QUANTALE-VALUED GAUGE SPACES

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Abstract. We introduce a quantale-valued generalization of approach spaces in terms of quantale-valued gauges. The resulting category is shown to be topological and to possess an initially dense object. Moreover we show that the category of quantale-valued approach spaces defined recently in terms of quantale-valued closures is a coreflective subcategory of our category and, for certain choices of the quantale, is even isomorphic to our category. Finally, the category of quantale-valued metric spaces is shown to be coreflectively embedded in our category.

1. Introduction

Approach spaces, introduced in [11, 12, 13], form a common supercategory of topological and metric spaces. Recently, a probabilistic generalization was considered [9]. In a recent paper, from the view point of *monoidal topology* [6] the definitions of an approach space and of a probabilistic approach space were generalized to the quantale-valued case by defining them with the help of quantale-valued closure operators [10]. Choosing $L = [0, \infty]$ with the opposite order and extended addition as quantale operation, one recovers Lowen's approach spaces. If one chooses as quantale the set of *distance distribution functions* $L = \Delta^+$ with a triangle function induced by a left-continuous t-norm as quantale operation, then probabilistic approach spaces are recovered. In [10, 9] furthermore these quantale-valued approach spaces were characterized by certain quantale-valued convergence structures, see also [8].

Classically, there are many different but equivalent ways of defining an approach space. One definition in terms of gauges is of particular interest. Such a gauge is an ideal of quasi-metrics that satisfies a so-called local saturation condition. In this paper, after collecting the lattice background and definitions and results about L-approach spaces and L-metric spaces in the next two sections, in section 4 we generalize this definition, by considering L-gauges, i.e. filters of L-metrics that satisfy a suitable generalization of the saturation condition. We show that the resulting category of L-gauge spaces is topological and has an initially dense object. Furthermore in section 5, following the classical lines of proof, we show that the category of L-approach spaces [9] is isomorphic to a coreflective subcategory of the category of L-gauge spaces. We give a condition on the quantale L which guarantees that both categories are isomorphic and show with two examples that

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we cannot omit this condition. In particular, we show that in the probabilistic case, probabilistic approach spaces and probabilistic gauge spaces are not the same. In the final section 6 we show that the category of L-metric spaces can naturally be embedded into our category as a coreflective subcategory.

2. **Preliminaries**

We consider in this paper completely distributive lattices, i.e. complete lattices L that satisfy the following distributive laws.

$$(CD1) \quad \bigvee_{j\in J}\left(\bigwedge_{i\in I_j}\alpha_{ji}\right) = \bigwedge_{f\in\prod_{j\in J} I_j}\left(\bigvee_{j\in J}\alpha_{jf(j)}\right),$$

$$(CD2) \quad \bigwedge_{j\in J}\left(\bigvee_{i\in I_j}\alpha_{ji}\right) = \bigvee_{f\in\prod_{j\in J} I_j}\left(\bigwedge_{j\in J}\alpha_{jf(j)}\right).$$

We assume that L is non-trivial in the sense that $\top \neq \bot$ for the top element $\top$ and the bottom element $\bot$. It is well known that, in any complete lattice L, (CD1) and (CD2) are equivalent. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$, α is *well-below* β, if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and $\alpha \lhd \bigvee_{j\in J}\beta_j$ iff $\alpha \lhd \beta_i$ for some $i \in J$. A complete lattice is completely distributive if and only if we have $\alpha = \bigvee\{\beta \;:\; \beta \lhd \alpha\}$ for any $\alpha \in L$, see e.g. Theorem 7.2.3 in [1]. Similarly, we can define the *well-above relation*, β is *well-above* α, $\alpha \prec \beta$ if for all subsets $D \subseteq L$ such that $\bigwedge D \leq \alpha$ there is $\delta \in D$ with $\delta \leq \beta$. Then $\alpha \prec \beta$ implies $\alpha \leq \beta$ and $\bigwedge_{j\in J}\beta_j \prec \alpha$ iff $\beta_j \prec \alpha$ for some $j \in J$. L is completely distributive iff $\alpha = \bigwedge\{\beta \in L \;:\; \alpha \prec \beta\}$ for any $\alpha \in L$. Clearly, in a complete lattice L we have $\alpha \lhd \beta$ iff $\beta \prec^{op} \alpha$ in the opposite order. For more results on lattices we refer to [4].

The triple $(L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *quantale* if $(L, *)$ is a semigroup, and $*$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{j\in J}\alpha_j) * \beta = \bigvee_{j\in J}(\alpha_j * \beta) \quad \text{and} \quad \beta * (\bigvee_{j\in J}\alpha_j) = \bigvee_{j\in J}(\beta * \alpha_j).$$

A quantale $(L, \leq, *)$ is called *commutative* if $(L, *)$ is a commutative semigroup and it is called *integral* if the top element of L acts as the unit, i.e. if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$. In any such quantale we can define an implication $\alpha \to \beta = \bigvee\{\gamma \in L \;:\; \alpha * \gamma \leq \beta\}$. Then $\alpha * \beta \leq \gamma$ iff $\alpha \leq \beta \to \gamma$. We give a list of properties of the implication.

Lemma 2.1. [7] *Let $(L, \leq, *)$ be an integral and commutative quantale and let $\alpha, \beta, \gamma, \beta_j \in L$ $(j \in J)$.*

(1) *If $\alpha \leq \beta$ then $\alpha \to \gamma \geq \beta \to \gamma$ and $\gamma \to \alpha \leq \gamma \to \beta$;*
(2) *$\alpha \leq (\alpha \to \beta) \to \beta$;*
(3) *$\alpha \to (\bigwedge_{j\in J}\beta_j) = \bigwedge_{j\in J}(\alpha \to \beta_j)$;*
(4) *$(\bigvee_{j\in J}\beta_j) \to \alpha = \bigwedge_{j\in J}(\beta_j \to \alpha)$.*

Example 2.2. A *triangular norm* or *t-norm* is a binary operation $*$ on the unit interval $[0,1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $([0,1], \leq, *)$ can be considered as a quantale if the t-norm is left-continuous. The three most commonly used (left-continuous) t-norms are:

- the minimum t-norm: $\alpha * \beta = \alpha \wedge \beta$,
- the product t-norm: $\alpha * \beta = \alpha \cdot \beta$,
- the Lukasiewicz t-norm: $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

Example 2.3. The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + a = \infty$ for all $\alpha, \beta \in [0, \infty]$) is a quantale, see e.g. [3]. In this quantale we have $\alpha \to \beta = (\beta - \alpha) \vee 0$. Furthermore $\bigvee_{j \in J}(\alpha_j \to \beta) = (\bigwedge_{j \in J} \alpha_j) \to \beta$ for all $\alpha_j, \beta \in L$.

Example 2.4. A function $\varphi : [0, \infty] \longrightarrow [0,1]$, which is non-decreasing, left-continuous on $(0, \infty)$ in the sense that $\varphi(x) = \bigvee\{\varphi(y) \ : \ y < x\}$ for all $x \in (0, \infty)$, and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$ is called a *distance distribution function* [17]. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq a < \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases} \qquad \text{and} \qquad \varepsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$

are in Δ^+. The set Δ^+ is ordered pointwise, i.e. for $\varphi, \psi \in \Delta^+$ we define $\varphi \leq \psi$ if for all $x \geq 0$ we have $\varphi(x) \leq \psi(x)$. The bottom element of Δ^+ is ε_∞ and the top element is ε_0 and the set Δ^+ with this order then becomes a complete lattice. We note that $\bigwedge_{i \in I} \varphi_i$ is in general not the pointwise infimum. It is shown in [3] that this lattice is completely distributive.

A binary operation, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and that satisfies the boundary condition $\varphi * \varepsilon_0 = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function* [15, 16, 17]. A triangle function is called *sup-continuous* [17], if $(\bigvee_{i \in I} \varphi_i) * \psi = \bigvee_{i \in I}(\varphi_i * \psi)$ for all $\varphi_i, \psi \in \Delta^+$, $(i \in I)$, i.e. if $(\Delta^+, \leq, *)$ is a quantale.

We will later use the triangle function τ_* induced by a t-norm $*$, defined by $\tau_*(\varphi, \psi)(x) = \bigvee_{u+v=x} \varphi(u) * \psi(v)$ for all $x \in [0, \infty]$, see [17].

Example 2.5. A frame is a quantale with $* = \wedge$.

Example 2.6. A commutative and integral quantale $(L, \leq, *)$ which satisfies $(\alpha \to \beta) \to \beta = \alpha \vee \beta$ for all $\alpha, \beta \in L$ is a *complete MV-algebra* [7]. In a complete MV-algebra we have the properties $\bigwedge_{j \in J}(\alpha * \beta_j) = \alpha * \bigwedge_{j \in J} \beta_j$ and $\bigvee_{j \in J}(\alpha_j \to \beta) = (\bigwedge_{j \in J} \alpha_j) \to \beta$ for all $\alpha_j, \beta \in L$.

A *value quantale* [3] is a commutative and integral quantale $(L, \leq, *)$ with an underlying completely distributive lattice $(L, \leq)$ such that $\bot \lhd \top$ and $\alpha \vee \beta \lhd \top$ whenever $\alpha, \beta \lhd \top$. Examples for value quantales are $([0, \infty], \geq, +)$ or $(\Delta^+, \leq, *)$ with a sup-continuous triangle function, see [3]. It should be noted that Flagg [3] uses the opposite order. The following result is shown in [3].

Lemma 2.7. [3] *Let $(L, \le, *)$ be a value quantale. If $\alpha \lhd \top$, then there is $\beta \lhd \top$ such that $\alpha \lhd \beta * \beta$.*

We will later need the following condition.

Definition 2.8. A quantale $(L, \le, *)$ satisfies the condition (I) if

$$(I) \text{ for all } \bot \prec \beta \text{ and all } \gamma \lhd \top \text{ we have } \beta \not\le \gamma * \beta.$$

Lemma 2.9. *If the quantale $(L, \le, *)$ is integral and satisfies the* strong cancellation law

$$(SCL) \text{ for all } \gamma, \alpha \in L, \bot \prec \beta : \gamma * \beta \le \alpha * \beta \text{ implies } \gamma \le \alpha$$

and if $\top \not\lhd \top$ then the condition (I) is satisfied.

