THE UNIFORM BOUNDEDNESS PRINCIPLE IN FUZZIFYING TOPOLOGICAL LINEAR SPACES

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Abstract. The main purpose of this study is to discuss the uniform boundedness principle in fuzzifying topological linear spaces. At first the concepts of uniformly boundedness principle and fuzzy equicontinuous family of linear operators are proposed, then the relations between fuzzy equicontinuous and uniformly bounded are studied, and with the help of net convergence, the characterization of fuzzy equicontinuous is proved. Finally, the famous theorem of the uniform boundedness principle is presented in fuzzifying topological linear spaces.

1. Introduction

As we have known, the uniform boundedness principle or resonance theorem is one of the three very important theorems in classical functional analysis. Thus the study of the uniform boundedness principle in fuzzy functional analysis is very important for the further development of fuzzy analysis. Ever since the notions of the fuzzy topological vector spaces and fuzzy normed linear spaces were introduced by Katsaras [1, 2], many authors investigated various aspects of fuzzy topological vector spaces. Some of them, there are several papers involved in the uniform boundedness principle in fuzzy functional analysis. The first effort in this direction was made by Yan and Fang [14], the principle of uniform boundedness for the family of L-fuzzy linear order-homomorphisms in L-topological vector spaces is proved by them. In [10], Xiao and Zhu studied the uniform boundedness principle of linear operators in Felbin′s type fuzzy normed linear spaces. Later, Song and Fang [7] gave a resonance theorem for the family of quasi-homogeneous operators in fuzzy normed linear spaces, and discussed some applications of the resonance theorem in Menger probabilistic metric spaces [9, 20].

By using the semantical method of Łukasiewicz logic, Ying [17, 18, 19] established a fundamental framework of fuzzifying topology. Influenced by Ying′s work, the notation of fuzzifying topological linear spaces was introduced by Qiu [3] originally, and the concepts of fuzzy boundedness and fuzzy complete boundedness were defined, some basic properties of them were studied. Based on the framework given by Qiu, Yan studied the theory of fuzzifying topological linear spaces systematically [11, 12, 13, 16].

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As a continuation of above work, the uniform boundedness principle in fuzzifying topological linear spaces is studied in present manuscript. First the notions of uniformly bounded and a family of fuzzy equicontinuous linear operators is introduced, then the relations between them and the characterization of fuzzy equicontinuous are obtained. Finally, the famous theorem of the uniform boundedness principle is generalized to fuzzifying topological linear spaces. Considering the extensive application of the uniform boundedness principle, this study may be beneficial for the research on fuzzy normed linear spaces and fuzzy control systems.

2. Preliminaries

In this section, we recall some concepts and results related to fuzzifying topology and a fuzzifying topological linear space, which will be used in the sequel. For further details, please refer to [17, 18, 19, 3]. First, we give some fuzzy logical notations.

For any formula φ, the symbol $[\varphi]$ denotes the truth value of φ, where the set of truth values is the unit interval $[0,1]$. A formula φ is valid and we write $\models \varphi$ if and only if $[\varphi]=1$ for every interpretation.

$$(\alpha \to \beta) \overset{\text{def}}{=} \min(1, 1-[\alpha]+[\beta]);$$

$$[\alpha \wedge \beta] \overset{\text{def}}{=} \min([\alpha],[\beta]);$$

$$\alpha \bigvee \beta \overset{\text{def}}{=} \neg(\neg\alpha \bigwedge \neg\beta).$$

Throughout this study, X always denotes a universe of discourse. According to the terminology introduced by Rodabaugh [4], the notation $f^{\to}$ denotes a special universal lifting of the original function f, and $f^{\leftarrow}$ denotes the unique right adjoint of the image operator f^{-1} of the original function guaranteed by the Adjoint Functor Theorem.

Definition 2.1. (Šostak [8], Ying [17]) For each non-empty set X and every mapping $\tau : 2^X \to I$, the pair (X,τ) is called a fuzzifying topological space if and only if τ fulfills the following conditions.

(1) $\models X \in \tau$;

(2) $\models (\forall A_1,\ A_2 \in 2^X)(A_1 \in \tau \bigwedge A_2 \in \tau \to A_1 \cap A_2 \in \tau)$;

(3) $\models (\forall \mathscr{A})(((\mathscr{A} \subseteq 2^X) \bigwedge (\forall A \in \mathscr{A})(A \in \tau)) \to \bigcup_{A\in\mathscr{A}} A \in \tau)$.

