-Henstock double integrability, and therefore any valid property for the double integral (FH) will be valid for the double integral (FR), too.

Definition 2.16. (see [30]). A function $f : [a,b] \times [c,d] \to \mathbb{R}_{\mathcal{F}}$ is called:
(i) continuous in $(x_0, y_0) \in [a,b] \times [c,d]$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $(x,y) \in [a,b] \times [c,d]$ with $|x - x_0| < \delta$, $|y - y_0| < \delta$ we have

$$D\left(f(x,y), f(x_0,y_0)\right) < \varepsilon.$$

The function f is continuous on $[a,b] \times [c,d]$ if it is continuous in each $(x,y) \in [a,b] \times [c,d]$.
(ii) bounded if there exists $M \geq 0$ such that

$$D\left(f(x,y), \widetilde{0}\right) \leq M, \quad \forall (x,y) \in [a,b] \times [c,d].$$

The set of all continuous functions $f : [a,b] \times [c,d] \to \mathbb{R}_{\mathcal{F}}$ is denoted by

$$C\left([a,b] \times [c,d], \mathbb{R}_{\mathcal{F}}\right).$$

Lemma 2.17. *(see [30]). If $f \in C\left([a,b] \times [c,d], \mathbb{R}_{\mathcal{F}}\right)$ then*

$$(FR)\int_c^d \left((FR)\int_a^b f(s,t)\,ds\right) dt$$

exists and

$$\left[(FR)\int_c^d \left((FR)\int_a^b f(s,t)\,ds\right) dt\right]^r = \left[\int_c^d \int_a^b f_-^r(s,t)\,dsdt, \int_c^d \int_a^b f_+^r(s,t)\,dsdt\right].$$

Lemma 2.18. *If $f, g : [a,b] \times [c,d] \to \mathbb{R}_{\mathcal{F}}$ are continuous fuzzy functions then the function $\varphi : [a,b] \times [c,d] \to \mathbb{R}_+$ defined by $\varphi(s,t) = D\left(f(s,t), g(s,t)\right)$ is continuous on $[a,b] \times [c,d]$ and*

$$D\left((FR)\int_c^d (FR)\left(\int_a^b f(s,t)\,ds\right) dt, (FR)\int_c^d (FR)\left(\int_a^b g(s,t)\,ds\right) dt\right)$$
$$\leq \int_c^d \int_a^b D\left(f(s,t), g(s,t)\right) dsdt.$$

Proof. Firstly, we prove that the function $\varphi(s,t) = D(f(s,t), g(s,t))$ be continuous in every point $(s_0,t_0) \in [a,b] \times [c,d]$, for this purpose, we let the sequence $\{(s_n,t_n)\}_{n\geq 1}$, $(s_n,t_n) \in [a,b] \times [c,d]$, such that $\lim_{n\to\infty}(s_n,t_n) = (s_0,t_0)$.

In this case, we have:

$$\begin{aligned} D\left(f(s_n,t_n), g(s_n,t_n)\right) \leq & D\left(f(s_n,t_n), f(s_0,t_0)\right) + D\left(f(s_0,t_0), g(s_0,t_0)\right) + \\ & + D\left(g(s_0,t_0), g(s_n,t_n)\right), \end{aligned}$$

and on the other hand, we have:

$$\begin{aligned} D\left(f(s_0,t_0), g(s_0,t_0)\right) \leq & D\left(f(s_0,t_0), f(s_n,t_n)\right) + D\left(f(s_n,t_n), g(s_n,t_n)\right) + \\ & + D\left(g(s_n,t_n), g(s_0,t_0)\right). \end{aligned}$$

Taking the limit when $n \to \infty$, and according to the continuity of f and g we obtain:

$$\lim_{n\to\infty} D\left(f(s_n,t_n), g(s_n,t_n)\right) = D\left(f(s_0,t_0), g(s_0,t_0)\right),$$

that is φ is continuous at each $(s_0,t_0) \in [a,b]\times[c,d]$. Now, let $P_n = \{([s_{n-1},s_n];\xi_n)\}$, and $Q_m = \{([t_{m-1},t_m];\eta_m)\},\ m,n \in N$ be two sequences of partitions of $[a,b]$ and $[c,d]$ with $\Delta(P_n) \to 0$, when $n \to \infty$ and $\Delta(Q_m) \to 0$, when $m \to \infty$, respectively. So, we have:

$$\begin{aligned}
D&\left((FR)\int_c^d (FR)\left(\int_a^b f(s,t)\,ds\right)dt, (FR)\int_c^d (FR)\left(\int_a^b g(s,t)\,ds\right)dt\right)\\
&\le D\Bigg((FR)\int_c^d (FR)\left(\int_a^b f(s,t)\,ds\right)dt,\\
&\qquad ,\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i,\eta_j)\Bigg)\\
&+D\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i,\eta_j),\\
&\qquad ,\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \odot g(\xi_i,\eta_j)\Bigg)+\\
&+D\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \odot g(\xi_i,\eta_j),\\
&\qquad ,(FR)\int_c^d (FR)\left(\int_a^b g(s,t)\,ds\right)dt\Bigg)\\
&\le D\Bigg((FR)\int_c^d (FR)\left(\int_a^b f(s,t)\,ds\right)dt,\\
&\qquad ,\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i,\eta_j)\Bigg)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{m} D\left(f(\xi_i,\eta_j), g(\xi_i,\eta_j)\right)(x_i - x_{i-1})(y_j - y_{j-1})\\
&+D\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \odot g(\xi_i,\eta_j),\\
&\qquad ,(FR)\int_c^d (FR)\left(\int_a^b g(s,t)\,ds\right)dt\Bigg)
\end{aligned}$$

