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A kind of fuzzy upper topology on L-preordered sets

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Abstract

Considering a commutative unital quantale L as the truth value table and using the tool of L-generalized convergence structures of stratified L-filters, this paper introduces a kind of fuzzy upper topology, called fuzzy S-upper topology, on L-preordered sets. It is shown that every fuzzy join-preserving L-subset is open in this topology. When L is a complete Heyting algebra, for every completely distributive L-ordered set, the fuzzy S-upper topology has a special base such that it looks like the usual upper topology on the set of real numbers. For every complete L-ordered set, the fuzzy S-upper topology coincides the fuzzy Scott topology.

Keywords: Commutative unital quantale, (Complete) L-(pre)ordered set, Stratified L-filter, Stratified L-topology, Fuzzy S-upper topology, Fuzzy Scott topology.

1 Introduction

The interrelation between order/lattice structures and topological structures is an important content in both order/lattice theory and topology, which makes it possible to mutually characterize the properties of related structures [12, 20].

For a topological space, we can define some ordered relations/sets. There are mainly two approaches on this topic:

(Approach 1) The specialization (pre)order on the underlying set defined by using open sets.

(Approach 2) The so-called pointfree approach, that is the inclusion order between open sets, without using the underlying set at the very beginning.

From an order/lattice structure we can define different kinds of topological spaces. There are also mainly two approaches on this topic:

(Approach 3) The intrinsic topology on the underlying set, for example upper topology, lower topology, interval topology, Alexandrov topology, Scott topology, Lawson topology, etc.

(Approach 4) The Zariski-like topology (also called the spectrum), which is a kind of topology defined on the set of all prime ideals/filters.

The correspondences of Approaches 1 and 3 usually yield certain categorical isomorphisms between order/lattice structures and topological structures, for example, the isomorphism between posets and topological spaces with T_0 axiom [24, 29], the isomorphism between continuous lattices and injective T_0 spaces [27]. While those of Approaches 2 and 4 usually yield certain categorical dualities between order/lattice structures and topological structures, for example, the duality between spatial frames and sober topological spaces [18, 25] and the duality between complete-generated lattices and Alexandrov T_0 spaces [5].

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When these contents meet fuzzy mathematics, we need the concept of fuzzy order firstly. Since Zadeh [40] introduced his fuzzy order in 1971, different kinds of fuzzy orders are introduced and studied by different authors [1, 2, 3, 4, 7, 8, 9, 11, 17, 30, 39]. In [3, 4], for T a t-norm on the unit interval [0,1], Bodenhofer introduced and studied T-E-orders w.r.t. a T-equivalence E. Based on complete residuated lattices and with a strong background of fuzzy logic, a kind of fuzzy orders, namely **L**-orders, was introduced and studied systematically by Bělohlávek in [1, 2]. In [7, 8], Demirci extended Bodenhofer's T-E-orders from [0,1] to an integral, commutative, complete, quasi monoidal-lattice [15] (an iccqm-lattice, for short) $(L, *, \leq)$ and then studied vague lattices for L an integral, commutative cl-monoid (an iccl-monoid, for short). An L-E-order unifies the notions of T-E-orders [3, 4], **L**-orders [1, 2] and partial orders on an L-underdeterminate set [17] under the same framework. Several kinds of categorical isomorphisms between fuzzy topological spaces and fuzzy ordered structures are studies in [21, 35]. Several categorical dualities between fuzzy topological spaces and fuzzy ordered structures are studied in [33, 34, 36].

In [9, 39], in order to study quantitative domain theory via fuzzy sets, for L a complete Heyting algebra, a kind of L-orders was defined and studied by Fan and Zhang. Indeed, it is routine to show that an L-order in the sense of Bělohlávek and that in the sense of Fan and Zhang are equivalent to each other [32]. In fuzzy domain theory, the most important topology is the fuzzy Scott topology. In [37], Yao and Shi use the tool of fuzzy convergence structures to establish a framework of fuzzy Scott topology in fuzzy domain.

In this paper, we would like to introduce a kind of fuzzy upper topology on L-preordered set, called the *S-upper topology*, as a generalization of the ordinary upper set in analysis as well as the Scott topology on complete lattices. Here we give some hints and motivations of this kind of topology.

On the set R of real numbers, every open set U in the usual topology can be equivalently described as:

(1) for every net ξ, if $\xi \to a \in U$, then ξ falls in U eventually;

(2) for every filter $\mathcal{F}$, if $\mathcal{F} \to a \in U$, then $U \in \mathcal{F}$.

The right-hand topology (or the upper topology in the terminology of order theory) on the real set R is the family

$$\mathscr{T}_r = \{(a, +\infty) |\ a \in R\} \cup \{\emptyset, R\}.$$

We also can use the above method to characterize it if we change definition of convergence by

(1) a net ξ is upper-ordered convergent to $x \in R$ iff there exists an upper bounded subset $M \subseteq R$ such that $x \leq \sup M$ and ξ is lager than or equal to every $m \in M$ eventually;

(2) a filter $\mathcal{F}$ is upper-ordered convergent to $x \in R$ iff there exists an upper bounded subset $M \subseteq R$ such that $x \leq \sup M$ and $M \in \mathcal{F}^l$ (see its definition below).

In a preordered set P, for a subset $M \subseteq P$, the symbols M^l and M^u denote the set of all lower bounds and the set of all upper bounds of M respectively. For a set-theoretic filter $\mathcal{F}$ on P, we put $\mathcal{F}^l = \bigcup_{A \in \mathcal{F}} A^l$.

The upper topology on R can be taken back from upper-ordered convergence of nets or filters. In details, a subset $A \subseteq R$ is open in the upper topology iff for every filter $\mathcal{F}$, if it is upper-ordered convergent into A then $A \in \mathcal{F}$, iff for every net ξ, if it is upper-ordered convergent into A then ξ is in A eventually. We encourage readers to prove this fact directly.

In a complete lattice P, for $x \in P$ and $M \subseteq P$, $x \in \bigvee M$ iff $x \in M^{ul}$; and a filter $\mathcal{F}$ is upper-ordered convergent $x \in P$ iff $x \in (\mathcal{F}^l)^{ul}$, which will induce a topology on P. We will call it the *S-upper topology* on P.

This paper is organized as follows. In Section 2, we recall some basic concepts and results on lattices, fuzzy orders, fuzzy topology and fuzzy convergence spaces. In Section 3, we introduce a concept of fuzzy S-upper topology on L-preordered sets and then study its properties. In Section 4, we show that for every complete L-ordered set, the fuzzy S-upper topology coincides with the fuzzy Scott topology when L is a complete Heyting algebra.

