Archive of SID

Iranian Journal of Fuzzy Systems

Volume 16, Number 3, (2019), pp. 189-197

University of Sistan and Baluchestan

Fuzzy transferable-utility games: a weighted allocation and related results

Y. H. Liao[1] and L. Y. Chung[2]

[1] Department of Applied Mathematics, National Pingtung University, Pingtung 900, Taiwan
[2] Ling-Yun Chung (corresponding author), Graduate School of Technological and Vocational Education, National Yunlin University of Science and Technology, Yunlin 64002, Taiwan

twincos@ms25.hinet.net, darlingyun1016@gmail.com

Abstract

By considering the supreme-utilities among fuzzy sets and the weights among participants simultaneously, we introduce the supreme-weighted value on fuzzy transferable-utility games. Further, we provide some equivalent relations to characterize the family of all solutions that admit a potential on weights. We also propose the dividend approach to provide alternative viewpoint for the potential approach. Based on these equivalent relations, several axiomatic results are also proposed to present the rationality for the supreme-weighted value.

Keywords: Fuzzy transferable-utility games, the supreme-weighted value, potential, dividend, axiomatic result.

1 Introduction

On standard *transferable-utility* (TU) games, a characteristic function is defined over all the set of participants. This means that the options available for each participant are either to participate fully in a coalition or not to participate at all. However, the behavior of a participant may be not described by finite certain numbers in real situations. Thus, we consider the framework of the *fuzzy TU games*. A fuzzy TU game is a natural extension of a standard TU game in which each participant is allowed to participate with infinite various activity levels respectively. Several fuzzy solutions and related results have been proposed in the literature, such as Abhishek et al. [1], Branzei et al. [4], Butnariu and Kroupa [6], Hwang and Liao [9], Li and Zhang [10], Liao [11], Tsurumi et al. [15] and so on.

A vector field H is said to be "conservative" if there exists a differentiable mapping h such that H is the gradient of h. The mapping h is said to be the potential function for H. The notion "conservative" asserts that the entirety of the kinetic energy and the potential energy of a mote moving over a conservative vector field is constant. The potential approach is a useful notion in various spheres. For example, by introducing the potential that associates to each economy a real number and determining marginal contributions according to the potential, the marginal contributions could be treated as the payoffs of all participants always. In the framework of fuzzy TU games, Liao et al. [12] introduced the *supreme-consistent value*. Inspired by Calvo and Santos [7], Hart and Mas-Colell [8] and Ortmann [13, 14], Liao et al. [12] also proposed the potential approach for the supreme-consistent value.

Different from the pre-existing results, we propose new results on fuzzy TU games. The main results are as follows.

1. In general, the participants should be dissimilar under different situations. It is reasonable that the *weights* could be assigned to modify the discrimination among the participants under different situations respectively. In Section 2, we adopt the *weight functions* to introduce the *supreme-weighted value* on fuzzy TU games.

2. Based on the viewpoint of marginal contributions, we show that there exists just one potential and the resulting payoff vector coincides with the supreme-weighted value in Section 3. Based on the viewpoint of marginal accumulations, we also introduce the *dividend approach on weights*, and investigate the coincidence between the potential approach and the dividend approach.

Corresponding Author: L. Y. Chung

Received: November 2017; Revised: May 2018; Accepted: December 2018.

www.SID.ir

Archive of SID

3. In order to present the rationality for a solution that admit a potential on weights, we firstly provide some equivalent relations among the *weighted fuzzy potentializability* of a solution, the *weighted fuzzy balanced contributions* and the *weighted fuzzy path independence* in Section 4. Further, we adopt the weighted potential to characterize the supreme-weighted value of a *fuzzy auxiliary game*. Based on these equivalent relations, two axiomatic results are also proposed to analyze the rationality for the supreme-weighted value.

2 Preliminaries

Let U be the universe of participants. For $i \in U$ and $d_i \in (0,1]$, $D_i = [0, d_i]$ could be treated as the action (decision) space of agent i and $D_i^+ = (0, d_i]$, where 0 denotes no participation. Let $D^N = \prod_{i \in N} D_i$ be the product set of the action spaces of all participants of N. For all $T \subseteq N$, we define $\theta^T \in D^N$ is the vector with $\theta_i^T = 1$ if $i \in T$, and $\theta_i^T = 0$ if $i \in N \setminus T$. Denote 0_N the zero vector in $\mathbb{R}^N$.

A **fuzzy TU game**[1] is a triple (N, d, v), where $N \neq \emptyset$ is a finite set of participants, $d = (d_i)_{i \in N} \in [0,1]^N$ describes the highest level for each participant, and $v : D^N \to \mathbb{R}$ is a utility function with $v(0_N) = 0$ which assigns to each $\alpha \in D^N$ the worth that the participants can gain when each participant i plays at level α_i. Given a fuzzy TU game (N, d, v) and $\alpha \in D^N$, we write (N, α, v) for the **fuzzy TU subgame** obtained by restricting v to $\{\beta \in D^N \mid \beta_i \leq \alpha_i \forall\ i \in N\}$. Let $A(\alpha) = \{i \in N | \alpha_i \neq 0\}$ and α_T be the restriction of α at T for each $T \subseteq N$.

Denote the class of all fuzzy TU games by Γ. A **solution** on Γ is a map ψ assigning to each $(N, d, v) \in \Gamma$ an element $\psi(N, d, v) = \big(\psi_i(N, d, v)\big)_{i \in N} \in \mathbb{R}^N$. Here $\psi_i(N, d, v)$ is the payoff of $i \in N$ when i participates in game v.