Definition 2.2. (Ying [17]) Let (X,τ) be a fuzzifying topological space, $x \in X$. The fuzzifying neighborhood system of x is denoted by $\mathscr{N}_x : 2^X \to [0,1]$ and defined as

$$A \in \mathscr{N}_x :\Leftrightarrow \ \exists B((B \in \tau) \wedge (x \in B \subseteq A)).$$

Clearly, for each $A \in 2^X$, $\mathscr{N}_x(A) = \bigvee_{x\in B\subseteq A} \tau(A)$. By [17, Theorem 3.2], the mapping of $\mathscr{N}_x$ has the following properties: for all $U,V \subseteq X$,

(1) $\mathscr{N}_x(X)=1$;

(2) $\mathscr{N}_x(U) > 0 \Rightarrow x \in U$;

(3) $\mathscr{N}_x(U \cap V) = \mathscr{N}_x(U) \wedge \mathscr{N}_x(V)$.

Definition 2.3. (Ying [19]) Let (X, τ), (Y, η) be two fuzzifying topological spaces. An unary fuzzy predicate $C : Y^X \to [0, 1]$ called the fuzzy continuity is given as follows:

$$f \in C :\Leftrightarrow (\forall U)((U \in \eta) \to (f^{\leftarrow}(U) \in \tau)).$$

Intuitively, the degree to which f is continuous is

$$[C(f)] = \bigwedge_{U \in X} \min\{1, 1 - \eta(U) + \tau(f^{\leftarrow}(U))\}.$$

Definition 2.4. (Ying [18]) Let (X, τ) be a fuzzifying topological space, $A \subseteq X$. The derived set of A is defined as follows:

$$x \in d(A) :\Leftrightarrow (\forall U)((U \in \mathscr{N}_x) \to (U \bigcap (A \setminus \{x\}) \neq \emptyset)), \ \forall x \in X.$$

Intuitively,

$$d(A)(x) = \bigwedge_{U \cap (A \setminus \{x\}) = \emptyset} (1 - \mathscr{N}_x(U)), \ \forall x \in X.$$

Definition 2.5. (Ying [18]) Let (X, τ) be a fuzzifying topological space, $A \subseteq X$. The closure set of A is defined as follows:

$$\overline{A} :\Leftrightarrow A \bigcup d(A).$$

Clearly,

$$\overline{A}(x) = 1 - \mathscr{N}_x(X \setminus A), \quad \forall \ x \in X.$$

Definition 2.6. (Qiu [3]) Let (X, τ) be a fuzzifying topological space and X be a linear space over the number field $\mathbb{K}$, then (X, τ) is called a fuzzifying topological linear space if it fulfills:

(A1) For any $x, \ y \in X$ and any $V \subseteq X$ with $x + y \in V$,

$$\models V \in \tau \to (\exists V_x, \ V_y \in 2^X)((V_x + V_y \subseteq V) \bigwedge (V_x \in \tau \bigwedge V_y \in \tau)),$$

(A2) For each $s \in \mathbb{K}$, any $x \in X$, and any $V \in s\dot{x}$,

$$\models V \in \tau \to (\exists \delta > 0)(\exists V_x \in 2^X)((V_x \in \tau) \bigwedge (\forall t \in \mathbb{K})(|t - s| < \delta \to tV_x \subseteq V)).$$

Definition 2.7. (Qiu [3], Yan[12]) Let (X, τ) be a fuzzifying topological linear space. The mapping of the bounded degree $Bd(\cdot) : 2^X \to [0, 1]$ is defined as follows:

$$A \in Bd :\Leftrightarrow (\forall U)(U \in \mathscr{N}_\theta \to (\exists s \in \mathbb{K})(A \subseteq sU)).$$

Obviously, for each $A \subseteq X$, $[Bd(A)] = \bigwedge_{U \subseteq X} \bigwedge_{\substack{s \in \mathbb{K} \\ A \not\subseteq sU}} (1 - \mathscr{N}_\theta(U))$.