Taking the limit when $n,\ m \to \infty$, we get:

$$D\left((FR)\int_c^d (FR)\left(\int_a^b f(s,t)\,ds\right)dt,(FR)\int_c^d (FR)\left(\int_a^b g(s,t)\,ds\right)dt\right)$$
$$\leq \int_c^d \int_a^b D\left(f(s,t),g(s,t)\right)dsdt.$$

Thus, the proof is complete. □

Moreover, it can be easily proved that any continuous function $f:[a,b]\times[c,d]\to\mathbb{R}_{\mathcal{F}}$ is bounded. In [12] is considered a fuzzy-number-valued function $f:[a,b]\times[c,d]\to\mathbb{R}_{\mathcal{F}}$ having the following Lipschitz property: there exist $L_1,L_2\geq 0$ such that

$$D\left(f(x_1,y_1),f(x_2,y_2)\right)\leq L_1|x_1-x_2|+L_2|y_1-y_2| \tag{3}$$

for all $x_1,x_3\in[a,b]$, $y_1,y_2\in[c,d]$. We use the inequality (3) in the proof of presented some theorems in this paper.

Similar to the one dimensional case, a set of two dimensional block pulse functions $\phi_{i,j}(s,t)$ $(i=1,2,...,m;\ j=1,2,...,n)$ is defined as follows.

Definition 2.19. (see [26]). Two dimensional block-pulse functions on the region $[0,1)\times[0,1)$ is defined as follows:

$$\phi_{i,j}(s,t)=\begin{cases}1\ (s,t)\in[\frac{i-1}{m},\frac{i}{m})\times[\frac{j-1}{n},\frac{j}{n}),\\ \\ 0\ otherwise,\end{cases} \tag{4}$$

where $i=1,2,...,m;\ j=1,2,...,n$ with a positive integer values for $m,\ n$. Also, $\phi_{i,j}$ is called $(i,j)^{th}$ block-pulse function .

Similar to 1D case, The 2D block pulse functions satisfy in the properties of disjointness, orthogonality and completeness.[16]

3. Function Approximation

Now, we define the two dimensional fuzzy block-pulse functions as follows:

Definition 3.1. For $f\in C_F\big([0,1)\times[0,1)\big)$, the two dimensional fuzzy block-pulse function like operator for $m,n\in N$ and $s,t\in[0,1)$ defined by

$$\Phi_{m,n}^{(F)}(f(s,t))=\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m},\frac{j-0.5}{n}\right)\ \odot\phi_{i,j}(s,t),$$

where $\phi_{i,j}(s,t)$ defined by (4).

It is obvious that $\phi_{i,j}(s,t) \geq 0$, for all $(s,t) \in [0,1) \times [0,1)$, , $\{\{\phi_{i,j}\}_{i=1}^{m}\}_{j=1}^{n}$ are linearly independent, and

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\phi_{i,j}(s,t) = 1. \tag{5}$$

Thus, the fuzzy function $f(s,t)$ can be approximated using fuzzy block functions like operator as

$$f(s,t) \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right) \odot \phi_{i,j}(s,t).$$

Hence, the approximate value of the integral of fuzzy function can be obtained as follows:

$$(FR)\int_0^1 (FR)\int_0^1 f(s,t)dsdt \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right) \odot \int_0^1\int_0^1 \phi_{i,j}(s,t)dsdt.$$

Since

$$\int_0^1\int_0^1 \phi_{i,j}(s,t)dsdt = \frac{1}{mn},\ \forall\, i,j, \tag{6}$$

we have:

$$(FR)\int_0^1 (FR)\int_0^1 f(s,t)dsdt \approx \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right).$$

Theorem 3.2. *Let $f \in C_F\big([0,1)\times[0,1)\big)$ be a L-Lipschitz function. Then we have:*

$$D\left((FR)\int_0^1 (FR)\int_0^1 f(s,t)dsdt, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right)\right) \leq \frac{L_1}{2m} + \frac{L_2}{2n}.$$

Proof. . According to (5) and (6), we have:

$$D\left((FR)\int_0^1 (FR)\int_0^1 f(s,t)dsdt, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right)\right) =$$

$$= D\Bigg((FR)\int_0^1 (FR)\int_0^1 \sum_{i=1}^{m}\sum_{j=1}^{n}\phi_{i,j}(s,t)f(s,t)dsdt,$$

$$\int_0^1\int_0^1 \sum_{i=1}^{m}\sum_{j=1}^{n}\phi_{i,j}(s,t) f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right) dsdt\Bigg).$$

Then, applying the parts of 2 and 3 of Lemma 2.5, we obtain:

$$D\left((FR)\int_0^1 (FR)\int_0^1 f(s,t)dsdt, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right)\right) \leq$$

$$\leq \int_0^1\int_0^1 \sum_{i=1}^{m}\sum_{j=1}^{n}\phi_{i,j}(s,t) D\left(f(s,t), f\left(\frac{i-0.5}{m}, \frac{j-0.5}{n}\right)\right) dsdt.$$

As regards, f satisfies in Lipschitz condition (3), we obtain:

$$D\left((FR)\int_0^1 (FR)\int_0^1 f(s,t)dsdt, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m},\frac{j-0.5}{n}\right)\right) \leq$$

$$\leq \int_0^1\int_0^1 \sum_{i=1}^{m}\sum_{j=1}^{n}\phi_{i,j}(s,t)\left(L_1\left|s-\frac{i-0.5}{m}\right| + L_2\left|t-\frac{j-0.5}{n}\right|\right)dsdt.$$

According to the $(s,t) \in [\frac{i-1}{m},\frac{i}{m}) \times [\frac{j-1}{n},\frac{j}{n})$, we get:

$$D\left((FR)\int_0^1 (FR)\int_0^1 f(s,t)dsdt, \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\frac{i-0.5}{m},\frac{j-0.5}{n}\right)\right) \leq \frac{L_1}{2m}+\frac{L_2}{2n}.$$

