ON Q-BITOPOLOGICAL SPACES

R. NOOR, A. K. SRIVASTAVA AND S. K. SINGH

Abstract. We study here T_0-Q-bitopological spaces and sober Q-bitopological spaces and their relationship with two particular Sierpinski objects in the category of Q-bitopological spaces. The epireflective hulls of both these Sierpinski objects in the category of Q-bitopological spaces turn out to be the category of T_0-Q-bitopological spaces. We show that only one of these Sierpinski objects is sober Q-bitopological space and its epireflective hull in the category of T_0-Q-bitopological spaces turns out to be the category of saturated T_0-Q-bitopological spaces.

1. Introduction

The notion of a lattice-valued (fuzzy) set has been generalized to Q-valued set, where Q is a fixed member of a variety of Ω-algebras and correspondingly, the notion of Q-topology on a set has been introduced and studied (S.A. Solovyov [13]). The notions of T_0-ness and sobriety have also been studied for Q-topological spaces in [13] and the use of a 'Sierpinski Q-topological space' has been made to study some aspects of these notions[1] (see [11] also for a characterization of the category of Q-topological spaces). In this paper, we carry on further with this study and study Q-bitopological spaces and the resulting category Q-**BTOP** of Q-bitopological spaces. We introduce two suitable Sierpinski Q-bitopological spaces and investigate their relationship with T_0-Q-bitopological spaces and sober Q-bitopological spaces. In particular, we show that both the categories Q-$\mathbf{BTOP}_0$ and Q-**BSOB** of T_0-Q-bitopological spaces and sober Q-bitopological spaces are epireflective in Q-**BTOP** and Q-$\mathbf{BTOP}_0$ respectively. Furthermore, Q-$\mathbf{BTOP}_0$ is shown to be the epireflective hull of both of these Sierpinski Q-bitopological spaces in Q-**BTOP** and the epireflective hull of one of these Sierpinski Q-bitopological spaces in Q-$\mathbf{BTOP}_0$ is shown to be the category of saturated T_0-Q-bitopological spaces.

2. Preliminaries

For the category-theoretic notions used here, [1] may be referred. All subcategories are assumed to be full and isomorphism closed. If $\mathscr{C}$ is a category, then

Received: May 2016; Revised: April 2017; Accepted: October 2017

Key words and phrases: Ω-algebra, Q-bitopological space, Sierpinski Q-bitopological space, Sober Q-bitopological space, T_0-Q-bitopological space, Saturated T_0-Q-bitopological space, Epireflective hull.

[1] In a recent work [12], additional linkages of a suitable Sierpinski object have been established with T_0-Q-topological spaces and sober Q-topological spaces.

$ob\mathscr{C}$ will denote the class of all $\mathscr{C}$-objects and $\mathscr{C}(A, B)$ will denote the family of all $\mathscr{C}$-morphisms from A to B.

We recall some definitions to make the paper self-contained.

Let $\Omega = (n_\lambda)_{\lambda \in I}$ be a class of cardinal numbers.

Definition 2.1. [13] An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in I})$ (also denoted by A) consisting of a set A and a family of maps $\omega_\lambda^A : A^{n_\lambda} \to A$, called n_λ-ary operations on A. Furthermore, a map $f : (A, (\omega_\lambda^A)_{\lambda \in I}) \to (B, (\omega_\lambda^B)_{\lambda \in I})$ between Ω-algebras is called an Ω-algebra homomorphism provided that for every $\lambda \in I$, the following diagram commutes:

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{f^{n_\lambda}} & B^{n_\lambda} \\ {\scriptstyle \omega_\lambda^A}\downarrow & & \downarrow{\scriptstyle \omega_\lambda^B} \\ A & \xrightarrow[f]{} & B \end{array}$$

Let $\mathbf{Alg}(\Omega)$ denote the category of Ω-algebras and Ω-algebra homomorphisms[2].

Given an Ω-algebra $(A, (\omega_\lambda^A)_{\lambda \in I})$, a subset $B \subseteq A$ is called a subalgebra of $(A, (\omega_\lambda^A)_{\lambda \in I})$ if $\omega_\lambda^A((b_i)_{i \in n_\lambda}) \in B$, for every $\lambda \in I$ and every $(b_i)_{i \in n_\lambda} \in B^{n_\lambda}$. It is known that an intersection of subalgebras remains a subalgebra. Given a subset $S \subseteq A$, $\langle S \rangle$ denotes the subalgebra of $(A, (\omega_\lambda^A)_{\lambda \in I})$ 'generated by S', i.e., $\langle S \rangle$ is the intersection of all subalgebras of $(A, (\omega_\lambda^A)_{\lambda \in I})$ containing S.

Definition 2.2. [13] A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, subalgebras, and homomorphic images.

Throughout this paper, Q denotes a fixed member of a fixed variety.

Let X be a set. Then Q^X is also an Ω-algebra with all the operations on Q lifting canonically to Q^X as follows: $(\omega_\lambda^{Q^X}((p_i)_{i \in n_\lambda}))(x) = \omega_\lambda^Q((p_i(x))_{i \in n_\lambda})$, for every $(p_i)_{i \in n_\lambda} \in Q^X$.

Definition 2.3. [13] A Q-topology on a set X is a subset $\tau \subseteq Q^X$, which is a subalgebra of Q^X and the pair (X, τ) is called a Q-topological space. Furthermore, a map $f : (X, \tau) \to (Y, \delta)$ between two Q-topological spaces is called Q-continuous if $\mu \circ f \in \tau$, for every $\mu \in \delta$.

Let Q-**TOP** denote the category of all Q-topological spaces and their Q-continuous maps.

Definition 2.4. A Q-bitopological space is a triple (X, τ_1, τ_2), where X is a set and τ_1, τ_2 are Q-topologies on X. Furthermore, a map $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ between two Q-bitopological spaces is called Q-bicontinuous if $f : (X, \tau_i) \to (Y, \delta_i)$ is Q-continuous for $i = 1, 2$.

The notions of *subspace*, *homeomorphism* and *embedding* for Q-bitopological spaces are on expected lines.

Let Q-**BTOP** denote the category of all Q-bitopological spaces and their Q-bicontinuous maps.

[2] The category $\mathbf{Alg}(\Omega)$ has products (cf. [5]).

Remark 2.5. Let $\mathscr{F} = \{f_j : X \to (Y_j, \delta_j, \delta'_j) \mid j \in J\}$ be a family of maps, where X is a set and $\{(Y_j, \delta_j, \delta'_j) \mid j \in J\}$ is a family of Q-bitopological spaces. Then the Q-bitopology (τ, τ') on X, where τ (resp. τ') is the Q-topology on X generated by $\{\mu \circ f_j \mid \mu \in \delta_j, j \in J\}$ (resp. $\{\mu' \circ f_j \mid \mu' \in \delta'_j, j \in J\}$), is initial with respect to the family $\mathscr{F}$.

The fact pointed out in the above remark goes on to show that the category Q-**BTOP** is a topological category over the category of sets.

Definition 2.6. Given a family $\{(X_j, \tau_j, \tau'_j) \mid j \in J\}$ of Q-bitopological spaces, the initial Q-bitopology on X $(= \prod_{j \in J} X_j)$ with respect to the family of all projection maps $\{p_j : X \to (X_j, \tau_j, \tau'_j) \mid j \in J\}$ is called the product Q-bitopology.

