Iranian Journal of Fuzzy Systems
Volume 16, Number 3, (2019), pp. 97-112
University of Sistan and Baluchestan

Multiple attribute decision making with triangular intuitionistic fuzzy numbers based on zero-sum game approach

J. Xu[1], J. Y. Dong[2], S. P. Wan[3] and J. Gao[4]

[1,4] *College of Modern Economics & Management, Jiangxi University of Finance and Economics, Nanchang 330013, China*
[2] *School of Statistics, Jiangxi University of Finance and Economics, Nanchang 330013, China*
[3] *College of Information Technology, Jiangxi University of Finance and Economics, Nanchang 330013, China*

xujun1028@126.com, jiuyingdong@126.com, shupingwan@163.com, gaojie02111@163.com

Abstract

For many decision problems with uncertainty, triangular intuitionistic fuzzy number is a useful tool in expressing ill-known quantities. This paper develops a novel decision method based on zero-sum game for multiple attribute decision making problems where the attribute values take the form of triangular intuitionistic fuzzy numbers and the attribute weights are unknown. First, a new value index is defined for triangular intuitionistic fuzzy numbers on the basis of the centroid. Thereby, a new ranking approach is presented for comparing triangular intuitionistic fuzzy numbers. We formulate a multiple attribute decision making problem as a two-person zero-sum game with payoffs of triangular intuitionistic fuzzy numbers. Then, following the new ranking approach, the fuzzy matrix game is converted as a pair of crisp linear programming models, and the optimal strategies are objectively derived by solving such models. Therefore, the ranking order of alternatives is determined by the expected scores of alternatives. An example of video monitoring system selection is demonstrated to illustrate the effectiveness of the proposed methodology.

Keywords: Multiple attribute decision making, Triangular intuitionistic fuzzy numbers, Two-person zero-sum game, Linear programming model, Video monitoring system selection.

1 Introduction

Multiple attribute decision making (MADM) is a procedure for finding the optimal alternative from feasible alternatives characterized by multiple attributes [39]. MADM problems are ubiquitous in real-life decision-making scenarios, and have thus been extensively studied [35]. Given that decision problems touch upon multiple unquantifiable indices, they cannot be effectively solved by existing conventional MADM methods. Fuzzy set theory [44] is well suited for addressing such ambiguity encountered in solving MADM problems. Thus, fuzzy MADM problems have elicited considerable attention in various methods, including entropy theory [37], TOPSIS method [4, 20], programming model [9, 8], AHP [6, 7], and aggregation operators [15, 38, 16].

In recent years, several researchers have applied two-person zero-sum (TPZS) game theory to fuzzy MADM with triangular fuzzy numbers [5, 10], interval numbers [25], and linguistic values [43]. Chen et al. [5] considered the two players of TPZS game as the decision maker who maximizes the expected payoff and Nature who minimizes the loss. Thus, the MADM problem can be viewed as a fuzzy matrix game based on the decision maker and Nature. Larbani [10] extended this approach to bi-matrix games with fuzzy payoffs by using $\alpha-$cuts and by introducing Nature as the second player that represents the uncertainty involved in such games. Similarly, Wan [25] proposed an interval value based TPZS game to solve the investment choice problem. Yang and Wang [43] constructed a fuzzy matrix game that addresses vague linguistic fuzzy MADM with unknown weights.

Intuitionistic fuzzy sets [3] are believed by many scholars to be superior to classical fuzzy sets [44] for describing hesitation information in decision. Many IF theories have been developed and applied to decision problems [14, 41, 19,

Corresponding Author: S. P. Wan
Received: December 2017; Revised: July 2018; Accepted: November 2018.

13, 17, 47, 40, 36, 46, 29, 30, 42], such as the score functions based on point operators [14], the dynamic IF weighted averaging operator [41], the IF Choquet integral operator [19], the prioritized operators [13], cosine-based similarity measures [17] and the extended VIKOR method under IF environment [47]. In addition, based on the advantages of interval-valued IF set [2], the multi-attributive border approximation area comparison method is extended to solve material selection problems [40], the hybrid MADM method [36] and multiplicative multi-objective optimization by ratio analysis method [46] are used to evaluate the risk of failure modes. Besides, some valuable approaches on aggregating decision data into intuitionistic fuzzy numbers have been developed to handle heterogeneous group decision making [29, 30, 42].

Other researchers [18, 11, 12, 45, 26, 27, 28] introduced the concept of triangular intuitionistic fuzzy number (TIFN), in which the membership degree and non-membership degree are denoted by a triangular fuzzy number. The TIFNs is therefore a more flexible and effective tool in expressing vagueness information. In the real-world video monitoring system selection problems, decision makers are requested to provide the confidence level and non-confidence, and decision makers can only provide the lower bounds, the most possible values and the upper bounds for attribute values. Thus, TIFNs are a reasonable means to evaluate the attributes of providers. Currently, a number of studies have been conducted on TIFNs. These studies mainly focus on two catalogs [18, 11, 12, 45, 26, 27, 28, 34, 22, 31, 32] as detailed below. (1) Ranking methods of TIFNs and their applications to decision making problems [18, 11, 12, 45, 26, 27, 28, 34]. For instance, lexicographic ranking method [18, 11], ratio ranking method [12], the ranking method based on TOPSIS [45] and possibility mean, variance and covariance of TIFNs [26, 27, 28, 34]. (2) Triangular IF aggregation operators and their applications to decision making problems [22, 31, 32] including triangular IF hybrid aggregation operator [22], triangular IF power average operator [31] and triangular IF Triple Bonferroni harmonic mean operators [32].

The aforementioned methods for fuzzy MADM problems that use zero-sum game theory [3, 14, 41, 19] are only suitable for fuzzy MADM problems in which attribute values take the form of triangular fuzzy numbers; they cannot handle MADM problems with TIFNs. Moreover, the attribute weights in most of the existing MADM methods with TIFNs [18, 11, 12, 45, 26, 34, 22] are given in advance. In this paper, we construct a new zero-sum game model that can solve MADM problems where the assessments of alternatives assume the form of TIFNs and weight vector of attributes is unknown. The primary contributions of this paper can be illuminated briefly as detailed below: (1) According to the idea of centroid in a triangle, a new value index of TIFNs is defined and used to compare TIFNs. (2) Based on the ranking approach of TIFNs, we construct a two-person zero-sum game with payoffs to objectively derive the optimal attribute weights. (3) We provide a new means to address MADM problems with TIFNs. (4) An example of video monitoring system selection is demonstrated to illustrate the effectiveness of the proposed methodology.

The rest of this paper is set up as follows. Section 2 briefly introduces basic concepts and definitions for TIFNs. Section 3 gives a value index of TIFNs. Section 4 proposes a game model with TIFNs and shows a framework of MADM problems. Section 5 provides a video monitoring system selection example and comparative analyses. The conclusions of the work are presented in Section 6.

2 Triangular intuitionistic fuzzy numbers

Some basic concepts and operations related to TIFNs are introduced in this section.

2.1 Definition and operations of TIFNs

Definition 2.1. [12] *A TIFN* $\tilde{a} = ((a, b, c); u_{\tilde{a}}, v_{\tilde{a}})$ *defined on the real number set, whose membership and non-membership functions indicate the following form:*

$$\mu_{\tilde{a}}(x) = \begin{cases} (x-a)u_{\tilde{a}}/(b-a) & \text{if} a \le x < b, \\ u_{\tilde{a}} & \text{if} x = b, \\ (c-x)u_{\tilde{a}}/(c-b) & \text{if} b < x \le c, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

$$v_{\tilde{a}}(x) = \begin{cases} [b-x+v_{\tilde{a}}(x-a)]/(b-a) & \text{if} a \le x < b, \\ \nu_{\tilde{a}} & \text{if} x = b, \\ [x-b+v_{\tilde{a}}(c-x)]/(c-b) & \text{if} b < x \le c, \\ 1 & \text{otherwise}, \end{cases} \tag{2}$$

respectively, as described in Figure 1, where $u_{\tilde{a}}$ *and* $v_{\tilde{a}}$ *are the maximum membership degree and the minimum non-membership, respectively, in such a way that* $0 \le u_{\tilde{a}} \le 1$, $0 \le v_{\tilde{a}} \le 1$, $0 \le u_{\tilde{a}} + \nu_{\tilde{a}} \le 1$.

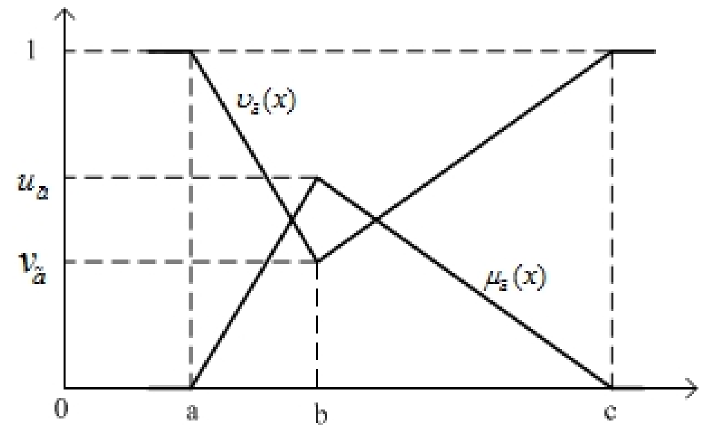

Figure 1: A TIFN $\tilde{a} = ((a, b, c); u_{\tilde{a}}, v_{\tilde{a}})$

Definition 2.2. [12] *Let* $\tilde{a}_1 = ((a_1, b_1, c_1); u_{\tilde{a}_1}, \nu_{\tilde{a}_1})$ *and* $\tilde{a}_2 = ((a_2, b_2, c_2); u_{\tilde{a}_2}, \nu_{\tilde{a}_2})$ *be two TIFNs and* λ *be a real number. Some operational laws are defined as follows:*

$\tilde{a}_1 + \tilde{a}_2 = ((a_1 + a_2, b_1 + b_2, c_1 + c_2); \min\{u_{\tilde{a}_1}, u_{\tilde{a}_2}\}, \max\{\nu_{\tilde{a}_1}, \nu_{\tilde{a}_2}\}),$

$\tilde{a}_1 - \tilde{a}_2 = ((a_1 - a_2, b_1 - b_2, c_1 - c_2); \min\{u_{\tilde{a}_1}, u_{\tilde{a}_2}\}, \max\{\nu_{\tilde{a}_1}, \nu_{\tilde{a}_2}\}),$

$$\lambda\tilde{a}_1 = \begin{cases} ((\lambda a_1, \lambda b_1, \lambda c_1); u_{\tilde{a}_1}, \nu_{\tilde{a}_1}) & \text{if} \lambda > 0, \\ ((\lambda c_1, \lambda b_1, \lambda a_1); u_{\tilde{a}_1}, \nu_{\tilde{a}_1}) & \text{if} \lambda < 0, \end{cases}$$

