Iranian Journal of Fuzzy Systems

Volume 16, Number 3, (2019), pp. 73-84

University of Sistan and Baluchestan

Categories of lattice-valued closure (interior) operators and Alexandroff L-fuzzy topologies

A. A. Ramadan[1] and L. Li[2]

[1] *Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt*
[2] *Department of Mathematics, Liaocheng University, Liaocheng, 252059 P.R. China and College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, P.R.China.*

aramadan58@hotmail.com, lilingqiang0614@126.com

Abstract

Galois connection in category theory play an important role in establish the relationships between different spatial structures. In this paper, we prove that there exist many interesting Galois connections between the category of Alexandroff L-fuzzy topological spaces, the category of reflexive L-fuzzy approximation spaces and the category of Alexandroff L-fuzzy interior (closure) spaces. This indicates that there is a close connection between the three structures.

Keywords: Complete residuated lattice, Alexandroff L-fuzzy topological space, L-fuzzy approximation space, Galois correspondence.

1 Introduction

Hájek [16] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [5] investigated information systems and decision rules over complete residuated lattices. Hence residuated lattices and their generalizations are the main structures of truth degree used in many-valued logic [5, 7, 11]. Höhle [20] introduced L-fuzzy topologies with algebraic structure L (cqm, quantales, MV-algebra). It has developed in many directions [12, 13, 14, 21, 22, 24, 32, 45]. Interior and closure operators are very useful tools in several areas of mathematical structures with direct applications, both mathematical (e.g. topology, logic) and extramathematical (e.g. data mining, knowledge representation). In fuzzy set theory, several particular case as well as general theory of interior operators which operate with fuzzy sets(so called fuzzy interior operators) are studied ([2, 4, 8, 10]). Closure operators, however, have appeared in a few studies only (Bianino and Gerla [8, 9]; Bondenhfer et al [10]) and it seems that no general theory of closure operators appeared so far. Recently, Bělohlávek [6] outlined a general theory of fuzzy interior (closure [3]) operators and fuzzy interior(closure)systems using the structure of the residuated lattice in place of the usual structure of truth value on $[0, 1]$. Fang and Yue [14] studied the relationship between L-fuzzy closure systems and L-fuzzy topological spaces from a category viewpoint for a complete residuated lattice L. Ramadan [36] studied the relationship between L-fuzzy interior systems and L-fuzzy topological spaces over complete residuated lattices. Yao and Han characterized Alexandroff L-topological spaces by means of L-orderings [46].

The rough set theory was originally proposed by Pawlak [29, 30] as a mathematical approach for handling imprecision and uncertainty in data analysis. In recent years, rough set theory has developed significantly due to its widespread applications. Various generalized rough set models have been established and their properties or structures have been investigated intensively [22, 25, 26, 27, 34, 35, 37, 39, 40, 41, 42, 43, 44, 49, 50]. Radzikowska [34, 35] developed fuzzy rough sets in complete residuated lattice. An interesting and natural research topic in rough set theory is the study of rough set theory via topology. Kortelainen [23] considered the relationship between modified sets, topological spaces, and rough sets based on a preorder. Subsequently, as generalizations of rough sets from the viewpoint of fuzzy sets, Qin and Pei [33] showed that there exists a one-to-one correspondence between the family of all the lower approximation

Corresponding Author: A. A. Ramadan
Received: June 2017; Revised: February 2018; Accepted: November 2018.

sets based on fuzzy preorder and the set of all fuzzy topologies that satisfy the so-called (TC) axiom. Pei et al. [31] observed that inverse serial relations are the weakest relations that can induce topological spaces, and that different relations based on generalized rough set models will induce different topological spaces. In addition, Hao and Li [18] determined a one-to-one correspondence between the set of all reflexive, transitive L-fuzzy relations and the set of all Alexandroff L-fuzzy topologies. Ma and Hu [28] investigated the topological and lattice structures of L-fuzzy rough sets determined by lower and upper sets. Qiao and Hu [32] studied the relationship between L-fuzzy pretopological spaces [48] and L-fuzzy approximation spaces based on the reflexive L-fuzzy relations from a category viewpoint.

In this paper, we investigate the relationships between the category of Alexandroff L-fuzzy topological spaces, the category of reflexive L-fuzzy approximation spaces and the category of Alexandroff L-fuzzy interior (closure) spaces. In particular, we obtain some interesting adjunctions between the considered categories. The content of the paper is organized as follows. In Section 2, we recall some fundamental concepts and related properties of L-fuzzy interior and closure operators. In Section 3, we investigate the Alexandroff L-fuzzy topologies induced by Alexandroff L-fuzzy interior(closure) operator. In Section 4, the relationships between Alexandroff L-fuzzy topological spaces and reflexive L-fuzzy approximation spaces are discussed from a categorical viewpoint. In Section 5, the relationship between reflexive L-fuzzy approximation spaces and Alexandroff L-fuzzy interior(closure) operator spaces are studied from a categorical viewpoint. In the final section, we present some conclusions of our research.

2 Preliminaries

Throughout this paper, L denotes a complete lattice. The greatest element of L is denoted by $\top$ and the least element of L is denoted by $\bot$. For $A \subseteq L$, we write $\bigvee A$ for the least upper bound of A and $\bigwedge A$ for the greatest lower bound of A. Specifically, $\bigvee L = \top$ and $\bigwedge L = \bot$ are respectively the universal upper and the universal lower bounds in L. We assume that $\top \neq \bot$, i.e. L has at least two elements.

Definition 2.1. [5, 11, 16] An algebra $(L, \wedge, \vee, \odot, \to, \bot, \top)$ is called a complete residuated lattice if it satisfies the following conditions:

(1) $(L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$;

(2) $(L, \odot, \top)$ is a commutative monoid;

(3) $x \odot y \leq z$ iff $x \leq y \to z$ for $x, y, z \in L$.

An operator $^* : L \to L$ defined by $a^* = a \to \bot$ is called a strong negation if $a^{**} = a$.

In this paper, we assume that $(L, \leq, \odot)$ is a complete residuated lattice unless otherwise specified. Some basic properties of the binary operation $\odot$ and residuated operation $\to$ are collected in the following lemma, and they can be found in many works.

Lemma 2.2. [5, 11, 16, 32] *Let L be a complete residuated lattice. Then the following properties hold for each $x, y, z, x_i, y_i \in L$,*

(1) *If $y \leq z$, $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*

(2) $x \odot y \leq x \wedge y$.

(3) $x \to y = \top$ *iff* $x \leq y$, $x \to \top = \top$ *and* $\top \to x = x$.

(4) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(5) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ *and* $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(6) $\bigvee_{i\in\Gamma}(x_i \to y) \leq (\bigwedge_{i\in\Gamma} x_i) \to y$ *and* $\bigvee_{i\in\Gamma}(x \to y_i) \leq x \to (\bigvee_{i\in\Gamma} y_i)$.

(7) $y \to z \leq x \odot y \to x \odot z$, $y \leq x \to (x \odot y)$.

(8) $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ *and* $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$

(9) $(x \to y) \odot x \leq y$ *and* $(x \to y) \odot (y \to z) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ *and* $x \to y \leq (z \to x) \to (z \to y)$.

(11) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ *and* $x \odot (y \to z) \leq y \to (x \odot z)$.

If the strong negation law is done, then L satisfies moreover

(12) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ *and* $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(13) $x \to y = y^* \to x^*$ *and* $x \odot y = (x \to y^*)^*$.

An L-subset [15] on a set X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X; in particular, the L-subsets $\top_X$ and $\bot_X$ defined by $\top_X(x) = \top$ and $\bot_X(x) = \bot$, $\forall x \in X$, are respectively the universal upper and lower bound in L^X. We denote the characteristic function of a subset $\{x\}$ of X by $\top_x$. We do not distinguish between an element $\alpha \in L$ and the constant function $\alpha : X \to L$ such that $\alpha(x) = \alpha$ for all $x \in X$. All algebraic operations on L can be extended pointwise to the power set L^X. That is, for all $\lambda, \mu \in L^X$, $\alpha \in L$ and $x \in X$,

(1) $\lambda \leq \mu$ if and only if $\lambda(x) \leq \mu(x)$,

(2) $(\lambda \odot \mu)(x) = \lambda(x) \odot \mu(x)$,
(3) $(\alpha \odot \lambda)(x) = \alpha \odot \lambda(x)$,
(4) $(\alpha \to \lambda)(x) = \alpha \to \lambda(x)$.

Definition 2.3. [5] An L-fuzzy relation R on X is an L-subset in $X \times X$. R is said to be
(1) reflexive if $R(x,x) = \top$ for all $x \in X$,
(2) symmetric if $R(x,y) = R(y,x)$ for all $x, y \in X$,
(3) transitive if $R(x,y) \odot R(y,z) \leq R(x,z)$ for all $x, y, z \in X$.

A reflexive and transitive L-fuzzy relation on X is called an L-partial order on X. There exists an inherent L-order S on L^X defined by $S(\lambda, \mu) = \bigwedge_{x \in X}(\lambda(x) \to \mu(x))$. The lemma below collects some properties of S used in this paper.

