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Degrees of M-fuzzy families of independent L-fuzzy sets

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Abstract

The present paper studies fuzzy matroids in view of degree. First we generalize the notion of (L, M)-fuzzy independent structure by introducing the degree of M-fuzzy family of independent L-fuzzy sets with respect to a mapping from L^X to M. Such kind of degrees is proved to satisfy some axioms similar to those satisfied by (L, M)-fuzzy independent structure. Then we define and study some special degrees (e.g. quotient degrees and isomorphism degrees) with respect to mappings between two (L, M)-fuzzy matroid-like spaces in details. Finally we give characterizations of these degrees and investigate relationships between them.

Keywords: Degree of M-fuzzy family of independent L-fuzzy sets, M-fuzzy family of independent L-fuzzy sets, (L, M)-fuzzy matroid.

1 Introduction

In 1935, Whitney introduced Matroid theory [42]. Since then, Birkhoff et al [4, 6, 27, 36] have made further development of matroid theory. Nowadays matroid theory has quite rich contents [41, 28]. There exists a deep connection between matroid theory and convexity theory. If we consider the close set of matroids as convex sets of a finite convex space, then matroids are actually finite convex spaces.

In the middle of the 20th, it was found that many mathematical structures can naturally introduce convex sets [2, 3, 37]. The emergence of axiomatic methods prompted mathematicians to abstractly summarize the properties of convex sets. This leads to generation and development of abstract convex structures theory [3, 5, 37].

With the development of fuzzy mathematics, matroid and convexity have been endowed with fuzzy set theory firstly introduced by Zadeh in 1965 [53].

In the fuzzy convexity of Rosa-Maruyama, the convex set is fuzzy, but the convexity is a crisp subset of I^X or L^X. In 2014, Shi and Xiu [35] proposed the concept of M-fuzzifying convexity which is a new approach to the fuzzification of convex structures. In addition, Shi and Xiu introduced the concept of (L, M)-fuzzy convexity in [50]. So far, there are many other praising work on fuzzy convexity [33, 51, 43, 29].

Goetsche and Voxmar [9] introduced the fuzzy matroid (We call it GV matroid) for the first time in 1988. Their processing method uses a distinct number to describe the potential of support set. In GV matroid theory, an independent structure is a crisp family of fuzzy sets. Subsequently, many concept of GV matroids were proposed and studied [10, 11, 12, 13, 14, 15]. In 2010, Li et al [19] defined refined GV matroids, established the transitivity theorem based on fuzzy circuits and studied the connectedness of GV matroid and refined GV matroids. In 2017 Li and Yi [22] studied some properties and relations of three different kinds of fuzzification for independent systems [9, 7, 17]. In the same year, Li et al [21] introduced three-way decision matroids and a fuzzy independent set system is generated from this.

In 2009, Shi proposed a new fuzzification of matroid theory, namely, M-fuzzifying matroid [31], whose independent structure is a fuzzy family of crisp sets. The study on this topic has achieved rich results. Many concepts in matroids theory have been successfully extended to the M-fuzzifying matroids [20, 23, 34, 39, 40, 26, 44, 48, 49, 52]. A more general fuzzy theory of matroids, (L, M)-fuzzy matroids, is given in 2009 by Shi [32]. Independent structures in this

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theory are fuzzy families of fuzzy sets. When $M = \{0, 1\}$, this kind of fuzzy matroids can be regarded as L-matroids. We notice that there are many studies on L-matroids [18, 24, 45, 46, 47].

The view of degree is widely used in many fields, such as fuzzification of topology category which requires to fuzzify objects and morphisms. Ghareeb and Al-omeri [8] discussed the notions of semi-openness, semi-continuity, preopenness, precontinuity, irresolutness and preirresolutness degree of functions in (L, M)-fuzzy topological spaces based on the implication operation. In 2018, Zhong and Shi [54] gave a definition of the degrees to which a mapping is continuous, open, closed or a quotient mapping with respect to the (L, M)-fuzzy topology degrees. In 2014, Liang and Shi [25] introduced the degree to which a mapping is continuous, open or closed in (L, M)-fuzzy topological spaces by using implication operation. In 2018, Al-Omeri, Khalil and Ghareeb [1] discussed the degree of semi-preopenness, semi-precontinuity, and semi-preirresoluteness for functions in (L, M)-fuzzy pretopology.

In this paper, we will define the degree of (L, M)-fuzzy independent structure to a mapping; this is a generalization of (L, M)-fuzzy independent structures. Then we prove this concept still satisfies some axioms of (L, M)-fuzzy matroids. Next, two kinds of special degrees to a mapping (i.e. the quotient degree to a surjective mapping and the isomorphism degree to a bijective mapping) are defined. Moreover we discuss their characterizations and relationships.