2.2 Cut sets of TIFNs

Definition 2.3. [6] *For a TIFN* $\tilde{a} = ((a, b, c); u_{\tilde{a}}, \nu_{\tilde{a}})$*, the* $(\alpha, \beta)-$cut *is defined as* $\tilde{a}_{\alpha,\beta} = \{x|\mu_{\tilde{a}}(x) \geq \alpha, \upsilon_{\tilde{a}}(x) \leq \beta\}$*, where* $0 \leq \alpha + \beta \leq 1$ *and* $0 \leq \alpha \leq u_{\tilde{a}}$*,* $\nu_{\tilde{a}} \leq \beta \leq 1$*.*

Definition 2.4. [6] *For a TIFN* $\tilde{a} = ((a, b, c); u_{\tilde{a}}, \nu_{\tilde{a}})$*, the* $\alpha-$cut *is defined as* $\tilde{a}_\alpha = \{x|\mu_{\tilde{a}}(x) \geq \alpha\}$*, where* $0 \leq \alpha \leq u_{\tilde{a}}$*, denoted by* $\tilde{a}_\alpha = [L_{\tilde{a}}(\alpha), R_{\tilde{a}}(\alpha)]$*. Calculation is as follows:*

$$[L_{\tilde{a}}(\alpha), R_{\tilde{a}}(\alpha)] = \left[a + \frac{\alpha\,(b-a)}{u_{\tilde{a}}}, c - \frac{\alpha\,(c-b)}{u_{\tilde{a}}}\right] \quad (3)$$

Definition 2.5. [6] *For a TIFN* $\tilde{a} = ((a, b, c); u_{\tilde{a}}, \nu_{\tilde{a}})$*,* $\beta-$cut *is defined as* $\tilde{a}_\beta = \{x|\upsilon_{\tilde{a}}(x) \leq \beta\}$*, where* $\nu_{\tilde{a}} \leq \beta \leq 1$*, denoted by* $\tilde{a}_\beta = [L_{\tilde{a}}(\beta), R_{\tilde{a}}(\beta)]$*. Calculation is as follows:*

$$[L_{\tilde{a}}(\beta), R_{\tilde{a}}(\beta)] = \left[\frac{(1-\beta)b + (\beta - \nu_{\tilde{a}})a}{1 - \nu_{\tilde{a}}}, \frac{(1-\beta)b + (\beta - \nu_{\tilde{a}})c}{1 - \nu_{\tilde{a}}}\right] \quad (4)$$

3 New value indexes for triangular intuitionistic fuzzy numbers

Yager [10] indicated that ranking fuzzy numbers is a problem that has been widely studied for years, and no unique best method exists. FNs cannot be ordered totally, because they are expressed by possibility distributions. To rank FNs, a widely used method is to convert FNs into real numbers that can be linearly ordered. The proposed method for ranking TIFNs in this paper also follows such a common practice. Li et al. [11] and Li [12] developed value indices for TIFN based on the mean and sum values of the cut set of TIFN, respectively. Wan [26] also proposed a value index based on the possibility mean value. In this section, a novel value index of a TIFN is proposed on the basis of the centroid concept of triangle [1]. Centroid is chosen as a point of reference, because it is a balancing point of the plane figure [21]. Therefore, the centroid point of the cut set of a TIFN is a better reference point than the middle point.

3.1 Centroid point of $\alpha-$cut set for a TIFN

The following statements are considered. (1) The smaller the confidence level α, the larger the uncertainty of the corresponding cut set contains. (2) The confidence level α divides ΔABC into two parts as follows: ΔBDE and trapezoid ADEC, where the ΔBDE corresponds to the area of the $\alpha-$cut set $\tilde{a}_\alpha$ (Figure 2). In the following, we need to calculate the centroid of $\tilde{a}_\alpha$ first.

Definition 3.1. [24] *For a TIFN* $\tilde{a} = ((a,b,c); u_{\tilde{a}}, \nu_{\tilde{a}})$, *we assume that the functions* $\mu^L_{\tilde{a}}(x) = (x-a)u_{\tilde{a}}/(b-a)$ *and* $\mu^R_{\tilde{a}}(x) = (c-a)u_{\tilde{a}}/(c-b)$ *represent the lines AB and BC in Figure 2, respectively. The inverse functions of* $\mu^L_{\tilde{a}}(x)$ *and* $\mu^R_{\tilde{a}}(x)$ *are* $f^L_{\tilde{a}}(y) = a + (b-a)y/u_{\tilde{a}}$ *and* $f^R_{\tilde{a}}(y) = c - (c-b)y/u_{\tilde{a}}$, *respectively. Based on calculus theories, the centroid* $G(\tilde{a}_\alpha) = (\bar{x}(\tilde{a}_\alpha), \bar{y}(\tilde{a}_\alpha))$ *of* ΔBDE *can be defined as follows:*

$$\bar{x}(\tilde{a}_\alpha) = \frac{\int_\alpha^{u_{\tilde{a}}} x(f^R_{\tilde{a}}(y) - f^L_{\tilde{a}}(y))\mathrm{d}y}{\int_\alpha^{u_{\tilde{a}}} (f^R_{\tilde{a}}(y) - f^L_{\tilde{a}}(y))\mathrm{d}y} \tag{5}$$

and

$$\bar{y}(\tilde{a}_\alpha) = \frac{\int_\alpha^{u_{\tilde{a}}} y(f^R_{\tilde{a}}(y) - f^L_{\tilde{a}}(y))\mathrm{d}y}{\int_\alpha^{u_{\tilde{a}}} (f^R_{\tilde{a}}(y) - f^L_{\tilde{a}}(y))\mathrm{d}y} \tag{6}$$

Compared with the middle point of the cut set of a TIFN $\tilde{a}$ proposed by Li's method [12], the horizontal coordinate $\bar{x}(\tilde{a}_\alpha)$ of the centroid $G(\tilde{a}_\alpha)$ may be more reasonably considered as the mean of $\alpha-$cut set $\tilde{a}_\alpha$. Given that ΔBDE corresponds to the area of the $\alpha-$cut set $\tilde{a}_\alpha$, the centroid of ΔBDE would be more suitable for reflecting the membership function under α.

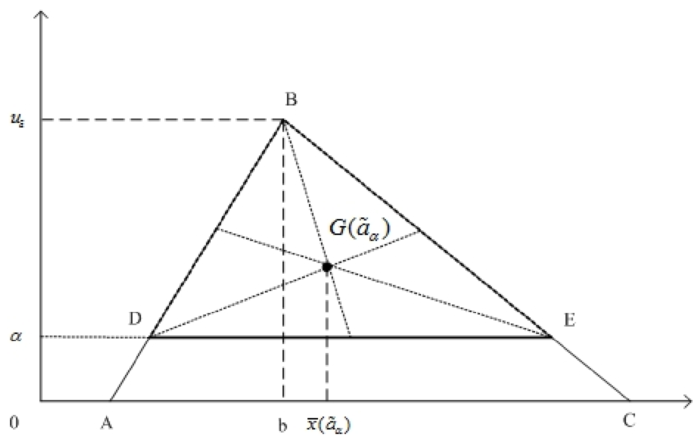

Figure 2: Centroid of $\alpha-$cut set

Remark 3.2. *Given that* $\mu^L_{\tilde{a}}(x) : [a,b] \to [0, u_{\tilde{a}}]$ *and* $\mu^R_{\tilde{a}}(x) : [b,c] \to [0, u_{\tilde{a}}]$ *are both continuous and strictly monotonic functions, the corresponding inverse functions exist and should also be continuous and strictly monotonic. The inverse functions of* $\mu^L_{\tilde{a}}(x)$ *and* $\mu^R_{\tilde{a}}(x)$ *can be denoted by* $f^L_{\tilde{a}}(y) : [0, u_{\tilde{a}}] \to [a,b]$ *and* $f^R_{\tilde{a}}(y)$:$[0, u_{\tilde{a}}] \to [b,c]$, *then* $f^L_{\tilde{a}}(y)$ *and* $f^R_{\tilde{a}}(y)$ *are integrable on* $[0, u_{\tilde{a}}]$. *As* $0 \le \alpha \le u_{\tilde{a}}$. *Thus,* $f^L_{\tilde{a}}(y)$ *and* $f^R_{\tilde{a}}(y)$ *are also integrable on* $[\alpha, u_{\tilde{a}}]$. *By equation (5), we have*

$$\bar{x}(\tilde{a}_\alpha) = \frac{1}{3}[a+b+c - \frac{(c+a-2b)}{u_{\tilde{\alpha}}}] \tag{7}$$

In a similar way, we get

$$\bar{y}(\tilde{a}_\alpha) = \frac{1}{3}(2\alpha + u_{\tilde{a}}) \tag{8}$$

3.2 Centroid point of $\beta-$cut set for a TIFN

As for the non-membership function, considering that (1) the larger the confidence level β, the greater the uncertainty of the corresponding cut set contains; (2) the confidence level β splits ΔOPQ into two parts, as follows: ΔOMN and trapezoid $MPQN$, where the ΔOMN corresponds to the area of the $\beta-$cut set $\tilde{a}_\beta$ (Figure 3). Thus, we also need to compute the centroid of $\tilde{a}_\beta$.

Definition 3.3. [24] *For a TIFN* $\tilde{a} = ((a,b,c); u_{\tilde{a}}, \nu_{\tilde{a}})$, *assume that the functions* $\upsilon^L_{\tilde{a}}(x) = [b - x + v_{\tilde{a}}(x-a)]/(b-a)$ *and* $\upsilon^R_{\tilde{a}}(x) = [x - b + v_{\tilde{a}}(c-x)]/(c-b)$ *represent the lines OP and OQ in Fig. 3, respectively. Clearly, the inverse functions of* $\upsilon^L_{\tilde{a}}(x)$ *and* $\upsilon^R_{\tilde{a}}(x)$ *are* $f^L_{\tilde{a}}(y) = [b - av_{\tilde{a}} - (b-a)y]/(1-v)$ *and* $f^R_{\tilde{a}}(y) = [b - cv_{\tilde{a}} + (c-b)y]/(1-v)$. *Based on the basic theories of calculus, the centroid* $G(\tilde{a}_\beta) = (\bar{x}(\tilde{a}_\beta), \bar{y}(\tilde{a}_\beta))$ *of the triangle (OMN) can be defined as follows:*

$$\bar{x}(\tilde{a}_\beta) = \frac{\int_{v_{\tilde{a}}}^{\beta} x(g^R_{\tilde{a}}(y) - g^L_{\tilde{a}}(y))\mathrm{d}y}{\int_{v_{\tilde{a}}}^{\beta} (g^R_{\tilde{a}}(y) - g^L_{\tilde{a}}(y))\mathrm{d}y} \tag{9}$$

$$\bar{y}(\tilde{a}_\beta) = \frac{\int_{\upsilon_{\tilde{a}}}^{\beta} y(g_{\tilde{a}}^R(y) - g_{\tilde{a}}^L(y))\mathrm{d}y}{\int_{\upsilon_{\tilde{a}}}^{\beta} (g_{\tilde{a}}^R(y) - g_{\tilde{a}}^L(y))\mathrm{d}y} \tag{10}$$

where the horizontal coordinate $\bar{x}(\tilde{a}_\beta)$ of the centroid $G(\tilde{a}_\beta)$ can be viewed as the mean of $\beta-$cut set $\tilde{a}_\beta$.