Lemma 2.4. [5, 13, 14] *Let $\lambda, \mu, \rho, \nu \in L^X$, and $\alpha \in L$. Then the following properties hold.*
(1) $\lambda \leq \mu \Leftrightarrow S(\lambda, \mu) = \top$.
(2) *If $\lambda \leq \mu$, then $S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \geq S(\mu, \rho)$.*
(3) *$S(\lambda, \mu) \odot S(\nu, \rho) \leq S(\lambda \odot \nu, \mu \odot \rho)$ and $S(\lambda, \alpha \odot \lambda) \geq \alpha$.*
(4) *$S(\lambda, \mu) \odot S(\mu, \rho) \leq S(\lambda, \rho)$ and $\lambda \odot S(\lambda, \mu) \leq \mu$.*
(5) *$S(\lambda, \alpha_X \to \mu) = S(\alpha_X \odot \lambda, \mu) = \alpha_X \to S(\lambda, \mu)$ and $S(\mu, \lambda) \to \lambda \geq \mu$.*
(6) *Let $\phi : X \to Y$ be an ordinary mapping. Define $\phi^{\to} : L^X \to L^Y$ and $\phi^{\leftarrow} : L^Y \to L^X$ by $\phi^{\to}(\lambda)(y) = \bigvee_{\phi(x)=y} \lambda(x)$, $\forall \lambda \in L^X$, $y \in Y$ and $\phi^{\leftarrow}(\mu)(x) = \mu(\phi(x)) = \mu \circ \phi(x)$, $\forall \mu \in L^Y$. Then, for $\lambda, \mu \in L^X$ and $\rho, \nu \in L^Y$, we have $S(\lambda, \mu) \leq S(\phi^{\to}(\lambda), \phi^{\to}(\mu))$ and $S(\rho, \nu) \leq S(\phi^{\leftarrow}(\rho), \phi^{\leftarrow}(\nu))$ and the equalities hold if ϕ is bijective.*

A concrete category is a pair $(\mathcal{C}, U)$, where $\mathcal{C}$ is a category and $U : \mathcal{C} \to$ **Set** is a faithful functor(or a forgetful functor). For each $\mathcal{C}$-object X, $U(X)$ is called the underlying set of X. Thus, every object in a concrete category can be regarded as a structured set. We write $\mathcal{C}$ for $(\mathcal{C}, U)$, if the concrete functor is obvious. All of the categories considered in this paper are concrete categories. A concrete functor between two concrete categories $(\mathcal{C}, U)$, and $(\mathcal{D}, V)$, is a functor $G : \mathcal{C} \to D$ with $U = V \circ G$, which means that G only changes the structures on the underlying sets. Hence, in order to define a concrete functor $G : \mathcal{C} \to D$, we only consider the following two requirements. First, we assign to each $\mathcal{C}$-object X, a $\mathcal{D}$-object $G(X)$ such that $V(G(X)) = U(X)$. Second, we verify that if a function $f : U(X) \to U(Y)$ is a $\mathcal{C}$-morphism $X \to Y$, then it is also a $\mathcal{D}$-morphism $G(X) \to G(Y)$.

Theorem 2.5. [1, 19] *Suppose that $F : \mathcal{D} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ are concrete functors. Then the following conditions are equivalent:*
(1) *$\{id_Y : F \circ G(Y) \to Y \mid Y \in \mathcal{C}\}$ is a natural transformation from the functor $F \circ G$ to the identity functor on $\mathcal{C}$, and $\{id_X : X \to G \circ F(X) \mid X \in \mathcal{D}\}$ is a natural transformation from the identity functor $\mathcal{D}$ to the functor $G \circ F$.*
(2) *For each $Y \in \mathcal{C}$, $id_Y : F \circ G(Y) \to Y$ is a $\mathcal{C}$-morphism, and for each $X \in \mathcal{D}$, $id_X : X \to G \circ F(X)$ is a $\mathcal{D}$-morphism.*

In this case, (F, G) is called a Galois correspondence between $\mathcal{C}$ and $\mathcal{D}$. If (F, G) is a Galois correspondence, then it is easy to check that F is a left adjoint of G, or equivalently that G is a right adjoint of F.

Definition 2.6.[34, 41] Let R be an L-fuzzy relation. Then the pair (X, R) is referred to as an L-fuzzy approximation space and the following operators $\underline{R}, \overline{R} : L^X \to L^X$ are defined as

$$\forall \lambda \in L^X, \forall x \in X, \underline{R}(\lambda)(x) = \bigwedge_{y \in X} R(x,y) \to \lambda(y), \ \overline{R}(\lambda)(x) = \bigvee_{y \in X} R(x,y) \odot \lambda(y)$$

are called lower and upper L-fuzzy approximation operators. The L-fuzzy approximation operators are precisely the quasi-approximation operators generated by L-relations in [17].

For any L-fuzzy approximation spaces (X, R_X) and (Y, R_Y), a mapping $\phi : (X, R_X) \to (Y, R_Y)$ is called an approximation mapping if, for all $\lambda \in L^Y$, $\phi^{\leftarrow}(\underline{R_Y}(\lambda)) \leq \underline{R_X}(\phi^{\leftarrow}(\lambda))$ and $\phi^{\leftarrow}(\overline{R_Y}(\lambda)) \geq \overline{R_X}(\phi^{\leftarrow}(\lambda))$.

The category of L-fuzzy approximation spaces based on reflexive L-fuzzy relations and morphisms defined above is denoted by **L-APP**.

Definition 2.7. [20, 38] A map $\mathcal{T} : L^X \to L$ is called an Alexandroff L-fuzzy topology on X if it satisfies the following conditions:
(LT1) $\mathcal{T}(\top_X) = \top$ and $\mathcal{T}(\bot_X) = \top$.
(LT2) $\mathcal{T}(\bigwedge_{i \in \Gamma} \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i)$ for all $\{\lambda_i\}_{i \in \Gamma} \subseteq L^X$.
(LT3) $\mathcal{T}(\bigvee_{i \in \Gamma} \lambda_i) \geq \bigwedge_i \mathcal{T}(\lambda_i)$ for all $\{\lambda_i\}_{i \in \Gamma} \subseteq L^X$.

The pair $(X, \mathcal{T})$ is called an Alexandroff L-fuzzy topological space. And $(X, \mathcal{T})$ is called dual if $\mathcal{T}(\lambda) = \mathcal{T}(\lambda^*)$ for all $\lambda \in L^X$.

A mapping $\phi : X \longrightarrow Y$ between Alexandroff L-fuzzy topological spaces is called continuous from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$ if it holds that $\mathcal{T}_X(\phi^{\leftarrow}(\lambda)) \geq \mathcal{T}_Y(\lambda)$ for all $\lambda \in L^Y$. The category of Alexandroff L-fuzzy topological spaces with continuous mappings as morphisms is denoted by **AL-FTop**. The subcategory consisting of dual objections in **AL-FTop** is denoted as **DAL-FTop**.

An Alexandroff L-fuzzy topological space is said to be

(R) enriched if $\mathcal{T}(\alpha_X \odot \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

(S) strong if $\mathcal{T}(\alpha_X \to \lambda) \geq \mathcal{T}(\lambda)$ for all $\lambda \in L^X$ and $\alpha \in L$.

Definition 2.8. [20, 47] A mapping $\mathcal{I} : L^X \to L^X$ is called an L-fuzzy interior operator if it satisfies:

(I1) $\mathcal{I}(\top_X) = \top_X$. (I2) $S(\mathcal{I}(\lambda), \lambda) = \top$ for all $\lambda \in L^X$.

(I3) $S(\lambda, \mu) \leq S(\mathcal{I}(\lambda), \mathcal{I}(\mu))$ for all $\lambda, \mu \in L^X$.

(I4) $\mathcal{I}(\mathcal{I}(\lambda)) = \mathcal{I}(\lambda)$ for all $\lambda \in L^X$.

A mapping $\phi : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ is called continuous if it satisfies $\mathcal{I}_X(\phi^{\leftarrow}(\lambda)) \geq \phi^{\leftarrow}(\mathcal{I}_Y(\lambda))$ for all $\lambda \in L^Y$. The category of L-fuzzy interior operators with continuous mappings as morphisms is denoted by **AL-FIOP**. An L-fuzzy interior operator is said to be Alexandroff if it satisfies moreover, $\mathcal{I}(\bigwedge_{i\in\Gamma} \lambda_i) = \bigwedge_{i\in\Gamma} \mathcal{I}(\lambda_i)$, for all $\{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

Definition 2.9. [20] A mapping $\mathcal{C} : L^X \to L^X$ is called an L-fuzzy closure operator if it satisfies:

(C1) $\mathcal{C}(\bot_X) = \bot_X$. (C2) $S(\lambda, \mathcal{C}(\lambda)) = \top$, for all $\lambda \in L^X$.

(C3) $S(\lambda, \mu) \leq S(\mathcal{C}(\lambda), \mathcal{C}(\mu))$, for all $\lambda, \mu \in L^X$.

(C4) $\mathcal{C}(\mathcal{C}(\lambda)) = \mathcal{C}(\lambda)$, for all $\lambda \in L^X$.

A mapping $\phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is called continuous if it satisfies $\phi^{\leftarrow}(\mathcal{C}_Y(\lambda)) \geq \mathcal{C}_X(\phi^{\leftarrow}(\lambda))$ for all $\lambda \in L^Y$. The category of L-fuzzy closure operators with continuous mappings as morphisms is denoted by **AL-FCOP**. An L-fuzzy closure operator is said to be Alexandroff if moreover it satisfies , $\mathcal{C}(\bigvee_{i\in\Gamma} \lambda_i) = \bigvee_{i\in\Gamma} \mathcal{C}(\lambda_i)$, for all$\{\lambda_i\}_{i\in\Gamma} \subseteq L^X$.

Let $(X, \mathcal{I})$ be an L-fuzzy interior space and $(X, \mathcal{C})$ an L-fuzzy closure space. Then the triple $(X, \mathcal{I}, \mathcal{C})$ is called an L-fuzzy operator space, and denoted as $(X, \mathcal{OP})$ for simplicity. And $(X, \mathcal{OP})$ is called dual if $\mathcal{C}(\lambda) = (\mathcal{I}(\lambda^*))^*$ and $\mathcal{I}(\lambda) = (\mathcal{C}(\lambda^*))^*$ for all $\lambda \in L^X$.