2 Preliminaries

In this paper, unless otherwise stated, $(L, \vee, \wedge)$ and $(M, \vee, \wedge)$ are completely distributive lattices [38]. The smallest element and the largest element in M are denoted by $\top_M$ and $\bot_M$, respectively. X is a non-empty set. We denote the set of all subsets of X by 2^X and denote the set of all L-subsets of X by L^X. The largest element and the smallest element in L^X are denoted by $\top_{L^X}$ and $\bot_{L^X}$ respectively.

An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $Cop(L)$.

The binary relation $\lhd$ in L is defined as follows: for $a, b \in L$, $a \lhd b$ if and only if $\forall D \subseteq L$, the relation $b \leq \sup D$ implies the $\exists d \in D$ s.t. $a \leq d$ [38]. $\beta(b) = \{a \in L \mid a \lhd b\}$ is called the greatest minimal family of b in the sense of [38], and $\beta^*(b) = \beta(b) \cap Cop(L)$. Moreover, the binary relation $\lhd^{op}$ in L is defined as follows: for $a, b \in L$, $a \lhd^{op} b$ if and only if $\forall D \subseteq L$, the relation $b \geq \inf D$ implies $\exists d \in D$ s.t. $a \geq d$. $\alpha(b) = \{a \in L \mid a \lhd^{op} b\})$ is called the greatest maximal family of b in the sense of [38]. In a completely distributive lattice L, $\alpha(b)$, $\beta(b)$ are exists, and $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ [38].

Lemma 2.1. [38] *Let L be a completely distributive lattice and $\{a_i \mid i \in I\} \subseteq L$. Then*

(1) $\beta\left(\bigvee_{i \in I} a_i\right) = \bigcup_{i \in I} \beta(a_i)$, *i.e., β is an union-preserving mapping.*

(2) $\alpha\left(\bigwedge_{i \in I} a_i\right) = \bigcup_{i \in \omega} \alpha(a_i)$, *i.e., α is an $\wedge - \cup$ mapping.*

Definition 2.2. [30] *Let $A \in L^X$, $a \in L$. Define*

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \quad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\},$$
$$A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}, \quad A^{(a)} = \{x \in X \mid (A(x) \not\leq a\}.$$

Definition 2.3. [32] *Let E be a finite set. A mapping $\mathcal{I} : L^E \to M$ is called an M-fuzzy family of independent L-fuzzy sets on E if it satisfies the following three conditions:*

(LMFI1) $\mathcal{I}(\chi_\emptyset) = \top_M$.

(LMFI2) For any $A, B \in L^E$, if $A \subseteq B$, then $\mathcal{I}(A) \geq \mathcal{I}(B)$.

(LMFI3) If $b = |B|(n) \not\leq |A|(n)$ for $A, B \in L^E$ and for some $n \in \mathbb{N}$, then

$$\bigvee_{e \in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big) \geq \mathcal{I}(A) \wedge \mathcal{I}(B).$$

If $\mathcal{I}$ is an M-fuzzy family of independent L-fuzzy sets on E, then the pair $(E, \mathcal{I})$ is called an (L, M)-fuzzy matroid.

Remark 2.4. *For an (L, M)-fuzzy matroid, if $L = \{0, 1\}$, then it become an M-fuzzifying matroid* [31]*; if $M = \{0, 1\}$, then it become an L-matroid* [32]*; if $L = M = \{0, 1\}$, then it become a matroid* [41].

Theorem 2.5. [32] *Let E be a finite set, and $\mathcal{I} : L^E \longrightarrow M$ a mapping. Then the following conditions are equivalent:*

(1) $(E, \mathcal{I})$ is an (L, M)-fuzzy matroid.

(2) For each $a \in Cop(M)$, $(E, \mathcal{I}_{[a]})$ is an L-matroid.

(3) For each $a \in P(M)$, $(E, \mathcal{I}^{(a)})$ is an L-matroid.

We can define a residual implication by $a \to b = \bigvee\{c \in M \mid a \wedge c \leq b\}$ in M. Also, we denote $a \leftrightarrow b = (b \to a) \wedge (a \to b)$. Some properties of the implication operation are listed in the following lemma.

Lemma 2.6. [16] *Let $(M, \vee, \wedge)$ be a completely distributive lattice, and $\to$ the implication operation corresponding to $\wedge$. Then the following statements hold ($a, b, c \in M$, $\{a_i\}_{i\in I}$, $\{b_i\}_{i\in I} \subseteq M$):*

(a) $\top_M \to a = a$.