Let (X, τ_1, τ_2) be a Q-bitopological space, Y be a set and $f : X \to Y$ be a surjective map. Then $\tau_1/f = \{\mu \in Q^Y \mid \mu \circ f \in \tau_1\}$ and $\tau_2/f = \{\mu \in Q^Y \mid \mu \circ f \in \tau_2\}$ are Q-topologies on Y. The pair $(Y, \tau_1/f, \tau_2/f)$ will be called the *quotient Q-bitopological space* with respect to (X, τ_1, τ_2) and f.

Definition 2.7. [8] Given a topological category $\mathscr{C}$ over the category of sets, $S \in ob\mathscr{C}$ is called a Sierpinski object if for every $X \in ob\mathscr{C}$, the family of all $\mathscr{C}$-morphisms from X to S is initial.

Remark 2.8. We point out that in [8], the notion of a Sierpinski object is defined for the so-called 'categories of sets with structures'.

Let $\mathscr{R}$ be a subcategory of $\mathscr{C}$.

Definition 2.9. [2] $\mathscr{R}$ is said to be epireflective in $\mathscr{C}$ if for each $X \in ob\mathscr{C}$, there exists an epimorphism $r_X : X \to RX$, with $RX \in ob\mathscr{R}$, such that for each $\mathscr{C}$-morphism $f : X \to Y$, with $Y \in ob\mathscr{R}$, there exist a (unique) $\mathscr{R}$-morphism $f^* : RX \to Y$, such that $f^* \circ r_X = f$. If moreover, for a class $\mathscr{H}$ of $\mathscr{C}$-morphisms , each $r_X \in \mathscr{H}$ and f^* is a $\mathscr{C}$-isomorphism whenever $f \in epi\mathscr{C} \bigcap \mathscr{H}$, then $\mathscr{R}$ is said to be an $\mathscr{H}$-firm epireflective subcategory of $\mathscr{C}$.

3. T_0-Q-bitopological Spaces

Recall that a Q-topological space (X, τ) has been called T_0 [13] if for every $x, y \in X$, with $x \neq y$, there exist $\mu \in \tau$ such that $\mu(x) \neq \mu(y)$.

Definition 3.1. A Q-bitopological space (X, τ_1, τ_2) is called T_0 if for every $x, y \in X$, with $x \neq y$, there exists $\mu \in \tau_1 \cup \tau_2$ such that $\mu(x) \neq \mu(y)$.

Let Q-$\mathbf{TOP}_0$ denote the subcategory of Q-**TOP** whose objects are T_0-Q-topological spaces and let Q-$\mathbf{BTOP}_0$ denote the subcategory of Q-**BTOP** whose objects are T_0-Q-bitopological spaces.

Proposition 3.2. $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}_0$ *if and only if* $(X, \langle \tau_1 \cup \tau_2 \rangle) \in obQ$-$\mathbf{TOP}_0$.

The proof, being trivial, is omitted.

Theorem 3.3. *Q-$\mathbf{BTOP}_0$ is an epireflective subcategory of Q-$\mathbf{BTOP}$.*

Proof. Let $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}$. Define a relation $\sim$ on X as follows: for every $x, y \in X$, $x \sim y$ if and only if $\mu(x) = \mu(y)$, for every $\mu \in \tau_1 \cup \tau_2$. It is easy to verify that $\sim$ is an equivalence relation on X. Let $\tilde{X} = X/\sim$ and let $\tilde{\tau}_i$ be the corresponding quotient Q-topologies on $\tilde{X}$ induced by the quotient map $q_X : X \to \tilde{X}$ and τ_i, $i = 1, 2$. Then one can easily see that $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2) \in obQ$-$\mathbf{BTOP}_0$ and the map $q_X : (X, \tau_1, \tau_2) \to (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2)$ is an epimorphism in Q-$\mathbf{BTOP}$. Now given $(Y, \delta_1, \delta_2) \in obQ$-$\mathbf{BTOP}_0$ and a Q-bicontinuous map $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$, define $\tilde{f} : (\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2) \to (Y, \delta_1, \delta_2)$ as $\tilde{f}(\tilde{x}) = f(x)$ (the T_0-ness of (Y, δ_1, δ_2) ensures that $\tilde{f}$ is well-defined). It can easily be verified that $\tilde{f}$ is Q-bicontinuous and $\tilde{f} \circ q_X = f$. □

By using Theorem 1 of [9], we get the following corollary.

Corollary 3.4. *Q-$\mathbf{BTOP}_0$ is closed under the formation of products and extremal subobjects in Q-$\mathbf{BTOP}$.*

Remark 3.5. We point out that the extremal subobjects of $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}$ are precisely the subspaces of (X, τ_1, τ_2) (cf. Proposition 21.13, [1]).

Proposition 3.6. *Q-$\mathbf{BTOP}_0$ is a well-powered and complete (Epi, Extremal mono)-category.*

Proof. Q-$\mathbf{BTOP}_0$ is an epireflective subcategory of Q-$\mathbf{BTOP}$ (Theorem 3.3). Also Q-$\mathbf{BTOP}$ is a topological category. Hence, by using Proposition 3.3 of [2], Q-$\mathbf{BTOP}_0$ is a well-powered and complete (Epi, Extremal mono)-category. □

4. Sierpinski Objects in Q-BTOP

The objective set out in this section is to obtain certain results (in the setting of Q-bitopological spaces), which are analogous to the corresponding results in [4] and [6] proved in the settings of bitopological spaces and fuzzy bitopological spaces.

In topology or fuzzy topology, the fact that the respective Sierpinski objects help 'identify' the open sets or open fuzzy sets, plays an important role in proving certain results. In bitopology or fuzzy bitopology, each open set or open fuzzy set also gives rise to a suitable bicontinuous map into respective Sierpinski objects which also, in some sense, helps 'identify' that open set or open fuzzy set (cf. [6], Propositions 2.2.2 and 2.2.8). A close examination of the proofs of these facts shows that the presence of 0 or 1 in the carrier sets, e.g., $\{0, 1\}$ or $[0, 1]$, or any frame L (which are just specific nullary operations, viewed algebraically) of the involved Sierpinski objects in the above-mentioned situations, plays a significant role. For Q-topological spaces, the carrier set Q of the Q-Sierpinski space, has no counterpart of 0 or 1. This makes it 'difficult' to prove the analogues of some results in [4] (or [6]) for Q-bitopological spaces. We shall see that the 'difficulty' is circumvented by equipping Q with a nullary operation. Accordingly, we make the following assumption.

Assumption: From now on, we will assume that Q has at least one 0-ary (nullary) operation, say ω. Further, let the distinguished element of Q corresponding to it be also denoted by ω. Then for any set X, ω, in turn, provides a distinguished element $\bar{\omega}$ of Q^X, where $\bar{\omega}$ is the ω-valued constant map from X to Q.

Let Q^2 denote the product $Q \times Q$ and let π_1, π_2 denote the two projection maps from Q^2 to Q. Consider the Q-bitopological space $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$.

We now proceed to show that the Q-bitopological space $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ is a counterpart of the so-called 'quad' in $\mathbf{BTOP}_0$ (of [4]) and the Sierpinski object I^2 in $\mathbf{BFTS}_0$ (of [6]).