2 Preliminaries

In this section, we will recall some basic concepts and results used throughout this paper.

2.1 Quantales

We refer to [26] for contents on quantales.

Definition 2.1. *Let L be a complete lattice and let $*$ be a semigroup operation on L. The pair $(L, *)$ is called a* quantale *if the operation $*$ is distributive over joins, that is,*

$$a * (\bigvee S) = \bigvee_{s \in S} (a * s) \text{ and } (\bigvee S) * a = \bigvee_{s \in S} (s * a)$$

for all $a \in L$ *and* $S \subseteq L$. *A quantale* $(L,*)$ *is called* commutative *(resp.,* unital*) if the operation* $*$ *is commutative (resp., has a unit element* ε*).*

For a commutative quantale $(L,*)$, the operation $*$ has a right adjoint $\rightarrow: L\times L \longrightarrow L$ given by

$$a \rightarrow b = \bigvee\{c \in L|\ a * c \le b\}\ (\forall a, b \in L).$$

A quantale $(L,*)$ is called a *complete residuated lattice* if it is commutative and strictly two-sided (i.e., the top element 1 is the unit of $*$). A *complete Heyting algebra* or a *frame* is a complete residuated lattice for $* = \wedge$.

Example 2.2. (1) *Let* $L = [0,\infty]$ *denote the extended interval of all non-negative real numbers with the same ordering as the real numbers. Let* $\times$ *be the usual multiplication on real numbers extended to the infinity such that* $x \times \infty = \infty$ *for every* $x \in [0,+\infty]$ *and* $0 \times \infty = 0$. *Then* $(L,\times,1)$ *is a commutative unital quantale.*

(2) *Let* $L = [0,\infty]^{op}$ *denote the extended interval of all non-negative real numbers with the opposite ordering as real numbers (so* 0 *is the greatest element). Let* $+$ *be the usual addition on real numbers extended to the infinity such that* $x + \infty = \infty$ *for every* $x \in L$. *Then* $(L,+,0)$ *is a commutative unital quantale.*

(3) *The unit interval* $[0,1]$ *equipped with a left continuous t-norm becomes a complete residuated lattice.*

Proposition 2.3. *Suppose that* $(L,*,\varepsilon)$ *is a commutative unital quantale. Then for all* $p,q,r \in L$, $\{p_i|\ i \in I\}$, $\{q_j|\ j \in J\} \subseteq L$,

(Q1) $\varepsilon \le p \rightarrow q \Longleftrightarrow p \le q$;
(Q2) $\varepsilon \rightarrow p = p$;
(Q3) $p * (p \rightarrow q) \le q$;
(Q4) $q \le p \rightarrow (p * q)$;
(Q5) $(p \rightarrow q) * (q \rightarrow r) \le p \rightarrow q$;
(Q6) $(\bigvee_i p_i) \rightarrow q = \bigwedge_i (p_i \rightarrow q)$;
(Q7) $p \rightarrow (\bigwedge_j q_j) = \bigwedge_j (p \rightarrow q_j)$;
(Q8) $(r \rightarrow p) \rightarrow (r \rightarrow q) \ge p \rightarrow q$;
(Q9) $(p \rightarrow r) \rightarrow (q \rightarrow r) \ge q \rightarrow p$;
(Q10) $p \rightarrow (q \rightarrow r) = (p * q) \rightarrow r$.

2.2 Fuzzy Posets

Let $(L,*,\varepsilon)$ be a commutative unital quantale. The materials of fuzzy posets are from [1, 2, 9, 32, 39].

Definition 2.4. *Let* X *be a nonempty set. A mapping* $e: X\times X \longrightarrow L$ *is called an* L-preorder *on* X *if*

(FO1) *Self-reflexivity:* $\forall x \in X,\ e(x,x) \ge \varepsilon$;
(FO2) *Transitivity:* $\forall x,y,z \in X,\ e(x,y) * e(y,z) \le e(x,z)$.
The pair (X,e) *is called an* L-preordered set.
An L*-preorder* e *is called an* L-order *if*
(FO3) *Anti-symmetry:* $\forall x,y \in X$, $e(x,y) \wedge e(y,x) \ge \varepsilon$ *implies* $x = y$.
The pair (X,e) *is called an* L-ordered set.

Definition 2.5. *Let* (X,e) *be an* L*-preordered set and* $A \in L^X$.

(1) A *is called an* upper set *if* $A(x) * e(x,y) \le A(y)$ *for all* $x,y \in X$;
(2) A *is called a* lower set *if* $A(x) * e(y,x) \le A(y)$ *for all* $x,y \in X$.

Definition 2.6. *Let* (X,e) *be an* L*-preordered set and* $A \in L^X$. *Define* $A^l, A^u \in L^X$ *by*

$$A^l(x) = \bigwedge_{y\in X} A(y) \rightarrow e(x,y),$$

$$A^u(x) = \bigwedge_{y\in X} A(y) \rightarrow e(y,x).$$

The values of $A^l(x)$ and $A^u(x)$ can be respectively considered as the element x being a lower bound and an upper bound of A. It is easily seen that A^l and A^u are a lower set and an upper set respectively.

Let (X,e) be an L-preordered set and $x \in X$. The L-subsets $\uparrow x,\ \downarrow x \in L^X$ are respectively given by $\uparrow x(y) = e(x,y)$ and $\downarrow x(y) = e(y,x)$ $(\forall y \in X)$. It is easily seen that $(\downarrow x)^u = \uparrow x$, $(\uparrow x)^l = \downarrow x$ and $(\downarrow x)^{ul} = \downarrow x$.

Definition 2.7. *Let (X,e) be an L-preordered set and $A \in L^X$.*

(1) *An element $a \in X$ is called a* supremum *of A is for every $x \in X$, it holds that $e(a,x) = A^u(x)$.*

(2) *An element $b \in X$ is called an* infimum *of A is for every $x \in X$, it holds that $e(x,b) = A^l(x)$.*

Proposition 2.8. *Let $A \in L^X$ and $a \in X$. Then*

(1) $e(x, \sqcup A) = A^{ul}(x)$, $e(\sqcap A, x) = A^{lu}(x)$;

(2) $a = \sqcup A$ *iff* $A^{ul}(a) \wedge A^u(a) \geq \varepsilon$;

(3) $a = \sqcap A$ *iff* $A^{lu}(a) \wedge A^l(a) \geq \varepsilon$.

Although we use $\sqcup A$ and $\sqcap A$ to denote the supremum and infimum of A respectively, please notice that in an L-preordered set (X,e) an L-subset maybe has more than one supermum and one infimum unless (X,e) is anti-symmetric [6].

