ON THE MATCHING NUMBER OF AN UNCERTAIN GRAPH

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ABSTRACT. Uncertain graphs are employed to describe graph models with indeterministic information that produced by human beings. This paper aims to study the maximum matching problem in uncertain graphs. The number of edges of a maximum matching in a graph is called matching number of the graph. Due to the existence of uncertain edges, the matching number of an uncertain graph is essentially an uncertain variable. Different from that in a deterministic graph, it is more meaningful to investigate the uncertain measure that an uncertain graph is k-edge matching (i.e., the matching number is greater than or equal to k). We first study the properties of the matching number of an uncertain graph, and then give a fundamental formula for calculating the uncertain measure. We further prove that the fundamental formula can be transformed into a simplified form. What is more, a polynomial time algorithm to numerically calculate the uncertain measure is derived from the simplified form. Finally, some numerical examples are illustrated to show the application and efficiency of the algorithm.

1. Introduction

Since the paper for solving the seven bridges problem of Königsberg was published by Euler in 1736, many problems on graphs have been studied by researchers. In classic graph theory, the edges and vertices are deterministic, and some properties on a graph can be verified. But in realistic life, there always exist indeterministic factors in graphs, which leads to new situations. That is to say, the traditional methods are no longer appropriated to verify some properties of the graphs with indeterministic factors.

As it is well known, probability theory, which was founded by Kolmogorov [19], is a branch of mathematics for studying the behavior of random phenomena. And then for a long time, probability theory has been steadily developed and has been widely applied in science and engineering. Random graphs were firstly put forward by Erdös and Rényi [7, 8] in 1959 and independently by Gilbert [17] at nearly the same time. According to their views, whether two vertices are joined can be described as a random variable. Since then, the random graph has been studied by many researches, such as Bollob*á*s [1], Gilbert [17], Mahmoud et al. [26], etc. EI Maftouhi and Gordones [5], EI Maftouhi [6] and Clark [3] investigated the matching number in a random graph. However, there are usually lots of vague phenomena that do not behave like randomness.

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As we know, a premise of applying probability theory is that the estimated probability distribution may be close enough to the real frequency. That is, we must have sufficient historical data. In practice, we are often lack of observed data about the unknown state of nature due to economical or technical reasons in some situations. In order to estimate the indeterminate quantity, we should rely on the subjective-intuitive opinions of experts. For example, the relationship between two persons may be expressed by human language like "potential friends".

There are two mathematical systems to model such subjective uncertainty. One is fuzzy set theory, which was initiated by Zadeh [37] via membership function. For scientists of fuzzy set theory, fuzzy set theory offers a powerful tool to deal with the incomplete information. Based on fuzzy set theory, fuzzy graphs were introduced by Rosenfeld [29] and Yeh and Bang [36] independently in 1975. Since then, lots of work on fuzzy graphs has been carried out. For instance, Mathew and Sunitha [27] studied fuzzy node connectivity and fuzzy arc connectivity. Samanta and Pal [30] introduced a fuzzy competition graph, and obtained several results of strong edges for fuzzy competition graphs. In addition, Sunitha and Mathew [33] introduced a survey of selected recent results on fuzzy graphs.

Uncertainty theory is another mathematical system to model subjective uncertainty. For the scientists of uncertainty theory, uncertainty theory offers a powerful tool to deal with the human's belief degree. The indeterminate quantity "potential friends" in a relationship network can be assumed to be an uncertain variable. Uncertainty theory was founded by Liu [20] in 2007 and refined by Liu [22] in 2010. Up to now, uncertainty theory has become a branch of mathematics for modeling human uncertainty. So far, there are so many relatively researches. Gao [13] gave the uncertainty distribution of the shortest path length, and investigated solutions to the α-shortest path and the most shortest path in an uncertain network. Yao [35] presented a sufficient condition as well as necessary condition for an uncertain mean-reverting stock model being no-arbitrage. Moreover, by constructing an uncertain programming model, Zhang and Peng [38] discussed Chinese postman problem with uncertain weights. Later, Sheng and Yao [31, 32] studied how to make an optimal transportation plan in an uncertain environment, and proposed some mathematical optimization models. Wu et al. [34] employed uncertainty theory to deal with multiple uncertain information for agency problem. Han et al. [18] solved the maximum network flow problem in an uncertain network within the framework of uncertainty theory. Recently, Zhou et al. [41] considered the uncertain minimum spanning tree problem with uncertain edge weights, and discussed the path optimality condition for two types of uncertain minimum spanning tree, namely uncertain expected minimum spanning tree and uncertain α-minimum spanning tree. Liu and Yao [25] provided an uncertain multilevel programming model to deal with a multilevel programming involving uncertain variables, and employed a genetic algorithm to solve the model.

Recently, the uncertainty theory is employed to deal with indeterministic factors in uncertain graphs. Gao and Gao [12] firstly proposed the concept of uncertain graph, and studied the connectivity index of an uncertain graph in 2013. Furthermore, Gao and Qin [15] studied the edge-connectivity of an uncertain graph. Later, Zhang and Peng discussed the Euler tour problem [39] in an uncertain graph. Gao [10, 11] introduced the concepts of cycle index and regularity index to study the properties of uncertain graph that are related to cycle and regularity, respectively. In addition, Gao et al. [14] discussed the properties of the diameter of an uncertain graph, and proposed an algorithm to determine the distribution function of the diameter. In 2013, Zhang and Peng [40] proposed the concept of matching index of an uncertain graph. In the following, we will prove that this paper extends the work of Zhang and Peng [40] from a special case to general cases.

As a matter of fact, matching is one of the basic concepts of the graph theory. In graph models, matching number is defined as the number of the edges of a maximum matching. In fact, due to uncertain edges, the matching number of an uncertain graph is essentially an uncertain variable, which means that it is more meaningful to determine the uncertain measure that a maximum matching contains k edges. To be more precise, this paper first proposes the concept of k-edge matching to describe that matching number is greater than or equal to k for an uncertain graph, then give a formula for calculating the uncertain measure that the graph is k-edge matching. In addition, a simplified form of the fundamental formula can be deduced. What is more, we explicitly design a polynomial time algorithm to numerically calculate the uncertain measure based on the simplified form and Gabow's matching algorithm. At last, the effectiveness and application of the algorithm are illustrated by some numerical examples.

The remainder of this paper is organized as follows. Section 2 first introduces some basic concepts and results of uncertainty theory and uncertain graph briefly, and then proposes the concept of k-edge matching. Section 3 investigates the theoretical formulas to calculate the uncertain measure that an uncertain graph is k-edge matching. In Section 4, a polynomial time algorithm is designed for calculating the uncertain measure and some numerical examples are performed to illustrate the application of the algorithm. In Section 5, we present the main innovations and advantages of the proposed work by comparing it with the existing work in the relevant literature. Section 6 concludes this paper with brief summary. Finally, the main differences between uncertainty theory and probability theory, as well as fuzzy set theory are presented in Appendix A for better understanding this paper.

2. **Preliminaries**

2.1. **Uncertainty Theory.** In this section, we will introduce some basic concepts of uncertainty theory for the completeness of this research. To do that, let us introduce the concepts of algebra, σ-algebra, Borel algebra, and Borel set. The credit references are not provided since these basic terminologies are well-known.