(b) $(a \to b) \geq c \Longleftrightarrow a \wedge c \leq b$.

(c) $a \to b = \top_M \Longleftrightarrow a \leq b$.

(d) $a \to \left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I}(a \to b_i)$, hence $a \to b \leq a \to c$ whenever $b \leq c$.

(e) $\left(\bigvee_{i\in I} a_i\right) \to b = \bigwedge_{i\in I}(a \to b_i)$, hence $a \to c \geq b \to c$ whenever $a \leq b$.

(f) $(a \to c) \wedge (c \to b) \leq a \to b$.

3 Degree of M-fuzzy family of independent L-fuzzy sets

In this section, we will define the degree to which a mapping $\mathcal{I} : L^E \longrightarrow M$ is an M-fuzzy family of independent L-fuzzy sets, and discuss its characterizations.

Definition 3.1. *Let E be a finite set, $\mathcal{I} : L^E \longrightarrow M$ a mapping, and $D(\mathcal{I}) = D_1(\mathcal{I}) \wedge D_2(\mathcal{I}) \wedge D_3(\mathcal{I})$, where $D_1(\mathcal{I}) = \mathcal{I}(\chi_\emptyset)$,*

$$D_2(\mathcal{I}) = \bigwedge_{\substack{A,B\in L^E \\ A\subseteq B}} \mathcal{I}(B) \to \mathcal{I}(A),$$

$$D_3(\mathcal{I}) = \bigwedge_{A,B\in L^E} \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\nleq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}(A) \wedge \mathcal{I}(B) \to \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big).$$

Then $D(\mathcal{I})$ is called the degree to which $\mathcal{I}$ is an M-fuzzy family of independent L-fuzzy sets, or the degree of an M-fuzzy family of independent L-fuzzy sets with respect to $\mathcal{I}$.

It is not difficult to verify that $D(\mathcal{I}) = \top_M$ if and only if $\mathcal{I}$ is an M-fuzzy family of independent L-fuzzy sets. Take $M = [0, 1]$ and $\mathcal{I}(A) = 0.5$ $(\forall A \in L^E)$, then we have $D(\mathcal{I}) \neq \top_M$.

Theorem 3.2. *Let E be a finite set, $\mathcal{I} : L^E \longrightarrow M$ a mapping. Then*

$$D(\mathcal{I}) = \bigvee\{a \in M \mid \forall u \in \beta(a), \mathcal{I}_{[u]} \text{ is a family of independent } L\text{-fuzzy sets}\}.$$

Proof. Assume $a \in M$ with $a \leq D(\mathcal{I})$. Then $a \leq D_i(\mathcal{I})$ $(i = 1, 2, 3)$. For each $u \lhd a$, as $a \leq D_1(\mathcal{I})$ and $u \lhd a$ (thus $u \leq a$), we have $u \leq D_1(\mathcal{I})$, which means $\chi_\emptyset \in \mathcal{I}_{[u]}$. Since $a \leq D_2(\mathcal{I})$, we have $a \leq \mathcal{I}(B) \to \mathcal{I}(A)$ for any $A, B \in L^E$ satisfying $A \subseteq B$. By Lemma 2.6, $\mathcal{I}(B) \wedge a \leq \mathcal{I}(A)$. If $B \in \mathcal{I}_{[u]}$, then $u \leq \mathcal{I}(B) \wedge a \leq \mathcal{I}(A)$, which means $A \in \mathcal{I}_{[u]}$. As $u \lhd a \leq D_3(\mathcal{I})$, we have $(\forall A, B \in L^E)$

$$a \leq \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\nleq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}(A) \wedge \mathcal{I}(B) \to \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big).$$

Analogously,

$$v \lhd \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\not\leq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}(A)\wedge\mathcal{I}(B)\to\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big) \quad (\forall A,B\in L^E).$$

As β is union-preserving (see Lemma 2.1), for any $A,B\in L^E$, there exists an $n\in\mathbb{N}$, a $b=|B|(n)\not\leq|A|(n)$, and $e\in F(A,B)$, such that $u\lhd\big(\mathcal{I}(A)\wedge\mathcal{I}(B)\to\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big)\big)$, which implies $u\leq\big(\mathcal{I}(A)\wedge\mathcal{I}(B)\to\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big)\big)$. By Lemma 2.6, $\mathcal{I}(A)\wedge\mathcal{I}(B)\wedge u\leq\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big)$. It follows that, for any $A,B\in\mathcal{I}_{[u]}$ (i.e. $u\leq\mathcal{I}(A)\wedge\mathcal{I}(B)$), there exists an $n\in\mathbb{N}$, a $b=|B|(n)\not\leq|A|(n)$, and an $e\in F(A,B)$, such that $u=u\wedge\mathcal{I}(A)\wedge\mathcal{I}(B)\leq\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big)$, i.e., $(b\wedge A_{[b]})\cup e_b\in\mathcal{I}_{[u]}$. Since a is arbitrary, $D(\mathcal{I})$ is not greater than right-hand side.