Thus, the proof is complete. □

4. Fuzzy Integral Equations

Here, we consider the two dimensional nonlinear fuzzy Fredholm integral equation (1), where H(x,y,s, t) is a crisp kernel function over $[0,1]^4$, f, F are continuous fuzzy-number-valued functions and $G : R_F \to R_F$ is a continuous fuzzy function. We assume that H is continuous and therefore it is uniformly continuous with respect to (s,t) and there exists $M > 0$, such that $M_H = \max_{0\leq x,y,s,t\leq 1} |H(x,y,s,t)|$. Let $\Omega = \{f : [0,1]^2 \to R_F; f \text{ is continuous}\}$ be the space of the two-dimensional fuzzy continuous functions with the metric $D^*(f,g) = \sup_{0\leq s,t\leq 1} D(f(s,t),g(s,t))$, for $f,g \in \Omega$. In the following theorem, sufficient conditions for the existence and uniqueness of the solution of equation (1) are given.

Theorem 4.1. *(See* [30]*). Let $H(x,y,s,t)$ be continuous and positive for $0 \leq x,y,s,t \leq 1$ and $f : [0,1)^2 \to R_F$ be continuous on $[0,1)^2$. Moreover assume that there exists $L > 0$, such that*

$$D(G(F_1(s,t)),G(F_2(s,t))) \leq L.D(F_1(s,t),F_2(s,t)), \forall (s,t) \in [0,1]^2,\ F_1,\ F_2 \in \Omega.$$

If $M_H L < 1$, then the iterative procedure for $k \in \mathbf{N}$

$$F_0(x,y) = f(x,y),$$

$$F_k(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t)\odot G(F_{k-1}(s,t))dsdt. \quad (7)$$

converges to the solution of F of (1). In addition, the following error bound holds:

$$D^*(F,F_k) \leq \frac{(M_H L)^k}{1-M_H L} D^*(F_1,F_0),\ \forall k \in \mathbf{N}. \quad (8)$$

Remark 4.2. If $F_0 = f$ the error estimate (8), becomes:

$$D^*(F,F_k) \leq \frac{(M_H L)^{k+1}}{L(1-M_H L)}\left(L\|f\|_F + M_0\right),\ \forall k \in \mathbf{N}, \quad (9)$$

where $M_0 = \sup_{0\leq x,y\leq 1} \|G(\tilde{0})\|_F$.

Theorem 4.3. *Suppose that:*

(i) $f : [0,1)^2 \to R_F$ *is fuzzy continuous and bounded such that* $M_f = \sup_{0 \le x,y < 1} \|f(x,y)\|_F$,
$[f(x,y)]^r = [f^r_-(x,y), f^r_+(x,y)],\ \forall r \in [0,1]$;
and $f^r_-(x,y),\ f^r_+(x,y),\ r \in [0,1]$ *are equicontinuous;*

(ii) G *is fuzzy continuous and bounded such that* $M_G = \sup_{0 \le x,y < 1} \|G(u(x,y))\|_F$ *for every* $u(x,y) \in R_F$*;*

(iii) H *is positive and uniformly continuous with respect to* (s,t) *and bounded is such that* $M_H = \max_{0 \le x,y,s,t < 1} H(x,y,s,t)$.

Then the unique solution of (1), can be obtained by solving the following integral equations for $r \in [0,1],\ (x,y) \in [0,1] \times [0,1]$*:*

$$[F(x,y)]^r = [f(x,y)]^r + \int_0^1 \int_0^1 H(x,y,s,t).[G([F(s,t)]^r)]^r dsdt; \tag{10}$$

Proof. Let $\Gamma_r = S_r(f_r)$ be the solution set of rth inclusion $r \in [0,1]\}$ where the subscript r indicates that the r-level set of a fuzzy set is included. First, according to Theorem 1 [13], the set Γ_r is nonempty, compact and connected. Thus the first condition of the Characterization Theorem holds. Second, since f, G are continuous we imply that these fuzzy functions are upper semi-continuous and so $[f]^{r_2} \subseteq [f]^{r_1}$ and $[G]^{r_2} \subseteq [G]^{r_1}$ for $0 \le r_1 \le r_2 \le 1$, consequently $\Gamma_{r_2} \subseteq \Gamma_{r_1}$, which is equivalence to the second condition of the Characterization Theorem. Third, Let (r_n) be a nondecreasing sequence in $[0,1]$ converging to $r \in (0,1]$. Now, we should prove that $\bigcap_{n=1}^{\infty} \Gamma_{r_n} = \Gamma_r$. We note that Γ_{r_n} is a nonincreasing sequence of nonempty, compact and connected sets and so, according to Theorem 1.11 [2], the set $\Gamma = \bigcap_{n=1}^{\infty} \Gamma_{r_n}$ is nonempty, compact and connected. Now, we show that, $\Gamma = S_r(f_r)$. Since $S_r(f_r) \subseteq S_{r_n}(f_{r_n})$ for each n we have $S_r(f_r) \subseteq \Gamma$. It remains to show $\Gamma \subseteq S_r(f_r)$. For this purpose, let $u_{r_n} \in S_{r_n}(f_{r_n})$ for each n, then there exist the continuous function $f_{r_n}(x,y)$ and an integrable function $v_{r_n}(x,y) \in G_{r_n}[u_{r_n}(x,y)]$ with

$$[u(x,y)]^{r_n} = [f(x,y)]^{r_n} + \int_0^1 \int_0^1 H(x,y,s,t).[v(s,t)]^{r_n} r dsdt. \tag{11}$$