Definition 2.1. Let Γ be a nonempty set. A collection $\mathcal{L}$ of a subset of Γ is called an algebra over Γ if it satisfies: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, then

$$\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}.$$

The collection $\mathcal{L}$ is called a σ-algebra over Γ if it satisfies (a), (b) and (c′) if the union of a countable collection of elements in $\mathcal{L}$ remains in $\mathcal{L}$, i.e., when $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, we have

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}.$$

Definition 2.2. The Borel algebra over the set of real numbers is the smallest σ-algebra $\mathcal{B}$ containing all open intervals. Any element in $\mathcal{B}$ is called a Borel set.

Uncertainty theory was founded by Liu [20] in 2007 as a new branch of axiomatic mathematic for dealing with human's belief degrees by using uncertain measure.

Definition 2.3. (Liu [20]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. Each element Λ in $\mathcal{L}$ is assigned a number $\mathcal{M}\{\Lambda\}$. The set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies:
Axiom 1.(Normality) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2.(Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any Λ;
Axiom 3.(Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

For simplicity, the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Later in 2009, Liu [21] studied the product uncertain measure, which leads to the product measure axiom of uncertainty theory:
Axiom 4.(Liu [21], Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2.4. (Liu [20]) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma | \xi(\gamma) \in B\}$$

is an event.

Definition 2.5. (Liu [20]) The uncertainty distribution of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any $x \in \Re$.

Theorem 2.6. *(Liu [22]) Let ξ be an uncertain variable with uncertainty distribution Φ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \le x\} = \Phi(x), \qquad \mathcal{M}\{\xi > x\} = 1 - \Phi(x).$$

An uncertain variable is said to be Boolean if it takes values either 0 or 1. For instance, let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space with $\Gamma = \{\gamma_1, \gamma_2\}$, and assume that $\mathcal{M}\{\gamma_1\} = \alpha, \mathcal{M}\{\gamma_2\} = 1 - \alpha$, then

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1, \\ 0, & \text{if } \gamma = \gamma_2 \end{cases}$$

is a boolean uncertain variable.

Definition 2.7. (Liu [20]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \cdots, B_n$ of $\Re$.

A real function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly increasing if f satisfies the following conditions:
(1) $f(x_1, x_2, \cdots, x_n) \geq f(y_1, y_2, \cdots, y_n)$ when $x_i \geq y_i$ for $i = 1, 2, \cdots, n$;
(2) $f(x_1, x_2, \cdots, x_n) > f(y_1, y_2, \cdots, y_n)$ when $x_i > y_i$ for $i = 1, 2, \cdots, n$.

For boolean uncertain variables $\xi_i, i = 1, 2, \cdots, m$, Gao and Qin [15] gave the following theorem:

Theorem 2.8. *(Gao and Qin [15]) Suppose that f is a monotone function and $\xi_1, \xi_2, \cdots, \xi_m$ are independent boolean uncertain variables with $\mathcal{M}\{\xi_i = 1\} = \alpha_i$, $\mathcal{M}\{\xi_i = 0\} = 1 - \alpha_i$ for $i = 1, 2, \cdots, m$. Let $\eta = f(\xi_1, \xi_2, \cdots, \xi_m)$. Then we have*

$$\mathcal{M}\{\eta \geq k\} = \sup_{f(B_1, B_2, \cdots, B_m) \subset [k, +\infty)} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\},$$

where B_i is the subset of $\{0, 1\}, i = 1, 2, \cdots, m$, respectively.

2.2. **Uncertain Graph and k-edge Matching.** In this subsection, some basic terminology of graph theory refers to Bondy and Murty [2] and Gibbons [16]. Assume V is a set of vertices, and E is a set of edges. Then a graph can be denoted by $G = (V, E)$.

A matching H in a graph G is essentially a subset of E such that every vertex is incident to at most one edge in the set. If a matching H satisfies $|H| \geq |H'|$ for any matching H', then H is said to be maximum, where $|H|$ denotes the number of edges in H. Figure 1 illustrates two classic graphs. Graph G_1 contains a matching $\{e_1\}$, which is also the maximum matching. Graph G_2 has an edge set $\{e_1, e_2, e_3\}$. We can obtain a maximum matching $\{e_1, e_3\}$ by removing e_2.

The number of edges of a maximum matching in a graph is said to be ***matching number*** of the graph. In this paper, we denote by $\alpha(G)$ the matching number of a graph G. The matching number in a classic graph is deterministic. However, it is not the case in a graph in the state of indeterminacy. Based on fuzzy set theory, the concept of fuzzy graph was proposed. A standardized definition of fuzzy graph can be referred as follows.

Definition 2.9. (Mordeson and Nair [28]) A fuzzy graph $G = (V, \mu, \rho)$ is a nonempty set V together with a pair of functions $\mu : V \to [0, 1]$ and $\rho : V \times V \to [0, 1]$ such

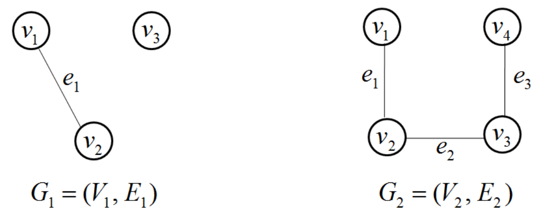

Figure 1. Classic Graphs G_1 and G_2

	Fuzzy graph	Uncertain graph
Indeterminacy factor	Vertex, edge	Edge
The tool of measure	Possibility measure	Uncertain measure
Indeterminate quantity	Fuzzy variable	Uncertain variable
Theoretical basis	Fuzzy set theory	Uncertainty theory

Table 1. Comparison Between the Fuzzy Graph and the Uncertain Graph

that $\rho(x, y) \leq \mu(x) \wedge \mu(y)$ for all x, y in V, where μ is a fuzzy subset of V and ρ is a symmetric fuzzy relation on μ.

Based on uncertainty theory, the concept of uncertain graph was defined by Gao and Gao [12] in 2013.

Definition 2.10. (Gao and Gao [12]) Assume $V = \{v_1, v_2, \cdots, v_n\}$ is a collection of vertices, $E = \{e_1, e_2, \cdots, e_m\}$ is a collection of uncertain edges, and $\xi = \{\xi_1, \xi_2, \cdots, \xi_m\}$ is a set of indicator function. Then the triple (V, E, ξ) is said to be an uncertain graph.

It follows from Definitions 2.9 and 2.10 that fuzzy graph and uncertain graph are two directions to study graphs with indeterminacy factors. Although fuzzy graph and uncertain graph look like each other in some ways, they are very different mathematical theories. Fuzzy graph is defined on fuzzy set theory, while uncertain graph is defined on uncertainty theory. That is, they use different measures, one uses the tool of possibility measure and another uses the tool of uncertain measure. From the topological structure, fuzzy graph contains indeterminacy both in vertices and edges, while uncertain graph just in edges. Briefly, the main differences between the fuzzy graph and the uncertain graph are summarized in Table 1 for more readable.