Definition 2.9. *An L-preordered set is called* complete *if every L-subset has a supremum, or equivalently, every L-subset has an infimum.*

Example 2.10. (1) *Let $e_L : L \times L \longrightarrow L$ by $e(a,b) = a \to b$. Then (L, e_L) is a complete L-ordered set, where $\sqcup A = \bigvee_{a \in L} A(a) * a$ and $\sqcap A = \bigwedge_{a \in L} A(a) \to a$ $(\forall A \in L^L)$.*

(2) *Define* $\mathrm{Sub}_X : L^X \times L^X \longrightarrow L$ *by* $\mathrm{Sub}_X(A,B) = \bigwedge\limits_{x \in X} A(x) \to B(x)$. *Then* Sub_X *is an L-order on L^X* [13] *and* (L^X, Sub_X) *is a complete L-ordered sets, where $\sqcup \mathcal{A} = \bigvee_{A \in L^X} \mathcal{A}(A) * A$ and $\sqcap \mathcal{A} = \bigwedge_{A \in L^X} \mathcal{A}(A) \to A$ $(\forall \mathcal{A} \in L^{(L^X)})$.*

2.3 Fuzzy Topology and Fuzzy Convergence Structures

We refer to [15, 16] for notions of fuzzy topology, to [32, 37] for fuzzy domain theory and to [10, 19, 31] for those of fuzzy convergence spaces.

Definition 2.11. *Let X be a nonempty set. A subfamily $\delta \subseteq L^X$ is called a* stratified L-topology *on X if*

(FT1) $1_X,\ 0_X \in \delta$;

(FT2) $A, B \in \delta$ *implies* $A \wedge B \in \delta$;

(FT3) $\{A_i |\ i \in I\} \subseteq \delta$ *implies* $\bigvee_i A_i \in \delta$;

(FTS) $A \in \delta,\ a \in L$ *imply* $a_X * A \in \delta$.

The pair (X,δ) is called a stratified L-topological space. *Members in δ are called* open sets *in the space.*

A subfamily $\mathcal{B} \subseteq \delta$ of a stratified L-topology is called a *base* if for every $A \in \delta$, there exists $\{B_j |\ j \in J\}$ together with $\{a_j |\ a_j \in L\}$ such that $A = \bigvee\limits_{j \in J} A_j * a_j$.

If L is a complete residuated lattice, then every stratified L-topology is *leveled*

(FTL) $a_X \in \delta$ for every $a \in L$.

The study of fuzzy domain theory is originated by Fan [9] and Zhang and Fan [39]. In 2010, Yao [32] and Yao and Shi [37] modified some definitions in [39] and then established a framework of fuzzy Scott topology.

Definition 2.12. *Let (X,e) be an L-preordered set. An L-subset $I \in L^X$ is called a* fuzzy ideal *of X if*

(FI1) $\bigvee\limits_{x \in X} I(x) \geq \varepsilon$;

(FI2) $\forall x, y \in X,\ I(y) * e(x,y) \leq I(x)$;

(FI3) $\forall x, y \in X,\ I(x) * I(y) \leq \bigvee\limits_{z \in X} I(z) * e(x,z) * e(y,z)$.

Definition 2.13. *An L-ordered set (X,e) is called an L-*fuzzy dcpo *if every fuzzy ideal has a supremum. An L-subset A in an L-fuzzy dcpo is called* fuzzy Scott open *if it is an upper set and $A(\sqcup I) \leq \bigvee_{x \in X} A(x) * I(x)$. The set of all fuzzy Scott open sets of (X,e) is denoted by $\sigma_L(X)$.*

For a commutative unital quantale L, the family of fuzzy Scott open sets needs not be an L-topology on X [28]. If L is a complete Heyting algebra, then $\sigma_L(X)$ is a leveled L-topology on X, which is defined by means of fuzzy convergence structures [37]. Using fuzzy convergence structures is an effective way to defining related fuzzy topological structure, especially for some more general lattices rather than a complete Heyting algebra. For example in [14], we successfully propose the definition of the so-called sd-quantale-valued fuzzy Scott topology by this approach.

So in the following, we will recall some materials on fuzzy convergence structures.

Definition 2.14. [10, 15] *Let X be a nonempty set. A mapping $\mathcal{F}: L^X \longrightarrow L$ is called a* stratified L-filter *on X if*
(FF1) $\mathcal{F}(1_X) = 1$, $\mathcal{F}(0_X) = 0$;
(FF2) $\mathcal{F}(A \wedge B) = \mathcal{F}(A) \wedge \mathcal{F}(B)$;
(FS) $a * \mathcal{F}(A) \leq \mathcal{F}(a * A)$.

The set of all stratified L-filters on X is denoted by $\mathbb{F}_L^s(X)$. For every $x \in X$, define $[x] : L^X \longrightarrow L$ by $[x](A) = A(x)$. Then $[x]$ is a stratified L-filter on X, called the *pointed L-filter* on X.

Proposition 2.15. *Let $\mathcal{F} \in \mathbb{F}_L^s(X)$. For all $A, B \in L^X$, we have*

$$\mathcal{F}(A) * \mathrm{Sub}_X(A, B) \leq \mathcal{F}(B).$$

Proof. $\mathcal{F}(A) * \mathrm{Sub}_X(A, B) \leq \mathcal{F}(A * \mathrm{Sub}_X(A, B)) \leq \mathcal{F}(B)$. □

Definition 2.16. [19, 10] *Let X be a nonempty set and let* $\lim : \mathbb{F}_L^s(X) \longrightarrow L^X$ *be a mapping. If*
(FC1) $\forall x \in X,\ \lim[x](x) \geq \varepsilon$;
(FC2) *if* $\mathcal{F} \leq \mathcal{G}$, *then* $\lim \mathcal{F} \leq \lim \mathcal{G}$,
then lim *is called an* L-generalized convergence structure *on X and the pair $(X, \lim)$ is called an* L-generalized convergence space.