Since the support of f, $[f]^0 \supseteq [f]^{r_n}$ is compact, by Arzelas Theorem the sequence $[f]^{r_n}$ is uniformly bounded and equicontinuous. Therefore, taking into account the conditions (ii)-(iii), we have:

$$|u^{r_n}_-(x,y)| \le |f^{r_n}_-(x,y)| + \int_0^1 \int_0^1 |H(x,y,s,t)|.|v^{r_n}_-(s,t)| dsdt \le M_f + M_H.M_G,$$

and similarly for the $u^r_+(x,y)$, we have:

$$|u^{r_n}_+(x,y)| \le M_f + M_H.M_G,$$

and so the sequence of $([u]^{r_n})$ is uniformly bounded. Now, we demonstrate the sequence of $([u]^{r_n})$ is equicontinuous. For this aim, we obtain:

$$\begin{aligned}|u_-^{r_n}(x_1,y_1)-u_-^{r_n}(x_2,y_2)| \leq & |f_-^{r_n}(x_1,y_1)-f_-^{r_n}(x_2,y_2)| \\ & + \int_0^1\int_0^1 |H(x_1,y_1,s,t)-H(x_2,y_2,s,t)|.|v_-^{r_n}(s,t)|dsdt \\ \leq & |f_-^{r_n}(x_1,y_1)-f_-^{r_n}(x_2,y_2)| + 2M_H.M_G,\end{aligned}$$

and analogously for the $|u_+^{r_n}(x_1,y_1)-u_+^{r_n}(x_2,y_2)|$, we get:

$$|u_+^{r_n}(x_1,y_1)-u_+^{r_n}(x_2,y_2)| \leq |f_+^{r_n}(x_1,y_1)-f_+^{r_n}(x_2,y_2)| + 2M_H.M_G.$$

As regards, H is uniformly continuous and $([f]^{r_n})$ is equicontinuous, it deduce that $([u]^{r_n})$ is equicontinuous and so compact. Consequently, there exists a subsequence $([u]^{r_{n1}}) \subseteq ([u]^{r_n})$ such that $[u]^{r_{n1}} \longrightarrow [u]^r$. From compactness of $([v]^{r_n})$, we conclude that $([v]^{r_{n1}})$ is also compact, hence there exists a further subsequence $[v]^{r_{n2}} \longrightarrow [v]^r$ and clearly $[u]^{r_{n2}} \longrightarrow [u]^r$. Since $\|v(x,y)\|_F \leq M_G$, the sequence of functions $[w(x,y)]^{r_{n2}} = [w(x,y)]^{r_{n2}}/M_G$ belongs to the unit ball of $L_\infty([0,1]\times[0,1])$, which is weakly compact by Theorem 1.10 [2], so there is subsequence $[w]^{r_{n3}}$ converges weakly to $[w]^r$. But the map $w \longrightarrow M_G w$ is a continuous map from $L_\infty([0,1]\times[0,1])$ to $L_1([0,1]\times[0,1])$ and thus the sequence $[v]^{r_{n3}}$ converges weakly to $[v]^r = M_G[w]^r$. We note that $L_\infty([0,1]\times[0,1])$ is the space of measurable functions from $[0,1]\times[0,1]$ to $\mathbb{R}$ with bounded almost everywhere on $[0,1]\times[0,1]$, with essential supremum norm and $L_\infty([0,1]\times[0,1])$ is the space of integrable functions from $[0,1]\times[0,1]$ to $\mathbb{R}$ with metric $\|f-g\|_1 = \int_0^1\int_0^1 |f(s,t)-g(s,t)|dsdt$. Now, from condition (iii), there is a subsequence $([v]^{r_{n4}}) \subseteq ([v]^{r_{n3}})$ such that

$$\int_0^1\int_0^1 H(x,y,s,t).[v(s,t)]^{r_4})dsdt \longrightarrow \int_0^1\int_0^1 H(x,y,s,t).[v(s,t)]^r dsdt.$$

The above expression and (11) implies

$$[u_(x,y)]^r = [f(x,y)]^r + \int_0^1\int_0^1 H(x,y,s,t).[v(s,t)]^r r dsdt,\ (x,y)\in[0,1]\times[0,1].$$

Then (10) is satisfied so $[F]^r \in S_r(f_r)$, thus $\Gamma \subseteq S_r(f_r)$. Finally, according to Characterization Theorem (Theorem 2.3) there exist fuzzy function $F(x,y)$ which $[F(x,y)]^r$ for $r\in[0,1]$ are its r-cuts. □

Now, we introduce the numerical method to find the approximate solution of the two dimensional nonlinear fuzzy Fredholm integral equation (1). In this way, we consider the following uniform partitions of the region $[0,1)^2$:

$$\begin{aligned} D_x &= 0 = s_1 < s_2 < ... < s_{m-1} < s_m = 1, \\ D_y &= 0 = t_1 < t_2 < ... < t_{n-1} < t_n = 1, \end{aligned} \tag{12}$$

with $s_i = \frac{i-0.5}{m}$, $t_j = \frac{j-0.5}{n}$, $1\leq i\leq m,\ 1\leq j\leq n.$

Then the following iterative procedure gives the approximate solution of equation (1) in the point $(x,y) \in [0,1)^2$ using two dimensional block pulse functions:

$$z_0(x,y) = f(x,y),$$

$$z_k(x,y) = f(x,y) \oplus \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} H(x,y,s_i,t_j) \odot G(z_{k-1}(s_i,t_j)), \ \forall\, k \in \mathbf{N}. \quad (13)$$

5. Convergence Analysis

In this section, we investigate the convergence of the iterative proposed method to the solution of equation (1) under the following conditions:

(i) $f : [0,1)^2 \to R_F$ is fuzzy continuous;

(ii) $H : [0,1)^4 \to R^+$ is continuous;

(iii) There exist $\alpha, \beta \geq 0$ such that

$$D(f(s',t'), f(s'',t'')) \leq \alpha |s' - s''| + \beta |t' - t''|,$$

for any $s', s'', t', t'' \in [0,1)$;

(iv) There exists $L > 0$ such that

$$D(G(F_1(s,t)), G(F_2(s,t))) \leq L.D(F_1(s,t), F_2(s,t)), \forall (s,t) \in [0,1)^2,$$

where $F_1, F_2 : [0,1)^2 \to R_F$;