Remark 3.4. *Given that $\upsilon_{\tilde{a}}^L(x) : [a,b] \to [1, \upsilon_{\tilde{a}}]$ and $\upsilon_{\tilde{a}}^R(x) : [b,c] \to [\upsilon_{\tilde{a}}, 1]$ are both continuous and strictly monotonic functions, the corresponding inverse functions exist and should also be continuous and strictly monotonic. The inverse functions of $\upsilon_{\tilde{a}}^L(x)$ and $\upsilon_{\tilde{a}}^R(x)$ can be denoted by $g_{\tilde{a}}^L(y) : [1, \upsilon_{\tilde{a}}] \to [a,b]$ and $g_{\tilde{a}}^R(y)$:$[\upsilon_{\tilde{a}}, 1] \to [b,c]$. Then, $g_{\tilde{a}}^L(y)$ and $g_{\tilde{a}}^R(y)$ are integrable on $[0, u_{\tilde{a}}]$. As $\upsilon_{\tilde{a}} \leq \beta \leq 1$. Thus, $g_{\tilde{a}}^L(y)$ and $g_{\tilde{a}}^R(y)$ are also integrable on $[\upsilon_{\tilde{a}}, \beta]$. Similarly, we have the following equations (11) and (12):*

$$\bar{x}(\tilde{a}_\beta) = \frac{1}{3}[\frac{2(1-\beta)b + (\beta - \upsilon_{\tilde{a}})a + (\beta - \upsilon_{\tilde{a}})c}{1 - \upsilon_{\tilde{a}}} + b] \tag{11}$$

$$\bar{y}(\tilde{a}_\beta) = \frac{1}{3}(2\beta + \upsilon_{\tilde{a}}) \tag{12}$$

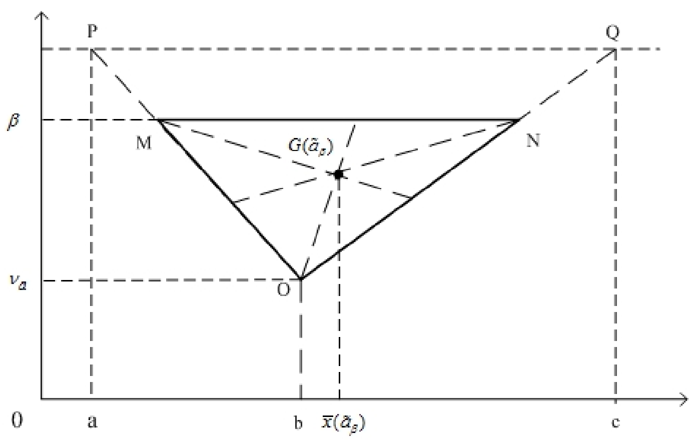

Figure 3: Centroid of $\beta-$cut set

3.3 Value indexes of membership and non-membership functions for a TIFN

Inspired by Li's method [12], this section introduces new value indexes of membership and non-membership functions for a TIFN based on the centroid.

Definition 3.5. *Let $\tilde{a}_\alpha$ and $\tilde{a}_\beta$ be the $\alpha-$cut set and $\beta-$cut set of a TIFN $\tilde{a} = ((a,b,c); \mu_{\tilde{a}}, \nu_{\tilde{a}})$. Then, the value indexes of $\mu_{\tilde{a}}(x)$ and $\upsilon_{\tilde{a}}(x)$ are defined as follows:*

$$V_\mu(\tilde{a}) = \int_0^{u_{\tilde{a}}} \bar{x}(\tilde{a}_\alpha) f(\alpha)\mathrm{d}\alpha \tag{13}$$

and

$$V_\nu(\tilde{a}) = \int_{\nu_{\tilde{a}}}^{1} \bar{x}(\tilde{a}_\beta) g(\beta)\mathrm{d}\beta \tag{14}$$

respectively, where $f(\alpha)$ is a non-decreasing weighting function satisfies: $f(\alpha) \geq 0$, $f(0) = 0$ and $\int_0^{u_{\tilde{a}}} f(\alpha)\mathrm{d}\alpha = u_{\tilde{a}}$; the $g(\beta)$ is a non-increasing weighting function and satisfies the conditions: $g(\beta) \geq 0$, $g(1) = 0$ and $\int_{\upsilon_{\tilde{a}}}^{1} g(\beta)\mathrm{d}\beta = 1 - \upsilon_{\tilde{a}}$.

Remark 3.6. *If the centroid of the full triangle ΔABC is defined as the value of $\mu_{\tilde{a}}(x)$ for a TIFN $\tilde{a}$, then we have $V_\mu(\tilde{a}) = \frac{1}{3}(a+b+c)$. $V_\mu(\tilde{a})$ ignores effect of the maximum membership degree. Likewise, for the full ΔOPQ triangle, we have $V_\nu(\tilde{a}) = \frac{1}{3}(a+b+c)$ which ignores the effect of the minimum non-membership degree.*

Remark 3.7. *The weighting functions $f(\alpha)$ and $g(\beta)$ indicate the importance of $\alpha-$cut set and $\beta-$cut set of a TIFN $\tilde{a}$, respectively. Generally, $f(\alpha)$ and $g(\beta)$ can be chosen in terms of practical application situations. Given considerable uncertainty about the values of $\mu_{\tilde{a}}(x)$ generated by $\alpha-$cut sets, $f(\alpha)$ can lessen the importance of the lower $\alpha-$cut sets by allocating different weights to elements in different $\alpha-$cut sets. Obviously, $V_\mu(\tilde{a})$ may be regarded as a mean value of the membership function, because it is a comprehensive reflection of all membership degrees. Likewise, because of considerable uncertainty about the values of $\upsilon_{\tilde{a}}(x)$ generated by $\beta-$cut sets, $g(\beta)$ can lessen the importance of the higher $\beta-$cut sets. Thus, $V_v(\tilde{a})$ may also be considered as a mean value of the non-membership function because it is a comprehensive reflection of all non-membership degrees.*

Remark 3.8. *If $f(\alpha)$ and $g(\beta)$ are respectively chosen as follows:*

$$f(\alpha)=\frac{2\alpha}{u_{\tilde{a}}}(\alpha\in[0,u_{\tilde{a}}]) \tag{15}$$

and

$$g(\beta)=\frac{2(1-\beta)}{1-v_{\tilde{a}}}(\beta\in[v_{\tilde{a}},1]) \tag{16}$$

according to equations (7), (9), (13) and (15), the value of the membership function of $\tilde{a}$ is calculated as follows:

$$V_\mu(\tilde{a})=\frac{1}{9}(a+7b+c)u_{\tilde{a}} \tag{17}$$

Further, by equations (8), (11), (14) and (16), the value of the non-membership function of $\tilde{a}$ is calculated as follows:

$$V_\nu(\tilde{a})=\frac{1}{9}(a+7b+c)(1-v_{\tilde{a}}) \tag{18}$$

according to the condition $0\le u_{\tilde{a}}+\nu_{\tilde{a}}\le 1$, it directly yields from equations (17) and (18) that $\frac{1}{9}(a+7b+c)u_{\tilde{a}}\le\frac{1}{9}(a+7b+c)(1-v_{\tilde{a}})$, i.e., $V_\mu(\tilde{a})\le V_v(\tilde{a})$. Thus, the value of $\tilde{a}=((a,b,c);u_{\tilde{a}},\nu_{\tilde{a}})$ can be expressed as an ordered pair $(V_\mu(\tilde{a}),V_v(\tilde{a}))$.

3.4 Comprehensive value index of a TIFN

Definition 3.9. *Let $\tilde{a}=((a,b,c);u_{\tilde{a}},\nu_{\tilde{a}})$ be a TIFN, a comprehensive value index of $\tilde{a}$ is defined as follows:*

$$V_\lambda(\tilde{a})=\lambda V_\mu(\tilde{a})+(1-\lambda)V_\nu(\tilde{a}) \tag{19}$$

where $\lambda\in[0,1]$ represents the decision maker's attitude $\lambda\in[0,1/2)$. means that the decision maker's attitude is negative; $\lambda\in(1/2,1]$ means that the decision maker's attitude is positive; $\lambda=1/2$ means that the decision maker's attitude maintains neutrality. Thus, the comprehensive value index may reflect the decision maker's subjectivity attitude.

Definition 3.10. *Let $\tilde{a}_1=((a_1,b_1,c_1);u_{\tilde{a}_1},\nu_{\tilde{a}_1})$ and $\tilde{a}_2=((a_2,b_2,c_2);u_{\tilde{a}_2},\nu_{\tilde{a}_2})$ be two TIFNs. Based on the comprehensive value index, for $\lambda\ge 0$, we have:*
(i) If $V_\lambda(\tilde{a}_1)<V_\lambda(\tilde{a}_2)$, then $\tilde{a}_1<\tilde{a}_2$;
(ii) If $V_\lambda(\tilde{a}_1)=V_\lambda(\tilde{a}_2)$, then $\tilde{a}_1=\tilde{a}_2$;
(iii) If $V_\lambda(\tilde{a}_1)>V_\lambda(\tilde{a}_2)$, then $\tilde{a}_1>\tilde{a}_2$.