Conversely, suppose that, for each $u\in\beta(a)$, $\mathcal{I}_{[u]}$ is a family of independent L-fuzzy sets. As $\chi_\emptyset\in\mathcal{I}_{[u]}$, $u\leq\mathcal{I}(\chi_\emptyset)$. If $A,B\in L^E$ with $A\subseteq B$ and $B\in\mathcal{I}_{[u]}$, then $A\in\mathcal{I}_{[u]}$. We have $u=\mathcal{I}(B)\wedge u\leq\mathcal{I}(A)$. By Lemma 2.6, $u\leq\mathcal{I}(B)\to\mathcal{I}(A)$, i.e.,

$$u\leq\bigwedge_{\substack{A,B\in L^E \\ A\subseteq B}}\mathcal{I}(B)\to\mathcal{I}(A).$$

If for any $A,B\in\mathcal{I}_{[u]}$ and $b=|B|(n)\not\leq|A|(n)$ (for some $n\in\mathbb{N}$), there exist an $e\in F(A,B)$ such that $(b\wedge A_{[b]})\cup e_b\in\mathcal{I}_{[u]}$. Then $u=\mathcal{I}(A)\wedge\mathcal{I}(B)\wedge u\leq\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big)$. By Lemma 2.6, $u\leq\mathcal{I}(A)\wedge\mathcal{I}(B)\to\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big)$. So we have

$$u\leq\bigwedge_{A,B\in L^E}\bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\not\leq|A|(n)}}\bigvee_{e\in F(A,B)}\mathcal{I}(A)\wedge\mathcal{I}(B)\to\mathcal{I}\big((b\wedge A_{[b]})\cup e_b\big).$$

By arbitrariness of u and $a=\bigvee\beta(a)$, we know $a\leq D_i(\mathcal{I})$ for $i=1,2,3$. Thus, the right-hand side is not greater than $D(\mathcal{I})$. □

In Definition 3.1 and Theorem 3.2, if let $L=\{0,1\}$, then LMF3 is equal to MF3 [32]. So we can define the degree of M-fuzzy family of independent sets and its characterization. The next concept is a kind of independent degree of an L-fuzzy set.

Definition 3.3. *Let $\mathcal{I}:L^E\longrightarrow M$. For each $A\in L^E$, $M_{\mathcal{I}}(A)=D(\mathcal{I})\wedge\mathcal{I}(A)$ is called the degree of A is an independent L-fuzzy set with respect to $\mathcal{I}$ or the independent L-fuzzy set degree of A with respect to $\mathcal{I}$.*

Remark 3.4. *If $D(\mathcal{I})=\top_M$ (it means $\mathcal{I}$ is an M-fuzzy family of independent L-fuzzy sets), then $M_{\mathcal{I}}(A)=\mathcal{I}(A)$. So we can regard it as a generalization of the degree of M-fuzzy family of independent L-fuzzy sets.*

Theorem 3.5. *Let $\mathcal{I}:L^E\longrightarrow M$ be a mapping, $M_{\mathcal{I}}(A)$ be defined as in Definition 3.3 ($A\in L^E$). Then*

(1) $M_{\mathcal{I}}(\{\chi_\emptyset\})=D(\mathcal{I})$.

(2) For any $A,B\in L^E$, if $A\subseteq B$, then $M_{\mathcal{I}}(B)\leq M_{\mathcal{I}}(A)$.

(3) For any $A,B\in L^E$, if $b=|B|(n)\not\leq|A|(n)$ for some $n\in\mathbb{N}$, then

$$\bigvee_{e\in F(A,B)}M_{\mathcal{I}}\big((b\wedge A_{[b]})\cup e_b\big)\geq M_{\mathcal{I}}(A)\wedge M_{\mathcal{I}}(B).$$

Proof. (1) Obviously.