(v) $M_H L < 1$, where L is as given in the above item and $M_H \geq 0$ is such that $M_H = \max_{0 \leq x,y,s,t < 1} H(x,y,s,t)$;

(vi) There exist $\mu, \lambda \geq 0$ such that

$$|H(x,y,s',t') - H(x,y,s'',t'')| \leq \mu |s' - s''| + \lambda |t' - t''|,$$

for any $x, y, s', t', s'', t'' \in [0,1)$;

(vii) There exist $\gamma, \eta \geq 0$ such that

$$|H(s',t',s,t) - H(s'',t'',s,t)| \leq \gamma |s' - s''| + \eta |t' - t''|,$$

for any $s, s', s'', t, t', t'' \in [0,1)$;

Firstly, we prove an interesting result about the satisfying functions $H(x,y,s,t) \odot G(F_k(s,t))$ in Lipschitz condition which is used in the proof of the main result.

Lemma 5.1. *Consider the iterative procedure (7). Under the conditions (i)-(vii), the functions $\varphi_k(s,t) = H(x,y,s,t) \odot G(F_k(s,t))$ are Lipschitzian.*

Proof. Using fuzzy Distance, we have:

$$D(\varphi_k(s',t'), \varphi_k(s'',t'')) =$$
$$D(H(x,y,s',t') \odot G(F_k(s',t')), H(x,y,s'',t'') \odot G(F_k(s'',t'')) \leq$$
$$\leq D(H(x,y,s',t') \odot G(F_k(s',t')), H(x,y,s',t') \odot G(F_k(s'',t''))+$$
$$+D(H(x,y,s',t') \odot G(F_k(s'',t'')), H(x,y,s'',t'') \odot G(F_k(s'',t'')).$$

Using part (2) of Lemma 2.5, condition (iv) and Lemma 2.6, we obtain:

$$D(\varphi_k(s',t'), \varphi_k(s'',t'')) \leq |H(x,y,s',t')| LD(F_k(s',t'), F_k(s'',t''))+$$
$$+|H(x,y,s',t') - H(x,y,s'',t'')| D\left(G(F_k(s'',t'')), \tilde{0}\right).$$

Using condition (vi), we get:

$$D(\varphi_k(s',t'),\varphi_k(s'',t'')) \leq M_H L D\left(F_k\left(s',t'\right),F_k\left(s'',t''\right)\right) + \\ +\left(\mu\left|s'-s''\right| + \lambda\left|t'-t''\right|\right) D^*\left(G\left(F_k\right)\right),\tilde{0}\big). \quad (14)$$

On the other hand, we have:

$$D\left(F_k\left(s',t'\right),F_k\left(s'',t''\right)\right) \leq D\left(f\left(s',t'\right),f\left(s'',t''\right)\right) + \\ +D\Big((FR)\int_0^1 (FR)\int_0^1 H(s',t',s,t)\odot G(F_{m-1}(s,t))dsdt, \\ (FR)\int_0^1 (FR)\int_0^1 H(s'',t'',s,t)\odot G(F_{m-1}(s,t))dsdt\Big) \\ \leq D\left(f\left(s',t'\right),f\left(s'',t''\right)\right) + \\ +\int_0^1\int_0^1 |H(s',t',s,t)-H(s'',t'',s,t)| D(G(F_{k-1}(s,t)),\tilde{0})dsdt.$$

Consequently, using conditions (iii) and (vii), we obtain:

$$D\left(F_k\left(s',t'\right),F_k\left(s'',t''\right)\right) \leq \\ \alpha|s'-s''| + \beta|t'-t''| + \left(\gamma\left|s'-s''\right| + \eta\left|t'-t''\right|\right) D^*(G(F_{k-1}),\tilde{0}) \quad (15)$$

Substituting the inequality (15), into the inequality (14), we get:

$$D(\varphi_k(s',t'),\varphi_k(s'',t'')) \leq M_H L\Big(\alpha\left|s'-s''\right| + \beta\left|t'-t''\right|\Big) + \\ + M_H L\Big(\gamma\left|s'-s''\right| + \eta\left|t'-t''\right|\Big) D^*(G(F_{k-1}),\tilde{0})) + \\ + \Big(\mu\left|s'-s''\right| + \lambda\left|t'-t''\right|\Big) D^*(G(F_k),\tilde{0})). \quad (16)$$

By applying the properties of the norm function $\|.\|_F$ in Remark 2.7, we obtain

$$D\left(G\left(F_k(s'',t'')\right),\tilde{0}\right) \leq D\left(G\left(F_k(s'',t'')\right),G\left(\tilde{0}\right)\right) + D\left(G\left(\tilde{0}\right),\tilde{0}\right) \\ \leq LD\left(F_k(s'',t''),\tilde{0}\right) + \sup_{0\leq s'',t''\leq 1}\|H(\tilde{0})\|_F \\ = LD\left(z_k(s'',t''),\tilde{0}\right) + M_0. \quad (17)$$

By using again the properties of the norm function $\|.\|_F$ in Remark 2.7, we have

$$D\left(F_k(s'',t''),\tilde{0}\right) = \|F_k(s'',t'')\|_F \leq D\left(f\left(s'',t''\right),\tilde{0}\right) + \\ + \int_0^1\int_0^1 D(H(s'',t'',s,t)\odot G(F_{k-1}(s,t)),\tilde{0})dsdt.$$

By applying the inequality (17) into the above inequality, we have:

$$\|F_k(s'',t''\|_F \leq \|f(s'',t'')\|_F + \int_0^1\int_0^1 |H(s'',t'',s,t)|\left(LD(F_{k-1}(s,t),\tilde{0}) + M_0\right) dsdt \\ \leq \|f(s'',t'')\|_F + M_H\int_0^1\int_0^1 \left(LD^*(F_{k-1},\tilde{0}) + M_0\right) dsdt.$$