Lemma 2.8. *(Qiu [3]) Let (X, τ) be a fuzzifying topological linear space. Then,*

$$\models (\forall x \in X)(\forall V \in 2^X)(V \in \mathscr{N}_\theta \leftrightarrow V + x \in \mathscr{N}_x);$$

$$\models (\forall \lambda \in \mathbb{K})(\forall V \in 2^X)(\lambda \neq 0 \to (V \in \mathscr{N}_\theta \leftrightarrow \lambda V \in \mathscr{N}_\theta).$$

Definition 2.9. (Ying [17]) Let (X, τ) be a fuzzifying topological space. Then for any $x \in X$ and any $S \in D(X)$, we define

$$S \rhd x \ :\Leftrightarrow \ (\forall V \in \mathscr{P}(X))(V \in \mathscr{N}_x \to S \lesssim V),$$

where $S \lesssim V$ means S " almost in " V, that is, there is $n_0 \in D$ such that $S(n) \in V$ for all $n \in D$ with $n_0 \prec n$, and

$$D(X) = \{ \ S \mid S : D \to X, \ (D, \prec) \text{ is a direct set}\}.$$

Intuitively, the value of S converges to x, i.e. $[S \rhd x]$ is

$$[S \rhd x] = \bigwedge_{S \not\lesssim V} (1 - \mathscr{N}_x(V)).$$

3. Main Results

Definition 3.1. Let (X, τ), (Y, η) be two fuzzifying topological linear spaces, $L(X, Y)$ the family of linear operators from X to Y. Then unary fuzzy predicates $EC \in \mathscr{F}(\mathscr{P}(L(X, Y)))$ is called fuzzy equicontinuous, if for each $\mathcal{H} \subseteq L(X, Y)$,

$$\mathcal{H} \in EC :\Leftrightarrow (\forall Q)((Q \in \mathscr{N}_\theta^Y) \to (\exists P)((P \in \mathscr{N}_\theta^X) \wedge (\forall f \in \mathcal{H})(f^\to(P) \subseteq Q))).$$

Remark 3.2. It is easy to see that

$$[EC(\mathcal{H})] = \bigwedge_{Q \subseteq Y} \min\{1, 1 - \mathscr{N}_\theta^Y(Q) + \bigvee_{\substack{P \subseteq X \\ \forall f \in \mathcal{H}, f^\to(P) \subseteq Q}} \mathscr{N}_\theta^X(P)\}. \quad (1)$$

Definition 3.3. Let (X, τ), (Y, η) be two fuzzifying topological linear spaces, $L(X, Y)$ be the family of linear operators from X to Y. Then unary fuzzy predicates $UBd \in \mathscr{F}(\mathscr{P}(L(X, Y)))$ is called fuzzy uniformly bounded, if for each $\mathcal{H} \subseteq L(X, Y)$,

$$\mathcal{H} \in UBd :\Leftrightarrow (\forall A)((A \in Bd_X) \to (\exists B)((B \in Bd_Y) \wedge (\forall f \in \mathcal{H})(f^\to(A) \subseteq B))).$$

Remark 3.4. It is easy to find that

$$[UBd(\mathcal{H})] = \bigwedge_{A \subseteq X} \min\{1, 1 - Bd_X(A) + \bigvee_{\substack{B \subseteq Y \\ \forall f \in \mathcal{H}, f^\to(A) \subseteq B}} Bd_Y(B)\}. \quad (2)$$

Example 3.5. For a real line $\mathbb{R}$, a fuzzifying topology $\tau : 2^{\mathbb{R}} \to [0, 1]$ is defined as follows:

$$\tau(A) = \begin{cases} \frac{1}{2}, & A = [a, b) \text{ or } A = \{a\} \\ \frac{3}{4}, & A = (a, b] \\ 1, & A = (a, b) \end{cases}.$$

Denote $\mathfrak{B} = \{(a, b) \mid a, b \in \mathbb{R}, a < b\} \bigcup \{[a, b) \mid a, b \in \mathbb{R}, a < b\} \bigcup \{(a, b] \mid a, b \in \mathbb{R}, a < b\} \bigcup \mathbb{R}$. Then, for each $A \in 2^{\mathbb{R}}$,

$$\tau(A) = \bigvee_{\bigcup A_i = A, A_i \in \mathfrak{B}} \bigwedge_i \tau(A_i).$$

By [3, Example 3.1], $(\mathbb{R}, \tau)$ is exactly a fuzzifying topological linear space. In addition, for each n, the linear mapping $f_n : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f_n(x) = \frac{1}{n}x, \ \forall x \in \mathbb{R}.$$

Then it is easy to verify that $[EC(\{f_n\})] = 1$. If we put $g_n(x) = nx$, then we may check $[EC(\{g_n\})] = 0$. In addition, $[UBd(\{f_n\})] = 1, \ [UBd(\{g_n\})] = 0$.