In the framework of fuzzy TU games, Liao et al. [12] proposed the supreme-consistent value as follows.

Definition 2.1. *The* **supreme-consistent value**, *ϕ, is the map which associates to each $(N, d, v) \in \Gamma$ and each $i \in N$ the value*

$$\phi_i(N, d, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \cdot \big[v^*(S) - v^*(S \setminus \{i\})\big].$$

For all $S \subseteq N$, $v^*(S) = \sup_{\alpha \in D^N} \{v(\alpha) | A(\alpha) = S\}$ is the supreme-utility[2] *among all action vector with $A(\alpha) = S$.*

Remark 2.2. *Without loss of generality, one could assume that $A(d) = N$ for all $(N, d, v) \in \Gamma$.*

Given $(N, d, v) \in \Gamma$ and $S \subseteq N$, let $|S|$ be the amount of elements in S. The **fuzzy unanimity game** for TU games, (N, d, u_S) with $S \neq \emptyset$, is defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

Liao et al. [12] showed that $v^* = \sum_{S \in 2^N \setminus \{\emptyset\}} C_S \cdot u_S$ for all $(N, d, v) \in \Gamma$, where $C_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} v^*(T)$ is called the **dividend** among the participants in S.

Let $w : U \to \mathbb{R}^+$ be a positive function, then w is called a **weight function**. Given $(N, d, v) \in \Gamma$ and a weight function w, we define $|S|_w = \sum_{i \in S} w(i)$ for all $S \subseteq N$. By applying dividends and weight functions, we introduce a different solution on fuzzy TU games as follows.

Definition 2.3. *The* **supreme-weighted value**, *ϕ^w, is the solution on Γ which associates with all $(N, d, v) \in \Gamma$, all weight function w and all participants $i \in N$ the value $\phi_i^w(N, d, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{w(i) \cdot C_S}{|S|_w}$. By the definition of ϕ^w, all participants allocate the dividends by applying weights proportionally. These weights could be treated as parameters to modify the differences among all participants.*

A solution ψ satisfies **efficiency (EFF)** if $\sum_{i \in N} \psi_i(N, d, v) = v^*(N)$ for all $(N, d, v) \in \Gamma$.

Lemma 2.4. *The supreme-weighted value satisfies EFF.*

[1] A fuzzy TU game, which is defined by Aubin [2, 3], is a pair (N, v^a), where v^a is a mapping such that $v^a : [0,1]^N \to \mathbb{R}$ and $v^a(0_N) = 0$. In fact, $(N, v^a) = (N, \theta^N, v)$.

[2] Here we consider bounded fuzzy TU games, defined as those games (N, d, v) such that, there exists $K_v \in \mathbb{R}$ such that $v(\alpha) \leq K_v$ for all $\alpha \in D^N$. We adopt it to ensure that $v^*(S)$ is well-defined.

www.SID.ir

Archive of SID

Proof. Let $(N,d,v)\in\Gamma$ and w be a weight function. By Definition 2.3,

$$\begin{aligned}
\phi_i^w(N,d,v) &= \sum_{\substack{S\subseteq N\\ i\in S}} \frac{w(i)\cdot C_S}{|S|_w} = \sum_{\substack{S\subseteq N\\ i\in S}} \frac{w(i)}{|S|_w}\sum_{T\subseteq S}(-1)^{|S|-|T|}v^*(T)\\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \frac{w(i)}{|S|_w}\Big[\sum_{\substack{T\subseteq S\\ i\in T}}(-1)^{|S|-|T|}v^*(T)+\sum_{\substack{T\subseteq S\\ i\notin T}}(-1)^{|S|-|T|}v^*(T)\Big]\\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \frac{w(i)}{|S|_w}\Big[\sum_{\substack{T\subseteq S\\ i\in T}}(-1)^{|S|-|T|}v^*(T)+\sum_{\substack{T\subseteq S\\ i\in T}}(-1)^{|S|-|T|-1}v^*(T\setminus\{i\})\Big]\\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \frac{w(i)}{|S|_w}\Big[\sum_{\substack{T\subseteq S\\ i\in T}}(-1)^{|S|-|T|}v^*(T)-\sum_{\substack{T\subseteq S\\ i\in T}}(-1)^{|S|-|T|}v^*(T\setminus\{i\})\Big]\\
&= \sum_{\substack{S\subseteq N\\ i\in S}} \frac{w(i)}{|S|_w}\sum_{\substack{T\subseteq S\\ i\in T}}(-1)^{|S|-|T|}\big[v^*(T)-v^*(T\setminus\{i\})\big].
\end{aligned} \tag{1}$$

Given a fixed $T\subseteq N$. For all $k\in N$, we have that

1. If $T\subseteq S\subseteq N$ with $T\neq N$ and $k\in S$, then the coefficient of $v^*(T)$ in equation (1) is

$$\sum_{\substack{S\subseteq N\\ T\subseteq S}}\frac{w(k)}{|S|_w}(-1)^{|S|-|T|}-\sum_{\substack{S\subseteq N\\ (T\cup\{k\})\subseteq S}}\frac{w(k)}{|S|_w}(-1)^{|S|-|T\cup\{k\}|}.$$