Taking supremum from the above inequality for $0 \leq s'', t'' \leq 1$, it follows that

$$\|F_k\|_F \leq \|f\|_F + M_H\left(L\|F_{k-1}\|_F + M_0\right) = \|f\|_F + M_H L\|F_{k-1}\|_F + (M_H L)\frac{M_0}{L}.$$

By successive substitutions on the above inequality, we obtain

$$\|F_k\|_F \leq \frac{1-(M_H L)^{k+1}}{1-M_H L}\cdot\|f\|_F + \frac{1-(M_H L)^k}{1-M_H L}\cdot M_H M_0.$$

Since, $\frac{1-(M_H L)^k}{1-M_H L} \leq \frac{1}{1-M_H L}$, for all $k \in \mathbf{N}$ we obtain:

$$\|F_k\|_F \leq \frac{\|f\|_F + M_H M_0}{1-M_H L}.$$

Now, by taking into account the above inequality, from (16) we get:

$$D(\varphi_k(s',t'),\varphi_k(s'',t'')) \leq$$
$$\leq \left(\alpha M_H L + (\gamma M_H L)\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right) + \mu\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right)\right)|s'-s''| +$$
$$+\left(\beta M_H L + (\eta M_H L)\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right) + \lambda\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right)\right)|t'-t''|.$$

By supposing $L_1' = \left(\alpha M_H L + (\gamma M_H L)\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right) + \mu\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right)\right)$ and $L_2' = \left(\beta M_H L + (\eta M_H L)\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right) + \lambda\left(\frac{\|f\|_F + M_H M_0}{1-M_H L}\right)\right)$, we have:

$$D(\varphi_k(s',t'),\varphi_k(s'',t'')) \leq L_1'|s'-s''| + L_2'|t'-t''|.$$

Thus, the functions $\varphi_k(s,t)$ for all k are Lipschitzian. □

Theorem 5.2. *Under the conditions (i)-(iiv) the iterative procedure equation (13) converges to the unique solution of equation (1), F, and its error estimate is as follows:*

$$D^*(F,z_m) \leq \frac{(M_H L)^{k+1}}{L(1-M_H L)}\left(L\|f\|_F + M_0\right) + \frac{1}{1-M_H L}\left(\frac{L_1'}{2m} + \frac{L_2'}{2n}\right),$$

where $M_0 = \sup_{0\leq x,y\leq 1} \|F(\tilde{0})\|_F$.

Proof. . Since

$$F_1(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t) \odot G(F_0(s,t)dsdt,$$

we have

$$D(F_1(x,y), z_1(x,y)) = D(f(x,y), f(x,y)) +$$
$$+D\Bigg((FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t)\odot G(F_0(s,t))dsdt,$$
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} H\left(x,y,s_i,t_j\right)\odot G\left(z_0\left(s_i,t_j\right)\right)\Bigg)$$

As regards $F_0(x,y)=z_0(x,y)=f(x,y)$, we have:

$$\leq D\Bigg((FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t)\odot G(f(s,t))dsdt,$$
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} H\left(x,y,s_i,t_j\right)\odot G\left(f\left(s_i,t_j\right)\right)\Bigg).$$

Regarding to Lemma 5.1 and Theorem 3.2, we have:

$$D(F_1(x,y),z_1(x,y))\leq \frac{L_1'}{2m}+\frac{L_2'}{2n}.$$

Taking supremum from the above inequality for $0\leq x,y\leq 1$, it deduces that

$$D^*(F_1,z_1)\leq \frac{L_1'}{2m}+\frac{L_2'}{2n}. \tag{18}$$

Now, since

$$F_2(x,y)=f(x,y)\oplus (FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t)\odot G(F_1(s,t))dsdt,$$

we conclude that

$$D(F_2(x,y),z_2(x,y))=D(f(x,y),f(x,y))+$$
$$+D\Bigg((FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t)\odot G(F_1(s,t))dsdt,$$
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} H\left(x,y,s_i,t_j\right)\odot G\left(z_1\left(s_i,t_j\right)\right)\Bigg)$$
$$\leq D\Bigg((FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t)\odot G(F_1(s,t))dsdt,$$
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} H\left(x,y,s_i,t_j\right)\odot G\left(F_1\left(s_i,t_j\right)\right)\Bigg)+$$
$$+D\Bigg(\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} H\left(x,y,s_i,t_j\right)\odot G\left(F_1\left(s_i,t_j\right)\right),$$
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} H\left(x,y,s_i,t_j\right)\odot G\left(z_1\left(s_i,t_j\right)\right)\Bigg).$$

Using again Lemma 5.1 and Theorem 3.2 for part 1 of the above inequality and applying part (2) and (3) of Lemma 2.5 for part of 2 the above inequality, we obtain:

$$D(F_2(x,y),z_2(x,y))\leq \frac{L_1'}{2m}+\frac{L_2'}{2n}+$$
$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|H\left(x,y,s_i,t_j\right)\right| D\left(G\left(F_1\left(s_i,t_j\right)\right),G\left(z_1\left(s_i,t_j\right)\right)\right).$$

Considering conditions (iv) and (v), we have:

$$D(F_2(x,y),z_2(x,y)) \leq \frac{L_1'}{2m}+\frac{L_2'}{2n}+M_H L D^*(F_1,z_1).$$

Then, according to (18), we get:

$$D(F_2(x,y),z_2(x,y)) \leq (1+M_H L)\left(\frac{L_1'}{2m}+\frac{L_2'}{2n}\right).$$

By induction, for $k \geq 3$, we obtain:

$$D(F_k(x,y),z_k(x,y)) \leq \left(1+M_H L+\ldots+(M_H L)^{k-1}\right)\cdot\left(\frac{L_1'}{2m}+\frac{L_2'}{2n}\right).$$