For TIFN $\tilde{a}=((a,b,c);u_{\tilde{a}},\nu_{\tilde{a}})$, Li [12] proposed a ratio between the ambiguity index and value index as follows:

$$R(\tilde{a},\lambda)=\frac{V_\mu(\tilde{a})+\lambda(V_v(\tilde{a})-V_\mu(\tilde{a}))}{1+A_v(\tilde{a})-\lambda(A_v(\tilde{a})-A_\mu(\tilde{a}))}$$

where

$$V_\mu(\tilde{a})=\int_0^{u_{\tilde{a}}}\tfrac{1}{2}(L_{\tilde{a}}(\alpha)+R_{\tilde{a}}(\alpha))f(\alpha)\mathrm{d}\alpha \text{ and } V_v(\tilde{a})=\int_{v_{\tilde{a}}}^{1}\tfrac{1}{2}(L_{\tilde{a}}(\alpha)+R_{\tilde{a}}(\alpha))g(\beta)\mathrm{d}\beta$$

$$A_\mu(\tilde{a})=\int_0^{u_{\tilde{a}}}(R_{\tilde{a}}(\alpha)-L_{\tilde{a}}(\alpha))f(\alpha)\mathrm{d}\alpha \text{ and } A_v(\tilde{a})=\int_{v_{\tilde{a}}}^{1}(R_{\tilde{a}}(\alpha)-L_{\tilde{a}}(\alpha))g(\beta)\mathrm{d}\beta$$

Nan et al. [18] defined the average indexes of $\mu_{\tilde{a}}(x)$ and $\upsilon_{\tilde{a}}(x)$ as follows:

$$s_\mu(\tilde{a}) = \int_0^{w_{\tilde{a}}} \tfrac{1}{2}(L_{\tilde{a}}(\alpha) + R_{\tilde{a}}(\alpha))\mathrm{d}\alpha = \tfrac{1}{4}w_{\tilde{a}}(\underline{a} + 2a + \bar{a})$$

$$s_v(\tilde{a}) = \int_{u_{\tilde{a}}}^{1} \tfrac{1}{2}(L_{\tilde{a}}(\alpha) + R_{\tilde{a}}(\alpha))\mathrm{d}\beta = \tfrac{1}{4}(1 - u_{\tilde{a}})(\underline{a} + 2a + \bar{a})$$

Li et al. [11] developed a lexicographic ranking by the following value-index and ambiguity-index:

$$V_\mu(\tilde{a}) = \frac{1}{6}(a + 4b + c)u_{\tilde{a}}^2 \ \text{ and } \ V_\upsilon(\tilde{a}) = \frac{1}{6}(a + 4b + c)(1 - v_{\tilde{a}})^2$$

$$A_\mu(\tilde{a}) = \frac{1}{6}(c - a)u_{\tilde{a}}^2 \ \text{ and } \ V_\upsilon(\tilde{a}) = \frac{1}{6}(c - a)(1 - v_{\tilde{a}})^2$$

Example 3.11. *Consider two TIFNs* $\tilde{a}_1 = ((0.3, 0.4, 0.9); 0.9, 0.1)$ *and* $\tilde{a}_2 = ((0.3, 0.5, 0.6); 0.9, 0.1)$.

According to Definition 2.1, $\tilde{a}_1$ means that the most possible value is 0.4 with the membership degree 0.9 and the non-membership degree 0.1, while the most possible value of $\tilde{a}_2$ is 0.5 with the same membership degree and non-membership. For these two TIFNs, their maximum membership degree and minimum non-membership degree are equal, and $0.5 > 0.4$, thus, $\tilde{a}_1 < \tilde{a}_2$ intuitively.

By using Li's method [12], we have $R(\tilde{a}_1, \lambda) = 0.356$ and $R(\tilde{a}_2, \lambda) = 0.399$. Thus, their rank is $\tilde{a}_1 < \tilde{a}_2$. By our method, we get $V_\lambda(\tilde{a}_1) = 0.44$ and $V_\lambda(\tilde{a}_2) = 0.44$, so the rank is also $\tilde{a}_1 < \tilde{a}_2$, which shows the validation of our method.

However, by Nan et al.'s method [18], $s_\mu(\tilde{a}_1) = s_\upsilon(\tilde{a}_1) = 0.45$ and $s_\mu(\tilde{a}_2) = s_\upsilon(\tilde{a}_2) = 0.427$. Thus, their rank is $\tilde{a}_1 > \tilde{a}_2$, which is contrary to the correct rank: $\tilde{a}_1 < \tilde{a}_2$ obtained by using our method and Li's method [12]. This is a typical example, in which Nan et al.'s method [18] leads to counter-intuitive results.

Example 3.12. *Consider two TIFNs* $\tilde{a}_1 = ((0.3, 0.4, 0.7); 0.7, 0.2)$ *and* $\tilde{a}_2 = ((0.1, 0.5, 0.7); 0.7, 0.2)$.

Intuitively, the ranking order is not equal. However, by Nan et al.'s method [18], $s_\mu(\tilde{a}_1) = 0.315$, $s_\upsilon(\tilde{a}_1) = 0.36$ and $s_\mu(\tilde{a}_2) = 0.315$, $s_\upsilon(\tilde{a}_2) = 0.36$, which yields their rank is $\tilde{a}_1 = \tilde{a}_2$. Namely, Nan et al.'s method [18] cannot discriminate these fuzzy numbers. Based on our method, we get $V_\lambda(\tilde{a}_1) = 0.295\lambda + 0.338(1-\lambda)$ and $V_\lambda(\tilde{a}_2) = 0.334\lambda + 0.382(1-\lambda)$, thus the ranking order is $V_\lambda(\tilde{a}_1) < V_\lambda(\tilde{a}_2)$. According to Definition 3.8, it can be derived that $\tilde{a}_1 < \tilde{a}_2$. In fact, these two TIFNs can be intuitively ranked according to Definition 2.1, since the most possible value of $\tilde{a}_1$ is 0.4, and the most possible value of $\tilde{a}_2$ is 0.5. Thus, $\tilde{a}_2$ is larger than $\tilde{a}_1$ intuitively.

Example 3.13. *Consider two TIFNs* $\tilde{a}_1 = ((0.3, 0.45, 0.9); 0.9, 0.1)$ *and* $\tilde{a}_2 = ((0.2, 0.5, 0.8); 0.9, 0.1)$.

Intuitively, the ranking order is not equal. However, by Li et al.'s method [11], we have $V_\mu(\tilde{a}_1) = V_\upsilon(\tilde{a}_1) = 0.405$, $V_\mu(\tilde{a}_2) = V_\upsilon(\tilde{a}_2) = 0.405$, $A_\lambda(\tilde{a}_1) = 0.162$ and $A_\lambda(\tilde{a}_2) = 0.162$, therefore, the ranking order is $\tilde{a}_1 = \tilde{a}_2$. That is to say, Li et al.'s method [11] cannot distinguish these fuzzy numbers. According to Definition 2.1, the ranking order is $\tilde{a}_1 < \tilde{a}_2$ intuitively. By our method, we get the ranking order is also $\tilde{a}_1 < \tilde{a}_2$, which shows the validation of our method.

The above examples indicate that our method can not only overcome the shortcomings of the Nan et al.'s method [18] and Li et al.'s method [11], but also is simpler than Li's method [12].

4 Two-person zero-sum game model for MADM with TIFNs

In this section, a two-person zero-sum game model will be proposed for MADM problems with TIFNs. The optimal strategies can be obtained by solving a pair of intuitionistic fuzzy mathematical programming models. The decision process of the proposed method can be summarized in detail as Algorithm 4.5.

4.1 Representation for MADM problems with TIFNs

Let $S_1, S_2, \ldots, S_m$ be m alternatives that include an attribute set $A = \{a_1, a_2, \ldots, a_n\}$, $w = \{w_1, w_2, \cdots, w_n\}$ is the weight vector of the attributes $a_j (j = 1, 2, \cdots, n)$, where $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. Let the value of rating be a TIFN, thus, we can formalize the MADM problem into the matrix $\tilde{A} = (\tilde{a}'_{ij})_{m\times n}$, where $\tilde{a}'_{ij} = ((a'_{ij}, b'_{ij}, c'_{ij}); u_{\tilde{a}'_{ij}}, \nu_{\tilde{a}'_{ij}})$ denotes the rating within an attribute a_j of an alternative S_i.

To eliminate the influence of cost attribute C and benefit attribute B, we use the following normalized transformation:

$$\tilde{r}_{ij} = ((\frac{a'_{ij}}{c_j^+}, \frac{b'_{ij}}{c_j^+}, \frac{c'_{ij}}{c_j^+}); u_{\tilde{a}_{ij}}, v_{\tilde{a}_{ij}})(i = 1, 2, \cdots, m; j \in B), \tag{20}$$

$$\tilde{r}_{ij} = ((\frac{a_j^-}{c'_{ij}}, \frac{a_j^-}{b'_{ij}}, \frac{a_j^-}{a'_{ij}}); u_{\tilde{a}_{ij}}, v_{\tilde{a}_{ij}})(i = 1, 2, \cdots, m; j \in C) \tag{21}$$

where $c_j^+ = \max\{c'_{ij}|i = 1, 2, \cdots, m\}$, $a_j^- = \min\{a'_{ij}|i = 1, 2, \cdots, m\}$, $u_{\tilde{a}_{ij}} = u_{\tilde{a}'_{ij}}$, $v_{\tilde{a}_{ij}} = \nu_{\tilde{a}'_{ij}}$. Therefore, the matrix $\tilde{A} = (\tilde{a}'_{ij})_{m\times n}$ can be normalized into the matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$, where $\tilde{r}_{ij} = ((a_{ij}, b_{ij}, c_{ij}); u_{\tilde{a}_{ij}}, \nu_{\tilde{a}_{ij}})$.

4.2 Two-person zero-sum game with TIFN payoffs

Let $S = \{S_1, S_2, \cdots, S_m\}$ and $A = \{A_1, A_2, \cdots, A_m\}$ be the sets of all pure strategies available for player I (decision maker) and player II (Nature) in a matrix game, respectively, where the payoffs are TIFNs, the expectation of the decision maker is to maximize the expected payoff and that of Nature is to minimize its loss. We denote this TPZS game with TIFN payoffs. Denote this fuzzy matrix game by $\tilde{\Gamma} = (I, II, S^1, S^2, \tilde{R})$, where $S^1 = \{X = (x_1, x_2, \cdots, x_m)| \sum_{i=1}^{m} x_i = 1, x_i \geq 0\}$ is the set of mixed strategies of the decision maker, $S^2 = \{Y = (y_1, y_2, \cdots, y_n)| \sum_{j=1}^{n} y_j = 1, y_j \geq 0\}$ is the set of mixed strategies of Nature, $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$ represents the payoff matrix whose entries are TIFNs. Each element $\tilde{r}_{ij} = ((a_{ij}, b_{ij}, c_{ij}); u_{\tilde{a}_{ij}}, \nu_{\tilde{a}_{ij}})$ informs us about the knowledge that decision maker and Nature indicate on their own payoffs provided that decision maker (the maximizer) chooses pure strategy $S_i \in S$ and Nature (the minimizer) chooses pure strategy. For simplicity, these TIFNs will be assumed positive TIFNs.

Inspired by Nan et al. [18], we give the following definitions.

Definition 4.1. *If X and Y are the mixed strategies given by the decision maker and Nature in TPZS game, respectively, then the expected payoff of the decision maker and Nature are defined as follows:*

$$\tilde{E}(\tilde{R}) = X^{\mathrm{T}}\tilde{R}Y = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j \tilde{r}_{ij}$$

According to Definition 2.2, the expectation payoff for decision maker and Nature can be computed as follows:

$$\tilde{E}(\tilde{R}) = ((\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j, \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}x_i y_j, \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_i y_j); \min_j u_{\tilde{a}_{ij}}, \max_j v_{\tilde{a}_{ij}})$$

which is a TIFN.