A mapping $\phi : (X, \mathcal{I}_X, \mathcal{C}_X) \to (Y, \mathcal{I}_Y, \mathcal{C}_Y)$ between L-fuzzy operator spaces is called continuous if $\phi : (X, \mathcal{I}_X) \to (Y, \mathcal{I}_Y)$ and $\phi : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ are all continuous. The category of L-fuzzy operator spaces with continuous mappings as morphisms is denoted by **L-FOP**. The subcategory consisting of Alexandroff L-fuzzy operator spaces in **L-FOP** is denoted as **AL-FOP**. The subcategory consisting of dual objections in **AL-FOP** is denoted as **DAL-FOP**.

Proposition 2.10. [14, 36] *Let $(X, \mathcal{T})$ be an Alexandroff L-fuzzy topological space. Define a mappings $\mathcal{I}_\mathcal{T}, \mathcal{C}_\mathcal{T} : L^X \to L^X$ as, $\mathcal{I}_\mathcal{T}(\lambda) = \bigvee_{\mu\in L^X} \mathcal{T}(\mu) \odot S(\mu, \lambda) \odot \mu$ and $\mathcal{C}_\mathcal{T}(\lambda) = \bigwedge_{\mu\in L^X} ((\mathcal{T}(\mu) \odot S(\lambda, \mu)) \to \mu)$. Then*

(1) $\mathcal{I}_\mathcal{T}$ (resp., $\mathcal{C}_\mathcal{T}$) is an L-fuzzy interior (resp., closure) operator on X.

(2) $\mathcal{I}_\mathcal{T}(\alpha_X \odot \lambda) \geq \alpha_X \odot \mathcal{I}_\mathcal{T}(\lambda)$ (resp., $\mathcal{C}_\mathcal{T}(\alpha_X \odot \lambda) \geq \alpha_X \odot \mathcal{C}_\mathcal{T}(\lambda)$) for all $\lambda \in L^X$ and $\alpha \in L$.

3 The Alexandroff L-fuzzy topologies induced by Alexandroff L-fuzzy interior(closure) operator

Given an Alexandroff L-fuzzy interior (resp., closure) operator, there is a natural way to obtain an Alexandroff L-fuzzy topology, $\mathcal{T}_\mathcal{I}(\lambda) = S(\lambda, \mathcal{I}(\lambda))$ and $\mathcal{T}_\mathcal{C}(\lambda) = S(\mathcal{C}(\lambda), \lambda)$, $\lambda \in L^X$. It follows that $\mathcal{T}_\mathcal{I}(\mathcal{I}(\lambda)) = \top$ and $\mathcal{T}_\mathcal{C}(\mathcal{C}(\lambda)) = \top$.

Proposition 3.1. *Let $(X, \mathcal{I})$ be an Alexandroff L-fuzzy interior space. Define a mapping $\mathcal{T}_\mathcal{I} : L^X \to L$ by $\mathcal{T}_\mathcal{I}(\lambda) = S(\lambda, \mathcal{I}(\lambda))$. Then $\mathcal{T}_\mathcal{I}$ is an enriched Alexandroff L-fuzzy topology on X.*

Proposition 3.2. *Let $(X, \mathcal{C})$ be an Alexandroff L-fuzzy closure space. Define a mapping $\mathcal{T}_\mathcal{C} : L^X \to L$ by, $\mathcal{T}_\mathcal{C}(\lambda) = S(\mathcal{C}(\lambda), \lambda)$. Then $\mathcal{T}_\mathcal{C}$ is a strong Alexandroff L-fuzzy topology on X.*

Proposition 3.3. *Let $(X, \mathcal{OP})$ be an Alexandroff L-fuzzy operator space. Define a mapping $\mathcal{T}_{\mathcal{OP}} : L^X \to L$ as, $\mathcal{T}_{\mathcal{OP}}(\lambda) = S(\mathcal{C}(\lambda), \mathcal{I}(\lambda))$. Then (1) $\mathcal{T}_{\mathcal{OP}}$ is an Alexandroff L-fuzzy topology.*

(2) If $\mathcal{C}(\alpha_X \odot \lambda) = \alpha \odot \mathcal{C}(\lambda)$, then $\mathcal{T}_{\mathcal{OP}}$ is enriched. If $\mathcal{I}(\alpha \to \lambda) = \alpha \to \mathcal{I}(\lambda)$, then $\mathcal{T}_{\mathcal{OP}}$ is strong.

(3) $\mathcal{I}_{\mathcal{T}_{\mathcal{OP}}} \leq \mathcal{I}$ and $\mathcal{C}_{\mathcal{T}_{\mathcal{OP}}} \geq \mathcal{C}$. (4) $\mathcal{T}_{\mathcal{OP}} \geq \mathcal{T}_\mathcal{C} \odot \mathcal{T}_\mathcal{I}$. (5) If $\mathcal{T} \odot \mathcal{T} = \mathcal{T}$, then $\mathcal{T}_{\mathcal{OP}} \geq \mathcal{T}$.

Proof. (1) (LT1) $\mathcal{T}_{\mathcal{OP}}(\top_X) = \mathcal{T}_{\mathcal{OP}}(\bot_X) = \top$.
(LT2) Since $\mathcal{I}(\bigvee_{i\in\Gamma}\lambda_i) \geq \bigvee_{i\in\Gamma}\mathcal{I}(\lambda_i)$, then, by Lemma 2.4(2), we have

$$\mathcal{T}_{\mathcal{OP}}(\bigvee_{i\in\Gamma}\lambda_i) = S(\mathcal{C}(\bigvee_{i\in\Gamma}\lambda_i), \mathcal{I}(\bigvee_{i\in\Gamma}\lambda_i)) \geq S(\bigvee_{i\in\Gamma}\mathcal{C}(\lambda_i), \bigvee_{i\in\Gamma}\mathcal{I}(\lambda_i) \geq \bigwedge_{i\in\Gamma} S(\mathcal{C}(\lambda_i), \mathcal{I}(\lambda_i)) = \bigwedge_{i\in\Gamma}\mathcal{T}_{\mathcal{OP}}(\lambda_i).$$

(LT3) Since $\mathcal{C}(\bigwedge_{i\in\Gamma}\lambda_i) \leq \bigwedge_{i\in\Gamma}\mathcal{C}(\lambda_i)$. Then, by Lemma 2.4(2), we have

$$\mathcal{T}_{\mathcal{OP}}(\bigwedge_{i\in\Gamma}\lambda_i) = S(\mathcal{C}(\bigwedge_{i\in\Gamma}\lambda_i), \mathcal{I}(\bigwedge_{i\in\Gamma}\lambda_i)) \geq S(\bigwedge_{i\in\Gamma}\mathcal{C}(\lambda_i), \bigwedge_{i\in\Gamma}\mathcal{I}(\lambda_i)) \geq \bigwedge_{i\in\Gamma} S(\mathcal{C}(\lambda_i), \mathcal{I}(\lambda_i)) = \bigwedge_{i\in\Gamma}\mathcal{T}_{\mathcal{OP}}(\lambda_i).$$

(2) For any $\alpha \in L$, $\lambda \in L^X$, by Lemma (2.4) and Proposition (2.10), we have

$$\begin{aligned}\mathcal{T}_{\mathcal{OP}}(\alpha_X \odot \lambda) &= S(\mathcal{C}(\alpha_X \odot \lambda), \mathcal{I}(\alpha_X \odot \lambda) = S(\alpha_X \odot \mathcal{C}(\lambda), \mathcal{I}(\alpha_X \odot \lambda)) \geq S(\alpha_X \odot \mathcal{C}(\lambda), \alpha_X \odot \mathcal{I}(\lambda)) \\ &\geq (\alpha_X \to \alpha_X) \odot S(\mathcal{C}(\lambda), \mathcal{I}(\lambda)) = \top \odot \mathcal{T}_{op}(\lambda) = \mathcal{T}_{\mathcal{OP}}(\lambda).\end{aligned}$$

$$\begin{aligned}\mathcal{T}_{\mathcal{OP}}(\alpha_X \to \lambda) &= S(\mathcal{C}(\alpha_X \to \lambda), \mathcal{I}(\alpha_X \to \lambda)) = S(\mathcal{C}(\alpha_X \to \lambda), \alpha_X \to \mathcal{I}(\lambda)) = S(\alpha_X \odot \mathcal{C}(\alpha_X \to \lambda), \mathcal{I}(\lambda)) \\ &\geq S(\mathcal{C}(\alpha_X \odot (\alpha_X \to \lambda)), \mathcal{I}(\lambda)) \geq S(\mathcal{C}(\lambda), \mathcal{I}(\lambda)) = \mathcal{T}_{\mathcal{OP}}(\lambda).\end{aligned}$$

(3) By Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{\mathcal{OP}}}(\lambda)(x) &= \bigvee_{\mu\in L^X}(\mathcal{T}_{\mathcal{OP}}(\mu) \odot S(\mu,\lambda) \odot \mu(x)) = \bigvee_{\mu\in L^X}(S(\mathcal{C}(\mu), \mathcal{I}(\mu)) \odot S(\mu,\lambda) \odot \mu(x)) \\ &\leq \bigvee_{\mu\in L^X} S(\mathcal{C}(\mu), \mathcal{I}(\mu)) \odot S(\mathcal{I}(\mu), \mathcal{I}(\lambda)) \odot \mathcal{C}(\mu)(x) \leq \bigvee_{\mu\in L^X} S(\mathcal{C}(\mu), \mathcal{I}(\lambda)) \odot \mathcal{C}(\mu)(x) \leq \mathcal{I}(\lambda)(x).\end{aligned}$$

Hence, $\mathcal{I}_{\mathcal{T}_{OP}} \leq \mathcal{I}$. Next,

$$\begin{aligned}\mathcal{C}_{\mathcal{T}_{OP}}(\lambda)(x) &= \bigwedge_{\mu\in L^X}\Big((\mathcal{T}_{\mathcal{OP}}(\mu) \odot S(\lambda,\mu)) \to \mu(x)\Big) = \bigwedge_{\mu\in L^X}\Big((S(\mathcal{C}(\mu), \mathcal{I}(\mu))) \odot S(\lambda,\mu)) \to \mu(x)\Big) \\ &\geq \bigwedge_{\mu\in L^X}\Big(S(\mathcal{C}(\mu), \mathcal{I}(\mu)) \odot S(\mathcal{C}(\lambda), \mathcal{C}(\mu))) \to \mu(x)\Big) \geq \bigwedge_{\mu\in L^X} S(\mathcal{C}(\lambda), \mathcal{I}(\mu)) \to \mathcal{I}(\mu)(x) \geq \mathcal{C}(\lambda)(x).\end{aligned}$$

Thus, it follows that $\mathcal{C}_{\mathcal{T}_{\mathcal{OP}}} \geq \mathcal{C}$.
(4) It can be proved by Lemma 2.4 (4).
(5) From Proposition 3.1 and Proposition 3.2, it follows from (4).