Therefore, we have:

$$D^*(F_k,z_k) \leq \left(\frac{1-(M_H L)^k}{1-M_H L}\right)\left(\frac{L_1'}{2m}+\frac{L_2'}{2n}\right).$$

Since $M_H L<1$, we conclude that

$$D^*(F_k,z_k) \leq \frac{1}{1-M_H L}\left(\frac{L_1'}{2m}+\frac{L_2'}{2n}\right). \tag{19}$$

Using equations (9) and (19), we obtain:

$$\begin{aligned} D^*(F,z_k) \leq & D^*(F,F_k)+D^*(F_k,z_k) \\ \leq & \frac{(M_H L)^{k+1}}{L(1-M_H L)}\left(L\|f\|_F+M_0\right)+\frac{1}{1-M_H L}\left(\frac{L_1'}{2m}+\frac{L_2'}{2n}\right). \end{aligned}$$

□

Remark 5.3. Since $M_H L<1$, it is easy to show that

$$\lim_{k\to\infty m\to\infty n\to\infty} D^*(F,z_k)=0$$

This result shows that the proposed method is convergent.

6. Numerical Stability Analysis

For the iterative numerical method, it is more suitable investigating the stability of the obtained numerical solution with respect to the choice of the first iteration. So, in order to study the numerical stability of the iterative method (13) with respect to small changes in the starting approximation, we consider another first iteration term $\hat{F}_0 \in \Omega$ such that there exists $\epsilon>0$ for which $D(F_0(x,y),\hat{F}_0(x,y)<\epsilon$, for all $(x,y)\in[0,1]\times[0,1]$. The new sequence of successive approximations is:

$$\hat{F}_k(x,y)=\hat{f}(x,y)\oplus(FR)\int_0^1(FR)\int_0^1 H(x,y,s,t)\odot G(\hat{F}_{k-1}(s,t))dsdt. \tag{20}$$

Using the same iterative method, the terms of produced sequence are:

$$\begin{aligned} &\hat{z}_0(x,y)=\hat{f}(x,y), \\ &\hat{z}_k(x,y)=\hat{f}(x,y)\oplus\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} H\left(x,y,s_i,t_j\right)\odot G\left(\hat{z}_{k-1}\left(s_i,t_j\right)\right). \end{aligned} \tag{21}$$

Definition 6.1. We say that the method of successive approximations applied to solve equation (1) is numerically stable with respect to the choice of the first iteration iff there exist positive numbers $p,\ q$ and constants $k_1, k_2, k_3 > 0$ which are independent by step-sizes $h_1 = \frac{1}{m}$ and $h_2 = \frac{1}{n}$ respectively, such that

$$D(z_k(x,y), \hat{z}_k(x,y)) < K_1\epsilon + K_2 h_1^p + K_3 h_2^q. \tag{22}$$

Theorem 6.2. *Under the conditions of Theorem 3, the iterative method (21) is numerically stable with respect to the choice of the first iteration.*

Proof. . Since Firstly, we observe that:

$$\begin{aligned} D(z_k(x,y), \hat{z}_k(x,y)) \leq D(z_k(x,y), F_k(x,y)) + D(F_k(x,y), \hat{F}_k(x,y)) + \\ + D(\hat{F}_k(x,y), \hat{z}_k(x,y)). \end{aligned} \tag{23}$$

Using equation (19), we obtain:

$$D(z_k(x,y), F_k(x,y)) \leq \frac{1}{1 - M_H L}\left(\frac{L_1'}{2m} + \frac{L_2'}{2n}\right). \tag{24}$$

Also, by similar reasoning we have:

$$D(\hat{F}_k(x,y), z_k(x,y)) \leq \frac{1}{1 - M_H L}\left(\frac{L_1''}{2m} + \frac{L_2''}{2n}\right). \tag{25}$$

For obtaining the bound of $D(F_k(x,y), \hat{F}_k(x,y))$, we observe that

$$D(F_0(x,y), \hat{F}_0(x,y)) < \epsilon, \quad \forall (x,y) \in [0,1] \times [0,1],$$

and thus

$$\begin{aligned} & D(F_1(x,y), \hat{F}_1(x,y)) = D(f(x,y), \hat{f}(x,y)) + \\ & D\Big((FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t) \odot G(F_0(s,t))dsdt, \\ & ,(FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t) \odot (G(\hat{F}_0(s,t))dsdt\Big) \\ & \leq \epsilon + \int_0^1 \int_0^1 |H(x,y,s,t)| D\left(G(f(s,t)), G(\hat{f}(s,t)\right) dsdt \\ & \leq \epsilon + M_H \int_0^1 \int_0^1 D\left(G(f(s,t)), G(\hat{f}(s,t))\right) dsdt \\ & \leq \epsilon + M_H L D^*(f, \hat{f}). \end{aligned}$$

Therefor, we obtain:

$$D(F_1(x,y), \hat{F}_1(x,y)) < \epsilon + M_H L \epsilon.$$

By induction for $k \geq 2$, we get:

$$\begin{aligned} D(F_k(x,y), \hat{F}_k(x,y)) & \leq \epsilon + M_H L \cdot \epsilon + (M_H L)^2 \cdot \epsilon + ... + (M_H L)^{k-1} \cdot \epsilon + (M_H L)^k \cdot \epsilon \\ & \leq \epsilon \cdot \frac{1 - (M_H L)^{k+1}}{1 - M_H L}, \quad \forall x, y \in [0,1],\ k \in \mathbf{N}. \end{aligned}$$

According to $M_H L < 1$, this inequality becomes:

$$D\left(F_k(x,y),\hat{F}_k(x,y)\right) < \frac{\epsilon}{1-M_H L},\ \forall x,y \in [0,1),\ k \in \mathbf{N}.$$