So,

$$\begin{aligned}
&\sum_{k\in S}\sum_{\substack{S\subseteq N\\ T\subseteq S}}\frac{w(k)}{|S|_w}(-1)^{|S|-|T|} = \sum_{\substack{S\subseteq N\\ T\subseteq S}}\frac{|S|_w}{|S|_w}(-1)^{|S|-|T|} = \sum_{\substack{S\subseteq N\\ T\subseteq S}}(-1)^{|S|-|T|}\\
=&\Big[\sum_{\substack{T\subseteq S\subseteq N\\ |S|-|T|=0}}+\sum_{\substack{T\subseteq S\subseteq N\\ |S|-|T|=1}}+\cdots+\sum_{\substack{T\subseteq S\subseteq N\\ |S|-|T|=|N|-|T|}}\Big](-1)^{|S|-|T|}\\
=&\sum_{t=0}^{|N|-|T|}C_t^{|N|-|T|}(-1)^t \qquad \text{(let } t=|S|-|T|\text{)}\\
=&\sum_{t=0}^{|N|-|T|}C_t^{|N|-|T|}(-1)^t(1)^{|N|-|T|-t}=(-1+1)^{|N|-|T|}=0. \qquad \text{(since } T\neq N\text{)}
\end{aligned}$$

Similarly, $\sum_{k\in S}\sum_{\substack{S\subseteq N\\ (T\cup\{k\})\subseteq S}}\frac{w(k)}{|S|_w}(-1)^{|S|-|T\cup\{k\}|}=0.$

2. If $T\subseteq S\subseteq N$, $T\neq N$ and $k\notin S$, then it is easy to see that the coefficient of $v^*(T)$ in equation (1) is 0.

3. If $T=N$, then the coefficient of $v^*(T)$ in equation (1) is $\frac{w(k)}{|N|_w}(-1)^{|N|-|N|}$. So,

$$\sum_{k\in N}\frac{w(k)}{|N|_w}(-1)^{|N|-|N|}=\frac{|N|_w}{|N|_w}(-1)^{|N|-|N|}=1$$

.

So we have that the coefficient a_T of $v^*\big(T\big)$ in $\sum_{k\in N}\phi_k^w(N,d,v)$ is $a_T = \begin{cases} 0 & \text{if } 0<|T|<|N|,\\ 1 & \text{if } |T|=|N|.\end{cases}$ Hence, $\sum_{k\in N}\phi_k^w(N,d,v)$ $=\sum_{S\subseteq N}a_S\cdot v^*\big(S\big)=v^*(N)$. Therefore, the supreme-weighted value ϕ^w satisfies EFF. □

3 Potential approach and dividend approach

Inspired by the pre-existing results of the Shapley value [16] due to Calvo and Santos [7], Hart and Mas-Colell [8] and Ortmann [13, 14], we also investigate the potential approach on fuzzy TU games by considering the supreme-utilities among fuzzy sets and the weights among participants simultaneously.

Given a function $P:\Gamma\longrightarrow\mathbb{R}$ which associates $P(N,d,v)\in\mathbb{R}$ to each $(N,d,v)\in\Gamma$. For each weight function w and for each $i\in N$, we define that $D_i^wP\big(N,d,v\big)=w(i)\cdot\big[P\big(N,d,v\big)-P\big(N,(d_{N\setminus\{i\}},0),v\big)\big]$.

www.SID.ir

Archive of SID

Definition 3.1. *A solution ψ on Γ admits a* **potential on weights** *if there exists a function $P : \Gamma \to \mathbb{R}$ satisfies for all $(N, d, v) \in \Gamma$, for all weight function w and for all $i \in N$, $\psi_i(N, d, v) = D_i^w P(N, d, v)$.*

Solutions that admit a potential on weights assign a scalar evaluation to each game in such a way that a participant's payoff is his weighted marginal contribution to this evaluation. A function $P : \Gamma \longrightarrow \mathbb{R}$ is said to be **0-normalized** *if $P(\emptyset, d, v) = 0$. P is* **efficient** *if for all weight function w and for all $(N, d, v) \in \Gamma$,*

$$\sum_{i \in N} D_i^w P(N, d, v) = v^*(N). \tag{2}$$

The existence of a potential and a 0-normalized potential are equivalent, since the function $P^0(N, d, v) = P(N, d, v) - P(N, 0_N, v)$ is a 0-normalized potential if P is a potential. Furthermore, a solution ψ admits one 0-normalized potential at most.

Theorem 3.2. *There exists a uniquely efficient and 0-normalized potential P such that supreme-weighted value admits the potential P on weights.*

Proof. Let $(N, d, v) \in \Gamma$ and w be a weight function. It is easy to check that equation (2) can be rewritten as

$$P(N, d, v) = \frac{1}{|N|_w} \cdot \big[v^*(N) + \sum_{i \in N} w(i) \cdot P\big(N, (d_{N\setminus\{i\}}, 0), v\big)\big]. \tag{3}$$

Starting with $P(N, 0_N, v) = 0$, it determines $P(N, d, v)$ recursively. This verifies the existence of the potential P, and moreover that $P(N, d, v)$ is uniquely determined by (3) applied to (N, α, v) for all $\alpha \in D^N$. Let

$$P(N, d, v) = \sum_{S \subseteq N} \frac{C_S}{|S|_w}. \tag{1}$$

Based on Lemma 2.4, it is easy to verify that (2) is satisfied by this P; hence (4) defines the unique potential function. Combining Definitions 2.3, 3.1 with equation (4), the result now follows since, for all $i \in N$,

$$w(i) \cdot \big[P\big(N, d, v\big) - P\big(N, (d_{N\setminus\{i\}}, 0), v\big)\big] = \phi_i^w(N, d, v)$$

□

Different from the viewpoint of marginal contributions, we adopt the viewpoint of marginal accumulation to introduce the dividend approach on weights. Given a function $e : \Gamma \longrightarrow \mathbb{R}$ which associates $e(N, d, v) \in \mathbb{R}$ to each $(N, d, v) \in \Gamma$. Then for all weight function w and for all $i \in N$, we define $\Delta_i^w e(N, d, v) = w(i) \cdot \sum_{\substack{S \subseteq N \\ i \in S}} e(N, (d_S, 0_{N\setminus S}), v)$.