Proof. Let $\bot \prec \beta$ and $\gamma \lhd \top$. If we assume $\beta = \top * \beta \le \gamma * \beta$, then $\gamma = \top$, a contradiction. □

Example 2.10. (1) The two-point chain $L = \{0, 1\}$ does not satisfy the condition (I) as $1 \lhd 1$.
(2) Let $L = [0, \infty]$ with the opposite order and extended addition as quantale operation. Then the strong cancellation law is valid and hence L satisfies the condition (I).
(3) Let $L = [0, 1]$ and multiplication as quantale operation. Then the strong cancellation law is satisfied and hence L satisfies the condition (I).
(4) A frame $(L, \le, \wedge)$ does in general not satisfy (I). If $\alpha \ge \beta$, then $\beta = \alpha \wedge \beta$.
(5) The 4-element Boolean algebra $\{\bot, \alpha, \beta, \top\}$ with $\alpha \wedge \beta = \bot$ and $\alpha \vee \beta = \top$ satisfies (I), as $\alpha, \beta \not\le \alpha \wedge \beta$, but does not satisfy the strong cancellation law since $\alpha \wedge \beta \le \beta \wedge \beta$ but $\alpha \not\le \beta$.
(6) In an MV-algebra $(L, \le, *)$ we have $\beta \le \alpha * \beta$ iff $\beta \wedge (\alpha \to \bot) = \bot$. Hence an MV-algebra satifies (I) if and only if $\beta \wedge (\alpha \to \bot) \ne \bot$ whenever $\alpha \not\lhd \top$ and $\bot \not\prec \beta$. In particular, if L has no zero-divisors for $\wedge$, then $(L, \le, *)$ satisfies (I).
(7) As a final example we consider the lattice Δ^+. For $0 < \delta < \infty$ and $0 < \epsilon \le 1$ we define $f_{\delta\epsilon} \in \Delta^+$ by

$$f_{\delta\epsilon}(x) = \begin{cases} 0 & \text{if} \quad 0 \le x \le \delta \\ \epsilon & \text{if} \quad \delta < x < \infty \\ 1 & \text{if} \quad x = \infty. \end{cases}$$

The following Lemma is then not difficult to show.

Lemma 2.11. (1) *$f_{\delta\epsilon} \le f_{\delta'\epsilon'} \iff \delta' \le \delta, \epsilon \le \epsilon'$;*
(2) *$f_{\delta\epsilon} \lhd f_{\delta'\epsilon'} \iff \delta' < \delta, \epsilon < \epsilon'$;*
(3) *$f_{\delta\epsilon} \lhd \varphi \iff \epsilon < \varphi(\delta)$;*
(4) *$\varphi = \bigvee\{f_{\delta\epsilon} : f_{\delta\epsilon} \lhd \varphi\}$ for all $\varphi \in \Delta^+$;*
(5) *If $\varphi \lhd \epsilon_0$ then there is $\epsilon < 1$ such that $\varphi \le f_{\delta\epsilon}$.*

As a consequence, we can show the following result.

Lemma 2.12. *Let $*$ be a t-norm on $[0, 1]$ that satisfies the property (I), i.e. $0 < \beta$ and $\epsilon < 1$ implies $\epsilon * \beta < \beta$. Then $(\Delta^+, \le, \tau_*)$ satisfies the condition (I).*

Proof. We first note that in Δ^+ we have $\varepsilon_\infty \not\prec \varepsilon_\infty$, because $\bigwedge\{\varepsilon_a : a > 0\} = \varepsilon_\infty$ but there is no $a > 0$ such that $\varepsilon_a = \varepsilon_\infty$. Let now $\epsilon_\infty \prec \psi$, then there is $x \in [0, \infty)$ such that $\psi(x) > 0$. If furthermore $\varphi \lhd \varepsilon_0$, then there is $\epsilon < 1$ such that $\varphi \leq f_{\delta\epsilon}$. Hence we conclude

$$\begin{aligned}\tau_*(\varphi, \psi)(x) &= \bigvee_u \varphi(u) * \psi(x-u) \leq \bigvee_u f_{\delta\epsilon}(u) * \psi(x-u) \\ &= \bigvee_{u>\delta} \epsilon * \psi(x-u) = \epsilon * \bigvee_{u>\delta} \psi(x-u) \leq \epsilon * \psi(x).\end{aligned}$$

So if $\psi \leq \tau_*(\varphi, \psi)$, then $\psi(x) \leq \epsilon * \psi(x)$, a contradiction. □

We will consider in this paper only commutative, integral quantales $(L, \leq, *)$ with completely distributive underlying lattices.

We assume some familiarity with category theory and refer to the textbooks [2] and [14] for more details and notation. A *construct* is a category $\mathcal{C}$ with a faithful functor $U : \mathcal{C} \longrightarrow SET$, from $\mathcal{C}$ to the category of sets. We always consider a construct as a category whose objects are structured sets (S, ξ) and morphisms are suitable mappings between the underlying sets. A construct is called *topological* if it allows *initial constructions*, i.e. if for every source $(f_i : S \longrightarrow (S_i, \xi_i))_{i \in I}$ there is a unique structure ξ on S, such that a mapping $g : (T, \eta) \longrightarrow (S, \xi)$ is a morphism if and only if for each $i \in I$ the composition $f_i \circ g : (T, \eta) \longrightarrow (S_i, \xi_i)$ is a morphism. We call such a source an *initial source*. An object (S, ξ) in a category $\mathcal{C}$ is called *initially dense in* $\mathcal{C}$ if for any object (T, η) in $\mathcal{C}$ there is an initial source $(f_i : (T, \eta) \longrightarrow (S, \xi))_{i \in I}$.

3. L-approach Spaces and L-metric Spaces

In the sequel, let $L = (L, \leq, *)$ be a commutative and integral quantale, where $(L, \leq)$ is completely distributive. For a set X we denote its power set by $P(X)$.

Definition 3.1. [10] An *L-approach space* is a pair (X, c) of a set and a closure operator $c : P(X) \longrightarrow L^X$ satisfying, for all $x \in X$, $A, B, A_j \subseteq X$ $(j \in J)$, the axioms

(LC1) $c(\{x\})(x) = \top$;

(LC2) $\left(\bigwedge_{y \in B} \bigvee_{j \in J} c(A_j)(y)\right) * c(B)(x) \leq c(\bigcup_{j \in J} A_j)(x)$;

(LC3) $c(\emptyset)(x) = \bot$;

(LC4) $c(A \cup B) = c(A) \vee c(B)$.

A mapping $f : (X, c) \longrightarrow (X', c')$ between two L-approach spaces is called an *L-approach morphism* if $c(A)(x) \leq c'(f(A))(f(x))$ for all $x \in X$ and all $A \subseteq X$. The category with objects the L-approach spaces and morphisms the L-approach morphisms is denoted by L-AP.

Clearly, a closure operator $c : P(X) \longrightarrow L^X$ can equivalently be described by an *L-valued point-set distance function* $\delta : X \times P(X) \longrightarrow L$, writing $\delta(x, A) = c(A)(x)$. With this in mind, we can give the following characterization, which is more closely related to Lowen's original definition [11].

Lemma 3.2. *A pair (X,δ) with a set X and an L-distance $\delta : X \times P(X) \longrightarrow L$ is an L-approach space if, for all $x \in X$, $A, B \subseteq X$, the following axioms are satisfied.*
(LD1) $\delta(x,\{x\}) = \top$;
(LD2) $\delta(x,\emptyset) = \bot$;
(LD3) $\delta(x,A) \vee \delta(x,B) = \delta(x,A\cup B)$ *for all* $A,B \subseteq X$;
(LD4) $\delta(x,A) \geq \delta(x,\overline{A}^{\alpha}) * \alpha$ *for all* $\alpha \in L$, *where* $\overline{A}^{\alpha} = \{x \in X \, : \, \delta(x,A) \geq \alpha\}$.
A mapping $f : (X,\delta) \longrightarrow (X',\delta')$ is an L-approach morphism if and only if $\delta(x,A) \leq \delta'(f(x), f(A))$ for all $x \in X, A \subseteq X$.

Proof. We need only show that (LD4) and (LC2) are equivalent. Let first (LD4) be satisfied. We define $\alpha = \bigwedge_{y\in B}\bigvee_{j\in J}\delta(y,A_j)$ and show that $B \subseteq \overline{\bigcup_{j\in J} A_j}^{\alpha}$. For $y \in B$ we have, as a consequence of (LD3), $\bigvee_{j\in J}\delta(y,A_j) \leq \delta(y,\bigcup_{j\in J}A_j)$ and hence also $\alpha = \bigwedge_{z\in B}\bigvee_{j\in J}\delta(z,A_j) \leq \delta(y,\bigcup_{j\in J}A_j)$. Hence $y \in \overline{\bigcup_{j\in J}A_j}^{\alpha}$. We conclude $\alpha * \delta(x,B) \leq \alpha * \delta(x,\overline{\bigcup_{j\in J}A_j}^{\alpha}) \leq \delta(x,\bigcup_{j\in J}A_j)$ by (LD4), which is (LC2).

The converse follows taking $A_j = A$ and $B = \overline{A}^{\alpha}$. Then $\bigwedge_{y\in B}\delta(y,A) \geq \alpha$ and $\alpha * \delta(x,\overline{A}^{\alpha}) \leq (\bigwedge_{y\in B}\delta(y,A)) * \delta(x,B) \leq \delta(x,A)$. □

We give a further characterization of (LD4).

Lemma 3.3. *Let $(X,\delta) \in |L\text{-}AP|$. Then (LD4) is equivalent to*
(LD4') $\delta(x,B) * \bigwedge_{b\in B}\delta(b,A) \leq \delta(x,A)$ *for all* $A, B \subseteq X$ *and all* $x \in X$.

Proof. Let first (LD4) be true. We define $\alpha = \bigvee\{\gamma \in L \, : \, B \subseteq \overline{A}^{\gamma}\}$. Then $x \in \overline{A}^{\alpha}$ iff $\delta(x,A) \geq \gamma$ for all $\gamma \in L$ such that $B \subseteq \overline{A}^{\gamma}$, i.e. iff $x \in \bigcap_{\gamma : B\subseteq \overline{A}^{\gamma}} \overline{A}^{\gamma} \supseteq B$. Moreover, we have $B \subseteq \overline{A}^{\gamma}$ iff $\bigwedge_{b\in B}\delta(b,A) \geq \gamma$. Hence $\alpha = \bigvee\{\gamma \in L \, : \, \gamma \leq \bigwedge_{b\in B}\delta(b,A)\} = \bigwedge_{b\in B}\delta(b,A)$ and we conclude from (LD4) $\delta(x,A) \geq \delta(x,\overline{A}^{\alpha}) * \alpha \geq \delta(x,B) * \bigwedge_{b\in B}\delta(b,A)$. For the converse, we take $B = \overline{A}^{\alpha}$. Then $\bigwedge_{b\in B}\delta(b,A) \geq \alpha$ and we conclude $\delta(x,A) \geq \bigwedge_{b\in B}\delta(b,A) * \delta(x,B) \geq \alpha * \delta(x,\overline{A}^{\alpha})$, which is (LD4). □

Definition 3.4. An *L-metric space* is a pair (X,d) of a set X and an *L-metric* $d : X \times X \longrightarrow L$ which satisfies the following properties.
(LM1) $d(x,x) = \top$ for all $x \in X$ *(reflexivity)*, and
(LM2) $d(x,y) * d(y,z) \leq d(x,z)$ for all $x,y,z \in X$ *(transitivity)*.
A mapping between two L-metric spaces, $f : (X,d_X) \longrightarrow (Y,d_Y)$ is called an *L-metric morphism* if $d_X(x_1,x_2) \leq d_Y(f(x_1),f(x_2))$ for all $x_1,x_2 \in X$.