Proposition 4.1. *If* $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}$*, then for each* $\mu \in \tau_1$ *(resp.* $\mu \in \tau_2$*), there exists a* Q*-bicontinuous map* $h_\mu : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ *such that* $\pi_1 \circ h_\mu = \mu$ *(resp.* $\pi_2 \circ h_\mu = \mu$*).*

Proof. Let $\mu \in \tau_1$. Define $h_\mu : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ as $h_\mu(x) = (\mu(x), \omega)$, for every $x \in X$. Now, for $x \in X$, $(\pi_1 \circ h_\mu)(x) = \pi_1(h_\mu(x)) = \pi_1(\mu(x), \omega) = \mu(x)$, implying that $\pi_1 \circ h_\mu = \mu \in \tau_1$. Also, $(\pi_2 \circ h_\mu)(x) = \pi_2(h_\mu(x)) = \pi_2(\mu(x), \omega) = \omega = \bar{\omega}(x)$, implying that $\pi_2 \circ h_\mu = \bar{\omega} \in \tau_2$. Hence, h_μ is Q-bicontinuous.

Similarly, for $\mu \in \tau_2$, define h_μ as $h_\mu(x) = (\omega, \mu(x))$, for every $x \in X$. Now it can be shown that h_μ is Q-bicontinuous and $\pi_2 \circ h_\mu = \mu$. □

Theorem 4.2. $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ *is a Sierpinski object in* Q-$\mathbf{BTOP}$.

Proof. Let $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}$ and let $\mathscr{F} = \{f \mid f : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ is Q-bicontinuous$\}$. Let $(Y, \delta_1, \delta_2) \in obQ$-$\mathbf{BTOP}$ and $g : Y \to X$ be a mapping such that $f \circ g : (Y, \delta_1, \delta_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ is Q-bicontinuous for every $f \in \mathscr{F}$. We wish to show that g is Q-bicontinuous. Let $\mu \in \tau_1$. Then the Q-bicontinuous map $h_\mu : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$, described in Proposition 4.1, is already in $\mathscr{F}$. Now, $\mu \circ g = (\pi_1 \circ h_\mu) \circ g = \pi_1 \circ (h_\mu \circ g)$. Hence, $\mu \circ g \in \delta_1$ (as $h_\mu \circ g$ is Q-bicontinuous).

Similarly, for every $\mu \in \tau_2$, $\mu \circ g \in \delta_2$. So, g is Q-bicontinuous. Thus, $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ is a Sierpinski object in Q-$\mathbf{BTOP}$. □

Let $\mathscr{C}$ be a category and epi$\mathscr{C}$ denote the class of all $\mathscr{C}$-epimorphisms. Further, let $\mathscr{H}$ denote some class of $\mathscr{C}$-morphisms.

Definition 4.3. [2] A $\mathscr{C}$-object X is called:

- $\mathscr{H}$-injective if for every $e : Y \to Z$ in $\mathscr{H}$ and every $\mathscr{C}$-morphism $f : Y \to X$, there exists a $\mathscr{C}$-morphism $g : Z \to X$ such that $g \circ e = f$.
- weakly $\mathscr{H}$-injective if it is (epi$\mathscr{C} \cap \mathscr{H}$)-injective.
- $\mathscr{H}$-saturated if each $\mathscr{C}$-morphism $f : X \to Y$ with $f \in$epi$\mathscr{C} \cap \mathscr{H}$, is an isomorphism.
- a cogenerator in $\mathscr{C}$ if for every pair of distinct $f, g \in \mathscr{C}(Y, Z)$, there exists $h \in \mathscr{C}(Z, X)$ such that $h \circ f \neq h \circ g$.

Definition 4.4. [2] We say that $\mathscr{C}$ is $\mathscr{H}$-cogenerated by a class $\mathscr{A}$ of $\mathscr{C}$-objects if every $\mathscr{C}$-object X is an $\mathscr{H}$-subobject (i.e., there is some $h : X \to Y$, with $h \in \mathscr{H}$) of a product of objects in $\mathscr{A}$.

The following observation provides an example of a T_0-Q-bitopological space, which is most used here for our purpose.

Proposition 4.5. $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ *is* T_0.

From now on, we will assume that $\mathscr{H}$ denotes the class of all embeddings in Q-$\mathbf{BTOP_0}$.

Proposition 4.6. $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ *is* $\mathscr{H}$*-injective in* Q-$\mathbf{BTOP_0}$.

Proof. Let $e : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ be an embedding in Q-$\mathbf{BTOP_0}$ and let $f : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ be a Q-bicontinuous map. Then $\pi_i \circ f \in \tau_i$ for every $i = 1, 2$. As e is an embedding, $e^{-1} : e(X) \to X$ is Q-bicontinuous. So, $(\pi_i \circ f) \circ e^{-1} \in \delta_i|_{e(X)}$, implying that there exists $\mu_i \in \delta_i$ such that $\mu_i|_{e(X)} = (\pi_i \circ f) \circ e^{-1}$ for every $i = 1, 2$. Define $g : (Y, \delta_1, \delta_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ as $g(y) = (\mu_1(y), \mu_2(y))$, for every $y \in Y$. Then it is easy to see that g is Q-bicontinuous and $g \circ e = f$. Hence, $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ is $\mathscr{H}$-injective in Q-$\mathbf{BTOP_0}$. □

Proposition 4.7. $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ *is a cogenerator in* Q-$\mathbf{BTOP_0}$.

Proof. Let $f, g : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ be two distinct morphisms in Q-$\mathbf{BTOP_0}$. Then for some $x \in X$, $f(x) \neq g(x)$. As (Y, δ_1, δ_2) is T_0, $\mu(f(x)) \neq \mu(g(x))$ for some $\mu \in \delta_1 \cup \delta_2$. If $\mu \in \delta_1$ (resp. $\mu \in \delta_2$), then by Proposition 4.1, there exists a Q-bicontinuous map $h_\mu : (Y, \delta_1, \delta_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ defined as $h_\mu(y) = (\mu(y), \omega)$ (resp. $h_\mu(y) = (\omega, \mu(y))$). Clearly in either case $h_\mu \circ f \neq h_\mu \circ g$. Thus, $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ is a cogenerator in Q-$\mathbf{BTOP_0}$. □

In the following, we give another Sierpinski object in Q-$\mathbf{BTOP}$. We believe that this Sierpinski object behaves in a similar manner as do the 'triad' of $\mathbf{BTOP_0}$ in [4] and the Sierpinski object $2I$ of $\mathbf{BFTS_0}$ in [6].

Let $2Q = (Q \times \{\omega\}) \cup (\{\omega\} \times Q)$. Let $p_1, p_2 : 2Q \to Q$ be the maps defined by $p_1(x) = q$, if $x = (q, \omega)$ and ω otherwise (similarly, $p_2(x) = q$, if $x = (\omega, q)$ and ω otherwise). Let $P_i = \langle p_i\rangle$, for $i = 1, 2$ (considered as subalgebras of the Ω-algebra Q^{2Q}). Consider the Q-bitopological space $(2Q, P_1, P_2)$.