Lemma 3.6. *Let (X, τ), (Y, η) be two fuzzifying topological linear spaces, $L(X, Y)$ the family of linear operators from X to Y. Then*

$$\mathcal{H} \in EC :\Leftrightarrow (\forall Q \subseteq Y)((Q \in \mathscr{N}_\theta^Y) \to (\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q) \in \mathscr{N}_\theta^X)).$$

Proof. It needs to show that

$$[EC(\mathcal{H})] = \bigwedge_{Q\subseteq Y} \min\{1, 1 - \mathscr{N}_\theta^Y(Q) + \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q))\}.$$

By Remark 3.2, we should prove

$$\bigvee_{\substack{P\subseteq X \\ \forall f\in\mathcal{H}, f^{\rightarrow}(P)\subseteq Q}} \mathscr{N}_\theta^X(P) = \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)).$$

Obviously,

$$\bigvee_{\substack{P\subseteq X \\ \forall f\in\mathcal{H}, f^{\rightarrow}(P)\subseteq Q}} \mathscr{N}_\theta^X(P) \geq \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)).$$

On the other hand, for any $a \in (0, 1]$ with $a < \bigvee_{\substack{P\subseteq X \\ \forall f\in\mathcal{H}, f^{\rightarrow}(P)\subseteq Q}} \mathscr{N}_\theta^X(P)$. Then there exists $P_0 \subseteq X, \ \forall f \in \mathcal{H}, \ f^{\rightarrow}(P_0) \subseteq Q$ such that $a < \mathscr{N}_\theta^X(P_0)$. Notice $f^{\rightarrow}(P_0) \subseteq Q$ for all $f \in \mathcal{H}$, we have $P_0 \subseteq \bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)$, By [17, Theorem 3.2], we can get that $\mathscr{N}_\theta^X(P_0) \leq \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q))$. So

$$a \leq \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)).$$

Thus by the arbitrariness of a,

$$\bigvee_{\substack{P\subseteq X \\ \forall f\in\mathcal{H}, f^{\rightarrow}(P)\subseteq Q}} \mathscr{N}_\theta^X(P) \leq \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)).$$

□

Lemma 3.7. *Let (X, τ), (Y, η) be two fuzzifying topological linear spaces, $L(X, Y)$ the family of linear operators from X to Y. Then*

$$UBd(\mathcal{H}) :\Leftrightarrow (\forall A \subseteq X)((A \in Bd_X) \to (\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A) \in Bd_Y)).$$

Proof. It needs to show that

$$[UBd(\mathcal{H})] = \bigwedge_{A\subseteq X} \min\{1, 1 - Bd_X(A) + Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A))\}.$$

By Remark 3.4, we should prove

$$\bigvee_{\substack{B\subseteq Y\\ \forall f\in\mathcal{H}, f(A)\subseteq B}} Bd_Y(B) = Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\to}(A)).$$

Obviously,

$$\bigvee_{\substack{B\subseteq Y\\ \forall f\in\mathcal{H}, f(A)\subseteq B}} Bd_Y(B) \geq Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\to}(A)).$$

On the other hand, for any $a \in (0,1]$ with $a < \bigvee_{\substack{B\subseteq Y\\ \forall f\in\mathcal{H}, f(A)\subseteq B}} Bd_Y(B)$. Then there exists $B_0 \subseteq Y$, $\forall f \in \mathcal{H}$, $f^{\to}(A) \subseteq B_0$ such that $a < Bd_Y(B_0)$. Notice $f^{\to}(A) \subseteq B_0$ for all $f \in \mathcal{H}$, we have $\bigcup_{f\in\mathcal{H}} f^{\to}(A) \subseteq B_0$. By Definition 2.7, we can get that $Bd_Y(B_0) \leq Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\to}(A))$. So

$$a \leq Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\to}(A)).$$

Thus

$$\bigvee_{\substack{B\subseteq Y\\ \forall f\in\mathcal{H}, f(A)\subseteq B}} Bd_Y(B) = Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\to}(A)).$$