We denote the category of L-metric spaces with L-metric morphisms by $L\text{-}MET$. We further denote the fibre over X in $L\text{-}MET$ by $L\text{-}MET(X)$. We note that for $d_j \in L\text{-}MET(X)$ $(j \in J)$, we have that the pointwise infimum $\bigwedge_{j\in J} d_j \in L\text{-}MET(X)$. As also there is a largest L-metric on X, namely $d(x,y) = \top$ for all $x,y \in X$, the set $L\text{-}MET(X)$ is a complete lattice.

In case $L = \{0,1\}$, an L-metric space is a preordered set. If $L = [0,\infty]$ with the opposite order and extended addition as quantale operation, an L-metric space is a quasimetric space. If $L = \Delta^+$ and $*$ is a sup-continuous triangle function, an L-metric space is a probabilistic quasimetric space, see [3].

For a value quantale $(L, \leq, *)$, L-metric spaces were introduced under the name *continuity spaces* and L-metric morphisms were called *nonexpansive*, a name which has its justification if one uses the opposite order, in [3]. Often, L-metric spaces are called *L-categories*, e.g. [6, 19], or *L-preordered sets*, see e.g. [18]. Our main examples being quasimetric spaces and probabilistic (quasi-)metric spaces and because we generalize approach spaces, the theory of which has a strong metrical flavour, we prefer to use the term L-metric space.

Example 3.5. An integral quantale $(L, \leq, *)$ becomes an L-metric space if we define, for $\alpha \in L$, $d_\alpha(x, y) = (\alpha \wedge x) \to (\alpha \wedge y)$, $(x, y \in L)$. In fact, $d_\alpha(x, x) = (\alpha \wedge x) \to (\alpha \wedge x) = \top$ and $d_\alpha(x, y) * d_\alpha(y, z) = ((\alpha \wedge x) \to (\alpha \wedge y)) * ((\alpha \wedge y) \to (\alpha \wedge z)) \leq (\alpha \wedge x) \to (\alpha \wedge z) = d_\alpha(x, z)$.

Lemma 3.6. *Let X be a set and let (X', d') be an L-metric space and let $f : X \longrightarrow X'$. Define $d_f(x, y) = d'(f(x), f(y))$ for all $x, y \in X$, i.e. $d_f = d' \circ (f \times f)$. Then (X, d_f) is an L-metric space.*

Proof. The proof is straightforward and left for the reader. □

We note that for $f : X \longrightarrow X'$ and $g : X' \longrightarrow X''$ and (X'', d'') an L-metric space, we have $d_{g \circ f} = (d_g)_f$.

An L-distance $\delta : X \times P(X) \longrightarrow L$ generates in a natural way an L-metric. This L-metric will be useful later.

Lemma 3.7. *Let $\delta : X \times P(X) \longrightarrow L$ be an L-distance and let $Z \subseteq X$. Then $d_Z(x, y) = \delta(y, Z) \to \delta(x, Z)$ is an L-metric.*

Furthermore, if L satisfies $(\bigwedge_{j \in J} \alpha_j) \to \beta = \bigvee_{j \in J}(\alpha_j \to \beta)$ for all $\alpha_j, \beta \in L$ $(j \in J)$, then for any $A \subseteq X$ we have $\delta(x, A) \leq \bigvee_{a \in A} d_Z(x, a)$.

Proof. We have $d_Z(x, x) = \delta(x, Z) \to \delta(x, Z) = \top$ and $d(x, y) * d(y, z) = (\delta(y, Z) \to \delta(x, Z)) * (\delta(z, Z) \to \delta(y, Z)) \leq \delta(z, Z) \to \delta(x, Z) = d_Z(x, z)$. Hence d is an L-metric on X. Furthermore, from Lemma 3.3 we obtain $\delta(x, A) * \bigwedge_{a \in A} \delta(a, Z) \leq \delta(x, Z)$. Using the condition in the lemma, we obtain $\delta(x, A) \leq \left(\bigwedge_{a \in A} \delta(a, Z)\right) \to \delta(x, Z) = \bigvee_{a \in A} (\delta(a, Z) \to \delta(x, Z)) = \bigvee_{a \in A} d_Z(x, a)$. □

We have noted above that e.g. the interval $[0, \infty]$ with the opposite order and extended addition as quantale operation, as well as complete MV-algebras satisfy the condition stated in the lemma.

Finally we are showing that the category L-MET can nicely be embedded into the category L-AP.

Theorem 3.8. *L-MET can be embedded into L-AP as a coreflective subcategory.*

Proof. Let (X, d) be an L-metric space. We define for $x \in X$ and $A \subseteq X$

$$\delta^d(x, A) = \bigvee_{a \in A} d(x, a).$$

Then (X, δ^d) is an L-approach space. (LD1), (LD2) and (LD3) are easy and left for the reader. We only provide a proof for (LD4). If $y \in \overline{A}^\alpha$, then $\alpha \leq$

$\delta^d(y,A) = \bigvee_{a\in A} d(y,a)$. Hence $\alpha * \delta^d(x,\overline{A}^\alpha) = \alpha * \bigvee_{y\in\overline{A}^\alpha} d(x,y) \leq \bigvee_{a\in A} d(y,a) * \bigvee_{y\in\overline{A}^\alpha} d(x,y) = \bigvee_{y\in\overline{A}^\alpha}\bigvee_{a\in A} d(x,y) * d(y,a) \leq \bigvee_{y\in\overline{A}^\alpha}\bigvee_{a\in A} d(x,a) = \delta^d(x,A)$.

Furthermore, let $(X,d_X),(Y,d_Y) \in |L\text{-}MET|$ and let $f : X \longrightarrow Y$. Then $f : (X,d_X) \longrightarrow (Y,d_Y)$ is an L-metric morphism if and only if $f : (X,\delta^{d_X}) \longrightarrow (Y,\delta^{d_Y})$ is an L-approach morphism. If $f : (X,d_X) \longrightarrow (Y,d_Y)$ is an L-metric morphism, then for $x \in X$ and $A \subseteq X$ we have $\delta^{d_X}(x,A) = \bigvee_{a\in A} d_X(x,A) \leq \bigvee_{a\in A} d_Y(f(x),f(a)) \leq \bigvee_{b\in f(A)} d_Y(f(x),b) = \delta^{d_Y}(f(x),f(A))$. Hence $f : (X,\delta^{d_X}) \longrightarrow (Y,\delta^{d_Y})$ is an L-approach morphism. The converse is obvious using $d(x,y) = \delta^d(x,\{y\})$.

We note that if $(X,d) \neq (X,d')$ for two L-metric spaces, then there are $x,y \in X$ such that $\delta^d(x,\{y\}) = d(x,y) \neq d'(x,y) = \delta^{d'}(x,\{y\})$, i.e. $(X,\delta^d) \neq (X,\delta^{d'})$. Thus the functor

$$G : \left\{ \begin{array}{ccc} L\text{-}MET & \longrightarrow & L\text{-}AP \\ (X,d) & \longmapsto & (X,\delta^d) \\ f & \longmapsto & f \end{array} \right.$$

is an embedding functor.

We define now for $(X,\delta) \in |L\text{-}AP|$

$$d^\delta(x,y) = \delta(x,\{y\}).$$

Then $(X,d^\delta) \in |L\text{-}MET|$. We have $d^\delta(x,x) = \delta(x,\{x\}) = \top$ for all $x \in X$. Furthermore, by (LD1), we have $y \in \overline{\{y\}}^{\delta(y,\{z\})}$ and hence with (LD4) $d^\delta(x,y) * d^\delta(y,z) \leq \delta(x,\overline{\{y\}}^{\delta(y,\{z\})}) * \delta(y,\{z\}) \leq \delta(x,\{y\}) = d^\delta(x,y)$.

It is furthermore not difficult to see that for an L-approach morphism $f : (X,\delta_X) \longrightarrow (Y,\delta_Y)$, $f : (X,d^{\delta_X}) \longrightarrow (X,d^{\delta_Y})$ is an L-metric morphism and that we have for $(X,\delta) \in |L\text{-}AP|$ that $\delta^{(d^\delta)}(x,A) \leq \delta(x,A)$ and for $(X,d) \in |L\text{-}MET|$ we have $d^{(\delta^d)}(x,y) = d(x,y)$. From this the claim follows. □

4. The Category of L-gauge Spaces

Definition 4.1. Let $\mathcal{H} \subseteq L\text{-}MET(X)$ and $d \in L\text{-}MET(X)$.

(1) d is called *locally supported by* $\mathcal{H}$ if for all $x \in X$, $\alpha \lhd \top$, $\bot \prec \omega$ there is $e_x^{\alpha,\omega} \in \mathcal{H}$ such that $e_x^{\alpha,\omega}(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega$;

(2) $\mathcal{H}$ is called *locally directed* if for all finite subsets $\mathcal{H}_0 \subseteq \mathcal{H}$, $\bigwedge_{d\in\mathcal{H}_0} d$ is locally supported by $\mathcal{H}$;

(3) $\mathcal{H}$ is called *locally saturated* if for $d \in L\text{-}MET(X)$ we have $d \in \mathcal{H}$ whenever d is locally supported by $\mathcal{H}$.

(4) The set

$$\widehat{\mathcal{H}} = \{d \in L\text{-}MET(X) \ : \ d \text{ is locally supported by } \mathcal{H}\}$$

is called the *local saturation of* $\mathcal{H}$.

For $L = [0,\infty]$ and the opposite order, Lowen [11, 12, 13] calls a locally supporting family *(locally) dominating*. This expression seems not suitable in our setting why we chose a new term.

We give two characterizations of local support.