Theorem 3.4. *If a mapping $\phi : (X, \mathcal{OP}_X) \to (Y, \mathcal{OP}_Y)$ is continuous, then $\phi : (X, \mathcal{T}_{\mathcal{OP}_X}) \to (Y, \mathcal{T}_{\mathcal{OP}_Y})$ is continuous.*

Proof. Since $\phi^{\leftarrow}(\mathcal{I}_Y(\lambda)) \leq \mathcal{I}_X(\phi^{\leftarrow}(\lambda))$ and $\phi^{\leftarrow}(\mathcal{C}_Y(\lambda)) \geq \mathcal{C}_X(\phi^{\leftarrow}(\lambda))$, it follows that

$$\mathcal{T}_{\mathcal{OP}_Y}(\lambda) = S(\mathcal{C}_Y(\lambda), \mathcal{I}_Y(\lambda)) \leq S(\phi^{\leftarrow}(\mathcal{C}_Y(\lambda)), \phi^{\leftarrow}(\mathcal{I}_Y(\lambda))) \leq S(\mathcal{C}_X(\phi^{\leftarrow}(\lambda)), \mathcal{I}_X(\phi^{\leftarrow}(\lambda))) = \mathcal{T}_{\mathcal{OP}_X}(\phi^{\leftarrow}(\lambda))$$

4 On the relationships between Alexandroff L-fuzzy topological spaces and reflexive L-fuzzy approximation spaces

We devote this section to the categorical aspect of the relationship between Alexandroff L-fuzzy topological spaces and reflexive L-fuzzy approximation spaces.

4.1 On the adjunction $(\underline{\Delta}, \Gamma)$: L-APP $\rightharpoonup$ AL-FTop.

Let $(X, \mathcal{T})$ be an Alexandroff L-fuzzy topological space. Then it is easily seen that the mapping $R_{\mathcal{T}} : X \times X \longrightarrow L$ defined by $R_{\mathcal{T}}(x,y) = \bigwedge_{\lambda\in L^X}(\mathcal{T}(\lambda) \to (\lambda(x) \to \lambda(y)))$ is a reflexive L-fuzzy relation. Indeed, for any $x \in X$,

$R_{\mathcal{T}}(x,x) = \bigwedge_{\lambda\in L^X}(\mathcal{T}(\lambda)\to(\lambda(x)\to\lambda(x))) = \top$, i.e., $R_{\mathcal{T}}$ is reflexive.

Theorem 4.1. *Let $f:(X,\mathcal{T}_X)\longrightarrow(Y,\mathcal{T}_Y)$ be continuous mapping between Alexandroff L-fuzzy topological spaces. Then $f:(X,R_{\mathcal{T}_X})\longrightarrow(Y,R_{\mathcal{T}_Y})$ is an approximation mapping.*

Proof. For any $x,z\in X$,

$$\begin{aligned} R_{\mathcal{T}_X}(x,z) &= \bigwedge_{\lambda\in L^X}(\mathcal{T}_X(\lambda)\to(\lambda(x)\to\lambda(z)))\le\bigwedge_{\mu\in L^Y}(\mathcal{T}_X(f^{\leftarrow}(\mu))\to(f^{\leftarrow}(\mu)(x)\to f^{\leftarrow}(\mu)(z)))\\ &\le \bigwedge_{\mu\in L^Y}(\mathcal{T}_Y(\mu)\to(\mu(f(x))\to\mu(f(z)))) = R_{\mathcal{T}_Y}(f(x),f(z)).\end{aligned}$$

It follows that for any $\lambda\in L^Y, x\in X$

$$\begin{aligned} \underline{R_{\mathcal{T}_X}}(f^{\leftarrow}(\lambda))(x) &= \bigwedge_{z\in X}(R_{\mathcal{T}_X}(x,z)\to f^{\leftarrow}(\lambda)(z))\ge\bigwedge_{z\in X}(R_{\mathcal{T}_Y}(f(x),f(z))\to\lambda(f(z)))\\ &\ge \bigwedge_{y\in Y}(R_{\mathcal{T}_Y}(f(x),y)\to\lambda(y)) = f^{\leftarrow}(\underline{R_{\mathcal{T}_Y}}(\lambda))(x).\end{aligned}$$

$$\begin{aligned} \overline{R_{\mathcal{T}_X}}(f^{\leftarrow}(\lambda))(x) &= \bigvee_{z\in X}(R_{\mathcal{T}_X}(x,z)\odot f^{\leftarrow}(\lambda)(z))\le\bigvee_{z\in X}(R_{\mathcal{T}_Y}(f(x),f(z))\odot\lambda(f(z)))\\ &\le \bigvee_{y\in Y}(R_{\mathcal{T}_Y}(f(x),y)\odot\lambda(y)) = f^{\leftarrow}(\overline{R_{\mathcal{T}_Y}}(\lambda))(x).\end{aligned}$$

The above theorem shows that the correspondence $(X,\mathcal{T}_X)\vdash(X,R_{\mathcal{T}_X})$ induces a concrete functor Γ : **AL-FTop** $\to$ **L-APP** with $\Gamma(X,\mathcal{T}_X)=(X,R_{\mathcal{T}_X})$, $\Gamma(\phi)=\phi$. Let (X,R) be a reflexive L-fuzzy approximation space. Then it is easily seen that the mapping $\mathcal{T}_{\underline{R}}: L^X\longrightarrow L$ defined by, for all $\lambda\in L^X$, $\mathcal{T}_{\underline{R}}(\lambda)=S(\lambda,\underline{R}(\lambda))$ is an enriched Alexandroff L-fuzzy topology.

Theorem 4.2. *Let $f:(X,R_X)\longrightarrow(Y,R_Y)$ be an approximation mapping. Then $f:(X,\mathcal{T}_{\underline{R_X}})\longrightarrow(Y,\mathcal{T}_{\underline{R_Y}})$ is continuous.*

Proof. Indeed, it follows that for any $\lambda\in L^Y$,

$$\mathcal{T}_{\underline{R_Y}}(\lambda)=S(\lambda,\underline{R_Y}(\lambda))\le S(f^{\leftarrow}(\lambda),f^{\leftarrow}(\underline{R_Y}(\lambda)))\le S(f^{\leftarrow}(\lambda),\underline{R_X}(f^{\leftarrow}(\lambda)))=\mathcal{T}_{\underline{R_X}}(f^{\leftarrow}(\lambda))$$

The above theorem shows that the correspondence $(X,R_X)\vdash(X,\mathcal{T}_{\underline{R_X}})$ induces a concrete functor $\underline{\Delta}$: **L-APP** $\to$ **AL-FTop** with $\underline{\Delta}(X,R_X)=(X,\mathcal{T}_{\underline{R_X}})$, $\underline{\Delta}(\phi)=\phi$.

Theorem 4.3. *Let $(X,\mathcal{T})$ be an Alexandroff L-fuzzy topological space and (X,R) be a reflexive L-fuzzy approximation space. Then $\mathcal{T}_{\underline{R}_{\mathcal{T}}}\ge\mathcal{T}$ and $R_{\mathcal{T}_{\underline{R}}}\ge R$.*

Proof. At first, we prove that $\mathcal{T}_{\underline{R}_{\mathcal{T}}}\ge\mathcal{T}$. For any $\lambda\in L^X$,

$$\begin{aligned} \mathcal{T}_{\underline{R}_{\mathcal{T}}}(\lambda) &= S(\lambda,\underline{R}_{\mathcal{T}}(\lambda))=\bigwedge_{x\in X}(\lambda(x)\to\bigwedge_{y\in X}(R_{\mathcal{T}}(x,y)\to\lambda(y)))=\bigwedge_{x,y\in X}(R_{\mathcal{T}}(x,y)\to(\lambda(x)\to\lambda(y)))\\ &\ge \bigwedge_{x,y\in X}((\mathcal{T}(\lambda)\to(\lambda(x)\to\lambda(y))\to(\lambda(x)\to\lambda(y)))\ge\mathcal{T}(\lambda).\end{aligned}$$

Now, we prove that $R_{\mathcal{T}_{\underline{R}}}\ge R$. For any $x,y\in X$,

$$\begin{aligned} R_{\mathcal{T}_{\underline{R}}}(x,y) &= \bigwedge_{\lambda\in L^X}(\mathcal{T}_{\underline{R}}(\lambda)\to(\lambda(x)\to\lambda(y)))=\bigwedge_{\lambda\in L^X}(S(\lambda,\underline{R}(\lambda))\to(\lambda(x)\to\lambda(y)))\\ &\ge \bigwedge_{\lambda\in L^X}((\lambda(x)\to\underline{R}(\lambda)(x))\to(\lambda(x)\to\lambda(y)))\ge\bigwedge_{\lambda\in L^X}((\lambda(x)\to(R(x,y)\to\lambda(y)))\to(\lambda(x)\to\lambda(y)))\\ &= \bigwedge_{\lambda\in L^X}((R(x,y)\to(\lambda(x)\to\lambda(y)))\to(\lambda(x)\to\lambda(y)))\ge R(x,y).\end{aligned}$$

Theorem 4.4. *$(\underline{\Delta},\Gamma)$ forms a Galois connection between the category **L-APP** and the category **AL-FTop**.*

Proof. The above theorem (1) shows that $\mathrm{id}_X : (X, \underline{\Delta}\circ\Gamma(\mathcal{T})) \longrightarrow (X,\mathcal{T})$ is continuous, and (2) shows that $\mathrm{id}_X : (X,R) \longrightarrow (X, \Gamma\circ\underline{\Delta}(R))$ is an L-approximation mapping. Thus $(\underline{\Delta},\Gamma)$ forms a Galois connection between the category **L-APP** and the category **AL-FTop**.