Proposition 4.8. $(2Q, P_1, P_2)$ *has the following properties:*

(1) *For every* $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}$ *and for every* $\mu \in \tau_1$ *(resp.* $\mu \in \tau_2$*), there exists a* Q*-bicontinuous map* $h_\mu : (X, \tau_1, \tau_2) \to (2Q, P_1, P_2)$ *with* $h_{\mu_Q}^{\leftarrow}(p_1) = \mu$ *(resp.* $h_{\mu_Q}^{\leftarrow}(p_2) = \mu$*).*

(2) $(2Q, P_1, P_2)$ *is* T_0.

(3) $(2Q, P_1, P_2)$ *is a Sierpinski object in* Q-$\mathbf{BTOP}$.

(4) $(2Q, P_1, P_2)$ *is a cogenerator in* Q-$\mathbf{BTOP_0}$.

(5) $X = (X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP_0}$ *if and only if the family* $\mathscr{F} = Q$-$\mathbf{BTOP_0}((X, \tau_1, \tau_2), (2Q, P_1, P_2))$ *separates points of* X.

Proof. The proof of these properties is quite similar to those given already in the case of $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ and hence are omitted. □

Proposition 4.9. $(2Q, P_1, P_2)$ *is not an* $\mathscr{H}$*-injective object in* Q-$\mathbf{BTOP_0}$.

Proof. Let $id : 2Q \to 2Q$ be the identity map and $i : 2Q \to Q^2$ be the inclusion map. We show that i is an epimorphism in Q-$\mathbf{BTOP_0}$. For this, let $f, g : (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle) \to (Y, \delta_1, \delta_2)$ be two distinct Q-$\mathbf{BTOP_0}$-morphisms. Then

$f(x) \neq g(x)$, for some $x = (q_1, q_2) \in Q^2$. Since Y is T_0, there exists some $\nu \in \delta_1 \cup \delta_2$, say $\nu \in \delta_1$, with $\nu(f(x)) \neq \nu(g(x))$, i.e., $f_Q^{\leftarrow}(\nu) \neq g_Q^{\leftarrow}(\nu)$. Now it is clear that $f_Q^{\leftarrow}(\nu), g_Q^{\leftarrow}(\nu) \in \langle \pi_1 \rangle$, so there exist $\lambda_1, \lambda_2 \in I$ such that $f_Q^{\leftarrow}(\nu) = \omega_{\lambda_1}^{Q^{Q^2}}(\langle \alpha_i \rangle_{i \in n_{\lambda_1}})$ and $g_Q^{\leftarrow}(\nu) = \omega_{\lambda_2}^{Q^{Q^2}}(\langle \beta_i \rangle_{i \in n_{\lambda_2}})$, where $\alpha_i = \pi_1$ or $\bar{\omega}$, for every $i \in n_{\lambda_1}$ and $\beta_i = \pi_1$ or $\bar{\omega}$, for every $i \in n_{\lambda_2}$. Since $f_Q^{\leftarrow}(\nu) \neq g_Q^{\leftarrow}(\nu)$, there exists some $z = (a, b) \in Q^2$ such that $f_Q^{\leftarrow}(\nu)(z) \neq g_Q^{\leftarrow}(\nu)(z)$. Take $z_1 = (a, \omega) \in 2Q$. Then it can be noticed also that $f_Q^{\leftarrow}(\nu)(z_1) \neq g_Q^{\leftarrow}(\nu)(z_1)$, which implies that $(f \circ i)(z_1) \neq (g \circ i)(z_1)$, i.e., $(f \circ i) \neq (g \circ i)$. Hence i is an epimorphism.

Now it is clear that, there cannot exist any Q-continuous map $h : Q^2 \to 2Q$ with $h \circ i = id$, for if such a map h exists then as id is an extremal monomorphism, i will have to be an isomorphism, which is not possible. Thus, $(2Q, P_1, P_2)$ is not an $\mathscr{H}$-injective object in Q-$\mathbf{BTOP}_0$. □

The 'significance' of the observation that the Sierpinski object $(2Q, P_1, P_2)$ is *not injective*, has, somewhat, already been indicated in the second paragraph of section 4. However, the same may be stated more clearly (following the observation of the referee), say as the following Remark, *after* the proof of Proposition 4.9.

Remark 4.10. Giuli and Salbany (cf. [4]) produced *two* counterparts of the usual two-point Sierpinski space 2_S (in the category $\mathbf{TOP}_0$ of T_0-topological spaces) in the category $\mathbf{BTOP}_0$ of T_0-bitopological spaces, and showed that, while both were Sierpinski objects in **BTOP** (like 2_S in **TOP**), only one of these was also injective in $\mathbf{BTOP_0}$ (like 2_S in $\mathbf{TOP}_0$). Analogous situation was shown to exist in the category $\mathbf{BFTS}_0$ (cf. [6]). It is natural to wonder if a similar situation also exists in the category Q-**BTOP**. Proposition 4.9 shows that this indeed is the case by showing that the Sierpinski object $(2Q, P_1, P_2)$ is not injective in Q-$\mathbf{BTOP}_0$.

5. Some Characterizations of T_0-Q-bitopological Spaces

We first present a few characterizations of T_0-Q-bitopological spaces which involve the role of the Sierpinski Q-bitopological space $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$.

Theorem 5.1. $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}_0$ *if and only if* $\mathscr{F} = Q$-$\mathbf{BTOP}_0$ $((X, \tau_1, \tau_2), (Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle))$ *separates points of* X.

Proof. Let $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP_0}$ and $x, y \in X$ with $x \neq y$. Then $\mu(x) \neq \mu(y)$, for some $\mu \in \tau_1 \cup \tau_2$. If $\mu \in \tau_1$ (resp. $\mu \in \tau_2$), then the Q-bicontinuous map $h_\mu : (X, \tau_1, \tau_2) \to (Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$, described in Proposition 4.1, is already in $\mathscr{F}$. Clearly, $h_\mu(x) \neq h_\mu(y)$. Thus, $\mathscr{F}$ separates points of X.

Conversely, let $\mathscr{F}$ separate points of X and let $x, y \in X$ with $x \neq y$. Then $f(x) \neq f(y)$, for some $f \in \mathscr{F}$ and hence, $\pi_1 \circ f \in \tau_1$ and $\pi_2 \circ f \in \tau_2$. As $f(x) \neq f(y)$, either $\pi_1(f(x)) \neq \pi_1(f(y))$ or $\pi_2(f(x)) \neq \pi_2(f(y))$, showing that (X, τ_1, τ_2) is T_0. □

Theorem 5.2. $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP_0}$ *if and only if* (X, τ_1, τ_2) *is homeomorphic to a subspace of a product of copies of* $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$.