Every L-generalized convergence space $(X, \lim)$ can induce a stratified L-topology [10] on X by means of

$$\delta_{\lim} = \{A \in L^X |\ A(x) * \lim \mathcal{F}(x) \leq \mathcal{F}(A)\}.$$

Definition 2.17. *Let X be a nonempty set and $\mathcal{F} \in \mathbb{F}_L^s(X)$. Define $\mathcal{F}^l \in L^X$ by $\mathcal{F}^l(x) = \bigvee_{A \in L^X} \mathcal{F}(A) * A^l(x)$, called the* set of lower bounds *of $\mathcal{F}$.*

Proposition 2.18. *Let (X, e) be an L-preordered set. Then for every $y \in X$, $[y]^l = \downarrow y$.*

Proof. First of all,

$$[y]^l(x) = \bigvee_{A \in L^X} A(y) * A^l(x) = \bigvee_{A \in L^X} A(y) * [\bigwedge_{z \in X} A(z) \to e(x, z)].$$

Firstly,

$$[y]^l(x) \leq \bigvee_{A \in L^X} A(y) * (A(y) \to e(x, y)) \leq e(x, y).$$

Secondly,

$$[y]^l(x) \geq \downarrow y(y) * [\bigwedge_{z \in X} \downarrow y(z) \to e(x, z)] \geq \varepsilon * \bigwedge_{z \in X} e(z, y) \to e(x, z) \geq e(x, y).$$

Hence, $[y]^l = \downarrow y$. □

Proposition 2.19. *Let (X, e) be an L-preordered set and $\mathcal{F} \in \mathbb{F}_L^s(X)$. Then for every $x \in X$, $\mathcal{F}^l(x) = \mathcal{F}(\uparrow x)$.*

Proof. Since $\mathcal{F}^l(x) = \bigvee_{A \in L^X} \mathcal{F}(A) * A^l(x) = \bigvee_{A \in L^X} \mathcal{F}(A) * \mathrm{Sub}_X(A, \uparrow x)$, we have $\mathcal{F}^l(x) \leq \mathcal{F}(\uparrow x)$ and $\mathcal{F}^l(x) \geq \mathcal{F}(\uparrow x) * \mathrm{Sub}_X(\uparrow x, \uparrow x) \geq \mathcal{F}(\uparrow x)$. Hence, $\mathcal{F}^l(x) = \mathcal{F}(\uparrow x)$. □

3 S-upper Topology on Poset: the Non-fuzzy Case

In this section, we make an attempt to propose the definition of S-upper topology on crisp posets and study some basic properties, so that it will be more easier to understand the related definitions in the next section.

Definition 3.1. *Let P be a poset. We call a filter $\mathcal{F}$ S-upper convergent to $a \in P$ if $a \in (\mathcal{F}^l)^{ul}$, in symbols, $\mathcal{F} \to_u a$.*

When P is a complete lattice, $\mathcal{F} \to_u a$ iff $a \leq \bigvee \mathcal{F}^l$. It is easy to verify that the S-upper convergence is a generalized convergence structure, which induces a topology $\mathfrak{u}(P)$ on P by

$$\mathfrak{u}(P) = \{A \subseteq P |\ (\forall \mathcal{F} \in \mathbb{F}(P),\ \forall a \in A)\ \mathcal{F} \to_u a \text{ always implies } A \in \mathcal{F}\}.$$

Proposition 3.2. *Every open set in $\mathfrak{u}(P)$ is an upper set.*

Proof. Suppose $x \in A$ and $x \leq y$. Then $[y] \to x$, which implies $A \in \mathcal{F}$ and then $y \in A$. Hence A is an upper set. □

Proposition 3.3. *For every poset P, it holds that $\omega(P) \subseteq \mathfrak{u}(P)$.*

Proof. Let $a \in L$ and $\mathcal{F} \to x \in P\backslash{\downarrow}a$. Then $x \leq \bigvee \mathcal{F}^l$ and $\bigvee \mathcal{F}^l \not\leq a$. Then $x \in (\mathcal{F}^l)^{ul}$ and we can find $y \in \mathcal{F}^l$ such that $y \not\leq a$ (Otherwise $\mathcal{F}^l \subseteq {\downarrow}a$ and $x \in (\mathcal{F}^l)^{ul} \subseteq ({\downarrow}a)^{ul} = {\downarrow}a$, a contradiction). Then ${\uparrow}y \in \mathcal{F}$ and $y \in P\backslash{\downarrow}a$. It follows that $P\backslash{\downarrow}a \in \mathcal{F}$. Hence, $P\backslash{\downarrow}a \in \mathfrak{u}(P)$. □

Theorem 3.4. *For every chain P, $\omega(P) = \mathfrak{u}(P)$. Thus, $\mathfrak{u}(R)$ just is the right-hand topology on R.*

Proof. Let $A \in \mathfrak{u}(P) - \{\emptyset, P\}$. Since it is an upper set then $A = {\uparrow}a$ or $A = P\backslash{\downarrow}b$. In the case of $A = P\backslash{\downarrow}b$, it is already in $\omega(P)$. In the case of $A = {\uparrow}a$, if there is a lower processor a^-, then $A = P\backslash{\downarrow}a^-$ is in $\omega(P)$. Otherwise $a = \bigvee\{x \in P|\ x < a\}$. Let $\mathcal{F} = {\uparrow}\{[x, a)|\ x < a\}$. Then $\mathcal{F}$ is a filter and $\mathcal{F}^l = {\downarrow}a - \{a\}$ and $(\mathcal{F}^l)^{ul} = {\downarrow}a$. Then $\mathcal{F} \to a \in A$, which implies $A \in \mathcal{F}$, but it is impossible. That is to say, $A \notin \mathfrak{u}(P)$. □

But for a poset which is not a chain, the two topologies $\omega(P)$ and $\mathfrak{u}(P)$ are not necessarily the same.

Example 3.5. *Let $P = \{\top, \overline{N}, \bot\}$, where $\overline{N}$ is the antichain of the set of natural numbers and $\top, \bot$ are the top and bottom elements respectively. A base of the upper topology of $\{{\uparrow}S|\ P - S \text{ is finite}\}$. Then the singleton $\{\top\}$ is not open in the upper topology of L. Suppose $\mathcal{F} \to_u \top$. Then $\top = \bigvee \mathcal{F}^l$. Then there exist at least two different natural number $m, n \in \mathcal{F}^l$. It follow that $\{T\} = {\uparrow}m \cap {\uparrow}n \in \mathcal{F}$. Hence $\{\top\}$ is open in $\mathfrak{u}(P)$.*

4 Fuzzy S-upper Topology on L-preordered Sets

Scott topology is the most important topology in domain theory, which is compatible with Scott convergence of filters and nets. A filter $\mathcal{F}$ is Scott convergent to $x \in P$ if there is an ideal $I \subseteq P$ such that $x \leq \bigvee I$ and $I \subseteq \mathcal{F}^l$. Historically, Scott topology are firstly defined on complete lattices. If P is a complete lattice, then a filter $\mathcal{F}$ is Scott convergent to $x \in P$ iff $x \leq \bigvee \mathcal{F}^l$ since $\mathcal{F}^l$ always is an ideal. For a subset $S \subseteq P$, the relation $x \leq \bigvee S$ is equivalent to $x \in S^{ul}$. This fact motivates us to introduce the following concept.