□

Definition 3.8. Let (X,τ), (Y,η) be two fuzzifying topological linear spaces, $L(X,Y)$ the family of linear operators from X to Y, $\mathcal{H} \subseteq L(X,Y), S \in D(X)$. Then $\mathcal{H}\circ S \rightrightarrows \theta^Y$ is defined as

$$\mathcal{H}\circ S \rightrightarrows \theta^Y :\Leftrightarrow (\forall U \subseteq Y)(U \in \mathscr{N}_\theta^Y(U) \to (\mathcal{H}\circ S \lesssim U).$$

Where $\mathcal{H}\circ S \widetilde{\subset} U$ means $\mathcal{H}\circ S$ " almost in " U uniformly, that is, there is $n_0 \in D$ such that $(f\circ S)(n) \in V$ for all $n \in D$ with $n_0 \prec n$ and any $f \in \mathcal{H}$.

Intuitively, the value of $\mathcal{H}\circ S$ uniformly converges to θ^Y, i.e. $[\mathcal{H}\circ S \rightrightarrows \theta^Y]$ is

$$[\mathcal{H}\circ S \rightrightarrows \theta^Y] = \bigwedge_{\mathcal{H}\circ S \not\lesssim U} (1 - \mathscr{N}_\theta^Y(U)).$$

Theorem 3.9. *Let (X,τ), (Y,η) be two fuzzifying topological linear spaces, $L(X,Y)$ the family of linear operators from X to Y. Then*

$$\mathcal{H} \in EC :\Leftrightarrow (\forall S \in D(X))((S \rhd \theta^X) \to (\mathcal{H}\circ S \rightrightarrows \theta^Y)).$$

i.e.,

$$[EC(\mathcal{H})] = \bigwedge_{S\in D(X)} \min\{1, 1 - [S \rhd \theta^X] + [\mathcal{H}\circ S \rightrightarrows \theta^Y]\}.$$

Proof. We first prove that

$$[EC(\mathcal{H})] \leq \bigwedge_{S\in D(X)} \min\{1, 1 - [S \rhd \theta^X] + [\mathcal{H}\circ S \rightrightarrows \theta^Y]\}.$$

For each $t < [EC(\mathcal{H})]$, then for every $Q \subseteq Y$, it satisfies $t < 1 - \mathscr{N}_\theta^Y(Q) + \bigvee_{\substack{P\subseteq X\\ \forall f\in\mathcal{H}, f^{\to}(P)\subseteq Q}} \mathscr{N}_\theta^X(P)$, there is $P \subseteq X$ with $f^{\to}(P) \subseteq Q$ for each $f \in \mathcal{H}$ such that

$t < 1 - \mathscr{N}_\theta^Y(Q) + \mathscr{N}_\theta^X(P)$. For each $S \in D(X)$, in the following we needs to prove that

$$t \leq 1 - [S \rhd \theta^X] + [\mathcal{H} \circ S \rightrightarrows \theta^Y] = 1 + \bigvee_{S \not\lesssim A} \mathscr{N}_\theta^X(A) - \bigvee_{\mathcal{H}\circ S \not\lesssim B} \mathscr{N}_\theta^X(B).$$

In fact, for any $b < \bigvee_{\mathcal{H}\circ S \not\lesssim B} \mathscr{N}_\theta^Y(B)$, there exists $B \subseteq Y$ with $b < \mathscr{N}_\theta^Y(B)$ such that $\mathcal{H} \circ S \not\lesssim B$. From the above proof, there is $P \subseteq X$ with $f^{\rightarrow}(P) \subseteq B$ for each $f \in \mathcal{H}$ such that $t < 1 - \mathscr{N}_\theta^Y(B) + \mathscr{N}_\theta^X(P)$. Then $b < \mathscr{N}_\theta^Y(B) < 1 - t + \mathscr{N}_\theta^X(P)$. Since $\mathcal{H} \circ S \not\lesssim B$, for each $n \in D$, there is $m \in D, f \in \mathcal{H}$ with $m \succeq n$ such that $f(S(m)) \notin B$. Again $f^{\rightarrow}(P) \subseteq B$, obviously, $S(m) \notin P$. This means that $S \not\lesssim P$. Hence

$$b < \mathscr{N}_\theta^Y(B) < 1 - t + \mathscr{N}_\theta^X(P) \leq 1 - t + \bigvee_{S \not\lesssim A} \mathscr{N}_\theta^X(A).$$

Furthermore, $\bigvee_{\mathcal{H}\circ S \not\lesssim B} \mathscr{N}_\theta^Y(B) \leq 1 - t + \bigvee_{S \not\lesssim A} \mathscr{N}_\theta^X(A)$. This deduces that $t \leq \bigwedge_{S\in D(X)} \min\{1, 1 - [S \rhd \theta^X] + [\mathcal{H} \circ S \rightrightarrows \theta^Y]\}$.