Definition 3.3. *A solution ψ on Γ admits a* **dividend on weights** *if there exists a function $e : \Gamma \to \mathbb{R}$ satisfies for all $(N, d, v) \in \Gamma$, for all weight function w and for all $i \in N$, $\psi_i(N, d, v) = \Delta_i^w e(N, d, v)$.*

Solutions that admit a dividend assign a scalar evaluation to each game in such a way that a participant's payoff is its *weighted marginal accumulation* to this evaluation. Moreover, a function $d : \Gamma \longrightarrow \mathbb{R}$ is said to be *0-normalized* if $e(N, 0_N, v) = 0$ for each $N \subseteq U$. And we say that it is *efficient* if it satisfies the following condition: For all $(N, d, v) \in \Gamma$ and for all weight function w, $\sum_{i \in N} \Delta_i^w e(N, d, v) = v^*(N)$.

Theorem 3.4. *A solution ψ on Γ admits a potential on weights if and only if ψ admits a dividend on weights.*

Proof. Assume that ψ is a solution and ψ admits a dividend e on weights. Define $P : \Gamma \longrightarrow \mathbb{R}$ to be that $P(N, d, v) = \sum_{S \subseteq N} e\big(N, (d_S, 0_{N\setminus S}), v\big)$ for all $(N, d, v) \in \Gamma$. Since ψ admits the dividend e, for all $(N, d, v) \in \Gamma$, for all weight function w and for all $i \in N$,

$$\begin{aligned}
\psi_i(N, d, v) = {} & \Delta_i^w e(N, d, v) = w(i) \sum_{\substack{S \subseteq N \\ i \in S}} e\big(N, (d_S, 0_{N\setminus S}), v\big) \\
= {} & w(i) \cdot \big[\sum_{S \subseteq N} e\big(N, (d_S, 0_{N\setminus S}), v\big) - \sum_{S \subseteq N\setminus\{i\}} e\big(N, (d_S, 0_{N\setminus S}), v\big)\big] \\
= {} & w(i) \cdot \big[P\big(N, d, v\big) - P\big(N, (d_S, 0_{N\setminus S}), v\big)\big] = D_i^w P(N, d, v).
\end{aligned}$$

Hence, ψ admits the potential P on weights.

www.SID.ir

Archive of SID

Assume that a solution ψ admits a potential P on weights. Define $e : \Gamma \longrightarrow \mathbb{R}$ to be that, for all $(N,d,v) \in \Gamma$, $e(N,d,v) = \sum_{S\subseteq N}(-1)^{|N|-|S|}P(N,(d_S,0_{N\setminus S}),v)$ It is easy to check that $P(N,d,v) = \sum_{S\subseteq N} e(N,(d_S,0_{N\setminus S}),v)$. Since ψ admits the potential P on weights, for all $(N,d,v) \in \Gamma$, for all weight function w and for all $i \in N$,

$$\begin{aligned}\psi_i(N,d,v) = \;& D_i^w P(N,d,v) = w(i)\cdot\big[P\big(N,d,v\big) - P\big(N,(d_{N\setminus\{i\}},0),v\big)\big] \\ =\;& w(i)\cdot\big[\sum_{S\subseteq N} e(N,(d_S,0_{N\setminus S}),v) - \sum_{S\subseteq N\setminus\{i\}} e(N,(d_S,0_{N\setminus S}),v)\big] \\ =\;& w(i)\sum_{\substack{S\subseteq N\\ i\in S}} e(N,(d_S,0_{N\setminus S}),v) = \Delta_i^w e(N,d,v).\end{aligned}$$

Hence, ψ admits the dividend e on weights. The proof is completed. □

Theorem 3.5. *There exists a uniquely efficient and 0-normalized dividend e such that supreme-weighted value admits the dividend e on weights.*

Proof. It is easy to derive this result by Theorems 3.2 and 3.4. □

Remark 3.6. *The dividend approach is a dual view of the potential approach. In fact, as some axioms are altered to fit solutions form focusing on "dividend", the executions for characterizations among solutions on fuzzy TU games are similar. The dividend forms not only offer interpretations for solutions but also provide motivations for axioms of solutions.*

4 Equivalent relations and axiomatic results

In this section, we propose some equivalent relations to characterize the family of all solutions that admit a potential on weights. In order to analyze the rationality for the supreme-weighted value, we adopt these equivalent relations to characterize the supreme-weighted value. We will make use of the following axioms. Let ψ be a solution on Γ. ψ satisfies **weighted fuzzy balanced contributions (WFBC)** if for all $(N,d,v) \in \Gamma$, for all weight function w and for all $i,j \in N$,

$$\tfrac{1}{w(i)}\cdot\Big[\psi_i\big(N,d,v\big) - \psi_i\big(N,(d_{N\setminus\{j\}},0),v\big)\Big] = \tfrac{1}{w(j)}\cdot\Big[\psi_j\big(N,d,v\big) - \psi_j\big(N,(d_{N\setminus\{i\}},0),v\big)\Big].$$