Lemma 4.2. *Let $\mathcal{H} \subseteq L\text{-}MET(X)$ and $d \in L\text{-}MET(X)$. Then d is locally supported by $\mathcal{H}$ iff $\bigwedge_{x\in X} \bigwedge_{\perp \prec \omega} \bigvee_{e\in\mathcal{H}} (e(x,\cdot) \to (d(x,\cdot) \vee \omega)) = \top$.*

Proof. Let first d be locally supported by $\mathcal{H}$. Then for $x \in X$, $\alpha \lhd \top$ and $\perp \prec \omega$ there is $e \in \mathcal{H}$ such that $\alpha \leq e(x,\cdot) \to (d(x,\cdot) \vee \omega)$. Hence, for all $\alpha \lhd \top$ we have $\alpha \leq \bigwedge_{x\in X} \bigwedge_{\perp\prec\omega} \bigvee_{e\in\mathcal{H}}(e(x,\cdot) \to (d(x,\cdot) \vee \omega))$ from which $\top = \bigvee_{\alpha\lhd\top} \alpha \leq \bigwedge_{x\in X} \bigwedge_{\perp\prec\omega} \bigvee_{e\in\mathcal{H}}(e(x,\cdot) \to (d(x,\cdot) \vee \omega))$ follows.

Conversely, let $\bigwedge_{x\in X} \bigwedge_{\perp\prec\omega} \bigvee_{e\in\mathcal{H}}(e(x,\cdot) \to (d(x,\cdot) \vee \omega)) = \top$. Then for all $x \in X$ and all $\perp \prec \omega$ we have $\bigvee_{e\in\mathcal{H}}(e(x,\cdot) \to (d(x,\cdot) \vee \omega)) = \top$. Hence, for $\alpha \lhd \top$, there is $e \in \mathcal{H}$ such that $e(x,\cdot) \to (d(x,\cdot) \vee \omega) \geq \alpha$ and this means that d is locally supported by $\mathcal{H}$. $\square$

For the following characterization, we define for a subset $\mathcal{H} \subset L\text{-}MET(X)$ and for $x \in X$, the set $\mathcal{H}(x) = \{f : X \longrightarrow L \ : \ f(\cdot) \geq d(x,\cdot), d \in \mathcal{H}\}$. The idea of this result goes back to [5].

Lemma 4.3. *Let $\mathcal{H} \subseteq L\text{-}MET(X)$ and $d \in L\text{-}MET(X)$. Then d is locally supported by $\mathcal{H}$ iff $\bigwedge_{x\in X} \bigwedge_{\perp\prec\omega} \bigvee\{\alpha \in L \ : \ \alpha \to (d(x,\cdot) \vee \omega) \in \mathcal{H}(x)\} = \top$.*

Proof. Let first d be locally supported by $\mathcal{H}$. Then for all $x \in X$, $\alpha\lhd\top$, $\perp \prec \omega$ there is $e \in \mathcal{H}$ such that $e(x,\cdot) \leq \alpha \to (d(x,\cdot) \vee \omega)$. Therefore $\alpha \to (d(x,\cdot) \vee \omega) \in \mathcal{H}(x)$ and we have $\bigvee\{\alpha \in L \ : \ \alpha \to (d(x,\cdot) \vee \omega) \in \mathcal{H}(x)\} \geq \bigvee_{\alpha\lhd\top} \alpha = \top$. This is true for all $x \in X$ and all $\perp \prec \omega$ and hence $\bigwedge_{x\in X} \bigwedge_{\perp\prec\omega} \bigvee\{\alpha \in L \ : \ \alpha \to (d(x,\cdot) \vee \omega) \in \mathcal{H}(x)\} = \top$.

Let now the condition of the Lemma be true. Then for all $x \in X$ and all $\perp \prec \omega$ we have $\bigvee\{\alpha \in L \ : \ \alpha \to (d(x,\cdot)\vee\omega) \in \mathcal{H}(x)\} = \top$. Let $\alpha\lhd\top$. Then there is $\beta \geq \alpha$ such that $\beta \to (d(x,\cdot)\vee\omega) \in \mathcal{H}(x)$ and because the set $\mathcal{H}(x)$ is an upper set, we find $\alpha \to (d(x,\cdot) \vee \omega) \in \mathcal{H}(x)$. Hence there is $e \in \mathcal{H}$ such that $e(x,\cdot) \leq \alpha \to (d(x,\cdot) \vee \omega)$ and this means that d is locally supported by $\mathcal{H}$. $\square$

Corollary 4.4. *Let $\mathcal{H} \subseteq L\text{-}MET(X)$. The following are equivalent.*

(1) *$\mathcal{H}$ is locally saturated.*
(2) *$\bigwedge_{x\in X} \bigwedge_{\perp\prec\omega} \bigvee_{e\in\mathcal{H}} (e(x,\cdot) \to (d(x,\cdot) \vee \omega)) = \top$ implies $d \in \mathcal{H}$.*
(3) *$\bigwedge_{x\in X} \bigwedge_{\perp\prec\omega} \bigvee\{\alpha \in L \ : \ \alpha \to (d(x,\cdot) \vee \omega) \in \mathcal{H}(x)\} = \top$ implies $d \in \mathcal{H}$.*

Definition 4.5. Let X be a set. $\mathcal{G} \subseteq L\text{-}MET(X)$ is called an *L-gauge* if $\mathcal{G}$ is a filter in $L\text{-}MET(X)$ and $\mathcal{G}$ is locally saturated. In particular, an L-gauge satisfies the axioms

(LG1) $\mathcal{G} \neq \emptyset$;
(LG2) $d \in \mathcal{G}$ and $d \leq e$ implies $e \in \mathcal{G}$;
(LG3) $d, e \in \mathcal{G}$ implies $d \wedge e \in \mathcal{G}$;
(LG4) $\mathcal{G}$ is locally saturated.

The pair $(X, \mathcal{G})$ is then called an *L-gauge space*. A mapping between two L-gauge spaces, $f : (X, \mathcal{G}) \longrightarrow (X', \mathcal{G}')$ is called an *L-gauge morphism* if $d' \circ (f \times f) \in \mathcal{G}$ whenever $d' \in \mathcal{G}'$.

It is not difficult to show that the class of L-gauge spaces together with the L-gauge morphisms forms a category which shall be denoted $L\text{-}GS$.

In case that the quantale L is the interval $[0,\infty]$ with the opposite order and extended addition as quantale operation, then $[0,\infty]$-gauge spaces are approach spaces defined by means of gauges, [13]. We will study the relation of L-approach spaces and L-gauge spaces in the next section.

Definition 4.6. Let $(X,\mathcal{G}) \in |L\text{-}GS|$ and let $\mathcal{H} \subseteq L\text{-}MET(X)$. If $\widehat{\mathcal{H}} = \mathcal{G}$, then $\mathcal{H}$ is called a *basis for the gauge* $\mathcal{G}$.

Proposition 4.7. *Let L be a value quantale. If $\emptyset \neq \mathcal{H} \subseteq L\text{-}MET(X)$ is locally directed, then $\mathcal{G} = \widehat{\mathcal{H}}$ is a gauge with $\mathcal{H}$ as basis.*

Proof. Clearly $\mathcal{H} \subseteq \widehat{\mathcal{H}}$, so that $\mathcal{G} \neq \emptyset$. If $d \in \widehat{\mathcal{H}}$ and $d \leq e$, then for $x \in X$, $\alpha \lhd \top$, $\bot \prec \omega$, there is $e_x^{\alpha,\omega} \in \mathcal{H}$ such that $e_x^{\alpha,\omega}(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega \leq e(x,\cdot) \vee \omega$. Hence e is locally supported by $\mathcal{H}$ and $e \in \widehat{\mathcal{H}}$. Let now $d, e \in \widehat{\mathcal{H}}$. We fix $x \in X$, $\alpha \lhd \top$ and $\bot \prec \omega$. Then there is $\beta \lhd \bot$ such that $\alpha \lhd \beta * \beta$ and hence there are $d_x^{\beta,\omega}, e_x^{\beta,\omega} \in \mathcal{H}$ such that $d_x^{\beta,\omega}(x,\cdot) * \beta \leq d(x,\cdot) \vee \omega$ and $e_x^{\beta,\omega}(x,\cdot) * \beta \leq e(x,\cdot) \vee \omega$. By local directedness then $d_x^{\beta,\omega} \wedge e_x^{\beta,\omega}$ is locally supported by $\mathcal{H}$ and hence there is $f_x^{\beta,\omega} \in \mathcal{H}$ such that $f_x^{\beta,\omega}(x,\cdot) * \beta \leq d_x^{\beta,\omega} \wedge e_x^{\beta,\omega}(x,\cdot) \vee \omega$. We conclude

$$\begin{aligned} f_x^{\beta,\omega}(x,\cdot) * \alpha &\leq f_x^{\beta,\omega}(x,\cdot) * \beta * \beta \leq ((d_x^{\beta,\omega} \wedge e_x^{\beta,\omega}(x,\cdot)) * \beta) \vee (\omega \vee \beta) \\ &\leq ((d^{\beta,\omega}(x,\cdot) * \beta) \wedge (e_x^{\beta,\omega}(x\cdot) * \beta)) \vee \omega \\ &\leq ((d(x,\cdot) \vee \omega) \wedge (e(x,\cdot) \vee \omega) \vee \omega \leq (d \wedge e)(x,\cdot) \vee \omega. \end{aligned}$$

Hence $d \wedge e$ is locally supported by $\mathcal{H}$, i.e. $d \wedge e \in \widehat{\mathcal{H}}$ and $\widehat{\mathcal{H}}$ is a filter.