Then, we conclude that

$$D(z_k(x,y),\hat{z}_k(x,y)) \leq \frac{\epsilon}{1-M_H L} + \frac{L_1' + L_1''}{2m(1-M_H L)} + \frac{L_2' + L_2''}{2n(1-M_H L)}. \tag{26}$$

By comparing inequalities (22) and (26), we deduce that

$$K_1 = \frac{1}{1-ML},\ K_2 = \frac{L_1' + L_1''}{2(1-M_H L)},\ K_2 = \frac{L_2' + L_2''}{2(1-M_H L)},\ p = q = 1.$$

So, the numerical stability of the proposed iterative method is proved. □

7. Numerical Examples

In this section, we apply the presented method in Section 5 for solving the two dimensional fuzzy integral equation (1) in two examples. The approximate solution is calculated for different values of k, m and n. Also, we compare the numerical solution obtained by using the proposed method with the exact solution. The computations associated with the examples were performed using Mathematica 7.

Example 7.1. Consider the following nonlinear two dimensional fuzzy integral equation:

$$F(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t) \odot (F(s,t))^2 dsdt,\ s,t,x,y \in [0,1],$$

where

$$H(x,y,s,t) = \frac{xyst}{2},$$

$$[f(x,y)]^r = [f_-^r(x,y), f_+^r(x,y)] = \left[\frac{(32r-r^2)}{32}xy, \frac{(60-28r-r^2)}{32}xy\right],$$

the exact solution is

$$[F(x,y]^r = \left[F_-^r(x,y), F_+^r(x,y)\right] = [rxy, (2-r)xy].$$

The comparison of the proposed iterative algorithm solution and the exact solution is shown in Table 1.

Example 7.2. Consider the following linear two dimensional fuzzy Fredholm integral equation:

$$F(x,y) = f(x,y) \oplus (FR)\int_0^1 (FR)\int_0^1 H(x,y,s,t) \odot F(s,t) dsdt,\ x,y,s,t \in [0,1],$$

where

$$H(x,y,s,t) = \frac{x^2+y^2+s^2+t^2}{5},$$

$$[f(x,y)]^r = \left[rxy - \frac{r}{20}(x^2+y^2+1), (2-r)xy - \frac{2-r}{20}(x^2+y^2+1)\right],$$

the exact solution is

$$[F(x,y^r] = \left[F_-^r(x,y), F_+^r(x,y)\right] = [rxy, (2-r)xy].$$

	k=10, m=n=10		k=15, m=n=50	
r-level	$\lvert\underline{F}-\underline{y}_m\rvert$	$\lvert\overline{F}-\overline{y}_m\rvert$	$\lvert\underline{F}-\underline{y}_m\rvert$	$\lvert\overline{F}-\overline{y}_m\rvert$
0	0.0000E-0	3.5573E-4	0.0000E-0	1.4283E-5
0.25	4.9471E-6	3.5573E-4	1.9840E-7	1.0744E-5
0.50	2.0104E-5	1.9328E-4	8.0640E-7	7.7575E-6
0.75	4.5969E-5	1.3197E-4	1.8440E-6	5.2959E-6
1	8.3069E-5	8.3069E-5	3.3329E-6	3.3329E-6

Table 1. The Accuracy on the Level Sets for Example 1 in $(x, y) = (0.5, 0.5)$

	k=10, m=n=10		k=12, m=n=50	
r-level	$\lvert\underline{F}-\underline{y}_m\rvert$	$\lvert\overline{F}-\overline{y}_m\rvert$	$\lvert\underline{F}-\underline{y}_m\rvert$	$\lvert\overline{F}-\overline{y}_m\rvert$
0	0.0000E-0	6.6570E-4	0.0000E-0	2.6675E-5
0.25	8.3212E-5	5.8249E-4	3.3344E-6	2.3341E-5
0.50	1.6642E-4	4.9927E-4	6.6688E-6	2.0006E-5
0.75	2.4964E-4	4.1606E-4	1.0003E-5	1.6672E-5
1	3.3285E-4	3.3285E-4	1.3338E-5	1.3338E-5

Table 2. The Accuracy on the Level Sets for Example 2 in $(x, y) = (0.5, 0.5)$

The comparison of the proposed iterative algorithm solution and the exact solution is shown in Table 2.

8. Conclusions

In this paper, we have presented an iterative procedure by using two dimensional fuzzy block-pulse functions to solve the two dimensional nonlinear Ferdholm fuzzy integral (1). The error estimation for approximating the solution of nonlinear two dimensional nonlinear fuzzy Fredholm integral equations is given in Theorem 5.1 in terms of supplementary Lipschitz condition which proves the convergence of the proposed method. The concept of numerical stability is defined based on the choice of the first iteration and then the numerical stability of the proposed iterative algorithm is proven. Finally, the illustrative numerical examples included in the study in order to test the accuracy and the convergence of the proposed method indicate that the proposed method performs well both for nonlinear and linear fuzzy two dimensional integral equations.

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ITERATIVE METHOD FOR SOLVING TWO-DIMENSIONAL NONLINEAR FUZZY INTEGRAL EQUATIONS USING FUZZY BIVARIATE BLOCK-PULSE FUNCTIONS WITH ERROR ESTIMATION

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روش تکراری جهت حل معادلات انتگرال فازی غیرخطی دوبعدی با استفاده از توابع بلاک پالس دومتغیره فازی همراه با تقریب خطا

چکیده. در این مقاله، فرآیندی تکراری براساس توابع بلا ک پالس فازی جهت حل معادلات انتگرال نوع دوم فردهلم غیرخطی فازی ارائه می نماییم. تقریب خطا و پایداری عددی روش ارائه شده برحسب شرط مکمل لیپشیتس بیان شده اند. سرانجام، به منظور نشان دادن دقت و همگرایی روش ارائه شده، مثالهای تشریحی لحاظ شده است.