Lemma 4.5. *Let L satisfy the strong negation law. Then*
(1) *if $\mathcal{T}$ is dual then, $R_{\mathcal{T}}$ is symmetric;*
(2) *if R is symmetric then, $\mathcal{T}_{\underline{R}}$ is dual.*

Proof. (1) If $\mathcal{T}$ is dual, then $R_{\mathcal{T}}(x,y) = \bigwedge_{\lambda\in L^X}(\mathcal{T}(\lambda^*) \to (\lambda^*(y) \to \lambda^*(x)))$. Note that $\lambda \in L^X$ iff $\lambda^* \in L^X$, it follows that $R_{\mathcal{T}}(x,y) = R_{\mathcal{T}}(y,x)$.

(2) If R is symmetric, then for any $\lambda \in L^X$,

$$\begin{aligned}
\mathcal{T}_{\underline{R}}(\lambda^*) &= S(\lambda^*, \underline{R}(\lambda^*)) = \bigwedge_{x\in X}(\lambda^*(x) \to \underline{R}(\lambda^*)(x)) = \bigwedge_{x\in X}\bigwedge_{y\in X}(\lambda^*(x) \to (R(x,y) \to \lambda^*(y)))) \\
&= \bigwedge_{x\in X}\bigwedge_{y\in X}(R(x,y) \to (\lambda^*(x) \to \lambda^*(y))) = \bigwedge_{x\in X}\bigwedge_{y\in X}(R(y,x) \to (\lambda(y) \to \lambda(x))) \\
&= \bigwedge_{y\in X}\bigwedge_{x\in X}(\lambda(y) \to (R(y,x) \to \lambda(x))) = \bigwedge_{y\in X}(\lambda(y) \to \underline{R}(\lambda)(y)) = S(\lambda, \underline{R}(\lambda)) = \mathcal{T}_{\underline{R}}(\lambda).
\end{aligned}$$

Corollary 4.6. *Let L satisfy the strong negation law. Then $(\underline{\Delta},\Gamma)$ forms a Galois connection between the category **L-APP** and the category **DAL-FTop**.*

4.2 On the adjunction $(\overline{\Delta},\Gamma)$: L-APP $\rightharpoonup$ DAL-FTop.

Let (X,R) be a reflexive L-fuzzy approximation space. Then it is easily seen that the mapping $\mathcal{T}_{\overline{R}} : L^X \longrightarrow L$ defined by, for all $\lambda \in L^X$, $\mathcal{T}_{\overline{R}}(\lambda) = S(\overline{R}(\lambda),\lambda)$ is a strong Alexandroff L-fuzzy topology.

Theorem 4.7. *Let $f : (X,R_X) \longrightarrow (Y,R_Y)$ be an approximation mapping continuous mapping. Then $f : (X, \mathcal{T}_{\overline{R_X}}) \longrightarrow (Y, \mathcal{T}_{\overline{R_Y}})$ is continuous.*

Proof. Indeed, it follows by that for any $\lambda \in L^Y$,

$$\mathcal{T}_{\overline{R_Y}}(\lambda) = S(\overline{R_Y}(\lambda),\lambda) \le S(f^{\leftarrow}(\overline{R_Y}(\lambda)), f^{\leftarrow}(\lambda)) \le S(\overline{R_X}(f^{\leftarrow}(\lambda)), f^{\leftarrow}(\lambda)) = \mathcal{T}_{\overline{R_X}}(f^{\leftarrow}(\lambda))$$

The above theorem shows that the correspondence $(X,R_X) \vdash (X, \mathcal{T}_{\overline{R_X}})$ induces a concrete functor $\overline{\Delta}$: **L-APP** $\to$ **AL-FTop** with $\overline{\Delta}(X,R_X) = (X,\mathcal{T}_{\overline{R_X}})$, $\overline{\Delta}(\phi) = \phi$.

In the following of this subsection, we assume that L satisfies the strong negation law.

Lemma 4.8. *If R is symmetric then, $\mathcal{T}_{\overline{R}}$ is dual.*

Proof. For any $\lambda \in L^X$, from R is symmetric we have

$$\begin{aligned}
\mathcal{T}_{\overline{R}}(\lambda^*) &= S(\overline{R}(\lambda^*),\lambda^*) = \bigwedge_{x\in X}(\overline{R}(\lambda^*)(x) \to \lambda^*(x)) = \bigwedge_{x\in X}\bigwedge_{y\in X}((R(x,y)\odot\lambda^*(y)) \to \lambda^*(x))) \\
&= \bigwedge_{x\in X}\bigwedge_{y\in X}(R(x,y) \to (\lambda^*(y) \to \lambda^*(x))) = \bigwedge_{x\in X}\bigwedge_{y\in X}(R(y,x) \to (\lambda(x) \to \lambda(y))) \\
&= \bigwedge_{y\in X}\bigwedge_{x\in X}(R(y,x)\odot\lambda(x)) \to \lambda(y))) = \bigwedge_{y\in X}(\overline{R}(\lambda)(y) \to \lambda(y)) = S(\overline{R}(\lambda),\lambda) = \mathcal{T}_{\overline{R}}(\lambda).
\end{aligned}$$

Theorem 4.9. *Let $(X,\mathcal{T})$ be a dual Alexandroff L-fuzzy topological space and (X,R) be a reflexive and symmetric L-fuzzy approximation space. Then $\mathcal{T}_{\overline{R_{\mathcal{T}}}} \ge \mathcal{T}$ and $R_{\mathcal{T}_{\overline{R}}} \ge R$.*

Proof. At first we prove that $\mathcal{T}_{\overline{R_{\mathcal{T}}}} \geq \mathcal{T}$. For any $\lambda \in L^X$,

$$\begin{aligned}\mathcal{T}_{\overline{R_{\mathcal{T}}}}(\lambda) &= S(\overline{R_{\mathcal{T}}}(\lambda), \lambda) = \bigwedge_{x\in X}((\bigvee_{y\in X}(R_{\mathcal{T}}(x,y) \odot \lambda(y))) \to \lambda(x)) = \bigwedge_{x,y\in X}(R_{\mathcal{T}}(x,y) \to (\lambda(y) \to \lambda(x)))\\ &\geq \bigwedge_{x,y\in X}((\mathcal{T}(\lambda^*) \to (\lambda^*(x) \to \lambda^*(y)) \to (\lambda^*(x) \to \lambda^*(y))) \geq \mathcal{T}(\lambda^*) = \mathcal{T}(\lambda).\end{aligned}$$

Now, we prove $R_{\mathcal{T}_{\overline{R}}} \geq R$. For any $x, y \in X$, by R is symmetric L-fuzzy relation, then

$$\begin{aligned}R_{\mathcal{T}_{\overline{R}}}(x,y) &= \bigwedge_{\lambda\in L^X}(\mathcal{T}_{\overline{R}}(\lambda) \to (\lambda(x) \to \lambda(y))) = \bigwedge_{\lambda\in L^X}(\mathcal{T}_{\overline{R}}(\lambda^*) \to (\lambda(x) \to \lambda(y))) = \bigwedge_{\lambda^*\in L^X}(S(\overline{R}(\lambda^*), \lambda^*) \to (\lambda(x) \to \lambda(y)))\\ &\geq \bigwedge_{\lambda\in L^X}(\overline{R}(\lambda^*)(x) \to \lambda^*(x)) \to (\lambda(x) \to \lambda(y))) \geq \bigwedge_{\lambda\in L^X}((R(x,y) \odot \lambda^*(y)) \to \lambda^*(x)) \to (\lambda(x) \to \lambda(y)))\\ &= \bigwedge_{\lambda\in L^X}((R(x,y) \to (\lambda(x) \to \lambda(y))) \to (\lambda(x) \to \lambda(y))) \geq R(x,y).\end{aligned}$$

Theorem 4.10. $(\overline{\Delta}, \Gamma)$ *forms a Galois connection between the category* ***L-APP*** *and the category* ***DAL-FTop****.*

Proof. The above theorem (1) shows that $\mathrm{id}_X : (X, \overline{\Delta} \circ \Gamma(\mathcal{T})) \longrightarrow (X, \mathcal{T})$ is continuous and (2) shows that $\mathrm{id}_X : (X, R) \longrightarrow (X, \Gamma \circ \overline{\Delta}(R))$ is an L-approximation mapping. Thus $(\overline{\Delta}, \Gamma)$ forms a Galois connection between the category **L-APP** and the category **DAL-FTop**.