In this section, (X, e) always denotes an L-preordered set unless otherwise stated.

Definition 4.1. *Define* $\lim_{\mathcal{S}} : \mathbb{F}_L^s(X) \longrightarrow L^X$ *by*

$$\lim{}_{\mathcal{S}}\mathcal{F} = (\mathcal{F}^l)^{ul}\ (\forall \mathcal{F} \in \mathbb{F}_L^s(X)).$$

We call $\lim_{\mathcal{S}}$ *the* S-upper convergence structure *on* (X, e).

If (X, e) is a complete L-ordered set, then

$$\lim{}_{\mathcal{S}}\mathcal{F}(x) = e(x, \sqcup\mathcal{F}^l)\ (\forall \mathcal{F} \in \mathbb{F}_L^s(X),\ \forall x \in X).$$

Proposition 4.2. *For all $\mathcal{F}, \mathcal{G} \in \mathbb{F}_L^s(X)$ and $x \in X$, we have*
(1) $\lim_{\mathcal{S}}[x] = {\downarrow}x$ *and* $\lim_{\mathcal{S}}[x](x) \geq \varepsilon$*;*
(2) *if $\mathcal{F} \leq \mathcal{G}$, then $\lim_{\mathcal{S}} \mathcal{F} \leq \lim_{\mathcal{S}} \mathcal{G}$.*

Proof. (1) By Proposition 2.18, $\lim_{\mathcal{S}}[x] = ([x]^l)^{ul} = ({\downarrow}x)^{ul} = {\downarrow}x$ and $\lim_{\mathcal{S}}[x](x) = {\downarrow}x(x) \geq \varepsilon$.
(2) If $\mathcal{F} \leq \mathcal{G}$, then $\mathcal{F}^l \leq \mathcal{G}^l$ and $\lim_{\mathcal{S}} \mathcal{F} = (\mathcal{F}^l)^{ul} \leq (\mathcal{G}^l)^{ul} \leq \lim_{\mathcal{S}} \mathcal{G}$. □

By Proposition 3.2, we know that $\lim_{\mathcal{S}}$ is an L-generalized convergence structure on X and the induced stratified L-toplogy is

$$\mathfrak{u}_L(X) = \{A \in L^X|\ \lim{}_{\mathcal{S}}\mathcal{F}(x) * A(x) \leq \mathcal{F}(A)\}.$$

We call it the fuzzy upper topology on (X, e).

Theorem 4.3. (1) *Every $A \in \mathfrak{u}_L(X)$ is an upper set;*
(2) *If L is a complete residuated lattice, then $\mathfrak{u}_L(X)$ is leveled.*

Proof. (1) Suppose $A \in \mathfrak{u}_L(X)$ and choose $\mathcal{F} = [y]$. Then $e(x, y) * A(x) \leq [y](A) = A(y)$.
(2) It is clear. □

In the classic setting, the standard upper topology on a poset P is generated by the family $\{L\backslash{\downarrow}x|\ x \in P\}$ as a subbase. If P is complete lattice, then it is easy to verify that $S \in \{L\backslash{\downarrow}x|\ x \in P\}$ iff χ_S, the characteristic function, is a join-preserving mapping from P to 2, the two-element lattice. That is to say, the standard upper topology on a complete lattice is generated by the set of all join-preserving subsets.

For complete lattices P and Q, a mapping $f: P \longrightarrow Q$ is join-preserving if $f(\bigvee S) = \bigvee f(S)$ for all $S \subseteq P$, or equivalently, f is monotone and $f(\bigvee S) \leq \bigvee f(S)$ for all $S \subseteq P$. Since $\bigvee f(S) = \max(f(S))^{ul}$. The inequality $f(\bigvee S) \leq \bigvee f(S)$ is equivalent to $f(\bigvee S) \in (f(S))^{ul}$ and further to $f(S^{ul}) \subseteq (f(S))^{ul}$. This fact motivates us to introduce the following concept.

Definition 4.4. *Let (X, e_1) and (Y, e_2) be two L-preordered sets. A mapping $f: X \longrightarrow Y$ is called* join-preserving *if it is monotone (i.e., $e_1(x_1, x_2) \leq e_2(f(x_1), f(x_2))$
$(\forall x_1, x_2 \in X)$) and $f_L^{\rightarrow}(S^{ul}) \leq (f_L^{\rightarrow}(S))^{ul}$ holds for every $S \in L^X$.*

Proposition 4.5. *Let (X, e) be an L-preordered set and considering (L, e_L) as a complete L-ordered set. An L-subset $A \in L^X$ is join-preserving if it is an upper set and $S^{ul}(x) * A(x) \leq \bigvee_{y \in X} A(y) * S(y)$ for all $S \in L^X$ and all $x \in X$.*

Proof. Since for every $y \in L$, $A_L^{\rightarrow}(S^{ul})(y) = \bigvee_{A(x)=y} S^{ul}(x)$, the inequality $A_L^{\rightarrow}(S^{ul}) \leq (A_L^{\rightarrow}(S))^{ul}$ is equivalent to $S^{ul}(x) \leq (A_L^{\rightarrow}(S))^{ul}(A(x))$. We know that (L, e_L) is a complete L-ordered set. Then

$$(A_L^{\rightarrow}(S))^{ul}(A(x)) = e_L(A(x), \sqcup A_L^{\rightarrow}(S)) = A(x) \rightarrow \sqcup A_L^{\rightarrow}(S).$$

Hence, the inequality $A_L^{\rightarrow}(S^{ul}) \leq (A_L^{\rightarrow}(S))^{ul}$ is equivalent to $S^{ul}(x) * A(x) \leq \sqcup A_L^{\rightarrow}(S) = \bigvee_{y \in X} A(y) * S(y)$. □

Theorem 4.6. *Every join-preserving map is open in $\mathfrak{u}_L(X)$.*

Proof. For all $(\mathcal{F}, x) \in \mathbb{F}_L^s(X) \times X$,

$$\lim{}_{\mathcal{S}}\mathcal{F}(x) * A(x) = (\mathcal{F}^l)^{ul}(x) * A(x) \leq \bigvee_{y \in X} A(y) * \mathcal{F}^l(y)$$

$$= \bigvee_{y \in X} A(y) * \mathcal{F}({\uparrow}y) \leq \mathcal{F}(\bigvee_{y \in X} A(y) * {\uparrow}y) = \mathcal{F}(A).$$

Hence, $A \in \mathfrak{u}_L(X)$. □

Definition 4.7. [22, 38] *Let (X, e) be a complete L-ordered set. For any $x \in X$, define an L-subset ∇x of X by*

$$\nabla x(y) = \bigwedge_{S \in \mathrm{Low}_L(X)} e(x, \sqcup S) \rightarrow S(y)\ (\forall y \in X).$$

where $\mathrm{Low}_L(X)$ is the set of all lower sets of (X, e). A complete L-ordered set (X, e) is called completely distributive *or a* completely distributive L-ordered set *if $x = \sqcup \nabla x$ for any $x \in X$.*

For every $x \in X$, the L-subset $\Delta x \in L^X$ is defined by $\Delta x(y) = \nabla y(x)$ $(\forall y \in X)$.