Proof. Let $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP_0}$ and let $\mathscr{F} = \{f \mid f : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ is Q-bicontinuous$\}$. Define $e : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)^{\mathscr{F}}$ as $e(x)(f) = f(x)$, for every $x \in X$ and for every $f \in \mathscr{F}$. For convenience, let $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)^{\mathscr{F}}$ be denoted as (P, Π_1, Π_2). Let for $f \in \mathscr{F}$, $p_f : (P, \Pi_1, \Pi_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ denote the f-th projection map. Then for every $x \in X$, $(p_f \circ e)(x) = p_f(e(x)) = e(x)(f) = f(x)$ implying that $p_f \circ e = f$. Thus, e is Q-bicontinuous. Let $x, y \in X$ with $x \neq y$. Then $\mu(x) \neq \mu(y)$, for some $\mu \in \tau_1 \cup \tau_2$. If $\mu \in \tau_1$ (resp. $\mu \in \tau_2$), then the Q-bicontinuous map $h_\mu : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$, described in Proposition 4.1, is already in $\mathscr{F}$. Clearly, $h_\mu(x) \neq h_\mu(y)$, showing that $e(x) \neq e(y)$. Thus, e is injective. Now, we show that $e^{-1} : e(X) \to X$ is Q-bicontinuous. Let $\mu \in \tau_1$. Then the Q-bicontinuous map $h_\mu : (X, \tau_1, \tau_2) \to (Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$, described in Proposition 4.1, is already in $\mathscr{F}$. So, $\pi_1 \circ p_{h_\mu} \in \Pi_1$. Now, for every $x \in X$, $(\mu \circ e^{-1})(e(x)) = \mu(x) = \pi_1(\mu(x), \omega) = \pi_1(h_\mu(x)) = \pi_1(e(x)(h_\mu)) = \pi_1(p_{h_\mu}(e(x))) = (\pi_1 \circ p_{h_\mu})(e(x))$, implying that $\mu \circ e^{-1} = (\pi_1 \circ p_{h_\mu})|_{e(X)}$. Similarly, for $\mu \in \tau_2$, $\mu \circ e^{-1} = (\pi_2 \circ p_{h_\mu})|_{e(X)}$. Thus, $e : X \to e(X)$ is a homeomorphism, i.e., e is an embedding and hence, (X, τ_1, τ_2) is homeomorphic to a subspace of product of copies of $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$.

The converse follows from Proposition 4.5, Corollary 3.4 and Remark 3.5 . □

Corollary 5.3. $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ *$\mathscr{H}$-cogenerates* Q-$\mathbf{BTOP_0}$.

By using Theorem 2 of [9] in conjunction with Remark 3.5 and Theorem 5.2 above, we get the following result:

Theorem 5.4. Q-$\mathbf{BTOP_0}$ *is the epireflective hull of* $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ *in* Q-$\mathbf{BTOP}$.

Theorem 5.5. $(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP_0}$ *if and only if* (X, τ_1, τ_2) *is homeomorphic to a subspace of a product of copies of* $(2Q, P_1, P_2)$.

Proof. The proof is similar to that of Theorem 5.2 above. □

Corollary 5.6. $(2Q, P_1, P_2)$ *$\mathscr{H}$-cogenerates* Q-$\mathbf{BTOP_0}$.

Again by using Theorem 2 of [9] in conjunction with Remark 3.5 and Theorem 5.5 above, we get the following result:

Theorem 5.7. Q-$\mathbf{BTOP_0}$ *is the epireflective hull of* $(2Q, P_1, P_2)$ *in* Q-$\mathbf{BTOP}$.

In [7], some characterizations of T_0-fuzzy bitopological spaces were given. Analogously, in this section, we give some characterization of T_0-Q-bitopological spaces. For this, we use a closure operator (cf. [4] and [3]) in the category Q-$\mathbf{BTOP}$.

Let $X = (X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}$ and $M \subseteq X$. Define $[M] = \cap\{Eq(f, g) \mid Y \in obQ$-$\mathbf{BTOP_0}$ and $f, g : X \to Y$ are Q-bicontinuous map with $f|_M = g|_M\}$, where $Eq(f, g) = \{x \in X \mid f(x) = g(x)\}$. It can be easily seen that [ ] is a hereditary closure operator. Call M to be [ ]-closed if $[M] = M$.

The following result follows from Theorem 1.11 of [3].

Proposition 5.8. *A morphism* $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ *in* Q-$\mathbf{BTOP_0}$ *is an epimorphism iff* $[f(X)] = Y$.

Theorem 5.9. *If $X = (X, \tau_1, \tau_2) \in obQ$-**BTOP**, then the following statements are equivalent:*

(1) $X \in obQ\text{-}\mathbf{BTOP}_0$.
(2) *for every $Y \in obQ$-**BTOP** and for every Q-bicontinuous map $f : Y \to X$, the graph $G_f = \{(y, f(y)) \mid y \in Y\}$ of f, is $[\ \ ]$-closed in $Y \times X$.*
(3) $D_X = \{(x, x) \mid x \in X\}$ *is $[\ \ ]$-closed in $X \times X$.*

Proof. (1) $\Rightarrow$ (2): Let $X \in obQ\text{-}\mathbf{BTOP}_0$. Clearly, $G_f \subseteq [G_f]$. Let $(y_0, x_0) \notin G_f$. Then $x_0 \neq f(y_0)$ and so, there exists $\mu \in \tau_1 \cup \tau_2$ such that $\mu(x_0) \neq \mu(f(y_0))$. Let $\mu \in \tau_1$. Define $g, h : Y \times X \to Q^2$ as $g(y, x) = (\mu(x), \omega)$ and $h(y, x) = (\mu(f(y)), \omega)$, for every $(y, x) \in Y \times X$. It is easy to see that both g and h are Q-bicontinuous. Clearly, $g|_{G_f} = h|_{G_f}$, but $g(y_0, x_0) \neq h(y_0, x_0)$, implying that $(y_0, x_0) \notin [G_f]$. Hence, $[G_f] \subseteq G_f$ and so, $[G_f] = G_f$.

(2) $\Rightarrow$ (3): Let (2) hold. If $f : X \to X$ is the identity map, then it is easily seen that $G_f = D_X$. Hence, applying (2) to the identity map f, D_X is seen to be $[\ \ ]$-closed in $X \times X$.

(3) $\Rightarrow$ (1): Let D_X be $[\ \ ]$-closed in $X \times X$. If possible, let (X, τ_1, τ_2) be not T_0. Then there exist $x, y \in X$ with $x \neq y$, such that $\mu(x) = \mu(y)$, for every $\mu \in \tau_1 \cup \tau_2$. This implies that $\tilde{\mu}(x, x) = \tilde{\mu}(x, y)$, for every $\tilde{\mu} \in \langle\{\nu \circ p_j \mid \nu \in \tau_i, j = 1, 2\}\rangle$, $i = 1, 2$, where p_1, p_2 are the two projection maps from $X \times X$ to X. As $x \neq y$, $(x, y) \notin D_X = [D_X]$. So, there exists $(Z, \delta_1, \delta_2) \in obQ\text{-}\mathbf{BTOP}_0$ and Q-bicontinuous maps $g, h : X \times X \to Z$ such that $g|_{D_X} = h|_{D_X}$, but $g(x, y) \neq h(x, y)$. Since (Z, δ_1, δ_2) is T_0, there exists $\sigma \in \delta_1 \cup \delta_2$ such that $\sigma(g(x, y)) \neq \sigma(h(x, y))$. Clearly, $\sigma \circ g, \sigma \circ h \in \langle\{\nu \circ p_j \mid \nu \in \tau_1, j = 1, 2\}\rangle$ or $\langle\{\nu \circ p_j \mid \nu \in \tau_2, j = 1, 2\}\rangle$. So, $(\sigma \circ g)(x, y) = (\sigma \circ g)(x, x)$ and $(\sigma \circ h)(x, y) = (\sigma \circ h)(x, x)$, which implies that $(\sigma \circ g)(x, y) = (\sigma \circ h)(x, y)$, i.e., $\sigma(g(x, y)) = \sigma(h(x, y))$, a contradiction. Hence, (X, τ_1, τ_2) is T_0. □

6. Epireflective Hull of Q^2 in Q-BTOP$_0$

In this section, we show that the epireflective hull of $(Q^2, \langle\pi_1\rangle, \langle\pi_2\rangle)$ in $Q\text{-}\mathbf{BTOP}_0$ consists precisely of the 'saturated' T_0-Q-bitopological space (defined below).