(2) For each $A,B\in L^E$ with $A\subseteq B$, since

$$M_{\mathcal{I}}(B)=D(\mathcal{I})\wedge\mathcal{I}(B)\leq D_2(\mathcal{I})\wedge\mathcal{I}(B)=\left(\bigwedge_{\substack{H,K\in L^E \\ H\subseteq K}}\mathcal{I}(K)\to\mathcal{I}(H)\right)\wedge\mathcal{I}(B)\leq\big(\mathcal{I}(B)\to\mathcal{I}(A)\big)\wedge\mathcal{I}(B),$$

$\big(\mathcal{I}(B)\to\mathcal{I}(A)\big)\wedge\mathcal{I}(B)\leq\mathcal{I}(A)$ by Lemma 2.6 ($a\wedge(a\to b)\leq b$). Thus, for each $A,B\in L^E$ with $A\subseteq B$, we have $M_{\mathcal{I}}(B)\leq\mathcal{I}(A)$. As $M_{\mathcal{I}}(B)\leq D(\mathcal{I})$, $M_{\mathcal{I}}(B)\leq M_{\mathcal{I}}(A)$.

(3) For each $A, B \in L^E$ with $|B|(n) \nleq |A|(n)$ for some $n \in \mathbb{N}$, we have

$$\begin{aligned} M_{\mathcal{I}}(A) \wedge M_{\mathcal{I}}(B) &= D(\mathcal{I}) \wedge \mathcal{I}(A) \wedge \mathcal{I}(B) \leq D_3(\mathcal{I}) \wedge \mathcal{I}(A) \wedge \mathcal{I}(B) \\ &= \mathcal{I}(A) \wedge \mathcal{I}(B) \wedge \\ &\quad \bigwedge_{H,K \in L^E} \bigvee_{\substack{n\in\mathbb{N} \\ b=|K|(n)\nleq|H|(n)}} \bigvee_{e\in F(H,K)} \mathcal{I}(H) \wedge \mathcal{I}(K) \to \mathcal{I}\big((b \wedge H_{[b]}) \cup e_b\big) \\ &\leq \mathcal{I}(A) \wedge \mathcal{I}(B) \wedge \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\nleq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}(A) \wedge \mathcal{I}(B) \to \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big) \\ &= \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\nleq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}(A) \wedge \mathcal{I}(B) \wedge \Big(\mathcal{I}(A) \wedge \mathcal{I}(B) \to \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big)\Big) \\ &\leq \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\nleq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big) \quad \text{(by Lemma 2.6).} \end{aligned}$$

For each $a \leq \big(M_{\mathcal{I}}(A) \wedge M_{\mathcal{I}}(B)\big)$ and each $u \lhd a$, we have

$$a \leq \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\nleq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big).$$

Thus

$$u \lhd \bigvee_{\substack{n\in\mathbb{N} \\ b=|B|(n)\nleq|A|(n)}} \bigvee_{e\in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big).$$

As β is union-preserving, there exists an $n \in \mathbb{N}$ and a $b = |B|(n) \nleq |A|(n)$ such that

$$u \lhd \bigvee_{e\in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big).$$

Thus

$$u \leq \bigvee_{e\in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big).$$

As $a = \bigvee \beta(a)$ and arbitrariness of u and a, we know

$$M_{\mathcal{I}}(A) \wedge M_{\mathcal{I}}(B) \leq \bigvee_{e\in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big).$$

As $M_{\mathcal{I}}(A) \wedge M_{\mathcal{I}}(B) \leq D(\mathcal{I})$, for some $n \in \mathbb{N}$ and $b = |B|(n) \nleq |A|(n)$,

$$M_{\mathcal{I}}(A) \wedge M_{\mathcal{I}}(B) \leq D(\mathcal{I}) \wedge \bigvee_{e\in F(A,B)} \mathcal{I}\big((b \wedge A_{[b]}) \cup e_b\big) = \bigvee_{e\in F(A,B)} M_{\mathcal{I}}\big((b \wedge A_{[b]}) \cup e_b\big).$$

□

It can be seen that a mapping $M_{\mathcal{I}} : L^E \longrightarrow M$ defined by Definition 3.3 satisfies (LMFI2) and (LMFI3).

Theorem 3.6. *Let E be a finite set, $\mathcal{I} : L^E \longrightarrow M$ a mapping, $M_{\mathcal{I}}$ the independent L-fuzzy set degree of A with respect to $\mathcal{I}$. Then*
$M_{\mathcal{I}}(D) = \bigvee\{a \in M \mid \forall u \in \beta(a), \mathcal{I}_{[u]}$ *is a family of independent L-fuzzy sets and* $A \in \mathcal{I}_{[a]}\}$.

Proof. It follows from Definition 3.3 and Theorem 3.2. □

4 Degree of two kinds of special mappings

Mappings from a matroid $(X, \mathcal{I}_X)$ to another matroid $(Y, \mathcal{I}_Y)$ are called weak mappings [41] provided that $B \in \mathcal{I}_Y$ implies $f^{\leftarrow}(B) \in \mathcal{I}_X$. Some mappings map independent sets to independent sets, i.e. $A \in \mathcal{I}_X$ implicates $f(A) \in \mathcal{I}_Y$. In this section, we define two kinds of degrees of these spacial properties. Unless otherwise stated, X, Y are finite sets.