Proposition 4.8. [38]*Let (X, e) be a complete L-ordered set and $x, y, u, v \in X$. Then*
(1) $\nabla x \leq {\downarrow}x,\ \Delta x \leq {\uparrow}x$;
(2) $e(u, x) * \nabla y(x) * e(y, v) \leq \nabla v(u)$.

Proposition 4.9. [22]*Let (X, e) be a completely distributive L-ordered set. Then for all $x, y \in X$,*

$$\nabla x(y) = \bigvee_{z \in X} \nabla x(z) * \nabla z(y).$$

Theorem 4.10. *If (X, e) is a completely distributive L-ordered set, then every $\Delta x \in \mathfrak{u}_L(X)$ $(\forall x \in X)$.*

Proof. Let $x \in X$ and $\mathcal{F} \in \mathbb{F}_L^s(X)$. For every $y \in X$, we have

$$\begin{aligned}
\lim_{\mathcal{S}}(\mathcal{F}, y) * \Delta x(y) &= e(y, \sqcup \mathcal{F}^l) * \bigvee_{z \in X} \nabla y(z) * \nabla z(x) \\
&= \bigvee_{z \in X} [e(y, \sqcup \mathcal{F}^l) * \nabla y(z)] * \nabla z(x) \\
&\leq \bigvee_{z \in X} [e(y, \sqcup \mathcal{F}^l) * (e(y, \sqcup \mathcal{F}^l) \to \mathcal{F}^l(z))] * \nabla z(x) \\
&\leq \bigvee_{z \in X} \mathcal{F}^l(z) * \nabla z(x) \\
&\leq \bigvee_{z \in X} \mathcal{F}(\uparrow z) * \Delta x(z) \\
&\leq \mathcal{F}(\bigvee_{z \in X} \uparrow z * \Delta x(z)) \\
&\leq \mathcal{F}(\Delta x).
\end{aligned}$$

Hence, $\Delta x \in \mathfrak{u}_L(X)$. □

For every $x \in X$, define $\mathcal{F}_x(A) = \bigvee_{y \in X} \nabla x(y) * \mathrm{sub}_X(\Delta y, A)$.

Lemma 4.11. $\mathcal{F}_x^l \geq \nabla x$.

Proof.

$$\begin{aligned}
\mathcal{F}_x^l(y) &= \bigvee_{A \in L^X} \mathcal{F}_x(A) * A^l(y) \\
&= \bigvee_{A \in L^X} \bigvee_{z \in X} \nabla x(z) * \mathrm{sub}_X(\Delta z, A) * A^l(y) \\
&= \bigvee_{z \in X} \nabla x(z) * [\bigvee_{A \in L^X} \mathrm{sub}_X(\Delta z, A) * A^l(y)] \\
&\geq \bigvee_{z \in X} \nabla x(z) * \mathrm{sub}_X(\Delta z, \uparrow z) * (\uparrow z)^l(y) \\
&\geq \bigvee_{z \in X} \nabla x(z) * \downarrow z(y) \\
&\geq \bigvee_{z \in X} \nabla x(z) * \nabla z(y) \\
&= \nabla x(y).
\end{aligned}$$

That is to say, $\mathcal{F}_x^l \geq \nabla x$. □

Lemma 4.12. *If L is a complete Heyting algebra. Then every $\mathcal{F}_x$ is an L-filter.*

Proof. (1)

$$\mathcal{F}_x(1_X) = \bigvee_{y \in X} \nabla x(y) \wedge \mathrm{sub}_X(\Delta y, 1_X) = \bigvee_{y \in X} \nabla x(y) = 1.$$

$$\mathcal{F}_x(0_X) = \bigvee_{y \in X} \nabla x(y) \wedge \mathrm{sub}_X(\Delta y, 0_X) = \bigvee_{y \in X} \nabla x(y) \wedge 0 = 0.$$

(2)

$$\begin{aligned}
&\mathcal{F}_x(A) \wedge \mathcal{F}_x(B) \\
= &[\bigvee_{y_1 \in X} \nabla x(y_1) \wedge \mathrm{sub}_X(\Delta y_1, A)] \wedge [\bigvee_{y_2 \in X} \nabla x(y_2) \wedge \mathrm{sub}_X(\Delta y_2, B)] \\
= &\bigvee_{y_1, y_2 \in X} \nabla x(y_1) \wedge \nabla x(y_2) \wedge \mathrm{sub}_X(\Delta y_1, A) \wedge \mathrm{sub}_X(\Delta y_2, B) \\
\leq &\bigvee_{y_1, y_2, y \in X} \nabla x(y) \wedge e(y_1, y) \wedge e(y_2, y) \wedge \mathrm{sub}_X(\Delta y_1, A) \wedge \mathrm{sub}_X(\Delta y_2, B) \\
\leq &\bigvee_{y_1, y_2, y \in X} \nabla x(y) \wedge \mathrm{sub}_X(\Delta y, A) \wedge \mathrm{sub}_X(\Delta y, B) \\
\leq &\bigvee_{y_1, y_2, y \in X} \nabla x(y) \wedge \mathrm{sub}_X(\Delta y, A \wedge B) \\
= &\mathcal{F}_x(A \wedge B).
\end{aligned}$$

(3)

$$\begin{aligned}
a \wedge \mathcal{F}_x(A) &= \bigvee_{y \in X} \nabla x(y) \wedge [a \wedge \mathrm{sub}_X(\Delta y, A)] \\
&\leq \bigvee_{y \in X} \nabla x(y) \wedge \mathrm{sub}_X(\Delta y, a \wedge A)] \\
&\leq \mathcal{F}_x(a \wedge A).
\end{aligned}$$