On the other hand, we prove the following

$$[EC(\mathcal{H})] \geq \bigwedge_{S\in D(X)} \min\{1, 1 - [S \rhd \theta^X] + [\mathcal{H} \circ S \rightrightarrows \theta^Y]\}.$$

For each $t \in ([EC(\mathcal{H})], 1)$, then there exists $U \subseteq Y$ such that $1 - \mathscr{N}_\theta^Y(U) + \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(U)) < t$. Denote $\lambda = t - 1 + \mathscr{N}_\theta^Y(U)$, clearly, $\mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(U)) < \lambda$. Denote $(\mathscr{N}_\theta^X)_{[\lambda]} = \{\ W \mid \mathscr{N}_\theta^X(W) \geq \lambda\}$, from the fact $X \in (\mathscr{N}_\theta^X)_{[\lambda]}$, we have $(\mathscr{N}_\theta^X)_{[\lambda]} \neq \emptyset$. For the nonempty set $(N_0^X)_{[\lambda]}$, we define a partial order " $\preceq$ " on it, $A \preceq B$ if and only if $B \subseteq A$. By the properties of fuzzifying neighborhood, it is easy to find that $((\mathscr{N}_\theta^X)_{[\lambda]}, \preceq)$ is a direct set. For each $V \in (\mathscr{N}_\theta^X)_{[\lambda]}$, since $\mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(U)) < \lambda$, we have $V \nsubseteq \bigcap_{f\in\mathcal{H}} f^{\leftarrow}(U)$. Then there exist $x_V \in V,\ f_V \in \mathcal{H}$ such that $x_V \notin f_V^{\leftarrow}(U)$. Then we let $S_U : (\mathscr{N}_\theta^X)_{[\lambda]} \to X, \quad V \to x_V$. For each $V \in (\mathscr{N}_\theta^X)_{[\lambda]},\ S_U \lesssim V$, this implies the relation $S_U \not\lesssim A \Rightarrow \mathscr{N}_\theta^X(A) < \lambda$. So $[S_U \rhd \theta^X] = \bigwedge_{S \not\lesssim A} (1 - \mathscr{N}_\theta^X(A)) \geq 1 - \lambda$. Since $\mathcal{H} \circ S_U \not\lesssim U$, then we have

$$[\mathcal{H} \circ S_U \rightrightarrows \theta^Y] = \bigwedge_{\mathcal{H}\circ S_U \not\lesssim B} (1 - \mathscr{N}_\theta^Y(B)) \leq 1 - \mathscr{N}_\theta^Y(U).$$

Thus $\min\{1, 1 - [S_U \rhd \theta^X] + [\mathcal{H} \circ S_U \rightrightarrows \theta^Y]\} \leq 1 - 1 + \lambda + 1 - \mathscr{N}_\theta^Y(U) = t$. Hence

$$\bigwedge_{S\in D(X)} \min\{1, 1 - [S \rhd \theta^X] + [\mathcal{H} \circ S \rightrightarrows \theta^Y]\} \leq [EC(\mathcal{H})].$$

This completes the proof. □

Theorem 3.10. *Let (X,τ), (Y,η) be two fuzzifying topological linear spaces, $L(X,Y)$ be the family of linear operators from X to Y. Then*

$$\models EC(\mathcal{H}) \to UBd(\mathcal{H}).$$

Proof. For each $a \in (0,1]$, if $a < [EC(\mathcal{H})]$, then for every $Q \subseteq Y$, $a < 1 - \mathscr{N}_\theta^Y(Q) + \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q))$. Thus we have that

$$\mathscr{N}_\theta^Y(Q) < 1 - a + \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)), \quad \forall Q \subseteq Y.$$

Hence we need to prove that for each $A \subseteq X$, $a \le 1 - Bd_X(A) + Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A))$, i.e. $a - Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A)) \le 1 - Bd_X(A)$.