We finally show that $\widehat{\mathcal{H}}$ is locally saturated. Let $d \in L\text{-}MET(X)$ be locally supported by $\widehat{\mathcal{H}}$ and let $x \in X$, $\alpha \lhd \top$ and $\bot \prec \omega$. There is $\beta \lhd \top$ such that $\alpha \lhd \beta * \beta$ and hence there is $e_x^{\beta,\omega} \in \widehat{\mathcal{H}}$ such that $e_x^{\beta,\omega}(x,\cdot) * \beta \leq d(x,\cdot) \vee \omega$. As $e_x^{\beta,\omega}$ is locally supported by $\mathcal{H}$ there is $f_x^{\beta,\omega} \in \mathcal{H}$ such that $f_x^{\beta,\omega}(x,\cdot) * \beta \leq e_x^{\beta,\omega}(x,\cdot) \vee \omega$ and we conclude

$$f^{\beta,\omega}(x,\cdot) * \alpha \leq f_x^{\beta,\omega}(x\cdot) * \beta * \beta \leq (e_x^{\beta,\omega}(x,\cdot) \vee \omega) * \beta \leq (e_x^{\beta,\omega}(x,\cdot) * \beta) \vee \omega \leq d(x,\cdot) \vee \omega.$$

Hence d is locally supported by $\mathcal{H}$, i.e. $d \in \widehat{\mathcal{H}}$. $\square$

Theorem 4.8. *Let L be a value quantale. Then the category L-GS is topological over SET.*

Proof. Let $f_j : X \longrightarrow X_j$ $(j \in J)$ be a family of mappings and let $(X_j, \mathcal{G}_j) \in |L\text{-}GS|$. We define

$$\mathcal{H} = \{ \bigwedge_{j \in K} d_j \circ (f_j \times f_j) \;:\; K \subseteq J \text{ finite }, d_j \in \mathcal{G}_j \forall j \in J \}.$$

Clearly $\mathcal{H}$ is locally directed, as finite meets of members of $\mathcal{H}$ belong to $\mathcal{H}$. For $d_j \in \mathcal{G}_j$ we have $d_j \circ (f_j \times f_j) \in \mathcal{H} \subseteq \widehat{\mathcal{H}}$, so that all mappings $f_j : (X, \widehat{\mathcal{H}}) \longmapsto (X_j, \mathcal{G}_j)$ are L-gauge morphisms. Let now $(Y,\mathcal{K}) \in |L\text{-}GS|$ and $g : Y \longrightarrow X$ be a mapping such that all $f_j \circ g : (Y,\mathcal{K}) \longrightarrow (X_j, \mathcal{G}_j)$ are L-gauge morphisms. Then for $d_j \in \mathcal{G}_j$ we know that $(d_{f_j})_g = d \circ (f_j \times f_j) \circ (g \times g) \in \mathcal{K}$. Let now $d \in \widehat{\mathcal{H}}$. Then for $\alpha \lhd \top$,

$\bot \prec \omega$ we have for all $x \in X$ that

$$(\bigwedge_{j\in K} d_j \circ (f_j \times f_j)(x,\cdot)) * \alpha \le d(x,\cdot) \vee \omega,$$

with some finite set $K \subseteq J$. We conclude for all $y_1, y_2 \in Y$ that

$$d_g(y_1,y_2) \vee \omega = d(g(y_1),g(y_2)) \vee \omega \ge (\bigwedge_{j\in K} d_j \circ (f_j \times f_j) \circ (g \times g)(y_1,y_2)) * \alpha.$$

As $\mathcal{K}$ is a filter, we conclude $\bigwedge_{j\in K} d_j \circ (f_j \times f_j) \circ (g \times g) \in \mathcal{K}$. Hence d_g is locally supported by $\mathcal{K}$ and therefore $d_g \in \mathcal{K}$ and $g : (Y,\mathcal{K}) \longrightarrow (X,\widehat{\mathcal{H}})$ is an L-gauge morphism. □

We finally show that L-GS has an initially dense object. To this end, we consider the L-metrics $d_\alpha : L \times L \longrightarrow L$ introduced in Example 3.4 and note that $\mathcal{H}_L = \{\bigwedge_{\alpha\in K} d_\alpha \;:\; K \subseteq L \text{ finite}\}$ is locally directed. Hence $(L,\widehat{\mathcal{H}_L})$ is an object in L-GS.

Theorem 4.9. *Let $(L,\le,*)$ be a value quantale and let $(X,\mathcal{G}) \in |L\text{-}GS|$. Then*

$$\Big(d_x(\cdot) = d(x,\cdot) : (X,\mathcal{G}) \longrightarrow (L,\widehat{\mathcal{H}_L})\Big)_{x\in X, d\in\mathcal{G}}$$

is an initial source.

Proof. We show that $\mathcal{G}$ is the initial gauge for the source. To this end, we first show that all d_x are L-gauge morphisms. Let $x \in X$ and $d \in \mathcal{G}$. Let further $e \in \widehat{\mathcal{H}_L}$. Then e is locally supported by $\mathcal{H}_L$, i.e. for all $\eta \in L$, $\alpha \lhd \top$ and $\bot \prec \omega$ there is a finite set $K = K_{\eta,\alpha,\omega} \subseteq L$ and $d_\gamma \in \mathcal{H}_L$ $(\gamma \in K)$ such that

$$\bigwedge_{\gamma\in K} d_\gamma(\eta,\cdot) * \alpha \le e(\eta,\cdot) \vee \omega.$$

We show that $e \circ (d_x \times d_x) \in \mathcal{G}$. For any $\kappa \in L$ we have $(\kappa \wedge d(x_1,x_2)) * d(x_1,x_2) \le \kappa \wedge (d(x,x_1) * d(x_1,x_2)) \le \kappa \wedge d(x,x_2)$. Hence $d(x_1,x_2) \le (\kappa \wedge d(x,x_1)) \to (\kappa \wedge d(x,x_2)) = d_\kappa(d(x,x_1), d(x,x_2))$.

Let now $x_1 \in X$, $\alpha \lhd \top$ and $\bot \prec \omega$. Then for all $x_2 \in X$ we have

$$\begin{aligned} e \circ (d_x \times d_x)(x_1,x_2) \vee \omega &= e(d(x,x_1), d(x,x_2)) \vee \omega \\ &\ge \bigwedge_{\gamma\in K_{d(x,x_1),\alpha,\omega}} d_\gamma(\eta,\cdot) * \alpha \ge d(x_1,x_2) * \alpha. \end{aligned}$$

Hence $e \circ (d_x \times d_x)$ is locally supported by $\mathcal{G}$, and therefore belongs to $\mathcal{G}$. Consequently, if we denote the initial L-gauge on X for the source $(d_x : X \longrightarrow (L,\widehat{\mathcal{H}_L}))_{x\in X,d\in\mathcal{G}}$ by $\mathcal{G}_{init}$, we have $\mathcal{G}_{init} \subseteq \mathcal{G}$.

Let now $d \in \mathcal{G}$. We show that d is locally supported by $\mathcal{G}_{init}$. Let $x \in X$, $\alpha \lhd \top$ and $\bot \prec \omega$. Then for $x_2 \in X$ we have

$$\begin{aligned} (d_\alpha \circ (d_x \times d_x)(x,x_2)) * \alpha &= ((\alpha \wedge d(x,x)) \to (\alpha \wedge d(x,x_2))) * \alpha \\ &= \alpha * (\alpha \to (\alpha \wedge d(x,x_2))) \le \alpha \wedge d(x,x_2) \le d(x,x_2) \vee \omega. \end{aligned}$$

Hence we have seen $d_\alpha \circ (d_x \times d_x)(x,\cdot) * \alpha \le d(x,\cdot) \vee \omega$ and because $d_\alpha \circ (d_x \times d_x) \in \mathcal{G}_{init}$ we conclude that d is locally supported by $\mathcal{G}_{init}$ and therefore $d \in \mathcal{G}_{init}$ and the proof is complete. □

5. L-approach Spaces as L-gauge Spaces

Proposition 5.1. *Let $(X,\delta) \in |L\text{-}AP|$. Define*

$$\mathcal{G}^\delta = \{d \in L\text{-}MET(X) \ : \ \forall A \subseteq X, x \in X \ : \delta(x,A) \leq \bigvee_{a \in A} d(x,a)\}.$$

Then $(X,\mathcal{G}^\delta) \in |L\text{-}GS|$.

Proof. We first show that $\mathcal{G}^\delta$ is a filter in $L\text{-}MET(X)$. Clearly $d \equiv \top \in \mathcal{G}^\delta$ and hence $\mathcal{G} \neq \emptyset$. If $d \in \mathcal{G}^\delta$ and $e \geq d$ then $\bigvee_{a\in A} e(x,a) \geq \bigvee_{a \in A} d(x,a) \geq \delta(x,A)$ and hence $e \in \mathcal{G}^\delta$. Finally, let $d_1, d_2 \in \mathcal{G}^\delta$. We denote $\mathcal{G}_0 = \{d_1, d_2\}$. By complete distributivity then

$$\bigvee_{a\in A} \bigwedge_{d \in \mathcal{G}_0} d(x,a) = \bigwedge_{\varphi \in \mathcal{G}_0^A} \bigvee_{a \in A} \varphi(a)(x,a).$$

Now, for $\varphi \in \mathcal{G}_0^A$ we have

$$\bigvee_{a\in A} \varphi(a)(x,a) = \bigvee_{d \in \mathcal{G}_0} \bigvee_{a \in \varphi^{\leftarrow}(d)} d(x,a) \geq \bigvee_{d \in \mathcal{G}_0} \delta(x, \varphi^{\leftarrow}(d))$$

$$= \delta(x, \bigcup_{d \in \mathcal{G}_0} \varphi^{\leftarrow}(d)) = \delta(x,A).$$

Hence $\bigvee_{a\in A} \bigwedge_{d \in \mathcal{G}_0} d(x,a) = \bigwedge_{\varphi \in \mathcal{G}_0^A} \bigvee_{a\in A} \varphi(a)(x,a) \geq \delta(x,A)$ and therefore $d_1 \wedge d_2 \in \mathcal{G}^\delta$.

Next we show that $\mathcal{G}^\delta$ is locally saturated. Let $d \in L\text{-}MET(X)$, let $x \in X$, $\alpha \lhd \top$ and $\bot \prec \omega$ and let $d_x^{\alpha,\omega} \in \mathcal{G}^\delta$ such that $d_x^{\alpha,\omega}(x,\cdot) * \alpha \leq d(x,\cdot) \vee \omega$. Then

$$\bigvee_{a\in A} d(x,a) \vee \omega \geq \bigvee_{a \in A} d_x^{\alpha,\omega}(x,a) * \alpha \geq \delta(x,A) * \alpha$$

and hence

$$\omega \vee \bigvee_{a \in A} d(x,a) \geq \bigvee_{\alpha \lhd \top} \delta(x,A) * \alpha = \delta(x,A) * \bigvee_{\alpha \lhd \top} \alpha = \delta(x,A) * \top = \delta(x,A).$$

This is true for any $\bot \prec \omega$ and we conclude

$$\delta(x,A) \leq \bigwedge_{\bot \prec \omega} \left(\omega \vee \bigvee_{a\in A} d(x,a)\right) = \left(\bigvee_{a \in A} d(x,a)\right) \vee \bigwedge_{\bot \prec \omega} \omega = \bigvee_{a\in A} d(x,a).$$

Hence $d \in \mathcal{G}^\delta$ and the proof is complete. □

Proposition 5.2. *Let $(X,\delta), (X',\delta') \in |L\text{-}AP|$ and let $f : (X,\delta) \longrightarrow (X',\delta')$ be an L-approach morphism. Then $f : (X,\mathcal{G}^\delta) \longrightarrow (X',\mathcal{G}^{\delta'})$ is an L-gauge morphism.*

Proof. Let $d' \in \mathcal{G}^{\delta'}$. Then for all $A' \subseteq X'$ and all $x' \in X'$ we have $\delta'(x',A') \leq \bigvee_{a' \in A'} d'(x',a')$. We want to show that $d_f \in \mathcal{G}^\delta$. Let $x \in X$ and let $A \subseteq X$. Then $\delta(x,A) \leq \delta'(f(x), f(A)) \leq \bigvee_{a \in A} d'(f(x), f(a)) = \bigvee_{a\in A} d_f(x,a)$. Hence $d_f \in \mathcal{G}^\delta$. □

Hence we can define a functor $E : \begin{cases} L\text{-}AP & \longrightarrow & L\text{-}GS \\ (X,\delta) & \longmapsto & (X,\mathcal{G}^{\delta}) \\ f & \longmapsto & f \end{cases}$. We will show in the sequel that in the case of a quantale that satisfies $(\bigwedge_{j\in J}\alpha_j)\to\beta = \bigvee_{j\in J}(\alpha_j\to\beta)$ for all $\alpha_j,\beta\in L$, this functor yields an embedding that is coreflective.