Definition 4.1. *Let $\mathcal{I}_X : L^X \longrightarrow M$, $\mathcal{I}_Y : L^Y \longrightarrow M$, and $f : X \longrightarrow Y$ be mappings. Then*

(1) The weak degree of f with respect to $\mathcal{I}_X$ and $\mathcal{I}_Y$ is defined by

$$RI(f) = \bigwedge_{B \in L^X} \Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f^{\leftarrow}(B)) \Big).$$

(2) The independent-to-independent degree of f with respect to $\mathcal{I}_X$ and $\mathcal{I}_Y$ is defined by

$$II(f) = \bigwedge_{A \in L^X} \Big(M_{\mathcal{I}_X}(A) \to M_{\mathcal{I}_Y}(f^{\rightarrow}(A)) \Big).$$

Theorem 4.2. *Given two mappings $\mathcal{I}_x : L^X \longrightarrow M$ and $\mathcal{I}_Y : L^Y \longrightarrow M$, Let $D(\mathcal{I}_X)$ and $D(\mathcal{I}_Y)$ denote the degree of M-fuzzy family of independent L-fuzzy sets with respect to $\mathcal{I}_X$ and $\mathcal{I}_Y$. Then*

(1) $RI(f) = \bigvee \left\{ a \in M \mid D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a \leq D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B)), \forall B \in L^X \right\}$.

(2) $RI(f) = \bigvee \left\{ a \in M \;\middle|\; \begin{array}{l} \forall b \leq D(\mathcal{I}_Y) \wedge a, \forall B \in (\mathcal{I}_Y)_{[b]}, \\ b \leq D(\mathcal{I}_X), f^{\leftarrow}(B) \in (\mathcal{I}_X)_{[b]} \end{array} \right\}$.

(3) $RI(f) = \bigvee \left\{ a \in M \;\middle|\; \begin{array}{l} \forall b \notin \alpha(D(\mathcal{I}_Y) \wedge a), \forall B \in (\mathcal{I}_Y)^{[b]}, \\ b \notin \alpha(D(\mathcal{I}_X)), f^{\leftarrow}(B) \in (\mathcal{I}_X)^{[b]} \end{array} \right\}$.

Proof. (1) For each $a \in M$, $a \leq RI(f)$ if and only if for each $B \in L^Y$, $a \leq M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f^{\leftarrow}(B))$, which means $M_{\mathcal{I}_Y}(B) \wedge a \leq M_{\mathcal{I}_X}(f^{\leftarrow}(B))$. Thus $D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a \leq D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B))$.

(2) By (1), for each a satisfies $D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a \leq D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B))$ (for any $B \in L^X$), if $b \leq D(\mathcal{I}_Y) \wedge a$ and $B \in (\mathcal{I}_Y)_{[b]}$ (i.e., $b \leq \mathcal{I}_Y(B)$), then $b \leq D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B))$, which means $b \leq D(\mathcal{I}_X)$ and $f^{\leftarrow}(B) \in (\mathcal{I}_X)_{[b]}$. Hence we have $RI(f)$ is not greater than the right-hand side.

Conversely, for each $a \in M$, if $b \leq D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a$, then $b \leq D(\mathcal{I}_Y) \wedge a$ and $b \leq \mathcal{I}_Y(B)$, i.e., $B \in (\mathcal{I}_Y)_{[b]}$. Thus $b \leq D(\mathcal{I}_X)$ and $f^{\leftarrow}(B) \in (\mathcal{I}_X)_{[b]}$, i.e., $\mathcal{I}_X(f^{\leftarrow}(B)) \geq b$, which means $D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B)) \geq b$. For arbitrariness of $b \in M$, we know $D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a \leq D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B))$, and combining with (1), the right-hand side is not greater than $RI(f)$.

(3) Suppose $a \in M$ satisfying $D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a \leq D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B))$ (for any $B \in L^X$), for each $B \in (\mathcal{I}_Y)^{[b]}$, if $b \notin \alpha(D(\mathcal{I}_Y) \wedge a)$, then $b \notin \alpha(D(\mathcal{I}_Y)) \cup \alpha(a)$ (by Lemma 2.1), and for $b \notin \alpha(\mathcal{I}_Y(B))$, by Lemma 2.1, $b \notin \alpha\big(D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a\big)$. Since (1) and α is order-reserving, $b \notin \alpha(D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B)))$, we can know $b \notin \alpha(D(\mathcal{I}_X)) \cup \alpha(\mathcal{I}_X(f^{\leftarrow}(B)))$, which means $RI(f)$ is not greater than the right-hand side.