For each $b \in (0,1]$, if $b < a - Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A))$, then

$$1 + b - a < 1 - Bd_Y(\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A)) = 1 - \bigwedge_{V\subseteq Y} \bigwedge_{\substack{s\in\mathbb{K} \\ \bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A)\not\subset sV}} (1 - \mathscr{N}_\theta^Y(V))$$

$$= \bigvee_{V\subseteq Y} \bigvee_{\substack{s\in\mathbb{K} \\ \bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A)\not\subset sV}} \mathscr{N}_\theta^Y(V).$$

So there exist $V_0 \subseteq Y$, $s_0 \in \mathbb{K}$ with $\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A) \not\subset s_0V_0$ such that $1 + b - a < \mathscr{N}_\theta^Y(V_0) < 1 - a + \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(V_0))$, i.e.

$$b < \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(V_0)).$$

From the fact $\bigcup_{f\in\mathcal{H}} f^{\rightarrow}(A) \not\subset s_0V_0$, we may obtain the relation $A \not\subset s_0(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(V_0))$. Thus $b < \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(V_0)) \le \bigvee_{s\in\mathbb{K}, A\not\subset sV} \mathscr{N}_\theta^X(V) = 1 - Bd_X(A)$. □

Definition 3.11. Let (X,τ) be a fuzzifying topological space, $A \subseteq X$. The unary fuzzy predicate $DS \in \mathscr{F}(\mathscr{P}(X))$ is said to be fuzzy dense if

$$DS(A) :\Leftrightarrow (\overline{A} \supset X). \text{ i.e., } [DS(A)] = \bigwedge_{x\in X} [1 - \mathscr{N}_x^X(X \setminus A)].$$

Example 3.12. Let $X = \{a,b,c\}$ and $\tau : 2^X \to [0,1]$ be defined as follows:

$$\tau(A) = \begin{cases} \frac{1}{2}, & A \ne \emptyset \text{ and } A \ne X \\ 1, & A = \emptyset \text{ or } A = X \end{cases}.$$

Then $[DS(\{a,b\})] = \frac{1}{2}$ and $[DS(X)] = 1$.

Definition 3.13. Let (X,τ) be a fuzzifying topological space, $A \subseteq X$. The unary fuzzy predicate $NDS \in \mathscr{F}(\mathscr{P}(X))$ is said to be fuzzy nowhere dense if

$$NDS(A) :\Leftrightarrow (\overline{X \setminus A} = X). \text{ i.e., } [NDS(A)] = \bigwedge_{x\in X} 1 - \mathscr{N}_x^X(A).$$

Definition 3.14. Let (X, τ) be a fuzzifying topological space, $A \subseteq X$, Λ be a countable set. The unary fuzzy predicate $FC_I \in \mathscr{F}(\mathscr{P}(X))$ is said to be fuzzy first category if

$$FC_I(A) :\Leftrightarrow (\exists A_i \subseteq X)(A = \bigcup_{i\in\Lambda} A_i) \bigwedge((\forall i \in \Lambda)(A_i \in NDS)).$$

Intuitively,

$$[FC_I(A)] = \bigvee_{A_i \subseteq X,\ A = \bigcup\limits_{i\in\Lambda} A_i} \bigwedge_{i\in\Lambda} \bigwedge_{x\in X} 1 - \mathscr{N}_x^X(A_i).$$

Definition 3.15. Let (X, τ) be a fuzzifying topological space, $A \subseteq X$, Λ be a countable set. The unary fuzzy predicate $FC_{II} \in \mathscr{F}(\mathscr{P}(X))$ is said to be fuzzy the second category if

$$FC_{II}(A) :\Leftrightarrow \neg(A \in FC_I).$$

Intuitively,

$$[FC_{II}(A)] = \bigwedge_{A_i \subseteq X,\ A = \bigcup\limits_{i\in\Lambda} A_i} \bigvee_{i\in\Lambda} \bigvee_{x\in X} \mathscr{N}_x^X(A_i).$$

Theorem 3.16. *Let (X, τ), (Y, η) be two fuzzifying topological linear spaces, $L(X, Y)$ be the family of linear operators from X to Y. Then*

$$\models (X \in FC_{II}) \bigwedge((\forall x \in X)(\bigcup_{f\in\mathcal{H}} fx \in Bd_Y)) \to EC(\mathcal{H}).$$

Proof. It needs to prove that $[FC_{II}(X)] \bigwedge[\bigwedge\limits_{x\in X} Bd_Y(\bigcup\limits_{f\in\mathcal{H}} fx)] \leq [EC(\mathcal{H})]$.