In the following, we propose a weighted analogue of the path independence property due to Ortmann [13, 14] on fuzzy TU games. A **simple order** for $(N,d,v) \in \Gamma$ is a bijection $\sigma : N \to N$. Let σ, σ' be two simple orders for (N,d,v). σ' is a **transposition** of σ if there exist $i,j \in N$ with $i \neq j$ and $\sigma(j) = \sigma(i)+1$, such that $\sigma'(i) = \sigma(j)$, $\sigma'(j) = \sigma(i)$ and $\sigma'(p) = \sigma(p)$ for all $p \in N\setminus\{i,j\}$. Let σ be a simple order. The activity level vector that is present after the t-th agent according to σ, denoted by $s^{\sigma,t}$, is defined by

$$s_i^{\sigma,t} = \begin{cases} d_i & \text{if } \sigma(i) \le t; \\ 0 & \text{otherwise.}\end{cases}$$

for all $i \in N$. Clearly, each simple order can be transformed to another simple order by applying transpositions. ψ satisfies **weighted fuzzy path independence (WFPI)** if for all $(N,d,v) \in \Gamma$, for all weight function w and for all simple orders σ, σ',

$$\sum_{i\in N}\frac{1}{w(i)}\cdot\psi_i(N,s^{\sigma,\sigma(i)},v) = \sum_{i\in N}\frac{1}{w(i)}\cdot\psi_i(N,s^{\sigma',\sigma'(i)},v).$$

In the following, we introduce a fuzzy extension of an auxiliary game due to Calvo and Santos [7]. Let $(N,d,v) \in \Gamma$ and ψ be a solution. The **auxiliary fuzzy game** (N,d,v_ψ) is defined as follows. For all $\alpha \in D^N$, $v_\psi^*(A(\alpha)) = \sum_{i\in N}\psi_i(N,\alpha,v)$, where $v_\psi^*(S) = \sup_{\alpha\in D^N}\{v_\psi(\alpha)|A(\alpha) = S\}$ for all $S \subseteq N$. Note that if ψ satisfies efficiency, then $v^* = v_\psi^*$.

Next, we state the main result as follows.

Theorem 4.1. *Let ψ be a solution on Γ. The following are equivalent:*

1. *ψ admits a potential on weights*
2. *ψ satisfies WFBC*
3. *ψ satisfies WFPI*

www.SID.ir

Archive of SID

4. $\psi(N,d,v) = \phi^w(N,d,v_\psi)$ *for all* $(N,d,v) \in \Gamma$ *and for all weight function* w.

Proof. Let ψ be a solution on Γ. To verify $1 \Rightarrow 2$, suppose ψ admits a potential P on weights. For all $(N,d,v) \in \Gamma$, for all weight function w and for all $i,j \in N$, $i \neq j$,

$$\begin{aligned} & \tfrac{1}{w(i)} \cdot \Big[\psi_i\big(N,d,v\big) - \psi_i\big(N,(d_{N\setminus\{j\}},0),v\big)\Big] \\ = & \tfrac{1}{w(i)} \cdot \Big[w(i) \cdot \big[P\big(N,d,v\big) - P\big(N,(d_{N\setminus\{i\}},0),v\big)\big] w(i) \cdot \big[P\big(N,(d_{N\setminus\{j\}},0),v\big) - P\big(N,(d_{N\setminus\{i,j\}},0,0),v\big)\big]\Big] \\ = & \big[P\big(N,d,v\big) - P\big(N,(d_{N\setminus\{i\}},0),v\big)\big]\big[P\big(N,(d_{N\setminus\{j\}},0),v\big) - P\big(N,(d_{N\setminus\{i,j\}},0,0),v\big)\big] \\ & = \big[P\big(N,d,v\big) - P\big(N,(d_{N\setminus\{j\}},0),v\big)\big]\big[P\big(N,(d_{N\setminus\{i\}},0),v\big) - P\big(N,(d_{N\setminus\{i,j\}},0,0),v\big)\big] \\ = & \tfrac{1}{w(j)} \cdot \Big[w(j) \cdot \big[P\big(N,d,v\big) - P\big(N,(d_{N\setminus\{j\}},0),v\big)\big] w(j) \cdot \big[P\big(N,(d_{N\setminus\{i\}},0),v\big) - P\big(N,(d_{N\setminus\{i,j\}},0,0),v\big)\big]\Big] \\ & = \tfrac{1}{w(j)} \cdot \Big[\psi_j\big(N,d,v\big) - \psi_j\big(N,(d_{N\setminus\{i\}},0),v\big)\Big]. \end{aligned}$$

Hence, ψ satisfies WFBC.

To verify $2 \Rightarrow 3$, suppose that ψ satisfies WFBC. Let $(N,d,v) \in \Gamma$ and σ, σ' be two simple orders for (N,d,v). Since each simple order can be transformed to another simple order by applying transpositions, we can assume that σ' is a transposition of σ. Let $i,j \in N$ with $i \neq j$ and $\sigma(j) = \sigma(i)+1$, such that $\sigma'(i) = \sigma(j)$, $\sigma'(j) = \sigma(i)$ and $\sigma'(p) = \sigma(p)$ for all $p \in N \setminus \{i,j\}$. Since σ' is a transposition of σ, for all $t \in N \setminus \{i,j\}$,

$$\psi_t(N, s^{\sigma,\sigma(t)}, v) = \psi_t(N, s^{\sigma',\sigma'(t)}, v). \tag{5}$$