The following definition is an analogue of the corresponding definition from [4].

Definition 6.1. A $Q\text{-}\mathbf{BTOP}_0$-object (X, τ_1, τ_2) is called saturated if each epimorphic-embedding $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ in $Q\text{-}\mathbf{BTOP}_0$ is an isomorphism.

We note that the term 'saturated' here coincides with the term '$\mathscr{H}$-saturated' (Definition 4.3), if $\mathscr{H}$ is the class of all embeddings in $Q\text{-}\mathbf{BTOP}_0$.

Let **Sat**-Q-$\mathbf{BTOP}_0$ denote the subcategory of $Q\text{-}\mathbf{BTOP}_0$ whose objects are saturated T_0-Q-bitopological spaces.

Proposition 6.2. $(X, \tau_1, \tau_2) \in ob\mathbf{Sat}\text{-}Q\text{-}\mathbf{BTOP}_0$ *if and only if for every embedding $e : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ in $Q\text{-}\mathbf{BTOP}_0$, $[e(X)] = e(X)$.*

Proof. Let $(X, \tau_1, \tau_2) \in ob\mathbf{Sat}\text{-}Q\text{-}\mathbf{BTOP}_0$ and $e : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ be an embedding in $Q\text{-}\mathbf{BTOP}_0$. Consider the inclusion map $i : e(X) \to [e(X)]$. As $[\ \ ]$ is a hereditary closure operator, i is an epimorphic-embedding. Since e is an embedding and $(X, \tau_1, \tau_2) \in ob\mathbf{Sat}\text{-}Q\text{-}\mathbf{BTOP_0}$ so, $e(X) \in ob\mathbf{Sat}\text{-}Q\text{-}\mathbf{BTOP_0}$. Hence, i is an isomorphism, i.e., $[e(X)] = e(X)$.

Conversely, let for every embedding $e : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ in $Q\text{-}\mathbf{BTOP_0}$, $[e(X)] = e(X)$. We have to show that $(X, \tau_1, \tau_2) \in ob\mathbf{Sat}\text{-}Q\text{-}\mathbf{BTOP_0}$. Let $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ be an epimorphic-embedding in $Q\text{-}\mathbf{BTOP_0}$. Then $[f(X)] = Y$, as f is an epimorphism in $Q\text{-}\mathbf{BTOP_0}$. Also $[f(X)] = f(X)$, as f is an embedding. Hence, $f(X) = Y$, i.e., f is an isomorphism. Thus, $(X, \tau_1, \tau_2) \in ob\mathbf{Sat}\text{-}Q\text{-}\mathbf{BTOP_0}$. □

We shall use the following results from [2] (Theorem 1.6 and Corollary 1.4) to obtain the epireflective hull of $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$ in the category $Q\text{-}\mathbf{BTOP_0}$.

Theorem 6.3. [2] *Let $\mathscr{C}$ be a complete and well-powered category. Then*

- *$\mathscr{C}$ admits an $\mathscr{H}$-firm epireflective subcategory $\mathscr{R}$ if and only if there exist a class $\mathscr{A}$ of weakly $\mathscr{H}$-injective objects of $\mathscr{C}$ which $\mathscr{H}$-cogenerates $\mathscr{C}$. In this case, $\mathscr{R}$ is the epireflective hull of $\mathscr{A}$ in $\mathscr{C}$.*
- *If $\mathscr{R}$ is an $\mathscr{H}$-firm epireflective subcategory $\mathscr{C}$ then $\mathscr{R}$ consists precisely of those objects which are $\mathscr{H}$-saturated and $\mathscr{R}$ also equals the subcategory consisting of weakly $\mathscr{H}$-injective objects of $\mathscr{C}$.*

Theorem 6.4. $\mathbf{Sat}\text{-}Q\text{-}\mathbf{BTOP_0}$ *is the epireflective hull of $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$ in Q-$\mathbf{BTOP_0}$.*

Proof. Recall that $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$ $\mathscr{H}$-cogenerates $Q\text{-}\mathbf{BTOP_0}$ (Corollary 5.3) and $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$ is an $\mathscr{H}$-injective object in $Q\text{-}\mathbf{BTOP_0}$ (Proposition 4.6). By using Theorem 6.3, the result follows immediately. □

7. Sober Q-bitopological Spaces

Let (X, τ) be a Q-topological space. Put $pt(\tau) = \{p : \tau \to Q \mid p \text{ is an } \Omega\text{-algebra homomorphism}\}$. Define $\phi : \tau \to Q^{pt(\tau)}$ as $\phi(\mu)(p) = p(\mu)$, for every $\mu \in \tau$ and for every $p \in pt(\tau)$. Then ϕ turns out to be an Ω-algebra homomorphism (cf. [13]). Consider the Q-topological space $(pt(\tau), \phi(\tau))$.

Definition 7.1. [13] A Q-topological space (X, τ) is called sober if $\psi_X : X \to pt(\tau)$, defined as $\psi_X(x)(\mu) = \mu(x)$, for every $x \in X$ and for every $\mu \in \tau$, is bijective.

For a Q-bitopological space (X, τ_1, τ_2), $(pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2))$ is also a Q-bitopological space.

For a Q-bitopological space (X, τ_1, τ_2), define a map $\eta_X : X \to pt(\langle \tau_1 \cup \tau_2 \rangle)$ as $\eta_X(x)(\mu) = \mu(x)$, for every $x \in X$ and for every $\mu \in \langle \tau_1 \cup \tau_2 \rangle$.

Proposition 7.2. *Given $(X, \tau_1, \tau_2) \in obQ\text{-}\mathbf{BTOP}$, $\eta_X : (X, \tau_1, \tau_2) \to (pt(\langle \tau_1 \cup \tau_2 \rangle), \phi_{\langle \tau_1 \cup \tau_2 \rangle}(\tau_1), \phi_{\langle \tau_1 \cup \tau_2 \rangle}(\tau_2))$ is Q-bicontinuous.*

Proof. Let $(X, \tau_1, \tau_2) \in obQ\text{-}\mathbf{BTOP}$. Let $\mu \in \tau_i$, $i = 1, 2$. Then $\forall x \in X$, $\eta_X{}_Q^{\leftarrow}(\phi_{\langle \tau_1 \cup \tau_2 \rangle}(\mu))(x) = (\phi_{\langle \tau_1 \cup \tau_2 \rangle}(\mu) \circ \eta_X)(x) = \phi_{\langle \tau_1 \cup \tau_2 \rangle}(\mu)(\eta_X(x)) = \eta_X(x)(\mu) = \mu(x)$, implying that $\eta_X{}_Q^{\leftarrow}(\phi_{\langle \tau_1 \cup \tau_2 \rangle}(\mu)) = \mu \in \tau_i$, $i = 1, 2$. Thus, η_X is Q-bicontinuous. □

Definition 7.3. A Q-bitopological space (X, τ_1, τ_2) is called sober (or bisober) if $\eta_X : X \to pt(\langle \tau_1 \cup \tau_2 \rangle)$ is bijective.