Hence, $\mathcal{F}_x$ is an L-filter. □

Theorem 4.13. *If L is a complete Heyting algebra and (X, e) is a completely distributive L-ordered set, then $\{\Delta x|\ x \in X\}$ is a base of $\mathfrak{u}_L(X)$.*

Proof. Since $\mathcal{F}_x^l \geq \nabla x$, we have $\lim_{\mathcal{S}}(\mathcal{F}_x, x) = e(x, \sqcup \mathcal{F}_x^l) \geq e(x, \sqcup \nabla x) = 1$.

$$\begin{aligned}\bigvee_{y\in X} \Delta y(x) \wedge \mathrm{sub}_X(\Delta y, A) &= \bigvee_{y\in X} \nabla x(y) \wedge \mathrm{sub}_X(\Delta y, A)\\ &= \mathcal{F}_x(A)\\ &\geq \lim_{\mathcal{S}}(\mathcal{F}_x, x) \wedge A(x)\\ &= 1 \wedge A(x)\\ &= A(x)\end{aligned}$$

Obviously, $\bigvee_{y\in X} \Delta y(x) \wedge \mathrm{sub}_X(\Delta y, A) \leq A(x)$. Hence, $A(x) = \bigvee_{y\in X} \Delta y(x) \wedge \mathrm{sub}_X(\Delta y, A)$ and $A = \bigvee_{y\in X} \Delta y \wedge \mathrm{sub}_X(\Delta y, A)$. Hence, $\{\Delta x|\ x \in X\}$ is a base of $\mathfrak{u}_L(X)$. □

Remark 4.14. *For the case of $L = \{0, 1\}$ and for the set R of the real numbers, the L-subset ∇x is the same as $(x, +\infty)$. That is to say, the fuzzy S-upper topology can be considered as a generalization of the usual upper topology on R, and even on every chain.*

5 Fuzzy S-upper Topology Coincides with Fuzzy Scott Topology on Complete L-ordered Sets

It is easy to verify that, if $\mathcal{F}$ is a filter on a poset X and if $x, y \in \mathcal{F}^l$, then $\uparrow x,\ \uparrow y \in \mathcal{F}$ and so $\uparrow x \cap \uparrow y \in \mathcal{F}$. If further X is a join semilattice, then $\uparrow x \cap \uparrow y = \uparrow(x \vee y)$ and $x \vee y \in \mathcal{F}^l$. That is to say, in a join semilattice X, for every filter $\mathcal{F}$, its lower bound $\mathcal{F}^l$ is a direct subset and consequently an ideal if X has a bottom (this condition guarantees $\mathcal{F}^l$ is nonempty). Then for a complete lattice, every filter $\mathcal{F}$ is Scott convergent to $\bigvee \mathcal{F}^l$.

Since a poset is a complete lattice if and only if it is simultaneously a dcpo and a join-semilattice with a bottom, we immediately know that for every complete lattice, the S-upper convergence and the classical Scott convergence coincide and so do S-upper topology and the classical Scott topology.

In this section, let L be a complete Heyting algebra and we assume (X, e) always is a complete L-ordered set. We will show that the fuzzy S-upper topology coincides with the fuzzy Scott topology.

For every $S \in L^X$, define $\langle S\rangle \in L^X$ by

$$\langle S\rangle(x) = \bigvee_{F\in 2^{(X)}} (\bigwedge_{a\in F} S(a)) \wedge e(x, \bigvee F),$$

where $2^{(X)}$ is the family of all finite subsets of X. It is easy to show that $\langle S\rangle$ is a lower set and $S \leq \langle S\rangle$.

Proposition 5.1. [23]*Let $I \in L^X$ be a lower set. Then I is a fuzzy ideal iff $I(0) = 1$ and $I(x) \wedge I(y) \leq I(x \vee y)$ for all $x, y \in X$.*

Proposition 5.2. *If $I \in L^X$ is a fuzzy ideal, then $\langle I\rangle = I$.*

Proof. We only need to show that $\langle I\rangle \leq I$. For every $x \in X$,

$$\langle I\rangle(x) = \bigvee_{F\in 2^{(X)}} (\bigwedge_{a\in F} I(a)) \wedge e(x, \bigvee F) \leq \bigvee_{F\in 2^{(X)}} I(\bigvee F) \wedge e(x, \bigvee F) \leq I(x).$$

□

Proposition 5.3. *If $S \in L^X$ is nonempty, then $\langle S\rangle$ is the minimal fuzzy ideal containing S.*

Proof. It is a routine to show that $\langle S\rangle$ is a fuzzy lower set containing S and $\langle S\rangle(0) = \bigvee_{F\in 2^{(X)}} (\bigwedge_{a\in F} S(a)) \wedge e(0, \bigvee F) \geq$

$\bigvee_{x\in X} S(x) = 1$. For all $x, y \in X$, we have

$$\begin{array}{ll} & \langle S\rangle(x) \wedge \langle S\rangle(y) \\ = & \bigvee_{F_1\in 2^{(X)}} (\bigwedge_{a\in F} S(a)) \wedge e(x, \bigvee F_1) \wedge \bigvee_{F_2\in 2^{(X)}} (\bigwedge_{b\in F_2} S(b)) \wedge e(y, \bigvee F_2) \\ = & \bigvee_{F_1,F_2\in 2^{(X)}} (\bigwedge_{a\in F_1} S(a)) \wedge (\bigwedge_{b\in F_2} S(b)) \wedge e(x, \bigvee F_1) \wedge e(y, \bigvee F_2) \\ \leq & \bigvee_{F_1,F_2\in 2^{(X)}} \bigwedge_{c\in F_1\cup F_2} S(c) \wedge e(x \vee y, \bigvee F_1 \cup F_2) \\ \leq & \bigvee_{F\in 2^{(X)}} (\bigwedge_{c\in F} S(c)) \wedge e(x\vee y, \bigvee F) \\ = & \langle S\rangle(x \vee y). \end{array}$$

Hence, $\langle S\rangle$ is a fuzzy ideal. The minimality is direct from Proposition 4.2. □

Theorem 5.4. *For every $S \in L^X$, we have $\sqcup S = \sqcup\langle S\rangle$.*

Proof. Let $s = \sqcup S$. We need to show that $e(s, x) = \bigwedge_{y\in X} \langle S\rangle(y) \to e(y, x)$.