Lemma 5.3. *Let L satisfy $(\bigwedge_{j\in J}\alpha_j)\to\beta = \bigvee_{j\in J}(\alpha_j\to\beta)$ for all $\alpha_j,\beta\in L$. Then the functor E is injective on objects.*

Proof. Let $(X,\delta),(X,\delta')\in|L\text{-}AP|$ with $\delta\neq\delta'$. Then there are $x\in X$ and $A\subseteq X$ such that $\delta(x,A)\neq\delta'(x,A)$. Without loss of generality we may assume $\delta(x,A)\not\leq\delta'(x,A)$. From Lemma 3.7 we know that $d_A\in\mathcal{G}^{\delta}$ where d_A is defined by $d_A(x,y)=\delta(y,A)\to\delta(x,A)$. Assume that $d_A\in\mathcal{G}^{\delta'}$. Then $\delta'(x,A)\leq\bigvee_{a\in A}d_A(x,a)=\bigvee_{a\in A}(\delta(a,A)\to\delta(x,A))=\delta(x,A)$, as for $a\in A$ we have $\delta(a,A)=\top$. This is a contradiction and hence $d_A\notin\mathcal{G}^{\delta'}$ and $(X,\mathcal{G}^{\delta})\neq(X,\mathcal{G}^{\delta'})$. □

Proposition 5.4. *Let $(X,\mathcal{G})\in|L\text{-}GS|$. If we define $\delta^{\mathcal{G}}: X\times P(X)\longrightarrow L$ by*

$$\delta^{\mathcal{G}}(x,A)=\bigwedge_{d\in\mathcal{G}}\bigvee_{a\in A}d(x,a),$$

then $(X,\delta^{\mathcal{G}})\in|L\text{-}AP|$.

Proof. (LD1) We have $\delta^{\mathcal{G}}(x,\{x\})=\bigwedge_{d\in\mathcal{G}}d(x,x)=\top$.

(LD2) We have $\delta^{\mathcal{G}}(x,\emptyset)=\bigwedge_{d\in\mathcal{G}}\bigvee\emptyset=\bot$.

(LD3) Clearly $\delta^{\mathcal{G}}(x,A\cup B)\geq\delta^{\mathcal{G}}(x,A)\vee\delta(x,B)$. For the converse inequality, let $\delta^{\mathcal{G}}(x,A)\vee\delta^{\mathcal{G}}(x,B)\prec\alpha$. Then there are $d_A,d_B\in\mathcal{G}$ such that $\bigvee_{a\in A}d_A(x,a)\prec\alpha$ and $\bigvee_{b\in B}d_B(x,b)\prec\alpha$. As $\mathcal{G}$ is an L-gauge we have $d_A\wedge d_B\in\mathcal{G}$ and by local saturation there is, for $\beta\lhd\top$, $\bot\prec\omega$ and $x\in X$ an L-metric $e_x^{\beta,\omega}\in\mathcal{G}$ such that $e_x^{\beta,\omega}(x,\cdot)*\beta\leq d_A\wedge d_B(x,\cdot)\vee\omega$. Hence we conclude

$$\begin{aligned}
\delta^{\mathcal{G}}(x,A\cup B)*\beta &= \left(\bigwedge_{d\in\mathcal{G}}\bigvee_{c\in A\cup B}d(x,c)\right)*\beta \leq \bigwedge_{d\in\mathcal{G}}\left(\bigvee_{c\in A\cup B}d(x,c)*\beta\right)\\
&\leq \bigvee_{c\in A\cup B}e_x^{\beta,\omega}(x,c)*\beta\\
&= \left(\bigvee_{a\in A}e_x^{\beta,\omega}(x,a)\vee\bigvee_{b\in B}e_x^{\beta,\omega}(x,b)\right)*\beta\\
&= \left(\bigvee_{a\in A}e_x^{\beta,\omega}(x,a)*\beta\right)\vee\left(\bigvee_{b\in B}e_x^{\beta,\omega}(x,b)*\beta\right)
\end{aligned}$$

$$\leq \left(\bigvee_{a\in A}d_A(x,a)\vee\omega\right)\vee\left(\bigvee_{b\in B}d_B(x,b)\vee\omega\right) \leq \alpha\vee\omega.$$

Hence we have seen that for all $\beta \lhd \top$ and all $\bot \prec \omega$ we have $\delta(x, A \cup B) * \beta \leq \alpha \vee \omega$. Therefore we conclude

$$\delta^{\mathcal{G}}(x, A \cup B) = \delta^{\mathcal{G}}(x, A \cup B) * \bigvee_{\beta \lhd \top} \beta = \bigvee_{\beta \lhd \top} \delta^{\mathcal{G}}(x, A \cup B) * \beta \leq \alpha \vee \omega$$

and consequently also $\delta^{\mathcal{G}}(x, A \cup B) \leq \bigwedge_{\bot \prec \omega} (\alpha \vee \omega) = \alpha \vee \bigwedge_{\bot \prec \omega} \omega = \alpha$. From this we obtain $\delta^{\mathcal{G}}(x, A \cup B) \leq \bigwedge \{\alpha \in L \,:\, \delta^{\mathcal{G}}(x, A) \vee \delta^{\mathcal{G}}(x, B) \prec \alpha\} = \delta(x, A) \vee \delta(x, B)$.

(LD4) Let $x \in X$, $A \subseteq X$, $\alpha \in L$ and $\beta \lhd \alpha$. For $b \in \overline{A}^{\alpha}$ we have $\bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(b, a) = \delta^{\mathcal{G}}(b, A) \geq \alpha$. Hence for all $d \in \mathcal{G}$ there is $a_\beta \in A$ such that $d(b, a_\beta) \rhd \beta$ and we conclude $d(x, a_\beta) \geq d(x, b) * d(b, a_\beta) \geq d(x, b) * \beta$. Therefore $\bigvee_{a \in A} d(x, a) \geq d(x, b) * \beta$. This is true for any $b \in \overline{A}^{\beta}$ and hence we obtain

$$\bigvee_{a \in A} d(x, a) \geq \bigvee_{b \in \overline{A}^{\beta}} (d(x, b) * \beta) = (\bigvee_{b \in \overline{A}^{\beta}} d(x, b)) * \beta.$$

As $\beta \lhd \alpha$ was arbitrary, we conclude, using $\overline{A}^{\alpha} \subseteq \overline{A}^{\beta}$,

$$\bigvee_{a \in A} d(x, a) \geq \bigvee_{\beta \lhd a} \left((\bigvee_{b \in \overline{A}^{\alpha}} d(x, b)) * \beta \right) = \bigvee_{b \in \overline{A}^{\alpha}} d(x, b) * \bigvee_{\beta \lhd \alpha} \beta = (\bigvee_{b \in \overline{A}^{\alpha}} d(x, b)) * \alpha.$$

This yields

$$\delta^{\mathcal{G}}(x, A) \geq \bigwedge_{d \in \mathcal{G}} \left((\bigvee_{b \in \overline{A}^{\alpha}} d(x, b)) * \alpha \right) \geq \left(\bigwedge_{d \in \mathcal{G}} (\bigvee_{b \in \overline{A}^{\alpha}} d(x, b)) \right) * \alpha = \delta^{\mathcal{G}}(x, \overline{A}^{\alpha}) * \alpha$$

and (LD4) is true. □

Proposition 5.5. *Let $(X, \mathcal{G}), (X', \mathcal{G}') \in |L\text{-}GS|$ and let $f : (X, \mathcal{G}) \longrightarrow (X', \mathcal{G}')$ be an L-gauge morphism. Then $f : (X, \delta^{\mathcal{G}}) \longrightarrow (X', \delta^{\mathcal{G}'})$ is an L-approach morphism.*

Proof. Let $x \in X$ and $A \subseteq X$. We have

$$\delta^{\mathcal{G}'}(f(x), f(A)) = \bigwedge_{d' \in \mathcal{G}'} \bigvee_{a \in A} d'(f(x), f(a)) = \bigwedge_{d' \in \mathcal{G}'} \bigvee_{a \in A} d_f(x, a).$$

As for $d' \in \mathcal{G}'$ we have $d_f \in \mathcal{G}$ we conclude

$$\delta^{\mathcal{G}'}(f(x), f(A)) \geq \bigwedge_{d \in \mathcal{G}} \bigvee_{a \in A} d(x, a) = \delta^{\mathcal{G}}(x, A).$$

□

Hence we can define a functor $K : \left\{ \begin{array}{ccc} L\text{-}GS & \longrightarrow & L\text{-}AP \\ (X, \mathcal{G}) & \longmapsto & (X, \delta^{\mathcal{G}}) \\ f & \longmapsto & f \end{array} \right.$.

We will need the following result.