Generally, in an uncertain graph, the vertices are all predetermined, while some edges are uncertain. That is to say, the existence of some edges is indeterministic, and the existence possibility is indicated by uncertain measure. For convenience, an indicator function ξ_i, which is a boolean uncertain variable, is used to indicate the existence of corresponding edge. That is, $\mathcal{M}\{\xi_i = 1\} = \alpha_i$ describes the existence possibility of edge e_i, where $\xi_i = 1$ means the edge e_i exists. If $\alpha_i = 0$, then the edge e_i does not exist at all; If $\alpha_i = 1$, then the edge e_i exists completely, or equivalently,

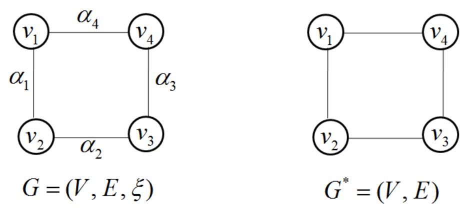

FIGURE 2. Uncertain Graph G and Its Underlying Graph G^*

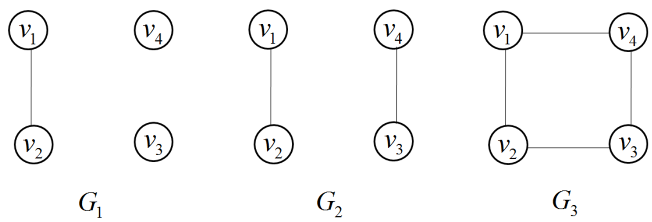

FIGURE 3. Some Models of the Uncertain Graph G That Presented in Figure 2

e_i is a deterministic edge. In a word, the larger the value of α_i is, the more the true the edge e_i exists.

Definition 2.11. (Zhang and Peng [40]) Let $G = (V, E, \xi)$ be an uncertain graph. The underlying graph $G^* = (V, E)$ of G is a graph obtained from G by replacing $0 < \alpha_i < 1$ with $\alpha_i = 1$, $i = 1, 2, \cdots, m$, respectively.

Briefly, the underlying graph contains all the vertices and edges of the original uncertain graph, and it is a deterministic graph. Usually, an uncertain graph G can be shown in a simpler form as in Figure 2, in which the existence chance α_i for each edge e_i is marked on the corresponding edge. Also, the underlying graph G^* of the uncertain graph is shown in Figure 2.

In fact, an uncertain graph can be considered as a function of ξ_i since different values of ξ_i lead to different models of uncertain graph. For an uncertain graph $G = (V, E, \xi)$, denote $\mathcal{E} = \{e_i | 0 < \alpha_i < 1\}$. Naturally, the uncertain graph G contains 2^m different models if $|\mathcal{E}| = m$. Each model is a deterministic graph. Figure 3 illustrates some models of the uncertain graph G shown in Figure 2 if all the uncertain measures α_i satisfy $0 < \alpha_i < 1, i = 1, 2, \cdots, 4$. It is clear that G_3 just is the underlying graph of the uncertain graph G.

For an uncertain graph, different models may lead to different maximum matchings. For instance, in Figure 3, $\alpha(G_1) = 1$, $\alpha(G_2) = \alpha(G_3) = 2$. In other words, for an uncertain graph G, $\alpha(G)$ is an uncertain variable rather than a crisp value.

Definition 2.12. An uncertain graph G is said to be k-edge matching if $\alpha(G) \geq k$ for a given positive integer k.

In order to indicate the possibility that an uncertain graph G is k-edge matching, it is meaningful to calculate the value of $\mathcal{M}\{\alpha(G) \geq k\}$. It is easy to verify that, in an uncertain graph $G = (V, E, \xi)$, $\alpha(G)$ can be considered as a function of ξ. That is, $\alpha(G) = \alpha(\xi_1, \xi_2, \cdots, \xi_m)$. In the following section, we will discuss the methods for calculating the value of $\mathcal{M}\{\alpha(\xi_1, \xi_2, \cdots, \xi_m) \geq k\}$ for a given positive integer k.

3. Calculating for k-edge Matching

For a deterministic graph, the matching number will not decrease when adding an edge. Since ξ_i are boolean uncertain variables, it is also clear that

$$\alpha(y_1, y_2, \cdots, y_m) < \alpha(z_1, z_2, \cdots, z_m),$$

where $y_i < z_i$. Thus, the matching number $\alpha(\xi_1, \xi_2, \cdots, \xi_m)$ is a strictly increasing function of $\xi_i, i = 1, 2, \cdots, m$. According to Theorem 2.8, we can easily formulate $\mathcal{M}\{\alpha(\xi_1, \xi_2, \cdots, \xi_m) \geq k\}$ as follows:

Theorem 3.1. *Let $G = (V, E, \xi)$ be an uncertain graph. For any positive integer k, we have*

$$\mathcal{M}\{\alpha(\xi_1, \xi_2, \cdots, \xi_m) \geq k\} = \sup_{\alpha(G) \subset [k, +\infty)} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\},$$

where B_i is the subset of $\{0, 1\}, i = 1, 2, \cdots, m$, respectively.

Based on Theorem 3.1, we get a fundamental formula for calculating the value of $\mathcal{M}\{\alpha(\xi_1, \xi_2, \cdots, \xi_m) \geq k\}$. However, it is still a fairly abstract formula for practical applications. For understanding convenience, we will reformulate it to a specifical form below.

Let $G = (V, E, \xi)$ be an uncertain graph and we define

$$\mathcal{N}(k) = \{H | H \text{ is a model of } G \text{ and } \alpha(H) \geq k\}.$$

By integrating the set $\mathcal{N}(k)$, we have

Theorem 3.2. *Let $G = (V, E, \xi)$ be an uncertain graph. The uncertain measure that G is k-edge matching is formulated as follows:*

$$\mathcal{M}\{\alpha(G) \geq k\} = \sup_{H \in \mathcal{N}(k)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} = \sup_{H \in \mathcal{N}(k)} \min_{e_i \in E(H)} \alpha_i$$

for any given positive integer k.

Proof. The proof can be divided into the following two steps.

Step 1: We prove that

$$\sup_{H \in \mathcal{N}(k)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} \geq \sup_{\alpha(G) \subset [k, +\infty)} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}. \quad (1)$$

Since m is a finite integer, there exists a series $\{B_1^*, B_2^*, \cdots, B_m^*\}$, where $B_i^* \in \{\{0\}, \{1\}, \{0, 1\}\}, i = 1, 2, \cdots, m$, such that

$$\alpha(B_1^*, \cdots, B_m^*) \subset \left[k, \frac{n}{2}\right],$$

and

$$\sup_{\alpha(B_1, B_2, \cdots, B_m) \subset [k, \frac{n}{2}]} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

Without loss of generality, suppose that $B_1^* = \{0\}$, we have

$$\alpha(\{0\}, B_2^*, \cdots, B_m^*) \subset \left[k, \frac{n}{2}\right].$$

Thus we get the fact that

$$\alpha(\{0,1\}, B_2^*, \cdots, B_m^*) \subset \left[k, \frac{n}{2}\right],$$

since $\alpha(\xi_1, \xi_2, \cdots, \xi_m)$ is strictly increasing with respect to ξ_i. On the one hand,

$$\min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i^*\} \le \mathcal{M}\{\xi_1 \in \{0,1\}\} \wedge \min_{2\le i\le m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