Since σ' is a transposition of σ, by equation (5),

$$\begin{aligned} & \sum_{p\in N} \tfrac{1}{w(p)} \cdot \psi_p(N, s^{\sigma',\sigma'(p)}, v) - \sum_{p\in N} \tfrac{1}{w(p)} \cdot \psi_p(N, s^{\sigma,\sigma(p)}, v) \\ = & \tfrac{1}{w(j)} \cdot \psi_j\big(N,(d_{N\setminus\{i\}},0),v\big) + \tfrac{1}{w(i)} \cdot \psi_i(N,d,v) \tfrac{1}{w(i)} \cdot \psi_i\big(N,(d_{N\setminus\{j\}},0),v\big) - \tfrac{1}{w(j)} \cdot \psi_j(N,d,v). \end{aligned} \tag{6}$$

Since ψ satisfies WFBC,

$$\tfrac{1}{w(i)} \cdot \big[\psi_i(N,d,v) - \psi_i\big(N,(d_{N\setminus\{j\}},0),v\big)\big] = \tfrac{1}{w(j)} \cdot \big[\psi_j(N,d,v) - \psi_j\big(N,(d_{N\setminus\{i\}},0),v\big)\big]. \tag{7}$$

By equations (6) and (7), $\sum_{p\in N} \frac{1}{w(p)} \cdot \psi_p(N, s^{\sigma,\sigma(p)}, v) - \sum_{p\in N} \frac{1}{w(p)} \cdot \psi_p(N, s^{\sigma',\sigma'(p)}, v) = 0$. Thus, $\sum_{p\in N} \frac{1}{w(p)} \cdot \psi_p(N, s^{\sigma,\sigma(p)}, v) = \sum_{p\in N} \frac{1}{w(p)} \cdot \psi_p(N, s^{\sigma',\sigma'(p)}, v)$. Hence, ψ satisfies WFPI.

To verify $3 \Rightarrow 2$, suppose that ψ satisfies WFPI. Let $(N,d,v) \in \Gamma$. It is trivial if $|N| = 1$. Assume that $|N| \geq 2$. Let $i,j \in N$. Let σ_i and σ_j be two simple orders with $\sigma_i(i) = \sigma_j(j) = |N|$, $\sigma_i(j) = \sigma_j(i) = |N| - 1$ and $\sigma_i(p) = \sigma_j(p)$ for all $p \notin \{i,j\}$. Since ψ satisfies WFPI,

$$\begin{aligned} 0 = & \sum_{p\in N} \tfrac{1}{w(p)} \cdot \psi_p(N, s^{\sigma_i,\sigma_i(p)}, v) - \sum_{p\in N} \tfrac{1}{w(p)} \cdot \psi_p(N, s^{\sigma_j,\sigma_j(p)}, v) \\ = & \tfrac{1}{w(j)} \cdot \psi_j\big(N,(d_{N\setminus\{i\}},0),v\big) + \tfrac{1}{w(i)} \cdot \psi_i(N,d,v) - \tfrac{1}{w(i)} \cdot \psi_i\big(N,(d_{N\setminus\{j\}},0),v\big) - \tfrac{1}{w(j)} \cdot \psi_j(N,d,v). \end{aligned}$$

Therefore, $\frac{1}{w(i)} \cdot \big[\psi_i(N,d,v) - \psi_i\big(N,(d_{N\setminus\{j\}},0),v\big)\big] = \frac{1}{w(j)} \cdot \big[\psi_j(N,d,v) - \psi_j\big(N,(d_{N\setminus\{i\}},0),v\big)\big]$. So, ψ satisfies WFBC. To verify $2 \Rightarrow 4$, suppose that ψ satisfies WFBC. Let $(N,d,v) \in \Gamma$. By Theorem 3.2 and result "$1 \Rightarrow 2$", the solution ϕ^w satisfies WFBC. The remaining proof proceeds by induction on $|N|$. Assume that $|N| = 1$ and $N = \{i\}$. By EFF of ϕ and definition of v_ψ, $\phi^w_i(N,d,v_\psi) = v_\psi(d) = \psi_i(N,d,v)$. Assume that $\psi(N,d,v) = \phi^w(N,d,v)$ if $|N| \leq l-1$, where $l \geq 2$. The case $|N| = l$: By WFBC of both ψ and ϕ^w, and induction hypotheses, for $i,j \in N$,

$$\begin{aligned} & \tfrac{1}{w(i)} \cdot \psi_i(N,d,v) - \tfrac{1}{w(j)} \cdot \psi_j(N,d,v) \\ = & \tfrac{1}{w(i)} \cdot \psi_i\big(N,(d_{N\setminus\{j\}},0),v\big) - \tfrac{1}{w(j)} \cdot \psi_j\big(N,(d_{N\setminus\{i\}},0),v\big) && \text{(by WFBC of } \psi) \\ = & \tfrac{1}{w(i)} \cdot \phi^w_i\big(N,(d_{N\setminus\{j\}},0),v\big) - \tfrac{1}{w(j)} \cdot \phi^w_j\big(N,(d_{N\setminus\{i\}},0),v\big) && \text{(by induction hypotheses)} \\ = & \tfrac{1}{w(i)} \cdot \phi^w_i(N,d,v_\psi) - \tfrac{1}{w(j)} \cdot \phi^w_j(N,d,v_\psi), && \text{(by WFBC of } \phi^w) \end{aligned}$$

www.SID.ir

Archive of SID

So, $w(j) \cdot \big[\psi_i(N,d,v) - \phi_i^w(N,d,v_\psi)\big] = w(i) \cdot \big[\psi_j(N,d,v) - \phi_j^w(N,d,v_\psi)\big]$ for all $i,j \in N$. By definition of v_ψ and EFF of ϕ^w,

$$\begin{aligned} 0 &= w(i) \cdot \big[v_\psi^*(N) - v_\psi^*(N)\big] = w(i) \cdot \big[\sum_{j\in N} \psi_j(N,d,v) - \sum_{j\in N} \phi_j^w(N,d,v_\psi)\big] \\ &= \sum_{j\in N} w(i) \cdot \big[\psi_j(N,d,v) - \phi_j^w(N,d,v_\psi)\big] = \sum_{j\in N} w(j) \cdot \big[\psi_i(N,d,v) - \phi_i^w(N,d,v_\psi)\big] \\ &= |N|_w \cdot \big[\psi_i(N,d,v) - \phi_i^w(N,d,v_\psi)\big]. \end{aligned}$$