Proposition 5.6. *Let L satisfy $(\bigwedge_{j \in J} \alpha_j) \to \beta = \bigvee_{j \in J} (\alpha_j \to \beta)$ for all $\alpha_j, \beta \in L$. Let $(X, \delta) \in |L\text{-}AP|$ and define $\mathcal{G}^{\delta}$ as in Proposition 5.1. Then for all $A \subseteq X$ and all $x \in X$ we have $\delta(x, A) = \bigwedge_{d \in \mathcal{G}^{\delta}} \bigvee_{a \in A} d(x, a)$.*

Proof. For $d \in \mathcal{G}^{\delta}$ we have $\bigvee_{a\in A} d(x,a) \geq \delta(x,A)$ and hence $\bigwedge_{d\in\mathcal{G}^{\delta}} \bigvee_{a\in A} d(x,a) \geq \delta(x,A)$. For the converse inequality we make use of Lemma 3.7. Then for any $Z \subseteq X$, $d_Z \in \mathcal{G}^{\delta}$, where $d_Z(x,y) = \delta(y,Z) \to \delta(x,Z)$. Hence we conclude

$$\begin{aligned}\bigwedge_{d\in\mathcal{G}^{\delta}} \bigvee_{a\in A} d(x,a) \;&\leq\; \bigvee_{Z\subseteq X} \bigvee_{a\in A} d_Z(x,a) \;=\; \bigwedge_{Z\subseteq X} \bigvee_{a\in A} (\delta(a,Z) \to \delta(x,Z)) \\ &\leq\; \bigvee_{a\in A} (\delta(a,A) \to \delta(x,A)) \;=\; \delta(x,A)\end{aligned}$$

as for $a \in A$ we have by (LD1) that $\delta(a,A) = \top$. $\square$

Corollary 5.7. *Let L satisfy $(\bigwedge_{j\in J} \alpha_j) \to \beta = \bigvee_{j\in J}(\alpha_j \to \beta)$ for all $\alpha_j, \beta \in L$. Let $(X,\delta) \in |L\text{-}AP|$. Then $\delta^{(\mathcal{G}^{\delta})} = \delta$, i.e. we have $K(E((X,\delta))) = (X,\delta)$.*

Proposition 5.8. *Let $(X,\mathcal{G}) \in |L\text{-}GS|$. Then $\mathcal{G} \subseteq \mathcal{G}^{(\delta^{\mathcal{G}})}$, i.e. we have $E(K((X,\mathcal{G}))) \geq (X,\mathcal{G})$.*

Proof. For $d \in \mathcal{G}$ we have $\delta^{\mathcal{G}}(x,A) \leq \bigvee_{a\in A} d(x,a)$ and hence $d \in \mathcal{G}^{(\delta^{\mathcal{G}})}$. $\square$

As a corollary, we obtain the following theorem.

Theorem 5.9. *Let L satisfy $(\bigwedge_{j\in J} \alpha_j) \to \beta = \bigvee_{j\in J}(\alpha_j \to \beta)$ for all $\alpha_j, \beta \in L$. Then the category L-AP is isomorphic to a coreflective subcategory of L-GS.*

In general, $\mathcal{G}^{(\delta^{\mathcal{G}})} \neq \mathcal{G}$, as is shown by the following two examples.

Example 5.10. Let $L = [0,1] \cup \{\bot = -1, \top = 2\}$ and the order inherited from $\mathbb{R}$ with $\wedge = *$ as the quantale operation. Then $\bot \prec \bot$ and $\top \lhd \top$. Let further $X = (0,1)$ and define, for $x \in X$, the L-metric $e_x : X \times X \longrightarrow L$ by

$$e_x(a,b) = \begin{cases} \top & \text{if} \quad a = b \\ x & \text{if} \quad a \neq b \end{cases}.$$

It is easily checked that e_x is an L-metric on X. Furthermore, we have for $A \subseteq X$ and $y \in X$

$$\bigwedge_{x\in X} \bigvee_{a\in A} e_x(y,a) = \bigvee_{a\in A} \bigwedge_{x\in X} e_x(y,a).$$

If $y \in A$, then we have $\bigwedge_{x\in X} \bigvee_{a\in A} e_x(y,a) \geq \bigvee_{a\in A} \bigwedge_{x\in X} e_x(y,a) \geq \bigwedge_{x\in X} e_x(y,y) = \top$. If $y \notin A$, then we have $y \neq a$ for all $a \in A$ and hence $\bigvee_{a\in A} \bigwedge_{x\in X} e_x(y,a) \leq \bigwedge_{x\in X} \bigvee_{a\in A} e_x(y,a) \leq \bigwedge_{x\in X} x = 0$.

We define now

$$\mathcal{H} = \{ \bigwedge_{x\in K} e_x \;:\; K \subseteq X \text{ finite } \}$$

Then $\mathcal{H}$ is locally directed and we denote $\mathcal{G} = \widehat{\mathcal{H}}$. We define $d_0 = \bigwedge_{x\in X} e_x$. For $A \subseteq X$ and $y \in X$ we have

$$\bigwedge_{d\in\mathcal{G}} \bigvee_{a\in A} d(y,a) \leq \bigwedge_{x\in X} \bigvee_{a\in A} e_x(y,a) = \bigvee_{a\in A} \bigwedge_{x\in X} e_x(y,a),$$

and hence $d_0 \in \mathcal{G}^{(\delta^{\mathcal{G}})}$. However, $d_0 \notin \mathcal{G}$. It is routine to verify that for $y \in X$, $\alpha = \top$ and $\beta = \perp$ there is no finite subset $K \subseteq X$ such that $\bigwedge_{x\in K} e_x(y,\cdot) = \bigwedge_{x\in K} e_x(y,\cdot) \wedge \top \le d_0(y,\cdot) \vee \perp = d_0(y,\cdot)$. Hence d_0 is not locally supported by $\mathcal{H}$, i.e. $d_0 \notin \mathcal{G}$. With regard to the following theorem we note that L is a linearly ordered value quantale but does not satisfy the property (I).

Example 5.11. Let $L = \Delta^+$. For $0 \le \alpha, \beta \le 1$ we define the distance distribution functions $\varphi_{\alpha\beta} \in \Delta^+$ by

$$\varphi_{\alpha\beta}(x) = \begin{cases} 0 & \text{if} \quad 0 \le x < 1-\alpha \\ \frac{1}{2\alpha}(x+\alpha-1) & \text{if} \quad 1-\alpha < x \le 1 \\ \frac{1}{2\beta}(x+\beta-1) & \text{if} \quad 1 < x \le 1+\beta \\ 1 & \text{if} \quad 1+\beta < x \end{cases}.$$

Furthermore, we put $\varphi_\alpha = \varphi_{\alpha\alpha}$ for short. Then $\varphi_\alpha \wedge \varphi_\beta = \varphi_{\alpha\wedge\beta,\alpha\vee\beta}$ and $\bigwedge_{0<\alpha<1} \varphi_\alpha = \varphi_{01}$. We consider now, for a set X and $0 < \alpha < 1$, the *equilateral space* [17] (X, d_α) with

$$d_\alpha(p,q) = \begin{cases} \varphi_\alpha & \text{if} \quad p \ne q \\ \varepsilon_0 & \text{if} \quad p = q \end{cases}.$$

It is shown in [17] that for any triangle function τ, an equilateral space is a (Δ^+, τ)-metric space.

For a non-empty $A \subseteq X$ and $p \in X$ we moreover have

$$\bigwedge_{0<\alpha<1} \bigvee_{a\in A} d_\alpha(p,a) = \begin{cases} \varphi_{01} & \text{if } p \notin A \\ \varepsilon_0 & \text{if } p \in A \end{cases},$$

and also

$$\bigvee_{a\in A} \bigwedge_{0<\alpha<1} d_\alpha(p,a) = \begin{cases} \varphi_{01} & \text{if } p \notin A \\ \varepsilon_0 & \text{if } p \in A \end{cases},$$

and the equality $\bigwedge_{0<\alpha<1} \bigvee_{a\in A} d_\alpha(p,a) = \bigvee_{a\in A} \bigwedge_{0<\alpha<1} d_\alpha(p,a)$ holds trivially if $A = \emptyset$. We define $\mathcal{H} = \{\bigwedge_{\alpha\in K} d_\alpha \ : \ K \subseteq (0,1) \text{ finite}\}$. Then $\mathcal{H}$ is locally directed and we define $\mathcal{G} = \widehat{\mathcal{H}}$. For $A \subseteq X$ and $p \in X$ we then have

$$\bigwedge_{d\in\mathcal{G}} \bigvee_{a\in A} d(p,a) \le \bigwedge_{0<\alpha<1} \bigvee_{a\in A} d_\alpha(p,a) = \bigvee_{a\in A} \bigwedge_{0<\alpha<1} d_\alpha(p,a),$$

and hence $d_0 = \bigwedge_{0<\alpha<1} d_\alpha \in \mathcal{G}^{(\delta^{\mathcal{G}})}$. However, for $\alpha = f_{1/2,1/2} \lhd \varepsilon_0$ (see Lemma 2.11) and $\beta = g_{1/4,2}$ where $g_{\delta,\gamma} = \begin{cases} \gamma & \text{if } 0 < x \le \delta \\ 1 & \text{if } \delta < x \end{cases}$, we have $\varepsilon_\infty \prec g_{1/4,2}$ but there is no finite subset $K \subseteq (0,1)$ such that

$$\left(\bigwedge_{\alpha\in K} d_\alpha(p,\cdot) \wedge f_{1/2,1/2}\right)(x) \le \left(d_0(p,\cdot) \vee g_{1/4,2}\right)(x)$$

for all $x \in [0,\infty]$. Indeed, for $p \ne q$ we have with $\delta = \bigwedge_{\alpha\in K} \alpha$ and $\gamma = \bigvee_{\alpha\in K} \alpha$ that $\bigwedge_{\alpha\in K} d_\alpha(p,q) = \varphi_{\delta\gamma}$ and for $1 - \delta/2 < x < 1$ we have $\frac{1}{4} < (\bigwedge_{\alpha\in K} d_\alpha(p,q) \wedge f_{1/2,1/2})(x) < \frac{1}{2}$ and $(d_0(p,q) \vee g_{1/4,2})(x) = \frac{1}{4}$. Therefore d_0 is not locally supported by $\mathcal{G}$ and hence $d_0 \notin \mathcal{G}$.

With regard to the following theorem, we note that if we choose the triangle function induced by the product t-norm, $L = \Delta^+$ satisfies the condition (I) but is not linearly ordered.

Under certain assumptions, however, we can guarantee that the categories L-AP and L-GS are isomorphic.