On the other hand, it is easy to verify that

$$\sup_{\alpha(B_1,B_2,\cdots,B_m)\subset[k,\frac{n}{2}]} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\} \ge \mathcal{M}\{\xi_1 \in \{0,1\}\} \wedge \min_{2\le i\le m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

Thus, we have

$$\begin{aligned} \sup_{\alpha(B_1,B_2,\cdots,B_m)\subset[k,\frac{n}{2}]} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\} &= \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i^*\} \\ &= \mathcal{M}\{\xi_1 \in \{0,1\}\} \wedge \min_{2\le i\le m} \mathcal{M}\{\xi_i \in B_i^*\}. \end{aligned}$$

That is to say, by extending B_i^* from $\{0\}$ to $\{0,1\}$, we can choose $\{B_1^*, B_2^*, \cdots, B_m^*\}$, where $B_i^* \in \{\{1\}, \{0,1\}\}, i = 1, 2, \cdots, m$, such that

$$\alpha(B_1^*, \cdots, B_m^*) \subset \left[k, \frac{n}{2}\right],$$

and

$$\sup_{\alpha(B_1,B_2,\cdots,B_m)\subset[k,\frac{n}{2}]} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\} = \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

By constructing a graph H^* with $V(H^*) = V(G)$, and $E(H^*) = \{e_i | e_i \in E(G), B_i^* = \{1\}\}$. Thus, $H^* \in \mathcal{N}(k)$ and

$$\begin{aligned} \sup_{H\in\mathcal{N}(k)} \min_{e_i\in E(H)} \mathcal{M}\{\xi_i = 1\} &\ge \min_{e_i\in E(H^*)} \mathcal{M}\{\xi_i = 1\} \\ &= \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i^*\} \\ &= \sup_{\alpha(G)\subset[k,+\infty)} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}. \end{aligned}$$

That is, the Inequality (1) holds.

Step 2: We prove that

$$\sup_{H\in\mathcal{N}(k)} \min_{e_i\in E(H)} \mathcal{M}\{\xi_i = 1\} \le \sup_{\alpha(G)\subset[k,+\infty)} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}. \tag{2}$$

Because of the finiteness of set $\mathcal{N}(k)$, there exists a model $H^{'} \in \mathcal{N}(k)$, satisfying

$$\sup_{H\in\mathcal{N}(k)} \min_{e_i\in E(H)} \mathcal{M}\{\xi_i = 1\} = \min_{e_i\in E(H')} \mathcal{M}\{\xi_i = 1\}.$$

Choose a series $\{B_1', B_2', \cdots, B_m'\}$ such that

$$B_i' = \begin{cases} \{1\}, & if \quad e_i \in E(H'), \\ \{0,1\}, & if \quad e_i \notin E(H'). \end{cases}$$

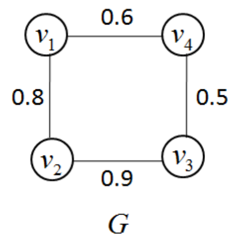

FIGURE 4. Uncertain Graph $G = (V, E, \xi)$

Since $H' \in \mathcal{N}(k)$, that is $\alpha(H') \geq k$. According to the choice of $\{B'_i\}$, it is easy to verify that

$$\alpha(B'_1, \cdots, B'_m) \subset \left[k, \frac{n}{2}\right],$$

and

$$\min_{e_i \in E(H')} \mathcal{M}\{\xi_i = 1\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B'_i\}.$$

Then, we have

$$\sup_{\alpha(B_1, \cdots, B_m) \subset [k, \frac{n}{2}]} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\} \geq \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B'_i\} = \min_{e_i \in E(H')} \mathcal{M}\{\xi_i = 1\}.$$

Thus the Inequality (2) holds immediately.

According to Inequalities (1) and (2), the proof of Theorem 3.2 is completed. □

4. **Algorithm and Examples**

In the following, we first give a simple example to illustrate the application of the method which is stated in Theorem 3.2.

Example 4.1. Assume that $G = (V, E, \xi)$ is an uncertain graph presented in Figure 4. Let $\mathcal{M}\{\xi_1 = 1\} = \alpha_1 = 0.8$, $\mathcal{M}\{\xi_2 = 1\} = \alpha_2 = 0.9$, $\mathcal{M}\{\xi_3 = 1\} = \alpha_3 = 0.5$, $\mathcal{M}\{\xi_4 = 1\} = \alpha_4 = 0.6$.

Let G^* be the underlying graph of G. According to Theorem 3.2, for any positive integer k, we just need to traverse the set

$$\mathcal{N}(k) = \{H | H \text{ is a model of } G \text{ and } \alpha(H) \geq k\}.$$

It is easy to verify that $\alpha(G^*) = 2$. That is, $\alpha(H) \leq \alpha(G^*) = 2$ holds for any model H of the uncertain graph G. Thus, for any positive integer $k \geq 3$,

$$\mathcal{N}(k) = \{H | H \text{ is a model of } G \text{ and } \alpha(H) \geq k\} = \emptyset,$$

which implies that $\mathcal{M}\{\alpha(G) \geq k\} = 0$.

Let $k = 2$. The set $\mathcal{N}(2)$ is illustrated in Figure 5. In order to calculate $\mathcal{M}\{\alpha(G) \geq 2\}$, we should first traverse $\mathcal{N}(2)$, i.e.,