Definition 4.2. *Let $\tilde{\upsilon} = ((a_\upsilon, b_\upsilon, c_\upsilon); u_{\tilde{\upsilon}}, v_{\tilde{\upsilon}})$ and $\tilde{\omega} = ((a_\omega, b_\omega, c_\omega); u_{\tilde{\omega}}, v_{\tilde{\omega}})$ be two TIFNs. We assume that $X^* \in S^1$ and $Y^* \in S^2$ exist. If the following inequalities are satisfied*

$$(X^*)^{\mathrm{T}}\tilde{R}Y \geq \tilde{\upsilon}\forall Y \in S^2, \quad \text{and} \quad X^{\mathrm{T}}\tilde{R}Y^* \leq \tilde{\omega}\forall X \in S^1,$$

then $(X^*, Y^*, \tilde{\upsilon}, \tilde{\omega})$ is regarded as a reasonable solution of the IF matrix game, where $\tilde{\upsilon}$ and $\tilde{\omega}$ are values of the reasonable game for decision maker and Nature, respectively.

Definition 4.3. *Let V and M be the sets of all $\tilde{\upsilon}$ and $\tilde{\omega}$, respectively. We assume that $\tilde{\upsilon}^* \in V$ and $\tilde{\omega}^* \in W$ exist. If the following inequalities are satisfied,*

$$\tilde{\upsilon}^* \geq \tilde{\upsilon}, \forall\tilde{\upsilon} \in V \quad \text{and} \quad \tilde{\omega}^* \leq \tilde{\omega}, \forall\tilde{\omega} \in W,$$

then $(X^*, Y^*, \tilde{\upsilon}^*, \tilde{\omega}^*)$ is a solution of $\tilde{\Gamma} = (I, II, S^1, S^2, \tilde{R})$, where $\tilde{\upsilon}^*$ and $\tilde{\omega}^*$ indicate decision maker's gain floor and Nature's loss ceiling, as denoted by $\tilde{\upsilon}^* = ((a_{\upsilon^*}, b_{\upsilon^*}, c_{\upsilon^*}); u_{\tilde{\upsilon}^*}, v_{\tilde{\upsilon}^*})$ and $\tilde{\omega}^* = ((a_{\omega^*}, b_{\omega^*}, c_{\omega^*}); u_{\tilde{\omega}^*}, v_{\tilde{\omega}^*})$, respectively; X^* and Y^* are called the optimal strategies.

4.3 Computation of the solution

According to Definitions 4.2 and 4.3, the optimal strategies (X^*, Y^*) can be solved by a pair of intuitionistic fuzzy mathematical programming models given by

$$\max\tilde{\upsilon} \quad \text{s.t} \quad \begin{cases} \sum_{i=1}^{m} \tilde{r}_{ij}x_i y_j \geq \upsilon, Y \in S^2 \\ \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, 2, \cdots, m \end{cases}$$

(22)

and

$$\min \tilde{\omega} \quad \text{s.t} \quad \begin{cases} \sum_{j=1}^{n} \tilde{r}_{ij} x_i y_j \le \tilde{\omega}, X \in S^1 \\ \sum_{j=1}^{n} y_j = 1, y_j \ge 0, j = 1, 2, \cdots, n \end{cases} \tag{23}$$

respectively.

By using Definition 2.2, the problems (22) and (23) can be transformed into

$$\max \tilde{\upsilon} \quad \text{s.t} \quad \begin{cases} \sum_{i=1}^{m} \tilde{r}_{ij} x_i \ge \upsilon, j = 1, 2, \cdots, n \\ \sum_{i=1}^{m} x_i = 1, x_i \ge 0, i = 1, 2, \cdots, m \end{cases} \tag{24}$$

and

$$\min \tilde{\omega} \quad \text{s.t} \quad \begin{cases} \sum_{j=1}^{n} \tilde{r}_{ij} y_j \le \tilde{\omega}, i = 1, 2, \cdots, m \\ \sum_{j=1}^{n} y_j = 1, y_j \ge 0, \text{j} = 1, 2, \cdots, n \end{cases} \tag{25}$$

respectively.

By using Definition 3.10, the problems (24) and (25) can be transformed into a pair of crisp linear programming problems as follows:

$$\max\{V_\lambda(\tilde{\upsilon})\} \quad \text{s.t} \quad \begin{cases} V_\lambda(\sum_{i=1}^{m} \tilde{r}_{ij}) x_i \ge V_\lambda(\tilde{\upsilon}), j = 1, 2, \cdots, n \\ \sum_{i=1}^{m} x_i = 1, x_i \ge 0, i = 1, 2, \cdots, m \end{cases} \tag{26}$$

and

$$\min\{V_\lambda(\tilde{\omega})\} \quad \text{s.t} \quad \begin{cases} V_\lambda(\sum_{j=1}^{n} \tilde{r}_{ij}) y_j \le V_\lambda(\tilde{\omega}), i = 1, 2, \cdots, m \\ \sum_{j=1}^{n} y_j = 1, y_j \ge 0, \text{j} = 1, 2, \cdots, n \end{cases} \tag{27}$$

respectively.

By using equations (17) and (18), the problems (26) and (27) can be rewritten to as follows:

$$\max\{\tfrac{1}{9}(a_\upsilon + 7b_\upsilon + c_\upsilon)[\lambda u_{\tilde{\upsilon}} + (1-\lambda)(1 - v_{\tilde{\upsilon}})]\} \quad \text{s.t} \quad \begin{cases} \sum_{i=1}^{m} \frac{a_{ij} + 7b_{ij} + c_{ij}}{9} [\lambda \bigwedge_i \{u_{\tilde{a}_{ij}}\} + (1-\lambda)(1 - \bigvee_i \{v_{\tilde{a}_{ij}}\}] x_i \\ \ge \frac{1}{9}(a_\upsilon + 7b_\upsilon + c_\upsilon)[\lambda u_{\tilde{\upsilon}} + (1-\lambda)(1 - v_{\tilde{\upsilon}})], j = 1, 2, \cdots, n \\ a_\upsilon < b_\upsilon \\ b_\upsilon < c_\upsilon \\ \sum_{i=1}^{m} x_i = 1, x_i \ge 0, i = 1, 2, \cdots, m \end{cases} \tag{28}$$

and

$$\min\{\tfrac{1}{9}(a_\omega + 7b_\omega + c_\omega)[\lambda u_{\tilde{\omega}} + (1-\lambda)(1 - v_{\tilde{\omega}})]\} \quad \text{s.t} \quad \begin{cases} \sum_{j=1}^{n} \frac{a_{ij} + 7b_{ij} + c_{ij}}{9} [\lambda \bigwedge_j \{u_{\tilde{a}_{ij}}\} + (1-\lambda)(1 - \bigvee_j \{v_{\tilde{a}_{ij}}\}] y_j \\ \le \frac{1}{9}(a_\omega + 7b_\omega + c_\omega)[\lambda u_{\tilde{\omega}} + (1-\lambda)(1 - v_{\tilde{\omega}})], i = 1, 2, \cdots, m \\ a_\omega < b_\omega \\ b_\omega < c_\omega \\ \sum_{j=1}^{n} y_j = 1, y_j \ge 0, \text{j} = 1, 2, \cdots, n \end{cases} \tag{29}$$

respectively, where $\underset{i}{\wedge}\{u_{\tilde{a}_{ij}}\} = \min\{u_{\tilde{a}_{1j}}, u_{\tilde{a}_{2j}}, \cdots, u_{\tilde{a}_{mj}}\}$, $\underset{i}{\vee}\{v_{\tilde{a}_{ij}}\} = \max\{v_{\tilde{a}_{1j}}, v_{\tilde{a}_{2j}}, \cdots, v_{\tilde{a}_{mj}}\}$, $\underset{j}{\wedge}\{u_{\tilde{a}_{ij}}\} = \min\{u_{\tilde{a}_{i1}}, u_{\tilde{a}_{i2}}, \cdots, u_{\tilde{a}_{in}}\}$, $\underset{j}{\vee}\{v_{\tilde{a}_{ij}}\} = \max\{v_{\tilde{a}_{i1}}, v_{\tilde{a}_{i2}}, \cdots, v_{\tilde{a}_{in}}\}$; $\lambda \in [0,1]$ is a parameter that represents the decision makers preference information; $a_\upsilon, b_\upsilon, c_\upsilon, a_\omega, b_\omega, c_\omega, x_i (i = 1, 2, \cdots, m)$ and $y_j (j = 1, 2, \cdots, m)$ are decision variables.

Given that computing the variables $a_\upsilon, b_\upsilon, c_\upsilon, a_\omega, b_\omega, c_\omega$ is not easy, the problems (28) and (29) can be solved by the method of variable transformation. Let $v_1 = \frac{1}{9}(a_\upsilon + 7b_\upsilon + c_\upsilon)[\lambda u_{\tilde{\upsilon}} + (1-\lambda)(1 - v_{\tilde{\upsilon}})]$ and $v_2 = \frac{1}{9}(a_\omega + 7b_\omega + c_\omega)[\lambda u_{\tilde{\omega}} + (1-\lambda)(1 - v_{\tilde{\omega}})]$, then, the problems (26) and (27) can be rewritten as follows:

$$\max\{v_1\} \quad \text{s.t} \quad \begin{cases} \sum_{i=1}^{m} \frac{a_{ij}+7b_{ij}+c_{ij}}{9}[\lambda \underset{i}{\wedge}\{u_{\tilde{a}_{ij}}\} + (1-\lambda)(1 - \underset{i}{\vee}\{v_{\tilde{a}_{ij}}\})]x_i \geq v_1, j = 1, 2, \cdots, n \\ \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, 2, \cdots, m \end{cases} \tag{30}$$

and

$$\min\{v_2\} \quad \text{s.t} \quad \begin{cases} \sum_{j=1}^{n} \frac{a_{ij}+7b_{ij}+c_{ij}}{9}[\lambda \underset{j}{\wedge}\{u_{\tilde{a}_{ij}}\} + (1-\lambda)(1 - \underset{j}{\vee}\{v_{\tilde{a}_{ij}}\})]y_j \leq v_2, i = 1, 2, \cdots, m \\ \sum_{j=1}^{n} y_j = 1, y_j \geq 0, \text{j} = 1, 2, \cdots, n \end{cases} \tag{31}$$

respectively, where $v_1, v_2, x_i, (i = 1, 2, \cdots, m)$ and $y_j (j = 1, 2, \cdots, n)$ are decision variables; $\lambda \in [0,1]$ is a parameter.

For the specific weight $\lambda \in [0,1]$, the optimal solution $(X^*(\lambda), v_1^*(\lambda))$ of the problem (30) and $(Y^*(\lambda), v_2^*(\lambda))$ of the problem (31) are obtained by Lingo software. Obviously, $(X^*(\lambda), v_1^*(\lambda))$ and $(Y^*(\lambda), v_2^*(\lambda))$ are also optimal solutions of the problems (26) and (27), respectively.

Definition 4.4. *Let (X^*, Y^*) be an equilibrium strategy for the fuzzy game $\tilde{\Gamma}$. The expected score of alternative S_i is defined as follows:*

$$z_i = x_i^* \cdot \sum_{j=1}^{n} [\tilde{r}_{ij} y_j^*] \tag{32}$$

Apparently, the expected score z_i is a TIFN. The higher the z_i, the better the alternative S_i. Thus, the expected scores of alternatives can be used to rank the alternatives.