For each $t \in (0, [FC_{II}(X)] \bigwedge[\bigwedge\limits_{x\in X} Bd_Y(\bigcup\limits_{f\in\mathcal{H}} fx)])$, and every $Q \subseteq Y$, it suffices to prove that $t \leq 1 - \mathscr{N}_\theta^Y(Q) + \mathscr{N}_\theta^X(\bigcap\limits_{f\in\mathcal{H}} f^{\leftarrow}(Q))$. For the case $\mathscr{N}_\theta^Y(Q) = 0$, it is trivial. In the following, we always suppose that $\mathscr{N}_\theta^Y(Q) > 0$, then for $a \in (0, \mathscr{N}_\theta^Y(Q))$, by [3, Theorem 4.1], there exists a balanced sets $U \subseteq Y$ with $a < \mathscr{N}_\theta^Y(Q)$ such that $U + U \subseteq Q$. From the fact $t < \bigwedge\limits_{x\in X} Bd_Y(\bigcup\limits_{f\in\mathcal{H}} fx)$, we distinguish the following two cases:

Case 1: For each $x \in X$, and all $s \in \mathbb{K}$, the relation $\bigcup\limits_{f\in\mathcal{H}} fx \nsubseteq sU$ holds, it follows that $t < 1 - \mathscr{N}_\theta^Y(U) < 1 - a$. By the arbitrariness of a, it implies that

$$t \leq 1 - \mathscr{N}_\theta^Y(Q) \leq 1 - \mathscr{N}_\theta^Y(Q) + \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)).$$

Case 2: For each $x \in X$, there exists $n_x \in \mathbb{N}$ such that $\bigcup\limits_{f\in\mathcal{H}} fx \subseteq n_x U$. Then for each $f \in \mathcal{H}$, we have $x \in n_x f^{\leftarrow}(U)$. Thus $x \in \bigcap\limits_{f\in\mathcal{H}} n_x f^{\leftarrow}(U) = n_x \bigcap\limits_{f\in\mathcal{H}} f^{\leftarrow}(U)$. Denote $W = \bigcap\limits_{f\in\mathcal{H}} f^{\leftarrow}(U)$, then $X = \bigcup\limits_{n\in\mathbb{N}} nW$. On the other hand $t < [FC_{II}(X)]$, then there are x_0 and n_0 such that $t < \mathscr{N}_{x_0}^X(n_0W) = \mathscr{N}_\theta^X(W - \frac{x_0}{n_0})$. Since $\frac{x_0}{n_0} \in W$, then

$$W - \frac{x_0}{n_0} \subseteq W - W = W + W = \bigcap_{f\in\mathcal{H}} f^{\leftarrow}(U) + \bigcap_{f\in\mathcal{H}} f^{\leftarrow}(U)$$
$$\subseteq \bigcap_{f\in\mathcal{H}} f^{\leftarrow}(U+U) \subseteq \bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q).$$

This follows that

$$t < \mathscr{N}_\theta^X(W - \frac{x_0}{n_0}) < \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q))$$
$$\leq 1 - \mathscr{N}_\theta^Y(Q) + \mathscr{N}_\theta^X(\bigcap_{f\in\mathcal{H}} f^{\leftarrow}(Q)).$$

Therefore the proof is completed. □

Remark 3.17. By the result made by [15, Remark 1], each classical topological linear space can be regarded as a special case of fuzzifying topological linear space. Applying Theorem 3.16, the uniform boundedness principle is automatically satisfied in classical topological linear spaces. That is, to say that Theorem 3.16 is a generalization of the classical uniform boundedness principle to fuzzy setting.

As Sadeqi and others [5, 6] showed any fuzzy normed linear space is a topological linear space, it is proved that the classical topologies induced by fuzzy norm are compatible with the vector structure. So, the conclusion of Theorem 3.16 holds in the frame of topological linear spaces which are introduced by Sadeqi and others [5, 6].

On the other hand, for a real line $\mathbb{R}$, a fuzzifying topology $\tau : 2^{\mathbb{R}} \to [0,1]$ is defined as Example 3.5. By [3, Example 3.1], $(\mathbb{R}, \tau)$ is exactly a fuzzifying topological linear space, but it is not a classical topological linear space. This means that the presented work is interesting.

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