$$\min_{e_i \in E(H_1)} \mathcal{M}\{\xi_i = 1\} = \min\{0.8, 0.5\} = 0.5,$$

$$\min_{e_i \in E(H_2)} \mathcal{M}\{\xi_i = 1\} = \min\{0.9, 0.6\} = 0.6,$$

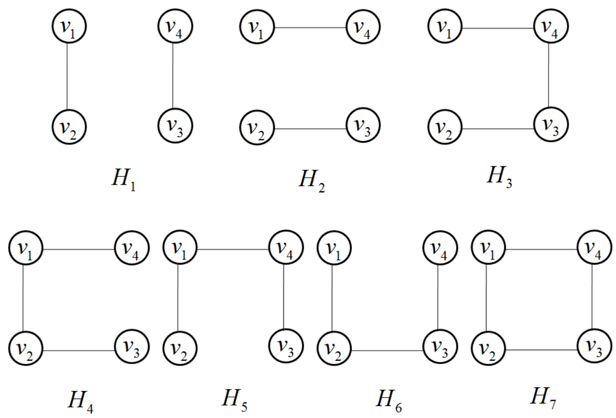

FIGURE 5. The Set $\mathcal{N}(2)$

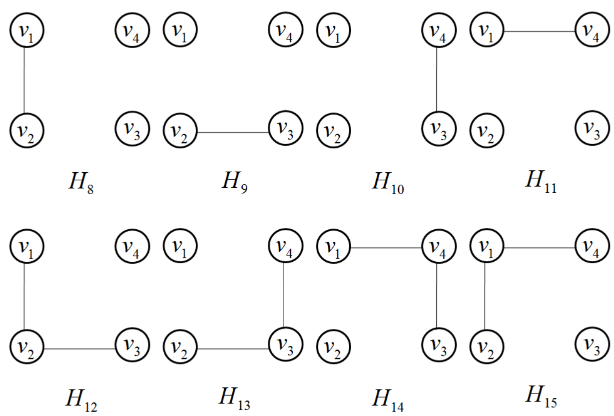

FIGURE 6. Some Elements of $\mathcal{N}(1)$

$$\min_{e_i\in E(H_3)} \mathcal{M}\{\xi_i=1\} = \min\{0.9, 0.5, 0.6\} = 0.5,$$
$$\min_{e_i\in E(H_4)} \mathcal{M}\{\xi_i=1\} = \min\{0.8, 0.9, 0.6\} = 0.6,$$
$$\min_{e_i\in E(H_5)} \mathcal{M}\{\xi_i=1\} = \min\{0.8, 0.5, 0.6\} = 0.5,$$
$$\min_{e_i\in E(H_6)} \mathcal{M}\{\xi_i=1\} = \min\{0.8, 0.9, 0.5\} = 0.5,$$
$$\min_{e_i\in E(H_7)} \mathcal{M}\{\xi_i=1\} = \min\{0.8, 0.9, 0.5, 0.6\} = 0.5.$$

It follows from Theorem 3.2 that

$$\begin{aligned}\mathcal{M}\{\alpha(G)\geq 2\} &= \sup_{H\in\mathcal{N}(2)} \min_{e_i\in E(H)} \mathcal{M}\{\xi_i=1\}\\ &= \max\{0.5, 0.6, 0.5, 0.6, 0.5, 0.5, 0.5\} = 0.6.\end{aligned}$$

Let $k = 1$. And the set of some elements of $\mathcal{N}(1)$ which is denoted by $\mathcal{N}'(1)$ is listed in Figure 6. Obviously, $\mathcal{N}(1) = \mathcal{N}'(1) \cup \mathcal{N}(2)$. To calculate $\mathcal{M}\{\alpha(G) \geq 1\}$ we should firstly traverse $\mathcal{N}(1)$. We can verify that

$$\sup_{H \in \mathcal{N}'(1)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} = 0.9.$$

Combining with $\mathcal{N}(2)$ and Theorem 3.2, we know

$$\begin{aligned} \mathcal{M}\{\alpha(G) \geq 1\} &= \sup_{H \in \mathcal{N}(1)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} \\ &= \sup_{H \in \mathcal{N}(2) \cup \mathcal{N}'(1)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} = \max\{0.6, 0.9\} = 0.9. \end{aligned}$$

Next, we introduce an algorithm for calculation $\mathcal{M}\{\alpha(G) \geq k\}$ based on Theorem 3.2. Let G be an uncertain graph with m uncertain edges. Here we need to point out that it is difficult to traverse a set $\mathcal{N}(k)$, since it needs to traverse all models of the uncertain graph. If $m = 8$, the uncertain graph contains $2^8 = 256$ models, which implies that the traversing set $\mathcal{N}(k)$ is inefficient. So it is essentially that we should propose a more efficient algorithm to calculate the value of $\mathcal{M}\{\alpha(G) \geq k\}$ based on Theorem 3.2 and Gabow's matching algorithm (Gabow [9]).

Let G be an uncertain graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$, and edge set $E\{e_1, e_2, \cdots, e_m\}$. Let $\mathcal{M}\{\xi_i = 1\} = \alpha_i$ be the existence possibility of edge e_i. In fact, Theorem 3.2 shows that $\mathcal{M}\{\alpha(G) \geq k\}$ takes values in the set of $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$. That is to say, we can traverse a much smaller set instead of traversing set $\mathcal{N}(k)$. The algorithm can be summarized in Algorithm 4.2.

Algorithm 4.2. Algorithm for Calculating the Value of $\mathcal{M}\{\alpha(G) \geq k\}$.

- **Step 1:** Rearrange the set $\{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ such that $1 = \alpha_0 \geq \alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_{m+1} = 0$. Set $t = 1$.
- **Step 2:** In the uncertain graph G, a new uncertain graph G_t can be obtained by removing the edge e_i that satisfies $\alpha_i < \alpha_t, i = 1, 2, \cdots, m$.
- **Step 3:** Denote the underlying graph of G_t by G_t^*. Find a maximum matching in the graph G_t^*, and calculate $\alpha(G_t^*)$. Set $k_t = \alpha(G_t^*)$.
- **Step 4:** If $k_t \geq k$, stop. Otherwise, let $t = t + 1$, if $t = m + 1$, stop; if $t < m + 1$, then go to **Step 2**.

In Step 4, if the algorithm is terminated by $t = m + 1$, then $\mathcal{M}\{\alpha(G) \geq k\} = 0$; if the algorithm is terminated by $k_t \geq k$, then $\mathcal{M}\{\alpha(G) \geq k\} = \alpha_t$.

In Step 3, the maximum matching of the the graph G_t^* can be obtained by any maximum matching algorithm, such as the Edmonds's matching algorithm (Edmonds [4]) and the Gabow's matching algorithm (Gabow [9]). As we know, the complexity of the Gabow's matching algorithm is $O(n^3)$. Thus, the complexity of the proposed algorithm is $O(m \cdot n^3)$, where m is the number of edges and n is the number of vertices. In the following, we show that Algorithm 4.2 gives exact results.

Proposition 4.3. *For a given positive integer k, Algorithm 4.2 gives an exact value of $\mathcal{M}\{\alpha(G) \geq k\}$.*

Proof. It follows from Step 1 that the less the value of index t is, the more the value of α_t is. If $t_1 > t_2$, according to Step 2, less edges are removed in the t_1-th iteration, and then uncertain graph G_{t_1} contains more edges. That is, $k_{t_1} \geq k_{t_2}$ if $t_1 > t_2$. In Step 4, there are two cases for termination condition. In the following, the discussion breaks down into the two cases.

Case 1: The algorithm is terminated by $t = m + 1$, which implies that $k_m < k$ in the m-th iteration. In fact, no edge will be removed in the m-th iteration. That is, $G_m = G$, and $G_m^* = G^*$. In addition, the algorithm is terminated by $k_m = \alpha(G_m^*) = \alpha(G^*) < k$ means that there exists no model H of G satisfying $\alpha(H) \geq k$. Thus, $\mathcal{M}\{\alpha(G) \geq k\} = 0$.

Case 2: The algorithm is terminated by $k_t \geq k$, which implies that $k_{t-1} < k$ in the $t-1$-th iteration. It follows from the proposed algorithm that $\alpha(G_t^*) = k_t$, i.e., $G_t^* \in \mathcal{N}(k_t)$. According to Theorem 3.2 and Algorithm 4.2, it is easy to verify that

$$\mathcal{M}\{\alpha(G) \geq k_t\} = \sup_{H \in \mathcal{N}(k_t)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} \geq \min_{e_i \in E(G_t^*)} \mathcal{M}\{\xi_i = 1\} = \alpha_t.$$

According to the termination condition $k_t \geq k$, we have

$$\mathcal{M}\{\alpha(G) \geq k\} \geq \mathcal{M}\{\alpha(G) \geq k_t\} \geq \alpha_t.$$

If

$$\mathcal{M}\{\alpha(G) \geq k\} > \alpha_t, \tag{3}$$

according to Algorithm 4.2, the value of $\mathcal{M}\{\alpha(G) \geq k\}$ is located in the set $\{\alpha_1, \alpha_2, \cdots, \alpha_{t-1}\}$. Without loss of generality, we can assume that $\mathcal{M}\{\alpha(G) \geq k\} = \alpha_{t-1}$.