4.4 Decision algorithm and complexity

To illustrate the decision process of the proposed method, Figure 4 shows the framework of MADM problems with TIFNs. Meanwhile, the decision procedure can be summarized in detail as Algorithm 4.5.

Algorithm 4.5. Algorithm of TPZS decision with TIFNs

Input:
Input the TIFN decision matrix $\tilde{A} = (\tilde{a}'_{ij})_{m\times n}$;
The specific λ;
The known attribute weight information Λ;
Output:
Ranking order of the alternatives
1: Compute the normalized decision matrix $\tilde{R}$ by equations (20) and (21);
2: Construct the two-person fuzzy zero-sum game $\tilde{\Gamma} = (I, II, S^1, S^2, \tilde{R})$;
3: Converted matrix game $\tilde{\Gamma}$ with TIFNs into the linear programming problems (30) and (31);
4: Solve problems (30) and (31) to get the optimal mixed strategies (X^*, Y^*);
5: **for** each alternative $S_i \in S$ **do**
6: Use equation (32) to compute the expected score z_i of the alternatives S_i;
7: Calculate the value of the membership function of z_i;
8: Compute the value of the non-membership function of z_i;
9: Derive the comprehensive value index $V_\lambda(z_i)$ of z_i;
10: **end for**
11: Rank the $V_\lambda(z_i), i = 1, 2, \cdots, m$;
12: **return** the ranking order of the alternatives;

Now the time complexity of the algorithm is analyzed as follows. Let n be the number of alternatives and m is the number of attributes in the MADM problem. In step 1, the normalized decision matrix $\tilde{R}$ is calculated by $O(mn)$ operations. In steps 2-3, after $O(n)+O(m)$ operations, the linear programming problems (30) and (31) can be constructed. In step 4, we use the simplex method to solve the problems (30) and (31), and the worst-case complexity is exponential time $O(2^n)+O(2^m)$ [23]. In steps 5-10, $O(mn)$ operations are needed to complete the calculation of the comprehensive value index of z_i. In step 11, we take $O(m^2)$ operations to rank the $V_\lambda(z_i)$ by the bubble sort algorithm. Obviously, the algorithm requires the most operations of exponential time. In most real decision situations, a few alternatives and attributes are found. Thus, the proposed decision algorithm is suitable for MADM problems.

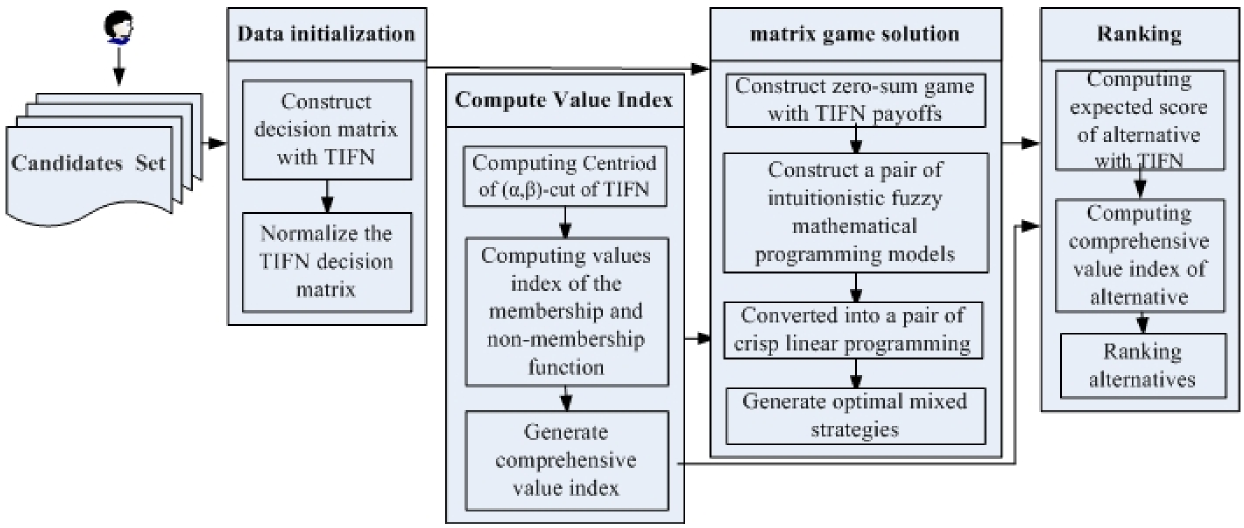

Figure 4: Framework of MADM problems with TIFNs

5 Illustrative example and comparative analyses

In this section, an example on video monitoring system evaluation is given to demonstrate the applicability of the proposed decision method. In addition, comparative analyses with other methods are made to demonstrate the advantages of the proposed method.

5.1 Video monitoring system selection example

Assuming that a school's campus security emergency work is rather weak, it is urgent to adopt a comprehensive and multi-channel, especially technical means to enhance campus safety and prevention. With the construction of colleges and universities, the modern video monitoring system (VMS) is not only the demand of campus security, but also an important part of the smart campus. In order to meet the requirements of the construction of the smart campus, the school plans to purchase a number of monitoring systems from existing VMS providers. After preliminary screening, four candidate VMS providers S_1, S_2, S_3 and S_4 remain to be further evaluated. Based on detailed provider ratings, the decision-maker panel evaluates the four candidate VMS providers according to the five factors, including product mobility (a_1), false positive rate (a_2),real-time (a_3), image quality (a_4) and security (a_5). The assessments of the candidate VMS providers with the attributes can be expressed by TIFNs, as shown in Table 1.

For example, the first TIFN ((3,4,9); 0.6,0.1) in Table 1 implies that the most possible rating of VMS provider S_1 on attribute a_1 is 4 with the maximum membership degree 0.6 and the minimum non-membership degree 0.1, the pessimistic value is 3 and the optimistic value is 9. We can get the rating TIFN ((3,4,9); 0.6,0.1) by using the following self-designed pre-processing:

(1) Assume that the ratings on attribute a_1 are TIFNs in ten-mark system (the grades from 0 up to 10 are used) to score.

(2) Decision maker provides the pessimistic value a, the optimistic value c and the most possible value b for each attribute value.

(3) For the most possible value, decision maker provides the maximum membership degree $u_{\tilde{a}}$ and the minimum non-membership degree $v_{\tilde{a}}$ in terms of decision makers experience or using statistical methods. Therefore, TIFN is generated.

To weight the attributes, the decision maker gives some information according to his experience, knowledge and judgment. It is more practical for attribute weights to take a pair-wise comparison [33]. For instance, image quality (a_4) is slightly more important than real-time (a_3). Here, the known attribute weight information given by the DM is as $\Lambda = \{w_1 + w_2 < 0.4, 2w_2 > w_1, w_3 > 0.05, w_4 > 2w_3, w_5 > 0.05\}$

Candidates	Attributes				
	a1	a2	a3	a4	a5
S1	((3,4,9); 0.6,0.1)	((3,5,7); 0.5,0.4)	((5,7,8); 0.5,0.2)	((6,8,9); 0.7,0.3)	((4,5,6); 0.5,0.3)
S2	((4,5,10); 0.5,0.2)	((5,6,7); 0.4,0.2)	((4,5,6); 0.5,0.1)	((6,7,8); 0.6,0.2)	((3,4,10); 0.6,0.4)
S3	((2,4,5); 0.5,0.3)	((6,8,10); 0.7,0.1)	((4,6,8); 0.5,0.1)	((4,6,10); 0.4,0.4)	((4,5,6); 0.6,0.2)
S4	((3,4,8); 0.6,0.1)	((4,5,6); 0.5,0.2)	((6,8,10); 0.4,0.3)	((4,6,8); 0.6,0.2)	((5,6,7); 0.6,0.3)

Table 1. TIFN decision matrix

The procedure for the VMS providers selection is elaborated below.

Step 1: Used the equations (20) and (21), the normalized decision matrix $\tilde{R}$ with TIFNs is yielded and listed in Table 2.

Candi dates	Attributes				
	a1	a2	a3	a4	a5
S1	((0.3,0.4,0.9) ;0.6,0.1)	((0.3,0.5,0.7) ;0.5,0.4)	((0.5,0.7,0.8) ;0.5,0.2)	((0.6,0.8,0.9) ;0.7,0.3)	((0.4,0.5,0.6) ;0.5,0.3)
S2	((0.4,0.5,1.0) ;0.5,0.2)	((0.5,0.6,0.7) ;0.4.0.2)	((0.4,0.5,0.6) ;0.5,0.1)	((0.6,0.7,0.8) ;0.6,0.2)	((0.3,0.4,1.0) ;0.6,0.4)
S3	((0.2,0.4,0.5) ;0.5,0.3)	((0.6,0.8,1.0) ;0.7,0.1)	((0.4,0.6,0.8) ;0.5,0.1)	((0.4,0.6,1.0) ;0.4,0.4)	((0.4,0.5,0.6) ;0.6,0.2)
S4	((0.3,0.4,0.8) ;0.6,0.1)	((0.4,0.5,0.6) ;0.5,0.2)	((0.6,0.8,1.0) ;0.4,0.3)	((0.4,0.6,0.8) ;0.6,0.2)	((0.5,0.6,0.7) ;0.6,0.3)

Table 2. Normalized TIFN decision matrix

Step 2: Construct the TPZS game $\tilde{\Gamma} = (I, II, S^1, S^2, \tilde{R})$, where $S^1 = \{X = (x_1, x_2, \cdots, x_m) | \sum_{i=1}^{m} x_i = 1, x_i \geq 0\}$ is the set of mixed strategies of the decision maker, $S^2 = \{Y = (y_1, y_2, \cdots, y_n) | \sum_{j=1}^{n} y_j = 1, y_j \geq 0\}$ is the set of mixed strategies of the Nature, $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$ represents the payoff matrix.

Step 3: According to the problems (30) and (31), a pair of linear programming model are obtained as follows:

$$\max\{v_1\} \quad \text{s.t} \quad \begin{cases} (0.444x_1 + 0.544x_2 + 0.389x_3 + 0.433x_4)[0.5\lambda + 0.7(1-\lambda)] \geq v_1, \\ (0.5x_1 + 0.6x_2 + 0.8x_3 + 0.5x_4)[0.4\lambda + 0.6(1-\lambda)] \geq v_1, \\ (0.689x_1 + 0.5x_2 + 0.6x_3 + 0.8x_4)[0.4\lambda + 0.7(1-\lambda)] \geq v_1, \\ (0.789x_1 + 0.7x_2 + 0.622x_3 + 0.6x_4)[0.4\lambda + 0.6(1-\lambda)] \geq v_1, \\ (0.5x_1 + 0.455x_2 + 0.5x_3 + 0.6x_4)[0.5\lambda + 0.6(1-\lambda)] \geq v_1, \\ x_1 + x_2 + x_3 + x_4 = 1, \\ x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0, \end{cases} \tag{33}$$

$$\min\{v_2\} \quad \text{s.t} \quad \begin{cases} (0.444y_1 + 0.5y_2 + 0.689y_3 + 0.789y_4 + 0.5y_5)[0.5\lambda + 0.6(1-\lambda)] \leq v_2, \\ (0.544y_1 + 0.6y_2 + 0.5y_3 + 0.7y_4 + 0.455y_5)[0.4\lambda + 0.6(1-\lambda)] \leq v_2, \\ (0.389y_1 + 0.8y_2 + 0.6y_3 + 0.622y_4 + 0.5y_5)[0.4\lambda + 0.6(1-\lambda)] \leq v_2, \\ (0.433y_1 + 0.5y_2 + 0.8y_3 + 0.6y_4 + 0.6y_5)[0.4\lambda + 0.7(1-\lambda)] \leq v_2, \\ y_1 + y_2 + y_3 + y_4 + y_5 = 1, \\ y_1 + y_2 < 0.4, 2y_2 > y_1, y_3 > 0.05, y_4 > 2y_3, y_5 > 0.05. \end{cases} \tag{34}$$

respectively.