4.3 On the adjunction (Δ, Γ) : L-APP $\rightharpoonup$ DAL-FTop.

Let (X, R) be an L-fuzzy approximation space. We define $\mathcal{T}_{R_X}(\lambda) = S(\overline{R_X}(\lambda), \underline{R_X}(\lambda))$. By Proposition 3.3, $(X, \mathcal{T}_{R_X}) \in$ **AL-FTop**. Let $f : (X, R_X) \to (Y, R_Y)$ be an approximation mapping. By Theorem 3.5, we have $f : (X, \mathcal{T}_{R_X}) \to (Y, \mathcal{T}_{R_Y})$ is continuous. Thus, Δ : **L-APP** $\to$ **AL-FTop** with $\Delta(X, R_X) = (X, \mathcal{T}_{R_X})$, $\Delta(f) = f$ define a concrete functor.

Lemma 4.11. *Let (X, R) be a reflexive L-fuzzy approximation space and $\odot = \wedge$. Then $\mathcal{T}_R = \mathcal{T}_{\underline{R}} \odot \mathcal{T}_{\overline{R}}$.*

Proof. For any $\lambda \in L^X$, it is obvious that $\mathcal{T}_R(\lambda) \leq \mathcal{T}_{\underline{R}}(\lambda), \mathcal{T}_{\overline{R}}(\lambda)$ and so $\mathcal{T}_R(\lambda) \leq \mathcal{T}_{\underline{R}}(\lambda) \wedge \mathcal{T}_{\overline{R}}(\lambda)$. On the other hand $\mathcal{T}_{\underline{R}}(\lambda) \wedge \mathcal{T}_{\overline{R}}(\lambda) = S(\overline{R}(\lambda), \lambda) \wedge S(\lambda, \underline{R}(\lambda)) \leq S(\overline{R}(\lambda), \underline{R}(\lambda)) = \mathcal{T}_R(\lambda)$. Let L satisfy the strong negation law and $\odot = \wedge$. Let $(X, \mathcal{T})$ be a dual Alexandroff L-fuzzy topological space and (X, R) be a reflexive and symmetric L-fuzzy approximation space. Then by Theorem 4.3 and Theorem 4.9 we have (1) $\mathcal{T}_{R_{\mathcal{T}}} \geq \mathcal{T}$ and (2) $R_{\mathcal{T}_R} \geq R$.

Theorem 4.12. *Let L satisfy the strong negative law and $\odot = \wedge$. Then (Δ, Γ) forms a Galois connection between the category* ***L-APP*** *and the category* ***DAL-FTop****.*

5 On the relationships between L-fuzzy approximation spaces and Alexandroff L-fuzzy interior(closure) operator

We devote this section to the categorical aspect of the relationship between reflexive L-fuzzy approximation spaces and Alexandroff L-fuzzy interior(closure) operator spaces.

5.1 On the adjunction $(\underline{\Theta}, \underline{\Phi})$: L-APP $\rightharpoonup$ AL-FIOP.

Let (X, R_X) be a reflexive L-fuzzy approximation space. Then it is not difficult to check that $\mathcal{I}_{R_X} = \underline{R_X}$ is an Alexandroff L-fuzzy interior operator, and then one can define a concrete functor $\underline{\Theta}$: **L-APP** $\to$ **AL-FIOP** by $\underline{\Theta}(X, R_X) = (X, \mathcal{I}_{R_X})$, $\underline{\Theta}(\phi) = \phi$.

Conversely, let $\mathcal{I}_X$ be an Alexandroff L-fuzzy interior operator. Then it is easily seen that $R_{\mathcal{I}_X}(x, y) = \mathcal{I}_X(\top_y^*)(x) \to \perp$ is a reflexive L-fuzzy relation.

Lemma 5.1. *Let $\phi:(X,\mathcal{I}_X)\to(Y,\mathcal{I}_Y)$ be a continuous mapping between Alexandroff L-fuzzy interior operator spaces Then $\phi:(X,R_{\mathcal{I}_X})\to(Y,R_{\mathcal{I}_Y})$ is an approximation mapping between L-fuzzy approximation spaces.*

Proof. For any $x,z\in X$,

$$R_{\mathcal{I}_X}(x,z)=\mathcal{I}_X(\top_z^*)(x)\to\bot\le\mathcal{I}_X(\phi^{\leftarrow}(\top_{\phi(z)}^*))(x)\to\bot\le\phi^{\leftarrow}(\mathcal{I}_Y(\top_{\phi(z)}^*))(x)\to\bot=\mathcal{I}_Y(\top_{\phi(z)}^*)(\phi(x))\to\bot=R_{\mathcal{I}_Y}(\phi(x),\phi(z)).$$

Similar to the proof of Theorem 4.1, we get that for any $\lambda\in L^Y, x\in X$

$$\underline{R_{\mathcal{I}_X}}(\phi^{\leftarrow}(\lambda))(x)\ge\phi^{\leftarrow}(\underline{R_{\mathcal{I}_Y}}(\lambda))(x),\overline{R_{\mathcal{I}_X}}(\phi^{\leftarrow}(\lambda))(x)\le\phi^{\leftarrow}(\overline{R_{\mathcal{I}_Y}}(\lambda))(x).$$

Thus $\phi:(X,R_{\mathcal{I}_X})\to(Y,R_{\mathcal{I}_Y})$ is an approximation mapping between L-fuzzy approximation spaces. The above lemma shows that the correspondence $(X,\mathcal{I}_X)\mapsto(X,R_{\mathcal{I}_X})$ induces a concrete functor $\underline{\Phi}$: **AL-FIOP** $\to$ **L-APP** by $\underline{\Phi}(X,\mathcal{I}_X)=(X,R_{\mathcal{I}_X})$, $\underline{\Phi}(\phi)=\phi$.

Theorem 5.2. *Let L satisfy the strong negation law. Then $(\underline{\Theta},\underline{\Phi})$ forms a Galois connection between the category **L-APP** and the category **AL-FIOP**. Moreover, $\underline{\Phi}$ is a left inverse of $\underline{\Theta}$, i.e., $\underline{\Phi}\circ\underline{\Theta}(X,R_X)=(X,R_X)$ for any $(X,R_X)\in$ **L-APP**.*

Proof. For any $(X,R_X)\in$ **L-APP**,

$$R_{\mathcal{I}_{R_X}}(x,y)=\mathcal{I}_{R_X}(\top_y^*)(x)\to\bot=\underline{R_X}(\top_y^*)(x)\to\bot=(\bigwedge_{z\in X}R_X(x,z)\to\top_y^*(z))\to\bot=(R_X(x,y)\to\bot)\to\bot=R_X(x,y).$$

It follows that $\underline{\Phi}(\underline{\Theta}(X,R_X))=(X,R_X)$ and then $id_X:(X,R_X)\to(X,\underline{\Phi}\circ\underline{\Theta}(R_X))$ is an approximation mapping.

For any $(X,\mathcal{I}_X)\in$ **AL-FIOP** and any $\lambda\in L^X$, $x\in X$,

$$\begin{aligned}&\mathcal{I}_{R_{\mathcal{I}_X}}(\lambda)(x)=\underline{R_{\mathcal{I}_X}}(\lambda)(x)=\textstyle\bigwedge_{y\in X}(R_{\mathcal{I}_X}(x,y)\to\lambda(y))=\bigwedge_{y\in X}((\mathcal{I}_X(\top_y^*)\to\bot)(x)\to\lambda(y))\\&=\textstyle\bigwedge_{y\in X}(\lambda^*(y)\to((\mathcal{I}_X(\top_y^*)(x)\to\bot)\to\bot))=\bigwedge_{y\in X}(\lambda^*(y)\to\mathcal{I}_X(\top_y^*)(x))\\&\ge\textstyle\bigwedge_{y\in X}\mathcal{I}_X((\lambda(y)\to\bot)\to\top_y^*)(x)=\bigwedge_{y\in X}\mathcal{I}_X(\lambda^*(y)\to\top_y^*)(x)\\&\ge\mathcal{I}_X(\textstyle\bigwedge_{y\in X}\lambda^*(y)\to\top_y^*)(x)=\mathcal{I}_X(\lambda)(x),\end{aligned}$$

where, $\lambda(x)=\bigwedge_{y\in X}(\lambda^*(y)\to\top_y^*)(x)$. It follows that $id_X:(X,\underline{\Theta}\circ\underline{\Phi}(\mathcal{I}_X))\to(X,\mathcal{I}_X)$ is continuous.

5.2 On the adjunction $(\overline{\Theta},\overline{\Phi})$: L-APP $\rightharpoonup$ AL-FCOP.

Let (X,R_X) be a reflexive L-fuzzy approximation space. Then it is not difficult to check that $\mathcal{C}_{R_X}=\overline{R_X}$ is an Alexandroff L-fuzzy closure operator, and then one can define a concrete functor $\overline{\Theta}$: **L-APP** $\to$ **AL-FCOP** by $\overline{\Theta}(X,R_X)=(X,\mathcal{C}_{R_X})$, $\overline{\Theta}(\phi)=\phi$.

Conversely, let $\mathcal{C}_X$ be an Alexandroff L-fuzzy closure operator. Then it is easily seen that $R_{\mathcal{C}_X}(x,y)=\mathcal{C}_X(\top_y)(x)$ is a reflexive L-fuzzy relation.