According to Theorem 3.2, there exists a graph $H \in \mathcal{N}(k)$ such that

$$\mathcal{M}\{\alpha(G) \geq k\} = \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} = \alpha_{t-1}.$$

In addition, Algorithm 4.2 tells us that all the edges $e_i \in E(G)$ with $\alpha_i \geq \alpha_{t-1}$ are contained in the graph G_{t-1}. That is, H is a spanning graph of G_{t-1}^*. So

$$\alpha(H) \leq \alpha(G_{t-1}^*) = k_{t-1} < k,$$

from which we can conclude $H \notin \mathcal{N}(k)$. Therefore, assumption (3) is not true, which means

$$\mathcal{M}\{\alpha(G) \geq k\} = \alpha_t.$$

Thus, the proof of the proposition is completed. □

Next, we shall use the proposed Algorithm 4.2 to test the numerical example given below.

Example 4.4. Consider an uncertain graph G with 7 vertices and 10 edges shown in Figure 7. We will employ the proposed Algorithm 4.2 to calculate the uncertain measure that G is 3-edge matching, i.e., $\mathcal{M}\{\alpha(G) \geq 3\}$.

According to Theorem 3.2, the value of $\mathcal{M}\{\alpha(G) \geq 3\}$ is located in the set $\{1, 0.9, 0.8, 0.7, 0.6, 0.4, 0.3\}$. As Step 1 of Algorithm 4.2 indicates, $\alpha_1 = 1$, $\alpha_2 = 0.9$, $\alpha_3 = 0.8$, $\alpha_4 = 0.7$, $\alpha_5 = 0.6$, $\alpha_6 = 0.4$, $\alpha_7 = 0.3$.

In the first iteration, i.e., $t = 1$ and $\alpha_1 = 1$. A new uncertain graph G_1 (see Figure 8) is obtained by removing the edges e_i that satisfy $\alpha_i < 1$ in the uncertain graph G.

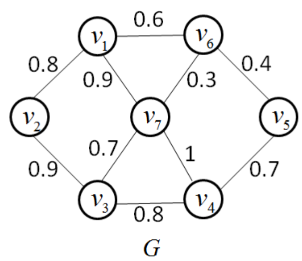

FIGURE 7. Uncertain Graph for Example 4.4

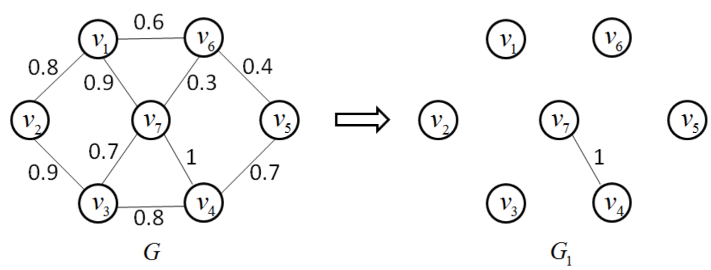

FIGURE 8. The First Iteration for Example 4.4

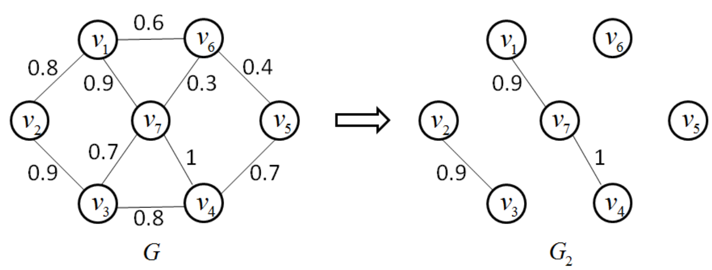

FIGURE 9. The Second Iteration for Example 4.4

Denote the underlying graph of G_1 is G_1^*, and we get $k_1 = \alpha(G_1^*) = 1$. According to Algorithm 4.2, we should continue the iteration.

In the second iteration, i.e., $t = 2$ and $\alpha_2 = 0.9$. After removing the edges e_i that satisfy $\alpha_i < 0.9$ in the uncertain graph G, it gives a new uncertain graph G_2 shown in Figure 9. Denote the underlying graph of G_2 is G_2^*. It is easy to verify that $k_2 = \alpha(G_2^*) = 2$. Thus, we should continue the iteration.

In the third iteration, i.e., $t = 3$ and $\alpha_3 = 0.8$. In the uncertain graph G, remove the edges e_i with $\alpha_i < 0.8$, and then obtain a new uncertain graph G_3 (see Figure 10) with underlying graph G_3^*. We can obtain that $k_3 = \alpha(G_3^*) = 2$.

In the fourth iteration, i.e., $t = 4$ and $\alpha_4 = 0.7$. In the uncertain graph G, remove the edges e_i with $\alpha_i < 0.7$, and we obtain a new uncertain graph G_4 shown in Figure 11.

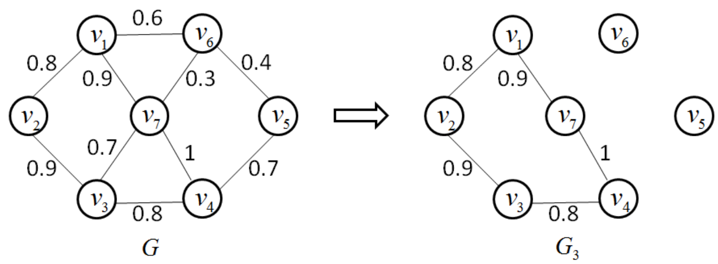

FIGURE 10. The Third Iteration for Example 4.4

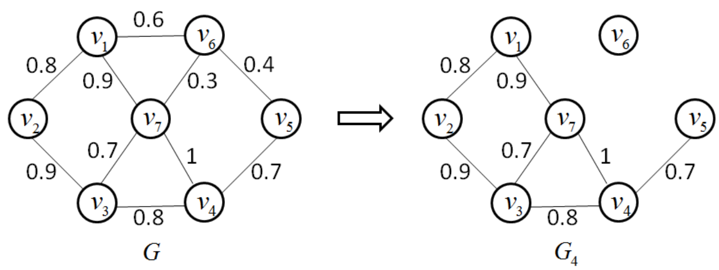

FIGURE 11. The Fourth Iteration for Example 4.4

Denote the underlying graph of G_4 is G_4^*. Clearly, $k_4 = \alpha(G_4^*) = 3$. According to Algorithm 4.2, the iteration can be terminated, and $\mathcal{M}\{\alpha(G) \geq 3\} = \alpha_4 = 0.7$.