$$\begin{array}{lll} \bigwedge_{y\in X} \langle S\rangle(y) \to e(y,x) & = & \bigwedge_{y\in X} [\bigvee_{F\in 2^{(X)}} (\bigwedge_{a\in F} S(a)) \wedge e(y, \bigvee F)] \to e(y,x) \\ & = & \bigwedge_{y\in X} \bigwedge_{F\in 2^{(X)}} [(\bigwedge_{a\in F} S(a)) \wedge e(y, \bigvee F)] \to e(y,x) \\ & = & \bigwedge_{F\in 2^{(X)}} (\bigwedge_{a\in F} S(a)) \to [\bigwedge_{y\in X} e(y, \bigvee F) \to e(y,x)] \\ & = & \bigwedge_{F\in 2^{(X)}} (\bigwedge_{a\in F} S(a)) \to e(\bigvee F, x) \\ & \leq & \bigwedge_{z\in X} S(z) \to e(z,x) \\ & = & e(s,x). \end{array}$$

Conversely, for all $y \in X$ and $F \in 2^{(X)}$, we need to show that

$$e(s, x) \leq \langle S\rangle(y) \to e(y, x).$$

In fact,

$$\begin{array}{lll} \langle S\rangle(y) \to e(y,x) & = & [(\bigwedge_{a\in F} S(a)) \wedge e(y, \bigvee F)] \to e(y,x) \\ & = & (\bigwedge_{a\in F} S(a)) \to [e(y, \bigvee F) \to e(y,x)] \\ & \geq & (\bigwedge_{a\in F} S(a)) \to e(\bigvee F, x) \\ & = & (\bigwedge_{a\in F} S(a)) \to (\bigwedge_{b\in F} e(b,x) \\ & \geq & \bigwedge_{a\in F} S(a) \to e(a,x) \\ & \geq & \bigwedge_{a\in X} S(a) \to e(a,x) \\ & = & e(s,x). \end{array}$$

Hence, $e(s, x) = \bigwedge_{y\in X} \langle S\rangle(y) \to e(y, x)$, as desired. □

For a fuzzy ideal I on X, define $\mathcal{F}_I : L^X \longrightarrow L$ by

$$\mathcal{F}_I(A) = \bigvee_{x\in X} I(x) \wedge sub_X(\uparrow x, A) \ (\forall A \in L^X).$$

Proposition 5.5. [37]*Let I be a fuzzy ideal on X. Then $\mathcal{F}_I$ is a stratified L-filter on X and $\mathcal{F}_I^l = I$.*

Corollary 5.6. *For every nonempty L-subset $S \in L^X$, we have $S \leq \mathcal{F}^l_{\langle S\rangle}$ and $\lim_{\mathcal{S}} \mathcal{F}_{\langle S\rangle}(\sqcup S) = 1$.*

Theorem 5.7. *For every $A \in L^X$, the following statements are equivalent:*

(1) *A is open in $\mathfrak{u}_L(X)$;*

(1′) *A is open in $\sigma_L(X)$;*

(2) *for any* $S \in L^X$, $A(\bigsqcup S) \leq \mathcal{F}_{\langle S \rangle}(A)$;
(2′) *for any* $I \in \mathcal{I}_L(X)$, $A(\bigsqcup I) \leq \mathcal{F}_I(A)$;
(3) A *is a fuzzy upper set and* $A(\bigsqcup S) \leq \bigvee_{x \in X} \langle S \rangle(x) \wedge A(x)$ *for all* $S \in L^X$;
(3′) A *is a fuzzy upper set and* $A(\bigsqcup I) \leq \bigvee_{x \in X} I(x) \wedge A(x)$ *for all* $I \in \mathcal{I}_L(X)$.
Consequently, $\mathfrak{u}_L(X) = \sigma_L(X)$.

Proof. (1′) ⇔ (2′) ⇔ (3′) are already known in fuzzy domain theory. (2) ⇒ (2′) and (3) ⇒ (3′) hold since $\langle I \rangle = I$ for $I \in \mathcal{I}_L(X)$. (2′) ⇒ (2) and (3′) ⇒ (3) hold since $\langle S \rangle \in \mathcal{I}_L(X)$ and $\sqcup S = \sqcup \langle S \rangle$.

(1)⇒(2):

$$A(\sqcup S) \leq \lim{}_{\mathcal{S}} \mathcal{F}_{\langle S \rangle}(\sqcup S) \to \mathcal{F}_{\langle S \rangle}(A) = 1 \to \mathcal{F}_{\langle S \rangle}(A) = \mathcal{F}_{\langle S \rangle}(A).$$

(3')⇒ (1): For any $x \in X$ and any $\mathcal{F} \in \mathbb{F}^s_L(X)$, by Lemma 3.2,

$$A(x) \wedge \lim{}_{\mathcal{S}} \mathcal{F}(x) = A(x) \wedge \mathcal{F}^l(x) = A(x) \wedge \mathcal{F}(\uparrow x) \leq \mathcal{F}(A(x) \wedge \uparrow x) \leq \mathcal{F}(A).$$

Hence, $\mathfrak{u}_L(X) = \sigma_L(X)$. □

6 Conclusions

Considering a commutative unital quantale L as the truth value table and using the tool of L-generalized convergence structures of stratified L-filters, this paper introduces the so-called fuzzy S-upper topology on L-preordered sets. It is shown that every fuzzy join-preserving L-subset is open in this topology. When L is a complete Heyting algebra, for every completely distributive L-ordered set, the fuzzy S-upper topology looks like the usual upper topology on the set of real numbers. For every complete L-ordered set, the fuzzy S-upper topology coincides the fuzzy Scott topology. The approach of using generalized convergence structure is an effective way to induce other types of intrinsic fuzzy topology on fuzzy posets.

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A KIND OF FUZZY UPPER TOPOLOGY ON L-PREORDERED SETS

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نوعی از توپولوژی بالا فازی روی مجموعه‌های L-پیش مرتب

چکیده. با در نظر گرفتن یک کوانتال یکه‌ای تعویض‌پذیر L به عنوان یک جدول ارزش راستی و به کار بردن ابزار ساختارهای همگرای L-تعمیم یافته از L-فیلترهای طبقه‌ای، این مقاله نوعی از توپولوژی بالا فازی، که توپولوژی S-بالا، روی مجموعه‌های L-پیش مرتب نامیده می‌شود معرفی می‌کند. نشان داده شده‌است که هر L-زیرمجموعه حافظ– الخلق در این توپولوژی باز است. هنگامی‌که L یک جبرHeyting تمام باشد، برای هر مجموعه‌ی L-مرتب بطور کامل توزیع‌پذیر، توپولوژی S-بالای فازی دارای مبنای خاصی است که شبیه توپولوژی بالای معمولی روی مجموعه اعداد حقیقی است. برای هر مجموعه L-مرتب کامل، توپولوژی S-بالای فازی با توپولوژی Scott فازی منطبق است.