Lemma 5.3. *Let $\phi:(X,\mathcal{C}_X)\to(Y,\mathcal{C}_Y)$ be a continuous mapping between Alexandroff L-fuzzy closure operator spaces. Then $\phi:(X,R_{\mathcal{C}_X})\to(Y,R_{\mathcal{C}_Y})$ is an approximation mapping between L-fuzzy approximation spaces.*

Proof. For any $x,z\in X$,

$$R_{\mathcal{C}_X}(x,z)=\mathcal{C}_X(\top_z)(x)\le\mathcal{C}_X(\phi^{\leftarrow}(\top_{\phi(z)}))(x)\le\phi^{\leftarrow}(\mathcal{C}_Y(\top_{\phi(z)}))(x)=\mathcal{C}_Y(\top_{\phi(z)})(\phi(x))=R_{\mathcal{C}_Y}(\phi(x),\phi(z)).$$

Similar to the proof of Theorem 4.1, we get that for any $\lambda\in L^Y, x\in X$,

$$\underline{R_{\mathcal{C}_X}}(\phi^{\leftarrow}(\lambda))(x)\ge\phi^{\leftarrow}(\underline{R_{\mathcal{C}_Y}}(\lambda))(x),\overline{R_{\mathcal{C}_X}}(\phi^{\leftarrow}(\lambda))(x)\le\phi^{\leftarrow}(\overline{R_{\mathcal{C}_Y}}(\lambda))(x).$$

Thus $\phi:(X,R_{\mathcal{C}_X})\to(Y,R_{\mathcal{C}_Y})$ is an approximation mapping between L-fuzzy approximation spaces. The above lemma shows that the correspondence $(X,\mathcal{C}_X)\mapsto(X,R_{\mathcal{C}_X})$ induces a concrete functor $\overline{\Phi}$: **AL-FCOP** $\to$ **L-APP** by $\overline{\Phi}(X,\mathcal{C}_X)=(X,R_{\mathcal{C}_X})$, $\overline{\Phi}(\phi)=\phi$.

Theorem 5.4. *$(\overline{\Theta},\overline{\Phi})$ forms a Galois connection between the category **L-APP** and the category **AL-FCOP**. Moreover, $\overline{\Phi}$ is a left inverse of $\overline{\Theta}$, i.e., $\overline{\Phi}\circ\overline{\Theta}(X,R_X)=(X,R_X)$ for any $(X,R_X)\in$ **L-APP**.*

Proof. For any $(X,R_X)\in$ **L-APP**,

$$R_{\mathcal{C}_{R_X}}(x,y)=\mathcal{C}_{R_X}(\top_y)(x)=\overline{R_X}(\top_y)(x)=\bigvee_{z\in X}R_X(x,z)\odot\top_y(z)=R_X(x,y).$$

It follows that $\overline{\Phi}(\overline{\Theta}(X,R_X))=(X,R_X)$ and then $id_X:(X,R_X)\to(X,\overline{\Phi}\circ\overline{\Theta}(R_X))$ is an approximation mapping. For any $(X,\mathcal{C}_X)\in$ **AL-FIOP** and any $\lambda\in L^X$, $x\in X$,

$$\begin{aligned}\mathcal{C}_{R_{\mathcal{C}_X}}(\lambda)(x) &= \overline{R_{\mathcal{C}_X}}(\lambda)(x) = \bigvee_{y\in X}\Big(R_{\mathcal{C}_X}(x,y)\odot\lambda(y)\Big)(x) = \bigvee_{y\in X}\Big(\mathcal{C}_X(\top_y)(x)\odot\lambda(y)\Big)(x)\\ &\leq \bigvee_{y\in X}\Big(\mathcal{C}_X(\top_y\odot\lambda(y))\Big)(x) = \mathcal{C}_X\Big(\bigvee_{y\in X}(\top_y\odot\lambda(y))\Big)(x) = \mathcal{C}_X(\lambda)(x).\end{aligned}$$

It follows that $id_X : (X, \overline{\Theta}\circ\overline{\Phi}(\mathcal{C}_X)) \to (X, \mathcal{C}_X)$ is continuous.

5.3 On the adjunction (Θ, Φ) : L-APP $\rightharpoonup$ DAL-FOP.

In this subsection, we assume that L satisfies the strong negation law. Let (X, R_X) be a reflexive L-fuzzy approximation space. Then it is not difficult to check that $(X, \mathcal{I}_{R_X} = \underline{R_X}, \mathcal{C}_{R_X} = \overline{R_X})$ is a dual Alexandroff L-fuzzy operator space, and then one can define a concrete functor Θ : **L-APP** $\to$ **DAL-FOP** by $\Theta(X, R_X) = (X, \mathcal{I}_{R_X}, \mathcal{C}_{R_X}),\ \Theta(\phi) = \phi$.

Lemma 5.5. *Let $(X, \mathcal{I}_X, \mathcal{C}_X)$ be a dual Alexandroff L-fuzzy operator space. Then $R_{\mathcal{C}_X} = R_{\mathcal{I}_X}$ is a reflexive L-fuzzy relation, and then both of them are denote as $R_{\mathcal{OP}_X}$.*

Proof. For any $x, y \in X,\ R_{\mathcal{I}_X}(x,y) = \mathcal{I}_X(\top_y^*)(x) \to \perp = (\mathcal{I}_X(\top_y^*))^*(x) = \mathcal{C}_X(\top_y)(x) = R_{\mathcal{C}_X}(x,y)$. By Lemma 5.1 and Lemma 5.3 it follows that the correspondence $(X, \mathcal{I}_X, \mathcal{C}_X) \mapsto (X, R_{\mathcal{OP}_X})$ induces a concrete functor Φ : **DAL-FOP** $\to$ **L-APP** by $\Phi(X, \mathcal{I}_X, \mathcal{C}_X) = (X, R_{\mathcal{OP}_X}),\ \Phi(\phi) = \phi$. Form Lemma 5.5, Theorem 5.2 and Theorem 5.4, we get the following theorem.

Theorem 5.6. *(Θ, Φ) forms a Galois connection between the category **L-APP** and the category **DAL-FOP**. Moreover, Φ is a left inverse of Θ.*

5.4 Further relationships

In this subsection, we assume that L satisfies the strong negation law.

Theorem 5.7. *Let $(X, \mathcal{T}_X)$ be a dual Alexandroff L-fuzzy topological space. Then $(X, \mathcal{I}_{\mathcal{T}_X}, \mathcal{C}_{\mathcal{T}_X})$ be a dual Alexandroff L-fuzzy operator space, and so by Lemma 5.5 we have that $R_{\mathcal{C}_{\mathcal{T}_X}} = R_{\mathcal{I}_{\mathcal{T}_X}}$.*

Proof. For any $\lambda \in L^X, x \in X$,

$$\begin{aligned}(\mathcal{I}_{\mathcal{T}_X}(\lambda^*))^*(x) &= (\bigvee_{\mu\in L^X}(\mathcal{T}_X(\mu)\odot S(\mu,\lambda^*)\odot\mu(x))) \to \perp = \bigwedge_{\mu\in L^X}[(\mathcal{T}_X(\mu)\odot S(\mu,\lambda^*)\odot\mu(x))] \to \perp\\ &= \bigwedge_{\mu\in L^X}[\mathcal{T}_X(\mu)\odot S(\lambda,\mu^*)] \to \mu^*(x) = \bigwedge_{\mu\in L^X}[\mathcal{T}_X(\mu^*)\odot S(\lambda,\mu^*)] \to \mu^*(x)\\ &= \bigwedge_{\nu\in L^X}[\mathcal{T}_X(\nu)\odot S(\lambda,\nu)] \to \nu(x) = \mathcal{C}_{\mathcal{T}_X}(\lambda)(x).\end{aligned}$$

Thus $(\mathcal{I}_{\mathcal{T}_X}(\lambda^*))^* = \mathcal{C}_{\mathcal{T}_X}(\lambda)$. That $(\mathcal{C}_{\mathcal{T}_X}(\lambda^*))^* = \mathcal{I}_{\mathcal{T}_X}(\lambda)$ can be proved similarly.

Theorem 5.8. *Let (X, R_X) be an L-fuzzy approximation space. Then $\mathcal{I}_{\mathcal{T}_{R_X}}(\lambda) \leq \mathcal{I}_{R_X}(\lambda)$ and $\mathcal{C}_{\mathcal{T}_{R_X}}(\lambda) \geq \mathcal{C}_{R_X}(\lambda)$ for all $\lambda \in L^X$.*

Proof. For any $\lambda \in L^X, x \in X$,

$$\begin{aligned}\mathcal{I}_{\mathcal{T}_{R_X}}(\lambda)(x) &= \bigvee_{\mu\in L^X}\mathcal{T}_{R_X}(\mu)\odot S(\mu,\lambda)\odot\mu(x) = \bigvee_{\mu\in L^X}S(\overline{R}_X(\mu),\underline{R_X}(\mu))\odot S(\mu,\lambda)\odot\mu(x)\\ &\leq \bigvee_{\mu\in L^X}S(\overline{R}_X(\mu),\underline{R_X}(\mu))\odot S(\underline{R_X}(\mu),\underline{R_X}(\lambda))\odot\mu(x) \leq \bigvee_{\mu\in L^X}S(\overline{R}_X(\mu),\underline{R_X}(\lambda))\odot\overline{R}_X(\mu)(x)\\ &\leq \underline{R_X}(\lambda)(x) = \mathcal{I}_{R_X}(\lambda)(x).\end{aligned}$$

$$\begin{aligned}\mathcal{C}_{\mathcal{T}_{R_X}}(\lambda) &= \bigwedge_{\mu\in L^X}((\mathcal{T}_{R_X}(\mu)\odot S(\lambda,\mu)) \to \mu\Big) = \bigwedge_{\mu\in L^X}((S(\overline{R}_X(\mu),\underline{R_X}(\mu))\odot S(\lambda,\mu))) \to \mu)\\ &\geq \bigwedge_{\mu\in L^X}(S(\overline{R}_X(\mu),\underline{R_X}(\mu))\odot S(\overline{R}_X(\lambda),\overline{R}_X(\mu))) \to \mu) \geq \bigwedge_{\mu\in L^X}(S(\overline{R}_X(\lambda),\underline{R_X}(\mu)) \to \mu)\\ &\geq \bigwedge_{\mu\in L^X}(S(\overline{R}_X(\lambda),\underline{R_X}(\mu)) \to \underline{R_X}(\mu)) \geq \overline{R}_X(\lambda)(x) = \mathcal{C}_{R_X}(\lambda)(x).\end{aligned}$$

6 Conclusions

In this paper, we systematically studied the relationships between the category of Alexandroff L-fuzzy topological space, the category of reflexive L-fuzzy approximation spaces and the category of Alexandroff L-fuzzy interior (closure) spaces. In particular, we obtain some interesting adjunctions between the considered categories.