Let Q-**SOB** denote the subcategory of Q-**TOP** whose objects are sober Q-topological spaces and let Q-**BSOB** denote the subcategory of Q-**BTOP** whose objects are sober Q-bitopological spaces.

The following result is easy to verify.

Proposition 7.4. *(X, τ_1, τ_2) is sober if and only if $(X, \langle \tau_1 \cup \tau_2 \rangle)$ is sober.*

Proposition 7.5. *$(X, \tau_1, \tau_2) \in obQ$-$\mathbf{BTOP}_0$ if and only if $\eta_X : (X, \tau_1, \tau_2) \to (pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2))$ is an embedding.*

Proof. Clearly, $\eta_X : (X, \tau_1, \tau_2) \to (pt(\langle \tau_1 \cup \tau_2 \rangle), \phi_{\langle \tau_1 \cup \tau_2 \rangle}(\tau_1), \phi_{\langle \tau_1 \cup \tau_2 \rangle}(\tau_2))$ is Q-bicontinuous (by Proposition 7.2). Let $x, y \in X$ with $x \neq y$. Then there exists $\mu \in \tau_1 \cup \tau_2$ such that $\mu(x) \neq \mu(y)$, which implies that $\eta_X(x) \neq \eta_X(y)$. Thus, η_X is injective. Now, we show that $\eta_X^{-1} : \eta_X(X) \to X$ is Q-bicontinuous. Let $\mu \in \tau_i$, $i = 1, 2$. Then for every $x \in X$, $(\mu \circ \eta_X^{-1})(\eta_X(x)) = \mu(x) = \eta_X(x)(\mu) = \phi(\mu)(\eta_X(x))$ implying that $\mu \circ \eta_X^{-1} = \phi(\mu)|_{\eta_X(X)} \in \phi(\tau_i)|_{\eta_X(X)}$, $i = 1, 2$. Hence, η_X is an embedding. The converse is easy to verify. □

Proposition 7.6. *$(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$ is sober.*

Proof. We show that $\eta_{Q^2} : Q^2 \to pt(\langle \langle \pi_1 \rangle \cup \langle \pi_2 \rangle \rangle)$ is bijective. As $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$ is T_0, so, by using Proposition 7.5, η_{Q^2} is injective. Clearly, $pt(\langle \langle \pi_1 \rangle \cup \langle \pi_2 \rangle \rangle) = pt(\langle \{\pi_1, \pi_2\} \rangle)$. Let $p \in pt(\langle \{\pi_1, \pi_2\} \rangle)$. Then $p : \langle \{\pi_1, \pi_2\} \rangle \to Q$ is an Ω-algebra homomorphism. Let $p(\pi_1) = \alpha$ and $p(\pi_2) = \beta$. Then $(\alpha, \beta) \in Q^2$. Now, $\eta_{Q^2}(\alpha, \beta)(\pi_1) = \pi_1(\alpha, \beta) = \alpha = p(\pi_1)$ and $\eta_{Q^2}(\alpha, \beta)(\pi_2) = \pi_2(\alpha, \beta) = \beta = p(\pi_2)$, showing that $\eta_{Q^2}(\alpha, \beta)|_{\{\pi_1, \pi_2\}}$
$= p|_{\{\pi_1, \pi_2\}}$. Hence, $\eta_{Q^2}(\alpha, \beta) = p$ and thus, η_{Q^2} is surjective. □

Proposition 7.7. *For each $(X, \tau_1, \tau_2) \in obQ$-**BTOP**, $(pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2))$ is sober.*

Proof. We know that, for any Q-topological space (X, τ), $(pt(\tau), \phi(\tau))$ is sober (cf. [13]). Now, consider the Q-topological space $(X, \langle \tau_1 \cup \tau_2 \rangle)$. Then $(pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\langle \tau_1 \cup \tau_2 \rangle))$ is sober. As ϕ is an Ω-algebra homomorphism, $\phi(\langle \tau_1 \cup \tau_2 \rangle) = \langle \phi(\tau_1) \cup \phi(\tau_2) \rangle$ and so, $(pt(\langle \tau_1 \cup \tau_2 \rangle), \langle \phi(\tau_1) \cup \phi(\tau_2) \rangle)$ is sober. By using Proposition 7.4, $(pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2))$ is sober. □

Theorem 7.8. *Q-**BSOB** is epireflective in Q-$\mathbf{BTOP}_0$.*

Proof. Let $(X, \tau_1, \tau_2) \in Q$-$\mathbf{BTOP}_0$. We show that $\eta_X : (X, \tau_1, \tau_2) \to (pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2))$, defined as $\eta_X(x)(\mu) = \mu(x)$, for every $x \in X$ and for every $\mu \in \langle \tau_1 \cup \tau_2 \rangle$, is the desired epireflection of (X, τ_1, τ_2) in Q-**BSOB**. Observe that, for any $\mu_1, \mu_2 \in \tau_1 \cup \tau_2$, $\phi(\mu_1) \circ \eta_X = \phi(\mu_2) \circ \eta_X$, implying that $\phi(\mu_1) = \phi(\mu_2)$. Let $g, h : (pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2)) \to (Z, \delta_1, \delta_2)$ be two distinct morphisms in Q-$\mathbf{BTOP}_0$. Then there exists $p \in pt(\langle \tau_1 \cup \tau_2 \rangle)$ such that $g(p) \neq h(p)$. As (Z, δ_1, δ_2) is T_0, there exists $\nu \in \delta_1 \cup \delta_2$ such that $\nu(g(p)) \neq \nu(h(p))$. Thus, $\nu \circ g \neq \nu \circ h$ and $\nu \circ g, \nu \circ h \in \phi(\tau_1) \cup \phi(\tau_2)$. This implies that $(\nu \circ g) \circ \eta_X \neq (\nu \circ h) \circ \eta_X$, i.e., $\nu \circ (g \circ \eta_X) \neq \nu \circ (h \circ \eta_X)$. Hence, $g \circ \eta_X \neq h \circ \eta_X$, showing that η_X is an epimorphism.

Let $(Y, \delta_1, \delta_2) \in obQ$-**BSOB** and let $f : (X, \tau_1, \tau_2) \to (Y, \delta_1, \delta_2)$ be a Q-bicontinuous map. We have to find a Q-bicontinuous map $f^* : (pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2)) \to (Y, \delta_1, \delta_2)$ such that $f^* \circ \eta_X = f$. Let $p \in pt(\langle \tau_1 \cup \tau_2 \rangle)$. Then $p : \langle \tau_1 \cup \tau_2 \rangle \to Q$ is an Ω-algebra homomorphism. Define $p' : \langle \delta_1 \cup \delta_2 \rangle \to Q$ as $p'(\nu) = p(\nu \circ f)$, for every $\nu \in \langle \delta_1 \cup \delta_2 \rangle$. It is easy to see that p' is an Ω-algebra homomorphism. As (Y, δ_1, δ_2) is sober, there exists a unique $y \in Y$ such that $\eta_Y(y) = p'$. Put $f^*(p) = y$. Let $\nu \in \delta_i$, $i = 1, 2$. Then for every $p \in pt(\langle \tau_1 \cup \tau_2 \rangle)$, $(\nu \circ f^*)(p) = \nu(f^*(p)) = \nu(y) = \eta_Y(y)(\nu) = p'(\nu) = p(\nu \circ f) = \phi(\nu \circ f)(p)$, implying that $(\nu \circ f^*) = \phi(\nu \circ f)$. As $\nu \in \delta_i$, $\nu \circ f \in \tau_i$, showing that $\nu \circ f^* \in \phi(\tau_i)$, $i = 1, 2$. Hence, f^* is Q-bicontinuous. Now, for every $x \in X$ and for every $\nu \in \langle \delta_1 \cup \delta_2 \rangle$, $\eta_Y(f(x))(\nu) = \nu(f(x)) = (\nu \circ f)(x) = \eta_X(x)(\nu \circ f) = (\eta_X(x))'(\nu)$. Hence, $\eta_Y(f(x)) = (\eta_X(x))'$ and so, $f^*(\eta_X(x)) = f(x)$, for every $x \in X$. Thus, $f^* \circ \eta_X = f$. □