Conversely, for each $a \in M$, if $b \notin \alpha(D(\mathcal{I}_Y) \wedge a)$ and $b \notin \alpha(\mathcal{I}_Y(B))$, then $b \notin \alpha(D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a)$, which implies $b \notin \alpha(D(\mathcal{I}_X))$ and $f^{\leftarrow}(B) \in (\mathcal{I}_X)^{[b]}$. Thus $b \notin \alpha(D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B)))$. As $x \notin \alpha(S)$ implies $x \notin \alpha(T)$ (i.e. $x \in \alpha(T)$ implies $x \in \alpha(S)$), $S \leq T$. It follows $D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(B) \wedge a \leq D(\mathcal{I}_X) \wedge \mathcal{I}_X(f^{\leftarrow}(B))$, which means the right-hand side is not greater than $RI(f)$. □

Theorem 4.3. *Given two mappings $\mathcal{I}_x : L^X \longrightarrow M$ and $\mathcal{I}_Y : L^Y \longrightarrow M$, Let $D(\mathcal{I}_X)$ and $D(\mathcal{I}_Y)$ denote the degree of M-fuzzy family of independent L-fuzzy sets with respect to $\mathcal{I}_X$ and $\mathcal{I}_Y$. Then*

(1) $II(f) = \bigvee \left\{ a \in M \mid D(\mathcal{I}_X) \wedge \mathcal{I}_X(A) \wedge a \leq D(\mathcal{I}_Y) \wedge \mathcal{I}_Y(f^{\rightarrow}(A)), \forall A \in L^X \right\}$.

(2) $II(f) = \bigvee \left\{ a \in M \;\middle|\; \begin{array}{l} \forall b \leq D(\mathcal{I}_X) \wedge a, \forall A \in (\mathcal{I}_X)_{[b]} \\ b \leq D(\mathcal{I}_Y), f^{\rightarrow}(A) \in (\mathcal{I}_Y)_{[b]} \end{array} \right\}$.

(3) $II(f) = \bigvee \left\{ a \in M \;\middle|\; \begin{array}{l} \forall b \notin \alpha(D(\mathcal{I}_X) \wedge a), \forall A \in (\mathcal{I}_X)^{[b]} \\ b \notin \alpha(D(\mathcal{I}_Y)), f^{\rightarrow}(A) \in (\mathcal{I}_Y)^{[b]} \end{array} \right\}$.

The proof is similar to Theorem 4.2.

Definition 4.4. *Given two mappings $\mathcal{I}_X : L^X \longrightarrow M$, $\mathcal{I}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then the isomorphism degree of f with respect to $\mathcal{I}_X$ and $\mathcal{I}_Y$ defined by $ISO(f) = RI(f) \wedge RI(f^{-1})$, where f^{-1} is the inverse mapping of f.*

Theorem 4.5. *Given two mappings $\mathcal{I}_X : L^X \longrightarrow M$, $\mathcal{I}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then $RI(f^{-1}) = II(f)$ and $ISO(f) = RI(f) \wedge RI(f^{-1}) = RI(f) \wedge II(f)$.*

Proof. As f is bijective, $(f^{-1})_L^{\leftarrow}(A) = f_L^{\rightarrow}(A)$ (for each $A \in L^X$), then

$$RI(f^{-1}) = \bigwedge_{A\in L^X} \Big(M_{\mathcal{I}_X}(A) \to M_{\mathcal{I}_Y}((f^{-1})^{\leftarrow}(A)) \Big) = \bigwedge_{A\in L^X} \Big(M_{\mathcal{I}_X}(A) \to M_{\mathcal{I}_Y}(f^{\rightarrow}(A)) \Big) = II(f).$$

Hence $ISO(f) = RI(f) \wedge RI(f^{-1}) = RI(f) \wedge II(f)$. □

Theorem 4.6. *Given three mappings $\mathcal{I}_X : L^X \longrightarrow M$, $\mathcal{I}_Y : L^Y \longrightarrow M$, $\mathcal{I}_Z : L^Z \longrightarrow M$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be mappings. Then*

(1) $II(f) \wedge RI(g) \leq RI(g \circ f)$.

(2) $II(f) \wedge II(g) \leq II(g \circ f)$.

(3) if f, g are bijective, then $ISO(f) \wedge ISO(g) \leq ISO(g \circ f)$.