Since $|N|_w \neq 0$, we have that $\psi_i(N,d,v) - \phi_i^w(N,d,v_\psi) = 0$ for all $i \in N$, i.e., $\psi_i(N,d,v) = \phi_i^w(N,d,v_\psi)$ for all $i \in N$. To verify $4 \Rightarrow 1$, suppose that $\psi(N,d,v) = \phi^w(N,d,v_\psi)$ for all $(N,d,v) \in \Gamma$ and for all weight function w. By Theorem 3.2, the solution ϕ^w admits a potential P_ϕ on weights. So we define a function of ψ as $P_\psi(N,d,v) = P_\phi(N,d,v_\psi)$ for all $(N,d,v) \in \Gamma$. Then for all $i \in N$,

$$\begin{aligned} \psi_i(N,d,v) &= \phi_i^w(N,d,v_\psi) = w(i) \cdot \big[P_\phi\big(N,d,v_\psi\big) - P_\phi\big(N,(d_{N\setminus\{i\}},0),v_\psi\big)\big] \\ &= w(i) \cdot \big[P\psi\big(N,d,v\big) - P_\psi\big(N,(d_{N\setminus\{i\}},0),v\big)\big]. \end{aligned}$$

Hence, ψ admits the potential P_ψ on weights. □

By applying Theorems 3.2 and 4.1, two axiomatic results of the supreme-weighted value can be proposed as follows.

Theorem 4.2.

1. *A solution ψ on Γ satisfies EFF and WFBC if and only if $\psi = \phi^w$ for all weight function w.*
2. *A solution ψ on Γ satisfies EFF and WFPI if and only if $\psi = \phi^w$ for all weight function w.*

Proof. The proof follows by definition of auxiliary fuzzy game, and Theorems 3.2, 4.1. □

The following examples show that each of the axioms adopted in Theorem 4.2 is logically independent of the remaining axioms.

Example 4.3. *Define a solution ψ by $\psi_i(N,d,v) = 0$ for all $(N,d,v) \in \Gamma$, for all weight function w and for all $i \in N$. It is easy to verify that ψ satisfies WFBC and WFPI, but it violates EFF.*

Example 4.4. *Define a solution ψ by for all $(N,d,v) \in \Gamma$, for all weight function w and for all $i \in N$, $\psi_i(N,d,v) = \frac{w(i)\cdot v^*(N)}{|N|_w}$. It is easy to verify that ψ satisfies EFF, but it violates WFBC and WFPI.*

5 Conclusions

1. In the framework of fuzzy TU games, there are several extensions of the Shapley value [16] in literature. Branzei et al. [4] proposed a Shapley value as the average of all marginal vectors which is generated by the orders of all participants. Butnariu [5] defined a Shapley function as a function which maps a fuzzy TU game to a function deriving the Shapley value from a fuzzy coalition, and showed the explicit form of the Shapley function on a limited class of fuzzy TU games. Later, Tsurumi et al. [15] followed Butnariu's approach to investigate a more natural class of fuzzy TU games. By considering supreme-utilities among fuzzy action vectors, Liao et al. [12] introduced the supreme-consistent value. By both considering the participants and their activity levels, Hwang and Liao [9] proposed different fuzzy extension of the Shapley value. One should compare our results with these pre-existing results. There are two major differences among these pre-existing studies and ours:
   (a) Differing from these pre-existing fuzzy extensions of the Shapley values, we propose the supreme-weighted value, the potential approach, the dividend approach and related results by considering supreme-utilities among fuzzy action vectors and weights among participants simultaneously.
   (b) We propose some equivalent relations to characterize the family of all fuzzy solutions that admit a potential on weights. Further, we adopt these equivalent relations to characterize the supreme-weighted value. These weighted equivalent relations and related axiomatic results do not appear in these pre-existing studies on fuzzy TU games.

www.SID.ir

Archive of SID

2. The advantages of our approach are that the supreme-weighted value of a bounded fuzzy TU game always exists and to compute a kind of global value for a given participant by considering supreme-utilities among fuzzy sets and the weights among participants simultaneously. In order to illustrate how the supreme-weighted value and related results can be used and to make its meaning more transparent, we quote the example introduced by Butnariu and Kroupa [6] as follows.
   (Butnariu and Kroupa [6]) Let $(N, [0,1]^N, v) \in \Gamma$ and N be a set of investors and suppose that the capital of each $i \in N$ is c_i. In this model the capital of a participant can be non-positive; in fact, some participants may be in need of capital (in this case an investment of a negative capital is a financing process). For all $\alpha \in [0,1]^N$, α could be treated as a fuzzy coalition. A fuzzy coalition α is seen as an organization meant to achieve some goals, which are common to its members. The endowment of a fuzzy coalition α with the capital it needs for its activities is done by the members and the degree of membership of participant $i \in N$ to fuzzy coalition α is measured by the percent of capital c_i participant i invests in the fuzzy coalition α. Observe that this way of measuring the degree of membership is different from the more usual one in which the degree of membership is measured by the share of coalitional capital a participant owns. It better reflects the risks participants are ready to take over when investing in a specific organization and also their personal interest in realizing the goals the organization is meant to achieve: if a participant with a capital of \$10000 and another participant with a capital of \$1000000 invest the same amount of \$10000 in organization α, it means that the first participant is much more interested in α and, consequently, more personally involved and assuming a higher risk than the second participant for the realization of the goals of α. In that follows we interpret the membership degree of a participant to a fuzzy coalition as a measure of the risk the participant assumes by transferring a part of his capital to the coalition considered as a collective decision maker.