Remark 7.9. At this point of writing, we are unable to prove/disprove that the Sierpinski object $(2Q, P_1, P_2)$ is sober, even though the objects in the categories $\mathbf{BTOP}_0$ of T_0-bitopological spaces and $\mathbf{BFTS}_0$ of T_0-fuzzy bitopological spaces, which correspond to $(2Q, P_1, P_2)$, are sober.

Proposition 7.10. **Sat**-Q-$\mathbf{BTOP}_0 \subseteq Q$-**BSOB**.

Proof. Let $(X, \tau_1, \tau_2) \in ob$**Sat**-$Q$-$\mathbf{BTOP}_0$. Then the map $\eta_X : (X, \tau_1, \tau_2) \to (pt(\langle \tau_1 \cup \tau_2 \rangle), \phi(\tau_1), \phi(\tau_2))$ is an epimorphic-embedding (Proposition 7.5 and Theorem 7.8). Hence, η_X is an isomorphism and so, $(X, \tau_1, \tau_2) \in obQ$-**BSOB**. Thus, **Sat**-Q-$\mathbf{BTOP}_0 \subseteq Q$-**BSOB**. □

Remark 7.11. The above containment of **Sat**-Q-$\mathbf{BTOP}_0$ in Q-**BSOB** is actually a proper containment. This follows from the facts that, if we specialize the Ω-algebra Q to the algebra $[0, 1]$ equipped with the usual join and meet operations, then **Sat**-Q-$\mathbf{BTOP}_0$ and Q-**BSOB**, respectively, get specialized to the categories **Sat**-$\mathbf{BFTS}_0$ and **BFSOB** of $2T_0$-saturated fuzzy bitopological spaces and sober fuzzy bitopological spaces, studied in [6], where it has been already shown that **Sat**-$\mathbf{BFTS}_0 \subsetneqq$ **BFSOB**.

8. Conclusion

In this paper, we have studied T_0-Q-bitopological spaces and sober Q-bitopological spaces and their close relations with two suitably identified Sierpinski objects in the category Q-**BTOP**. We have also found out the epireflective hull of one of these Sierpinski objects, viz., $(Q^2, \langle \pi_1 \rangle, \langle \pi_2 \rangle)$ in the category Q-$\mathbf{BTOP}_0$ which turns out to be the category **Sat**-Q-$\mathbf{BTOP}_0$. Moreover, the category Q-**BSOB** has also been shown to be epireflective in Q-$\mathbf{BTOP}_0$.

We would like to end with the following remark. It may be noted that the study of Q-topological/ bitopological spaces may eventually turn out to be quite different from that of L-topological/ bitopological spaces (where L is a sutaible *lattice*). Thus, while L has *both* an *order* structure and an *algebraic* structure, Q has *only* an algebraic structure. As a consequence, many results for L-topological/ bitopological space may not have any (or at least *analogous*) counterparts for Q-topological/ bitopological spaces (e.g., there is no smallest, i.e., indiscrete, Q-topology on a set in general). In particular, Rodabaugh's view (cf. [10]) that the study of L-valued bitopology is "categorically redundant", once L-valued topologies have been well-understood, does not automatically and immediately apply to Q-bitopological spaces, vis-a-vis Q-topological spaces (though it may be interesting to examine any possible relationship between Q-bitopological spaces and Q-topological spaces, on the lines of [10], but which clearly is outside the scope of the present paper). In any case, a close look at [10] shows that L, considered therein, is *at least* a semiquantale, where L is *necessarily* a complete lattice (which Q is *not*), with a binary operation and many observations in [10] make *crucial* use of the order structure of L.

Acknowledgements. The authors are grateful to the referee for making suggestions for improving the paper.

The first author would like to thank the University Grants Commission, New Delhi, India, for financial support through its Senior Research Fellowship.

References

[1] J. Adámek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, Wiley, 1990.
[2] G. C. L. Brümmer, E. Giuli and H. Herrlich, *Epireflections which are completions*, Cahiers Topo. Géom. Diff. Catégoriques, **33** (1992), 71–93.
[3] G. Castellini, *Closure operators, monomorphisms and epimorphisms in categories of groups*, Cahiers Topo. Géom. Diff. Catégoriques, **27** (1986), 151–167.
[4] E. Giuli and S. Salbany, *$2T_0$ spaces and closure operators*, Seminarberichte Fachbereich Math. und Inform., Fern Universität, Hagen, **29** (1988), 11–40.
[5] N. Jacobson, *Basic Algebra II*, W.H. Freeman and Company, 1980.
[6] A. S. Khastgir and A. K. Srivastava, *On two epireflective subcategories of the category of $2T_0$-fuzzy bitopological spaces*, Quaest. Math., **23** (2000), 163–177.
[7] A. S. Khastgir and A. K. Srivastava, *On T_0-objects in FTS and BFTS*, Fuzzy Sets and Systems, **109** (2000), 301–304.
[8] E. G. Manes, *Compact Hausdorff objects*, General Topology Appl., **4** (1974), 341–360.
[9] T. Marny, *On epireflective subcategories of topological categories*, General Topology Appl., **10** (1979), 175–181.
[10] S. E. Rodabaugh, *Functorial comparisons of bitopology with topology and the case for redundancy of bitopology in lattice-valued mathematics*, Appl. Gen. Topol., **9** (2008), 77–108.
[11] S. K. Singh and A. K. Srivastava, *A characterization of the category Q-**TOP***, Fuzzy Sets and Systems, **227** (2013), 46–50.
[12] S. K. Singh and A. K. Srivastava, *On Q-sobriety*, Quaest. Math., **39** (2016), 179–188.
[13] S. A. Solovyov, *Sobriety and spatiality in varieties of algebras*, Fuzzy Sets and Systems, **159** (2008), 2567–2585.

Rana Noor, Department of Mathematics, University of Allahabad, Allahabad-211 002, India
E-mail address: rananoor4@gmail.com

Arun K. Srivastava, DST-Centre for Interdisciplinary Mathematical Sciences, Banaras Hindu University, Varanasi-221 005, India
E-mail address: arunksrivastava@gmail.com

Sheo Kumar Singh*, Department of Mathematics, Mahatma Gandhi Central University Bihar, East Champaran-845 401, India
E-mail address: sheomathbhu@gmail.com

*Corresponding author