5. Innovations and Comparisons

In order to highlight the contributions of the proposed work to uncertain graphs, this section presents the innovations and advantages of the work by comparing the proposed work with the existing work in the relevant literature.

5.1. **Innovations.** The innovations mainly include the consideration on the uncertain information of a graph by uncertainty theory. In the real world, some indeterminacy factors might occur in graphs. Probability theory is a powerful tool to deal with random phenomenon. However, it is not suitable to regard every indeterminacy phenomenon as random phenomenon. When we employ probability theory implies that we should obtain a large amount of historical data. In many cases, we are frequently lack of observed data to estimate a probability distribution. Perhaps we have no choice but to rely on some domain experts to evaluate the belief degree that each event will occur.

Liu [24] pointed out that human beings usually estimate a much wider range of values than the object actually takes. That is, the belief degrees deviate far from the real frequencies. Thus, the belief degrees cannot be dealt with by probability theory. A counterexample was presented by Liu [23]. To deal with the experts' belief degrees, uncertainty theory was founded by Liu [20] in 2007.

	Matching number in random graphs [3, 5, 6]	The proposed work
Type of indeterminacy	Stochastic factors	Belief degrees
Uncertain parameter	Random variable	Uncertain variable
Theoretical tool	Probability theory	Uncertainty theory

TABLE 2. Innovations of the Proposed Work

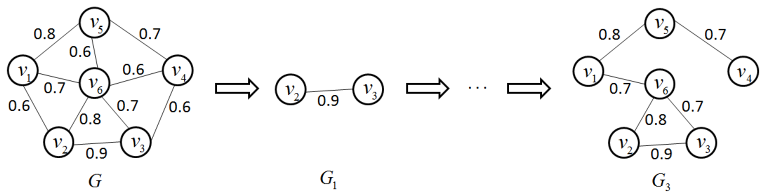

FIGURE 12. The Total Iteration Process for Example 5.2

This paper employs the uncertainty theory to study the matching number in a graph with uncertain edges. Compared to random graphs [3, 5, 6], the main innovations of this research are demonstrated clearly in Table 2.

5.2. **Comparisons.** In 2013, Zhang and Peng [40] proposed the concept of matching index of an uncertain graph in the framework of uncertainty theory.

Definition 5.1. (Zhang and Peng [40]) Let G be an uncertain graph with an underlying graph G^*, and H^* be a maximum matching of G^*. The matching index $\kappa(G)$ of G is the uncertain measure that G contains a matching H such that $|H| = |H^*|$.

If $|H^*| = l$, then the matching index is essentially the uncertain measure that the uncertain graph is l-edge matching. So, this paper extends the work of Zhang and Peng [40] from l-edge matching, i.e., the matching index, to k-edge matching in which k is an arbitrarily positive integer. That is, matching index is a special case of this paper.

Example 5.2. Still consider the uncertain graph $G = (V, E, \xi)$ that presented in Zhang and Peng [40]. We want to calculate the uncertain measure that G is 3-edge matching, i.e., $\mathcal{M}\{\alpha(G) \geq 3\}$.

We use the proposed Algorithm 4.2 on the uncertain graph, the total iteration process for calculating $\mathcal{M}\{\alpha(G) \geq 3\}$ is shown in Figure 12. That is, after the third iteration, the new uncertain graph G_3 is obtained, and $k_3 = \alpha(G_3) = 3$ can be verified. Thus, the algorithm should be terminated, and $\mathcal{M}\{\alpha(G) \geq 3\} = 0.7$. The result is the same with the matching index that of Zhang and Peng [40].

In addition, according to Algorithm 4.2, we can obtain $\mathcal{M}\{\alpha(G) \geq 1\} = 0.9$, $\mathcal{M}\{\alpha(G) \geq 2\} = 0.8$, and $\mathcal{M}\{\alpha(G) \geq k\} = 0$ for any positive integer with $k \geq 4$. From Theorem 2.6, we know,

$$\mathcal{M}\{\alpha(G) \leq 0\} = 0.1, \quad \mathcal{M}\{\alpha(G) \leq 1\} = 0.2,$$
$$\mathcal{M}\{\alpha(G) \leq 2\} = 0.3, \quad \mathcal{M}\{\alpha(G) \leq 3\} = 1.$$

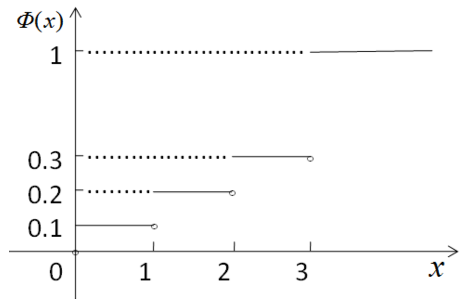

FIGURE 13. The Uncertainty Distribution of $\alpha(G)$ for Example 5.2

	Zhang and Peng [40]	The proposed work
Distribution function can be obtained	$\times$	$\surd$
Aim	l-edge matching (suppose $\vert H^*\vert = l$)	k-edge matching for any positive integer k

TABLE 3. Comparison Between the Existing Work and the Proposed Work

Figure 13 illustrates the uncertainty distribution of $\alpha(G)$. For any x, we can get $\mathcal{M}\{\alpha(G) \leq x\}$ by Figure 13. Table 3 highlights the advantages of the proposed work by comparing it with the existing research Zhang and Peng [40] in the framework of the uncertainty theory.

6. Conclusion

In this paper, we studied the properties of the maximum matching in uncertain graphs in which whether two vertices are joined by an edge cannot be completely determined. Specifically, we mainly concerned with how to obtain the possibility (or uncertain measure) that an uncertain graph contains a maximum matching with k edges (i.e., the matching number equals to k). The main contributions can be summarized as the following three aspects. Firstly, we discussed the characteristics of the matching number, and then proposed the concept of k-edge matching. Secondly, we presented a fundamental formula for calculating the uncertain measure that an uncertain graph is k-edge matching, and further proposed a simplified form of the fundamental formula. Thirdly, we designed an algorithm for solving the problem. We also proved that the algorithm is a polynomial time algorithm with computational complexity $O(m \cdot n^3)$, where m is the number of the edges, and n is the number of the vertices. At last, a numerical example was presented to show the application of the proposed algorithm.

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7. Appendix A

Generally, there are three mathematical theories to deal with the indeterminate world, i.e., probability theory, fuzzy set theory, and uncertainty theory. This appendix introduces the differences among uncertainty theory, probability theory, and fuzzy set theory. For more details, we can refer to the reference Liu [24].

In order to deal with random phenomena, a probability theory was founded by Kolmogorov [19]. Based on an uncertain measure which satisfies normality, duality, subadditivity and product axioms, an uncertainty theory was then founded by Liu [20]. A fuzzy set was defined by Zadeh [37] via its membership function μ which assigns to each element x a real number $\mu(x) \in [0, 1]$. Based on fuzzy set, a fuzzy set theory was then developed.

The most essential different between uncertainty theory and probability theory is that the measure of a product of events. For uncertain measure, the measure of a product of events is the minimum of the uncertain measures of the individual events. For probability measure, the measure of a product of events is the product of the probability measures of the individual events. Uncertainty is interpreted as belief degree, while probability is interpreted as frequency.