3. Based on the supreme-utilities of fuzzy action vectors and the weights among participants, we introduce an extended Shapley value and related results in this paper. It is reasonable that some traditional solutions could be extended to fuzzy TU games by applying the supreme-utilities of fuzzy action vectors and the weights among participants. This is left to the reader.

Acknowledgement

The authors are grateful to the editor, the associate editor and the anonymous referees for very helpful suggestions and comments.

References

[1] K. Abhishek, S. Rajneesh, V. Pragya, *Lyapunov fuzzy Markov game controller for two link robotic manipulator*, Journal of Intelligent and Fuzzy Systems, **34** (2018), 1479–1490.

[2] J. P. Aubin, *Coeur et valeur des jeux flous à paiements latèraux*, Comptes Rendus de l'Académie des Sciences, **279** (1974), 891–894.

[3] J. P. Aubin, *Cooperative fuzzy games*, Mathematics of Operations Research, **6** (1981), 1–13.

[4] R. Branzei, D. Dimitrov, S. Tijs, *Egalitarianism in convex fuzzy games*, Journal of Mathematics Mathematical Social Sciences, **47** (2004), 313–325.

[5] D. Butnariu, *Stability and Shapley value for an n-persons fuzzy game*, Fuzzy Sets and Systems, **4** (1980), 63–72.

[6] D. Butnariu, T. Kroupa, *Shapley mappings and the cumulative value for n-person games with fuzzy coalitions*, European Journal of Operational Research, **186** (2008), 288–299.

[7] E. Calvo, J.C. Santos, *Potential in cooperative TU-games*, Mathematical Social Sciences, **34** (1997), 175–190.

[8] S. Hart, A. Mas-Colell, *Potential, value and consistency*, Econometrica, **57** (1989), 589–614.

[9] Y. A. Hwang, Y.H. Liao, *The consistent value of fuzzy games*, Fuzzy Sets and Systems, **160** (2009), 644–656.

[10] S. Li, Q. Zhang, *A simplified expression of the Shapley function for fuzzy game*, European Journal of Operational Research, **196** (2009), 234–245.

www.SID.ir

Archive of SID

[11] Y. H. Liao, *Fuzzy games: a complement-consistent solution, axiomatizations and dynamic approaches*, Fuzzy Optimization and Decision Making, **16** (2017), 257–268.

[12] Y. H. Liao, W. Y. Hsieh, L. Y. Chung, *Fuzzy games: a complement-consistent solution, axiomatizations and dynamic approaches*, Journal of Intelligent and Fuzzy Systems, **30** (2016), 3167–3175.

[13] K. M. Ortmann, *Preservation of differences, potential, conservity*, Working paper **236**, Inst. of Math. Ec., University of Bielefeld, 1995. In: Kleinschmidt P., Bachem A., Derigs U., Fischer D., Leopold-Wildburger U., Mhring R. (Eds), Operations Research Proceedings (Operations Research Proceedings **1995**), Springer, Berlin, Heidelberg, 1995, pp. 270-275.

[14] K. M. Ortmann, *Conservation of energy in nonatomic games*, Center for Mathematical Economics Working Papers **237**, Center for Mathematical Economics, Bielefeld University, 1995.

[15] M. Tsurumi, T. Tanino, M. Inuiguchi, *A Shapley function on a class of cooperative fuzzy games*, European Journal of Operational Research **129** (2001), 596–618.

[16] L. S. Shapley, *A value for n-person game*, In: Kuhn, H. W., Tucker, A. W. (Eds.), Contributions to the Theory of Games II (Annals of Mathematics Studies **28**), Princeton, 1953, pp. 307-317.

www.SID.ir

Iranian Journal of Fuzzy Systems
Volume 16, Number 3, (2019), pp. 214
University of Sistan and Baluchestan

Fuzzy transferable-utility games: a weighted allocation and related results

Y. H. Liao and L. Y. Chung

بازی‌های سودمند انتقال‌پذیر : یک توزیع وزین و نتایج آن

چکیده. با در نظر گرفتن فواید برتر در بین مجموعه‌های فازی و وزن‌ها در بین شرکت‌کننده‌ها بطور همزمان، مقدار وزین روی بازی‌های سودمند انتقال‌پذیر را معرفی می‌کنیم. علاوه‌بر آن، برخی از روابط معادل برای توصیف خانواده تمام جواب‌هایی که پتانسیل پذیرش روی وزن‌ها را داراست فراهم می‌آوریم. همچنین روش سود سهام جهت فراهم آوردن نقطه نظر دیگری جهت روش پتانسیل را پیشنهاد می‌کنیم. براساس این روابط معادل، نتایج اصول موضوعه‌ی متعددی جهت ارائه گویایی مقدار وزن دار برتر پیشنهاد می‌کنیم.