Step 4: Assume that the risk preference parameter is $\lambda = 0.5$. By solving the problems (33) and (34), we can derive optimal strategy (X^*, Y^*), where $Y^* = (0.2667, 0.1333, 0.05, 0.1992, 0.3508)$ and $X^* = (0, 0.46, 0.10, 0.44)$.

Step 5-10: According to equation (32), the expected score of the alternatives are respectively obtained as follows:

$$z_1 = ((0,0,0); 0.5, 0.4), z_2 = ((0.144, 0.174, 0.254); 0.4, 0.4)$$

$$z_3 = ((0.023, 0.036, 0.050); 0.4, 0.4), z_4 = ((0.107, 0.146, 0.221); 0.4, 0.3)$$

Then, the comprehensive value indexes of the expected scores are computed by equations (17-19) as follows:

$$V_{0.5}(z_1) = 0, V_{0.5}(z_2) = 0.089, V_{0.5}(z_3) = 0.018, V_{0.5}(z_4) = 0.083$$

Step 11-12: Thus, $V_{0.5}(z_2) > V_{0.5}(z_4) > V_{0.5}(z_3) > V_{0.5}(z_1)$. According to Definition 10, the ranking order of expected scores is $z_2 > z_4 > z_3 > z_1$.

Based on the above steps, the ranking orders with different decision maker's preference information are obtained as in Table 3.

Table 3 indicates that the rank of candidate VMS providers may be different for diverse values of decision maker's preference parameter λ. When the decision maker is pessimistic (i.e., $0 \le \lambda < 0.5$), the ranking order of VMS providers is $S_4 \succ S_2 \succ S_3 \succ S_1$ (symbol "$\succ$" means "is preferred to"), the provider S_4 is the best VMS provider. When the decision maker is neutral (i.e. $\lambda = 0.5$), the best VMS provider is still S_4. However, when the decision maker is optimistic (i.e., $0.5 < \lambda \le 1$), the best VMS provider changes from S_4 to S_2. Obviously, the proposed method can make decisions according to the decision maker's subjective preferences. Thus, it is reasonable and necessary to take the decision maker's subjective preferences into consideration during the evaluation process.

λ	Evaluated score of alternatives				Ranking results
	S_1	S_2	S_3	S_4	
$\lambda = 0$	0	0.116	0.014	0.215	$S_4 \succ S_2 \succ S_3 \succ S_1$
$\lambda = 0.1$	0	0.129	0.020	0.213	$S_4 \succ S_2 \succ S_3 \succ S_1$
$\lambda = 0.2$	0	0.121	0.087	0.154	$S_4 \succ S_2 \succ S_3 \succ S_1$
$\lambda = 0.3$	0	0.107	0.018	0.149	$S_4 \succ S_2 \succ S_3 \succ S_1$
$\lambda = 0.4$	0	0.099	0.018	0.106	$S_4 \succ S_2 \succ S_3 \succ S_1$
$\lambda = 0.5$	0	0.089	0.018	0.083	$S_2 \succ S_4 \succ S_3 \succ S_1$
$\lambda = 0.6$	0	0.080	0.017	0.062	$S_2 \succ S_4 \succ S_3 \succ S_1$
$\lambda = 0.7$	0	0.077	0.018	0.049	$S_2 \succ S_4 \succ S_3 \succ S_1$
$\lambda = 0.8$	0	0.077	0.019	0.040	$S_2 \succ S_4 \succ S_3 \succ S_1$
$\lambda = 0.9$	0	0.081	0.021	0.031	$S_2 \succ S_4 \succ S_3 \succ S_1$
$\lambda = 1$	0	0.068	0.021	0.033	$S_2 \succ S_4 \succ S_3 \succ S_1$

Table 3. Fuzzy score of each alternative and the ranking results

5.2 Comparative analysis with Li et al.'s method

In this subsection, the advantages of the proposed method can be shown by using Li et al.'s method [11] to solve the example in Section 4.1.

In Li et al.'s method [11], the attribute weights are known as $w = (0.14, 0.3, 0.12, 0.3, 0.14)^{\mathrm{T}}$ in advance. The values index and ambiguity index of alternatives can be obtained as follows:

$$V_\lambda(S_1) = 0.151\lambda + 0.217(1-\lambda), V_\lambda(S_2) = 0.096\lambda + 0.215(1-\lambda)$$

$$V_\lambda(S_3) = 0.100\lambda + 0.225(1-\lambda), V_\lambda(S_4) = 0.092\lambda + 0.281(1-\lambda)$$

Hence, based on Li et al.'s method [33], the ranking results of the four VMS providers are $S_4 \succ S_3 \succ S_1 \succ S_2$ for $\lambda \in [0, 0.136]$, $S_4 \succ S_1 \succ S_3 \succ S_2$ for $\lambda \in (0.136, 0.52]$, $S_1 \succ S_4 \succ S_3 \succ S_2$ for $\lambda \in (0.52, 0.88]$, $S_1 \succ S_3 \succ S_4 \succ S_2$ for $\lambda \in (0.88, 0.94]$, $S_1 \succ S_3 \succ S_2 \succ S_4$ for $\lambda \in (0.94, 1]$.

Clearly, the ranking results obtained by Li et al.'s method [11] and the proposed method are considerably different. We can see that the proposed method has the following advantages over Li et al.'s method [11]:

(1) The proposed method formulates a MADM problem as a TPZS game with payoffs of TIFNs and then constructs a pair of intuitionistic fuzzy mathematical programming models to determine the attribute weights objectively. By contrast, Li et al.'s method [11] artificially gives the weights of attributes and cannot avoid subjective randomness. Therefore, the proposed method is more objective than Li et al.'s method [11].

(2) The proposed method gives different decision maker's risk preference information λ in advance, and each entity in the decision matrix includes the corresponding risk preference information. Thus, the decision results obtained by the proposed method are more consistent with the actual situation. By contrast, Li et al.'s method [11] gives the risk preference information λ after the attribute values are integrated. Therefore, the generated values of the alternatives may easily result in preference losses.

(3) The ranking value index of TIFNs adapted in Li et al.'s method [11] is based on the middle point of the cut set of a TIFN. Such value index is not always feasible and effective (Example 3.13).

5.3 Comparative analysis with the extended TOPSIS method

This subsection uses the extended TOPSIS method proposed by Zhang et al. [45] to solve the above VMS providers evaluation problem for further illustrating the superiority of the current method.

(1) Let the attribute weight vector be $w = (0.36, 0.18, 0.05, 0.11, 0.30)^{\mathrm{T}}$. By using the Zhang et al.'s method [45], the relative closeness of the alternatives is calculated as $\rho_1 = 0.356$, $\rho_2 = 0.371$, $\rho_3 = 0.358$, $\rho_4 = 0.371$. So, the ranking result is $S_4 \sim S_2 \succ S_3 \succ S_1$ and the best VMS provider is S_4 or S_2. Given that the school possesses limited money, it can choose only one provider to trade. Further evaluation is necessary to choose the best one from providers S_4 or S_2. In other words, Zhang et al.'s method [45] cannot discriminate which provider is best. The current method utilizes the TPZS game model to derive the attribute weights and obtain the best provider objectively. Thus, the current method is more reasonable and reliable.

(2) In the current method, the decision maker's risk preferences are sufficiently considered during the evaluation process. Thus decision makers can decide according to their own risk preference. Namely, decision makers can flexibly select the best provider under their different risk preferences. By contrast, Zhang et al.'s method [45] ignored the decision maker's risk preference. Therefore, compared with the Zhang et al.s method [45], the current method is more effective and flexible.

(3) The current method applies the expected score to evaluate each provider, which is a TIFN and can avoid the loss of uncertain information in original decision data. Zhang et al.'s method [45] uses the relative closeness to evaluate each provider, which is a crisp value and cannot describe uncertainty inherent of decision data. So, the current method is not easy to lose information.

6 Conclusions

This paper developed a new game method for handling multiple attribute decision making problems with TIFNs where the attribute weighs are unknown. First, a new value index was defined for TIFNs on the basic theories of calculus, and a ranking approach for TIFNs was proposed based on the value index. Then, a multiple attribute decision making problem with TIFNs was formulated as a two-person zero-sum game with payoffs of TIFNs. A pair of intuitionistic fuzzy optimization models was constructed and converted as multi-objective linear programming models based on the ranking order relations of TIFNs. Furthermore, the optimal strategies of two players are obtained by solving the multi-objective linear programming model. The rank of the alternatives is generated by the expected score of each alternative. Finally, example analyses verify the effectiveness and feasibility of our method. The analyses can take the decision makers subjective preferences into account and thus provide more choices for the decision maker. However, the proposed game method indicates two limitations. (1) Given that the complexity of linear programming is significantly related to the number of constraints and variables, it is not suitable for application environments with large-scale data. (2) The method cannot address multiple attribute group decision making problems with hybrid information. For future research, we intend to investigate the method that aggregates large-scale hybrid information into trapezoid intuitionistic fuzzy numbers. We will further extend the proposed method to multiple attribute group decision making under the circumstance of the trapezoid intuitionistic fuzzy numbers.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (Nos. 71740021, 11461030, 61602219, 11861034 and 71662014), Thirteen five Programming Project of Jiangxi province Social Science (2018) (Nos. 18GL19 and 18GL13), the Colleges Humanities Social Science Research Project of Jiangxi Province of China (No. GL162047) and the Science and Technology Project of Jiangxi Province of China (No. GJJ170424).

References

[1] T. Allahviranloo, R. Saneifard, *Defuzzification method for ranking fuzzy numbers based on center of gravity*, Iranian Journal of Fuzzy Systems, **9**(6) (2012), 57–67.

[2] K. Atanassov, G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy sets and systems, **31**(3) (1989), 343–349.

[3] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Systems, **20**(1) (1996), 87–96.

[4] A. Baykasoğlu, Ilker Gölcük, *Fuzzy sets*, Information and control, **8**(3) (1965), 338–353.