Acknowledgement

The author thank the reviewers and the editor for their valuable comments and suggestions. The second author also thank the National Natural Science Foundation of China (11501278, 11801248).

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, Wiley, New York, 1990.

[2] W. Bandler, L.Kohout, *Special properties, closures and interiors of crisp and fuzzy relations*, Fuzzy sets and Systems, **26**(3) (1988), 317–331.

[3] R. Bělohlávek, *Fuzzy closure operators*, Journal of Mathematical Analysis and Applications, **262** (2001), 473–489.

[4] R. Bělohlávek, *Fuzzy closure operators II*, Soft Computing, **7**(1) (2002), 53–64.

[5] R. Bělohlávek, *Fuzzy Relational Systems*, Kluwer Academic Publishers, New York, 2002.

[6] R. Bělohlávek, T. Funioková, *Fuzzy interior operators*, International Journal of General Systems, **33**(4) (2004), 415–430.

[7] R. Bělohlávek, M. Krupka, *Central points and approximation in residuated lattices*, International Journal of Approximate Reasoning, **66** (2015), 27–38.

[8] L. Biacino, G. Gerla, *Closure systems and L-subalgebras*, Information Sciences, **33** (1984), 181–195.

[9] L. Biacino, G. Gerla, *An extension principle for closure operators*, Journal of Mathematical Analysis and Applications, **198** (1996), 1–24

[10] U. Bodenhofer, M. De Cock, E.E. Kerre, *Openings and closures of fuzzy preorderings:theoretical basics and applications to fuzzy rule-based systems*, International Journal of General Systems, **32**(4) (2003), 343–360.

[11] K. Blount, C. Tsinakis, *The structure of residuated lattices*, International Journal of Algebra and Computation, **13** (2003), 437–461.

[12] J. Fang, *I-fuzzy Alexandrov topologies and specialization orders*, Fuzzy Sets and Systems, **128** (2007), 2359–2374.

[13] J. Fang, *The relationship between L-ordered convergence structures and strong L-topologies*, Fuzzy Sets and Systems, **161** (2010), 2923–2944.

[14] J . Fang, Y. Yue, *L -fuzzy closure systems*, Fuzzy sets and Systems, **161** (2010), 1242–1252.

[15] J. A. Goguen, *L-fuzzy sets*, Journal of Mathematical Analysis and Applications, **18** (1967), 145–174.

[16] P. Hájek, *Metamathematices of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[17] S. E. Han, IS. Kim, A. P. Šostak, *On approximate-type systems generated by L-relations*, Information Sciences, **281** (2014), 8–20.

[18] J. Hao, Q. Li, *The relationship between L-fuzzy rough set and L-topology*, Fuzzy Sets and Systems, **178** (2011), 74–83.

[19] H. Herrlich, M. Hušek, *Galois connections categorically*, Journal of Pure and Applied Algebra, **68** (1990), 165–180.

[20] U. Höhle, S.E. Rodabaugh, *Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series 3*, Kluwer Academic Publishers, Boston, 1999.

[21] Q. Jin, L. Q. Li, *One-axiom characterizations on lattice-valued closure (interior) operators*, Journal of Intelligent & Fuzzy Systems, **31**(3) (2016), 1679–1688.

[22] Q. Jin, L. Q. Li, G. W. Meng, *On the relationships betwween types of L-conergence spaces*, Iranian Journal of Fuzzy Systems, **13**(1) (2016), 93–103.

[23] J. Kortelainen, *On relationship between modified sets, topological space and rough sets*, Fuzzy Sets and Systems, **61** (1994), 91–95.

[24] H. Lai, D. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems, **157** (2006), 1865–1885.

[25] L.Q. Li, Q. Jin, K. Hu, F.F. Zhao, *The axiomatic characterizations on L-fuzzy covering-based approximation operators*, International Journal of General Systems, **46** (2017), 332–353.

[26] L. Q. Li, Q. G. Li, *On enriched L-topologies: Base and subbase*, Journal of Intelligent & Fuzzy Systems, **28**(6) (2015), 2423–2432.

[27] Z. Li, T. Xie, Q. Li, *Topological structure of generalized rough sets*, Computers & Mathematics with Applications, **63** (2012), 1066–1071.

[28] Z. M. Ma, B. Q. Hu, *Topological and lattice structures of L-fuzzy rough sets determined by lower and upper sets*, Information Sciences, **218** (2013), 194–204.

[29] Z. Pawlak, *Rough sets*, International Journal of Computing and Information Sciences, **11** (1982), 341–356.

[30] Z. Pawlak, *Rough Set: Theoretical Aspects of Reasoning About Data*, Kluwer Academic Publishers, Boston, 1991.

[31] Z. Pei, D. Pei, L. Zheng, *Topology vs generalized rough sets*, International Journal of Approximate Reasoning, **52** (2011), 231–239.

[32] J. Qiao, B. Q. Hu, *A short note on L-fuzzy approximation spaces and L-fuzzy pretopological spaces*, Fuzzy Sets and Systems, **312** (2017), 126–134.

[33] K. Y. Qin, Z. Pei, *On the topological properties of fuzzy rough sets*, Fuzzy Sets and Systems, **151** (2005), 601–613.

[34] A. M. Radzikowska, E. E. Kerre, *A comparative study of fuzzy rough sets*, Fuzzy Sets and Systems, **126** (2002), 137–155.

[35] A. M. Radzikowska, E. E. Kerre, *Fuzzy rough sets based on residuated lattices, in: Transactions on Rough Sets II*, Lecture Notes in Computer Science, **3135** (2004), 278–296.

[36] A. A. Ramadan, *L-fuzzy interior systems*, Computers & Mathematics with Applications, **62** (2011), 4301–4307.

[37] A. A. Ramadan, E. H. Elkordy, M. El-Dardery, *L-fuzzy approximition spaces and L-fuzzy toplogical spaces*, Iranian Journal of Fuzzy Systems, **13**(1) (2016), 115–129.

[38] S.E. Rodabaugh, E.P. Klement, *Topological and Algebraic Structures In Fuzzy Sets, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.

[39] Y.H. She, G.J. Wang, *An axiomatic approach of fuzzy rough sets based on residuated lattices*, Computers & Mathematics with Applications, **58** (2009), 189–201.

[40] A. Šostak, *Basic structures of fuzzy topology*, Journal of Mathematical Sciences, **78**(6) (1996), 662–701.

[41] E. Turunen, *Mathematics Behind Fuzzy Logic*, A Springer-Verlag Co., Heidelberg, 1999.

[42] C. Y. Wang, *Topological characterizations of generalized fuzzy rough sets*, Fuzzy Sets and Systems, **312** (2017), 109–125.

[43] C. Y. Wang, *Topological structures of L-fuzzy rough sets and similarity sets of L-fuzzy relations*, International Journal of Approximate Reasoning, **83** (2017), 160–175.

[44] C. Y. Wang, B. Q. Hu, *Granular variable precision fuzzy rough sets with general fuzzy relations*, Fuzzy Sets and Systems, **275** (2015), 39–57.

[45] Z. Wang, Y. Wang, K. Tang, *Some properties of L-fuzzy approximation spaces based on bounded integral residuated lattices*, Information Sciences, **278** (2014), 110–126.

[46] W. Yao, S.E. Han, *A Stone-type duality for sT0 stratified Alexandroff L-topological spaces*, Fuzzy Sets and Systems, **282** (2016), 1–20.

[47] W. Yao, B. Zhao, *Kernel systems on L-ordered sets*, Fuzzy Sets and Systems, **182** (2011), 101–109.

[48] D. Zhang, *Fuzzy pretopological spaces*, an extensitional topological extension of FTS, Chinese Annals of Mathematics, **3** (1999), 309–316.

[49] F. Zhao, Q. Jin, L. Li, *The axiomatic characterizations on L-generalized fuzzy neighborhood system-based approximation operators*, International Journal of General Systems, **47**(2) (2018), 155–173.

[50] F. Zhao, L. Li, *Axiomatization on generalized neighborhood system-based rough sets*, Soft Computing, **22**(18) (2018), 6099–6110.

Iranian Journal of Fuzzy Systems
Volume 16, Number 3, (2019), pp. 206
University of Sistan and Baluchestan

Categories of lattice-valued closure (interior) operators and Alexandroff *L*-fuzzy topologies

A. A. Ramadan and L. Li

رسته‌های عملگرهای (داخلی) بستار شبکه-مقدار و توپولوژی‌های L- فازی الکساندروف

چکیده. همبندی گالوایی در نظریه رسته نقش مهمی در برقراری ارتباط بین ساختارهای خاص متفاوت ایفا می‌کند. در این مقاله، ثابت می‌کنیم که همبندی‌های گالوایی جالب بسیاری بین رسته فضاهای توپولوژیکی L- فازی الکساندروف، رسته فضاهای تقریب L- فازی انعکاسی و فضاهای (بستار) داخلی L-فازی الکساندروف وجود دارد. این اشاره به این دارد که یک همبندی بسته‌ای بین سه ساختار وجود دارد.