Uncertainty theory and fuzzy set theory are two mathematical theories to deal with subjective uncertainty. Although they look like each other in some ways, they are very different mathematical systems. Uncertainty theory based on uncertain measure and fuzzy set theory based on possibility measure, which leads to the main difference between of them. First, uncertain measure satisfies self-dual, but possibility measure is not. In addition, the following equation

$$\text{Pos}\{\Lambda_1 \cup \Lambda_2\} = \text{Pos}\{\Lambda_1\} \vee \text{Pos}\{\Lambda_2\}$$

holds for any events Λ_1 and Λ_2 no matter if they are independent or not based on possibility measure. But,

$$\mathcal{M}\{\Lambda_1 \cup \Lambda_2\} = \mathcal{M}\{\Lambda_1\} \vee \mathcal{M}\{\Lambda_2\}$$

holds only for independent events Λ_1 and Λ_2 based on uncertain measure.

References

[1] B. Bollobás, *Degree sequences of random graphs*, Discrete Mathematics, **33** (1981), 1-19.

[2] J. Bondy and U. Murty, *Graph Theory with Applications*, Elsevier, New York, 1976.

[3] L. Clark, *The strong matching number of a random graph*, Canadian Journal of Mathematics, **24** (2001), 47-57.

[4] J. Edmonds, *Paths, trees, and flowers*, Canadian Journal of Mathematics, **17(3)** (1965), 449-467.

[5] A. EI Maftouhi and L. M. Gordones, *The maximum size of a strong matching in a random graph*, Australasian Journal of Combinatorics, **10** (1994), 97-104.

[6] A. EI Maftouhi, *The minimum size of a maximal strong matching in a random graph*, Australasian Journal of Combinatorics, **12** (1995), 77-80.

[7] P. Erdös and A. Rényi, *On random graph I*, Publicationes Mathematicae Debrecen, **6** (1959), 290-297.

[8] P. Erdös and A. Rényi, *On the strength of connectedness of a random graph*, Acta Mathematica Hungarica, **12(1-2)** (1964), 261-267.

[9] H. N. Gabow, *An efficient implementation of Edmonds' algorithm for maximum matching on graphs*, The Journal of the ACM, **23(2)** (1976), 221-234.
[10] X. Gao, *Cycle index of uncertain graph*, Information: An International Interdisciplinary Journal, **16(2(A))** (2013), 1131-1138.
[11] X. Gao, *Regularity index of uncertain graph*, Journal of Intelligent and Fuzzy Systems, **27(4)** (2014), 1671-1678.
[12] X. Gao and Y. Gao, *Connectedness index of uncertain graphs*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **21(1)** (2013), 127-137.
[13] Y. Gao, *Shortest path problem with uncertain arc lengths*, Computers and Mathematics with Applications, **62(6)** (2011), 2591-2600.
[14] Y. Gao, L. Yang, S. Li and S. Kar, *On distribution function of the diameter in uncertain graph*, Information Sciences, **296** (2015), 61-74.
[15] Y. Gao and Z. Qin, *On computing the edge-connectivity of an uncertain graph*, IEEE Transactions on Fuzzy Systems, **24(4)** (2015), 981-991.
[16] A. Gibbons, *Algorithmic Graph Theory*, Cambridge University Press, Cambridge, 1985.
[17] E. Gilbert, *Random graphs*, Annals of Mathematical Statistics, **30(4)** (1959), 1141-1144.
[18] S. Han, Z. Peng and S. Wang, *The maximum flow problem of uncertain network*, Information Sciences, **265** (2014), 167-175.
[19] A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Julius Springer, Berlin, 1933.
[20] B. Liu, *Uncertainty Theory*, second ed., Springer-Verlag, Berlin, 2007.
[21] B. Liu, *Some research problems in uncertainty theory*, Journal of Uncertain Systems, **3(1)** (2009), 3-10.
[22] B. Liu, *Uncertainty theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin, 2010.
[23] B. Liu, *Why is there a need for uncertainty theory*? Journal of Uncertain Systems, **6(1)** (2012), 3-10.
[24] B. Liu, *Uncertainty Theory*, fourth ed., Springer-Verlag, Berlin, 2015.
[25] B. Liu and K. Yao, *Uncertain multilevel programming: Algorithm and applications*, Computers & Industrial Engineering, **89** (2015), 235-240.
[26] H. M. Mahmoud, R. T. Smythe and J. Szymański, *On the structure of random plane-oriented recursive trees and their branches*, Random Structures & Algorithm, **4(2)** (1993), 151-176.
[27] S. Mathew and M. S. Sunitha, *Node connectivity and arc connectivity of a fuzzy graph*, Information Sciences. **180(4)** (2010), 519-531.
[28] J. N. Mordeson and P. S. Nair, *Fuzzy Graphs and Fuzzy Fypergraphs*, Physica-Verlag, Heidelberg, 2000.
[29] A. Rosenfeld, *Fuzzy Graphs*, In: L. A. Zadeh, K. S. Fu and M. Shimura (Eds.), Fuzzy sets and their applications to cognitive and decision processes. Academic Press, New York, 1975.
[30] S. Samanta and M. Pal, *Fuzzy k-competition graphs and p-competition fuzzy graphs*, Fuzzy Information and Engineering, **5(2)** (2013), 191-204.
[31] Y. Sheng and K. Yao, *Fixed charge transportation problem in uncertain environment*, Industrial Engineering and Management Systems, **11(2)** (2012), 183-187.
[32] Y. Sheng and K. Yao, *A transportation model with uncertain costs and demands*, Information: An International Interdisciplinary Journal, **15(8)** (2012), 3179-3186.
[33] M. S. Sunitha and S. Mathew, *Fuzzy graph theory: A survey*, Annals of Pure and Applied Mathematics, **4(1)** (2013), 92-110.
[34] X. Wu, R. Zhao and W. Tang, *Optimal contracts for the agency problem with multiple uncertain information*, Knowledge-Based Systems, **59** (2014), 161-172.
[35] K. Yao, *No-arbitrage determinant theorems on mean-reverting stock model in uncertain market*, Knowledge-Based Systems, **35** (2012), 259-263.
[36] R. T. Yeh and S. Y. Bang, *Fuzzy Relations, Fuzzy Graphs and Their Applications to Clustering Analysis*, In: L. A. Zadeh, K. S. Fu and M. Shimura (Eds.), Fuzzy sets and their applications to cognitive and decision processes. Academic Press, New York, 1975.
[37] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

[38] B. Zhang and J. Peng, *Uncertain programming model for Chinese postman problem with uncertain weights*, Industrial Engineering and Management Systems, **11(1)** (2012), 18-25.
[39] B. Zhang and J. Peng, *Euler index in uncertain graph*, Applied Mathematics and Computation, **218(20)** (2012), 10279-10288.
[40] B. Zhang and J. Peng, *Matching index of an uncertain graph: Concept and algorithm*, Applied and Computational Mathematics, **12(3)** (2013), 381-391.
[41] J. Zhou, L. Chen and K. Wang, *Path optimality conditions for minimum spanning tree problem with uncertain edge weights*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **23(1)** (2015), 49-71.

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