[5] Y. W. Chen, M. Larbani, *Two-person zero-sum game approach for fuzzy multiple attribute decision making problems*, Fuzzy Sets Systems, **157**(1) (2006), 34–51.

[6] R. Csutora, J. J. Buckley, *Fuzzy hierarchical analysis: the Lambda-Max method*, Fuzzy sets Systems, **120**(2) (2001), 181–195.

[7] M. Erdogan, I. Kaya, *An integrated multi-criteria decision-making methodology based on type-2 fuzzy sets for selection among energy alternatives in turkey*, Iranian Journal of Fuzzy Systems, **12**(1) (2015), 1–25.

[8] O. Jadidi, S. Zolfaghari, S. Cavalieri, *A new normalized goal programming model for multi-objective problems: a case of supplier selection and order allocation*, International Journal of Production Economics, **148**(1) (1965), 158–165.

[9] J. Jiang, X. Li, Y. W. Chen, D. W. Tang, *A goal programming approach to group decision making with incomplete fuzzy and interval preference relations*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **20**(3) (2012), 355–367.

[10] M. Larbani, *Solving bimatrix games with fuzzy payoffs by introducing Nature as a third player*, Fuzzy Sets Systems, **160**(5) (2009), 657–666.

[11] D. F. Li, J. X. Nan, M. J. Zhang, *A ranking method of triangular intuitionistic fuzzy numbers and application to decision making*, International Journal of Computational Intelligence Systems, **3**(5) (2010), 522–530.

[12] D. F. Li, *A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems*, Computers & Mathematics with Applications, **60**(6) (2010), 522–530.

[13] R. Lin, X. F. Zhao, G. W. Wei, *Fuzzy number intuitionistic fuzzy prioritized operators and their application to multiple attribute decision making*, Journal of Intelligent and Fuzzy Systems, **24**(4) (2013), 879–888.

[14] H. W. Liu, G. J. Wang, *Multi-criteria decision-making methods based on intuitionistic fuzzy sets*, European Journal of Operational Research, **179**(1) (2007), 220–233.

[15] P. D. Liu, F. Jin, *The trapezoid fuzzy linguistic bonferroni mean operators and their application to multiple attribute decision making*, Scientia Iranica, **19**(6) (2015), 1947–1959.

[16] Y. X. Ma, J. Wang, J. Q. Wang, X. H. Chen, Y. X. Ma, J. Wang, et al, *Two-tuple linguistic aggregation operators based on subjective sensation and objective numerical scales for multi-criteria group decision-making problem*, Scientia Iranica, **23**(3) (2016), 1399–1417.

[17] A. R. Mishra, P. Rani, D. Jain, *Information measure based TOPSIS for multicriteria decision making problem in intuitionistic fuzzy environment*, Iranian Journal of Fuzzy Systems, **14**(6) (2017), 41–63.

[18] J. X. Nan, D. F. Li, M. J. Zhang, *A lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers*, International Journal of Computational Intelligence Systems, **3**(3) (2010), 280–289.

[19] Z. Pei, L. Zheng, *A novel approach to multi-attribute decision making based on intuitionistic fuzzy sets*, Expert Systems with Applications, **39**(3) (2012), 2560–2566.

[20] Z. Pei, *A note on the TOPSIS method in MADM problems with linguistic evaluations*, Applied Soft Computing, **36**(C) (2015), 24–35.

[21] P. Rao, N. R. Shankar, *Ranking fuzzy numbers with a distance method using circumcenter of centroids and an index of modality*, Advances in Fuzzy Systems, **3** (2011), 1–7.

[22] J. Robinson, *Multiple attribute group decision analysis for intuitionistic triangular and trapezoidal fuzzy numbers*, International Journal of Fuzzy System Applications, **5**(3) (2016), 42–76.

[23] A. Schrijver, *Theory of linear and integer programming*, John Wiley & Sons, England, 1998.

[24] G. B. Thomas, R. L. Finney, M. D. Weir, *Calculus and analytic geometry*, Reading Addison-Wesley, Massachusetts, 1984.

[25] S. P. Wan, *Interval Multi-attribute Decision-making Method Based on Game Theory*, Systems Engineering, **28**(1) (2010), 17–20.

[26] S. P. Wan, *Multi-attribute decision making method based on possibility variance coefficient of triangular intuitionistic fuzzy numbers*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **21**(2) (2013), 223–243.

[27] S. P. Wan, Q. Y. Wang, J. Y. Dong, *The extended VIKOR method for multi-attribute group decision making with triangular intuitionistic fuzzy numbers*, Knowledge-Based Systems, **52**(6) (2013), 65–77.

[28] S. P. Wan, D. F. Li, *Possibility mean and variance based method for multi-attribute decision making with triangular intuitionistic fuzzy numbers*, J. Intelligent and Fuzzy Systems, **24**(4) (2013), 743–754.

[29] S. P. Wan, D. F. Li, *Fuzzy mathematical programming approach to heterogeneous multiattribute decision-making with interval-valued intuitionistic fuzzy truth degrees*, Information Sciences, **325** (2015), 484–503.

[30] S. P. Wan, J. Xu, J. Y. Dong, *Aggregating decision information into interval-valued intuitionistic fuzzy numbers for heterogeneous multi-attribute group decision making*, Knowledge-Based Systems, **113**(C) (2016), 155–170.

[31] S. P. Wan, Z. H. Yi, *Power average of trapezoidal intuitionistic fuzzy numbers using strict t-norms and t-conorms*, IEEE Transactions on Fuzzy Systems, **24**(5) (2016), 1035–1047.

[32] S. P. Wan, Y. J. Zhu, *Triangular Intuitionistic Fuzzy Triple Bonferroni Harmonic Mean Operators and Application to Multi-attribute Group Decision Making*, Iranian Journal of Fuzzy Systems, **13(5)**(2016), 117–145.

[33] S. P. Wan, G. L. Xu, F. Wang, J. Y. Dong, *A new method for Atanassovs interval-valued intuitionistic fuzzy MAGDM with incomplete attribute weight information*, Information Sciences, **316**(C) (2016), 329–347.

[34] S. P. Wan, J. Xu, *A method for multi-attribute group decision making with triangular intuitionistic fuzzy numbers and application to trustworthy service selection*, Scientia Iranica, **24**(2) (2017), 794–807.

[35] J. J. Wang, Y. Y. Jing, C. F. Zhang, J. H. Zhao, *Review on multi-criteria decision analysis aid in sustainable energy decision-making*, **13**(9) (2009), 2263–2278

[36] L. E. Wang, H. C. Liu, M. Y. Quan, *Evaluating the risk of failure modes with a hybrid MCDM model under interval-valued intuitionistic fuzzy environments*, Computer Industrial Engineering, **102** (2016), 175–185.

[37] X. Wang, J. Wang, X. Chen, *Fuzzy multi-criteria decision making method based on fuzzy structured element with incomplete weight information*, Iranian Journal of Fuzzy Systems, **13**(2) (2016), 1–17.

[38] G. Wei, X. Zhao, R. Lin, H. Wang, *Generalized triangular fuzzy correlated averaging operator and their application to multiple attribute decision making*, Applied Mathematical Modelling, **36**(7) (2012), 2975–2982.

[39] D. J. White, *Fuzzy multiple criteria decision making: Recent developments*, **78(2)**(1996), 139–153.

[40] Y. X. Xue, J. X. You, X. D. Lai, H. C. Liu, *An interval-valued intuitionistic fuzzy MABAC approach for material selection with incomplete weight information*, Applied Soft Computing, **38** (2016), 703–713.

[41] Z. H. Xu, R. R. Yager, *Dynamic intuitionistic fuzzy multi-attribute decision making*, Springer Berlin Heidelberg, **48**(1) (2012), 246–262.

[42] J. Xu, S. P. Wan, J. Y. Dong, *Aggregating decision information into Atanassov's intuitionistic fuzzy numbers for heterogeneous multi-attribute group decision making*, Applied Soft Computing, **41** (2016), 331–351.

[43] W. E. Yang, J. Q. Wang, *Vague linguistic matrix game approach for multi-criteria decision making with uncertain weights*, Journal of Intelligent & Fuzzy Systems, **25**(2) (2013), 315–324.

[44] L. A. Zadeh, *Fuzzy sets*, Information and control, **8**(3) (1965), 338–353.

[45] M. J. Zhang, J. X. Nan, D. F. Li, Y. X. Li, *TOPSIS for MADM with triangular intuitionistic fuzzy numbers*, Operations Research and Management Science, **21** (2012), 96–101.

[46] H. Zhao, J. X. You, H. C. Liu, *Failure mode and effect analysis using MULTIMOORA method with continuous weighted entropy under interval-valued intuitionistic fuzzy environment*, Soft Computing, **21**(18) (2017), 5355–5367.

[47] J. Zhao, X. Y. You, H. C. Liu, S. M. Wu, *An extended VIKOR method using intuitionistic fuzzy sets and combination weights for supplier selection*, Symmetry, **9**(9) (2017), 169.

Iranian Journal of Fuzzy Systems
Volume 16, Number 3, (2019), pp. 208
University of Sistan and Baluchestan

Multiple attribute decision making with triangular intuitionistic fuzzy numbers based on zero-sum game approach

J. Xu, J. Y. Dong, S. P. Wan and J. Gao

تصمیم‌گیری وابسته چندگانه با اعداد فازی شهودی مثلثی با روش بازی صفر-جمع

چکیده. برای بسیاری از مسائل تصمیم با عدم قطعیت، عدد فازی شهودی مثلثی ابزار مفیدی در بیان مقادیر مجهول است. این مقاله یک روش تصمیم جدید براساس بازی صفر-جمع برای مسائل تصمیم‌گیری وابسته چند گانه که مقدار وابسته به صورت اعداد فازی شهودی مثلثی است و وزن وابسته مجهول می‌باشدگسترش می‌دهد. ابتدا یک اندیس مقدار برای اعداد فازی شهودی مثلثی براساس مرکز ثقل تعریف شده‌است . به موجب آن، یک روش رتبه‌بندی جدید برای مقایسه اعداد فازی شهودی مثلثی ارائه گردیده‌است. یک مسئله تصمیم‌گیری وابسته چندگانه را به عنوان یک بازی صفر- جمع دو نفره با نتایج اعداد فازی شهودی مثلثی فرمول بندی می‌کنیم. سپس، به دنبال روش رتبه‌بندی جدید، بازی ماتریس فازی به عنوان یک جفت از مدل‌های برنامه‌ریزی خطی تبدیل شده‌است و با حل چنین مدل‌هایی بطور هدفمند تمهیدات بهینه‌ای اندیشیده شده‌است. بنابرای، ترتیب رتبه بندی گزینه‌ها توسط نمرات مورد انتظار گزینه‌ها تعیین شده‌است. مثالی از انتخاب سیستم دیده بانی تصویری برای بیان تأثیر روش پیشنهادی نشان داده شده‌است.