Theorem 5.12. *Let $(L, \leq, *)$ be a linearly ordered value quantale that satisfies the condition (I). Let further $\mathcal{G} \subseteq L$-$MET(X)$ be an L-gauge. Then $\mathcal{G}^{(\delta^{\mathcal{G}})} = \mathcal{G}$.*

Proof. We have seen above that $\mathcal{G} \subseteq \mathcal{G}^{(\delta^{\mathcal{G}})}$. Now we show that $\mathcal{G}^{(\delta^{\mathcal{G}})} \subseteq \mathcal{G}$. Let $d_0 \in \mathcal{G}^{(\delta^{\mathcal{G}})}$ and assume $d_0 \notin \mathcal{G}$. Then d_0 is not locally supported by $\mathcal{G}$ and hence there is an $x \in X$, $\alpha \lhd \top$, $\perp \prec \omega$ such that for all $e \in \mathcal{G}$ we have $e(x, \cdot) * \alpha \not\leq d_0(x, \cdot) \vee \omega$. As L is a value quantale, there is $\beta \lhd \top$ such that $\alpha \lhd \beta * \beta$ and hence we have for all $e \in \mathcal{G}$

$$e(x, \cdot) * (\beta * \beta) \not\leq d_0(x, \cdot) \vee \omega.$$

Consider a finite subset $\mathcal{D}_0 \subseteq \mathcal{G}$ and define

$$A(\mathcal{D}_0) = \{y \in X \;:\; \bigwedge_{d \in \mathcal{D}_0} d(x, y) * \beta \not\leq d_0(x, y) \vee \omega\}.$$

As $\mathcal{G}$ is locally directed, there is $e_0 \in \mathcal{G}$ such that

$$e_0(x, y) * \beta \leq \bigwedge_{d \in \mathcal{D}_0} d(x, y) \vee \omega.$$

As a consequence, if $e_0(x, y) * (\beta * \beta) \not\leq d_0(x, y) \vee \omega$, then $\bigwedge_{d \in \mathcal{D}_0} d(x, y) * \beta \not\leq d_0(x, y) \vee \omega$. For otherwise we had

$$e_0(x, y) * (\beta * \beta) \leq \left(\left(\bigwedge_{d \in \mathcal{D}_0} d(x, y) \right) * \beta \right) \vee \omega \leq d_0(x, y) \vee \omega,$$

a contradiction. It follows that

$$\emptyset \neq \{y \in X \;:\; e(x, y) * (\beta * \beta) \not\leq d_0(x, y) \vee \omega\} \subseteq A(\mathcal{D}_0).$$

Moreover we have for finite subsets $\mathcal{D}_0, \mathcal{D}_1 \subset \mathcal{G}$ that $A(\mathcal{D}_0 \cup \mathcal{D}_1) \subseteq A(\mathcal{D}_0) \cap A(\mathcal{D}_1)$ and hence the system $\{A(\mathcal{D}_0) \;:\; \mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}\}$ is a filter basis on X. We conclude, using $\delta^{(\mathcal{G}^{(\delta^{\mathcal{G}})})} = \delta^{\mathcal{G}}$,

$$\begin{aligned} & \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \delta^{\mathcal{G}}(x, A(\mathcal{D}_0)) \vee \omega \right) * \beta \\ = \; & \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \bigwedge_{e \in \mathcal{G}} \bigvee_{a \in A(\mathcal{D}_0)} e(x, a) \right) * \beta \end{aligned}$$

$$\geq \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \bigwedge_{e \in \mathcal{G}} \bigvee_{a \in A(\mathcal{D}_0 \cup \{e\})} \left(\bigwedge_{d \in \mathcal{D}_0} (d \wedge e)(x,a) \right) \right) * \beta$$
$$= \left(\bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \bigvee_{a \in A(\mathcal{D}_0)} \bigwedge_{d \in \mathcal{D}_0} d(x,a) \right) * \beta$$
$$= \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \bigvee_{a \in A(\mathcal{D}_0)} \left(\bigwedge_{d \in \mathcal{D}_0} d(x,a) * \beta \right).$$

As L is linearly ordered, the last expression is

$$\geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \bigvee_{a \in A(\mathcal{D}_0)} (d_0(x,a) \vee \omega)$$
$$\geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \bigwedge_{e \in \mathcal{G}^{(\delta^{\mathcal{G}})}} \bigvee_{a \in A(\mathcal{D}_0)} (e(x,a) \vee \omega)$$
$$\geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \delta^{(\mathcal{G}^{(\delta^{\mathcal{G}})})}(x, A(\mathcal{D}_0) \vee \omega \geq \bigwedge_{\mathcal{D}_0 \subseteq \mathcal{G} \text{ finite}} \delta^{\mathcal{G}}(x, A(\mathcal{D}_0) \vee \omega.$$

As L satisfies the property (I), this is a contradiction and hence $d_0 \in \mathcal{G}$. □

We obtain from Corollary 5.7 and Theorem 5.12 the following result.

Theorem 5.13. *Let $(L, \leq, *)$ be a linearly ordered value quantale that satisfies the condition (I) and $(\bigwedge_{j \in J} \alpha_j) \to \beta = \bigvee_{j \in J}(\alpha_j \to \beta)$ for all $\alpha_j, \beta \in L$. Then the categories L-GS and L-AP are isomorphic.*

In case of $L = [0, \infty]$ and the opposite order and extended addition as quantale operation, we see that in the case of approach spaces [11] the conditions on L are satisfied and hence ($[0,\infty]$-) gauges and ($[0,\infty]$-) approach distances are equivalent concepts. However, as can be seen with Example 5.11, probabilistic approach spaces [9] cannot equivalently be described by Δ^+-gauges.

6. L-metric Spaces as L-gauge Spaces

Theorem 6.1. *The category L-MET is isomorphic to a coreflective subcategory of L-GS.*

Proof. Let $(X, d) \in |L\text{-}MET|$ and define $\mathcal{G}^d = [d] = \{e \in L\text{-}MET(X) \ : \ d \leq e\}$. As $\mathcal{G}^d = [d]$ is a principal filter, it is naturally locally saturated and hence $(X, \mathcal{G}^d) \in |L\text{-}GS|$. Furthermore, let $f : (X, d) \longrightarrow (X', d')$ be an L-metric morphism and let $e' \in \mathcal{G}^{d'}$. Then $d' \leq e'$ and hence $e_f(x,y) = e'(f(x), f(y)) \geq d'(f(x), f(y)) \geq d(x,y)$. Hence $e_f \in \mathcal{G}^d$ and $f : (X, \mathcal{G}^d) \longrightarrow (X', \mathcal{G}^{d'})$ is an L-gauge morphism. Hence we can define a functor $F : \begin{cases} L\text{-}MET & \longrightarrow & L\text{-}GS \\ (X,d) & \longmapsto & (X, \mathcal{G}^d) \\ f & \longmapsto & f \end{cases}$. This functor is clearly injective on objects, for if we have two different L-metrics on X, we may assume $d(x,y) \not\leq d'(x,y)$ for $x,y \in X$. But then $d' \notin \mathcal{G}^d$ whereas $d' \in \mathcal{G}^{d'}$. Let now $(X,\mathcal{G}) \in |L\text{-}GS|$ and define $d^{\mathcal{G}} : X \times X \longrightarrow L$ by $d^{\mathcal{G}}(x,y) = \bigwedge_{d \in \mathcal{G}} d(x,y)$. Then $(X, d^{\mathcal{G}}) \in |L\text{-}MET|$. For $(X,\mathcal{G}), (X',\mathcal{G}') \in |L\text{-}GS|$ and an L-gauge morphism $f : (X,\mathcal{G}) \longrightarrow (X',\mathcal{G}')$ then $f : (X, d^{\mathcal{G}}) \longrightarrow (X', d^{\mathcal{G}'})$ is an L-metric morphism. To see this, let $x,y \in X$. Then, because for $d' \in \mathcal{G}'$ we have $d_f \in \mathcal{G}$, we conclude $d^{\mathcal{G}'}(f(x),f(y)) = \bigwedge_{d' \in \mathcal{G}'} d'(f(x),f(y)) = \bigwedge_{d' \in \mathcal{G}'} d_f(x,y) \geq \bigwedge_{d \in \mathcal{G}} d(x,y) = d^{\mathcal{G}}(x,y)$. Hence we can define a functor $H : \begin{cases} L\text{-}GS & \longrightarrow & L\text{-}MET \\ (X,\mathcal{G}) & \longmapsto & (X, d^{\mathcal{G}}) \\ f & \longmapsto & f \end{cases}$. For $(X,d) \in |L\text{-}MET|$ and $x,y \in X$ we have $d^{(\mathcal{G}^d)}(x,y) = \bigwedge_{e \in \mathcal{G}^d} e(x,y) = \bigwedge_{e \geq d} e(x,y)$ $= d(x,y)$. This shows $d^{(\mathcal{G}^d)} = d$, i.e. $F(H((X,d))) = (X,d)$. For $(X,\mathcal{G}) \in |L\text{-}GS|$ and $e \in \mathcal{G}$ we have $d^{\mathcal{G}}(x,y) \leq e(x,y)$ for all $x,y \in X$ and therefore $e \in \mathcal{G}^{(d^{\mathcal{G}})}$. Hence $\mathcal{G} \subseteq \mathcal{G}^{(d^{\mathcal{G}})}$, i.e. $H(F((X,\mathcal{G}))) \geq (X,\mathcal{G})$. □

Lemma 6.2. *Let $(X,d) \in |L\text{-}MET|$. Then $\mathcal{G}^d = \mathcal{G}^{\delta^d}$, i.e. we have $F = E \circ G$.*

Proof. We have $e \in \mathcal{G}^{\delta^d}$ if and only if for all $x \in X$ and all $A \subseteq X$ we have $\delta^d(x,A) \leq \bigvee_{a \in A} e(x,a)$, i.e. if and only if for all $x \in X$ and all $A \subseteq X$ we have $\bigvee_{a \in A} d(x,a) \leq \bigvee_{a \in A} e(x,a)$. Taking for A the one-point sets, we see $d \leq e$, i.e. $e \in \mathcal{G}^d$. Conversely, if $e \in \mathcal{G}^d$, then $d \leq e$ and hence $\delta^d(x,A) \leq \bigvee_{a \in A} e(x,a)$ for all $x \in X$ and all $A \subseteq X$, i.e. $e \in \mathcal{G}^{(\delta^d)}$. □

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QUANTALE-VALUED GAUGE SPACES

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فضاهای اندازه کوانتال - مقدار

چکیده. یک تعمیم کوانتال – مقدار از فضاهای رویکرد بر اساس اندازه های کوانتال – مقدار معرفی می کنیم. نشان داده می شود که رسته حاصل ، توپولوژیکی است وشامل یک شیء در ابتدا چگال است. بعلاوه نشان می دهیم که رسته فضاهای رویکرد کوانتال – مقدار که اخیراً بر اساس بستارهای کوانتال – مقدارتعریف شده اند یک زیر رسته هم بازتابی رسته ما است. در آخر نشان داه شده است که رسته فضاهای متریک کوانتال – مقدار بطور هم بازتابی قابل نشاندن در رسته ما است .