Proof. (1) $(g \circ f)_L^{\leftarrow}(D) = f_L^{\leftarrow}(g_L^{\leftarrow}(D))$ for any $D \in L^Z$ By Lemma 2.6, we know

$$\begin{aligned}
RI(f) \wedge RI(g) &= \left(\bigwedge_{B\in 2^Y} \Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B)) \Big) \right) \wedge \left(\bigwedge_{C\in L^Z} \Big(M_{\mathcal{I}_Z}(C) \to M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C)) \Big) \right) \\
&\leq \left(\bigwedge_{D\in L^Z} \Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(D)) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(g_L^{\leftarrow}(D))) \Big) \right) \wedge \left(\bigwedge_{C\in L^Z} \Big(M_{\mathcal{I}_Z}(C) \to M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C)) \Big) \right) \\
&= \left(\bigwedge_{D\in L^Z} \Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(D)) \to M_{\mathcal{I}_X}(g \circ f)_L^{\leftarrow}(D) \Big) \right) \wedge \left(\bigwedge_{C\in L^Z} \Big(M_{\mathcal{I}_Z}(C) \to M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C)) \Big) \right) \\
&= \bigwedge_{C\in L^Z} \left(\Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C)) \to M_{\mathcal{I}_X}((g \circ f)_L^{\leftarrow}(C)) \Big) \wedge \Big(M_{\mathcal{I}_Z}(C) \to M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C)) \Big) \right) \\
&\leq \bigwedge_{C\in L^X} \Big(M_{\mathcal{I}_Z}(C) \to M_{\mathcal{I}_X}(g \circ f)_L^{\leftarrow}(C) \Big) = RI(g \circ f).
\end{aligned}$$

The proof to (2) and (3) is similar. □

Theorem 4.7. *Given three mappings $\mathcal{I}_X : L^X \longrightarrow M$, $\mathcal{I}_Y : L^Y \longrightarrow M$, $\mathcal{I}_Z : L^Z \longrightarrow M$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be mappings. Then*

(1) If f is surjective,then $RI(f) \wedge II(g \circ f) \leq II(g)$.

(2) If g is injective, then $RI(g) \wedge II(g \circ f) \leq II(f)$.

Proof. (1) f is surjective, so $f_L^{\rightarrow}(f_L^{\leftarrow})(D) = D$ $\forall D \in L^Y$. Then$(g \circ f)_L^{\rightarrow}(f_L^{\leftarrow}(D)) = g_L^{\rightarrow}(f_L^{\rightarrow}(f_L^{\leftarrow}(D))) = g_L^{\rightarrow}(D)$. By Lemma 2.6, we know

$$\begin{aligned}
RI(f) \wedge II(g \circ f) &= \left(\bigwedge_{B\in L^Y} \Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B)) \Big) \right) \wedge \left(\bigwedge_{A\in L^x} \Big(M_{\mathcal{I}_X}(A) \to M_{\mathcal{I}_Z}((g \circ f)_L^{\rightarrow}(A)) \Big) \right) \\
&\leq \left(\bigwedge_{B\in L^Y} \Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B)) \Big) \right) \wedge \left(\bigwedge_{D\in L^Y} \Big(M_{\mathcal{I}_X}(f_L^{\leftarrow}(D)) \to M_{\mathcal{I}_Z}\big((g \circ f)_L^{\rightarrow}(f_L^{\leftarrow}(D))\big) \Big) \right) \\
&= \left(\bigwedge_{B\in L^Y} \Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B)) \Big) \right) \wedge \left(\bigwedge_{D\in L^Y} \Big(M_{\mathcal{I}_X}(f_L^{\leftarrow}(D)) \to M_{\mathcal{I}_Z}(g_L^{\rightarrow}(D)) \Big) \right) \\
&= \bigwedge_{B\in L^Y} \left(\Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B)) \Big) \wedge \Big(M_{\mathcal{I}_X}(f_L^{\leftarrow}(B)) \to M_{\mathcal{I}_Z}(g_L^{\rightarrow}(B)) \Big) \right) \\
&\leq \bigwedge_{B\in L^Y} \Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_Z}(g_L^{\rightarrow}(B)) \Big) = II(g).
\end{aligned}$$

(2) g is injective, so $g_L^{\leftarrow}(g_L^{\rightarrow}(B)) = B$ $(\forall B \in L^Y)$, then $g_L^{\leftarrow}((g \circ f)_L^{\rightarrow}(D)) = f_L^{\rightarrow}(D)$ $(\forall D \in L^X)$. By Lemma 2.6, we know

$$
\begin{aligned}
RI(g)\wedge II(g\circ f) &= \Big(\bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C)\to M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\Big)\Big)\wedge\Big(\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_X}(A)\to M_{\mathcal{I}_Z}((g\circ f)_L^{\rightarrow}(A))\Big)\Big)\\
&\le \Big(\bigwedge_{D\in L^X}\Big(M_{\mathcal{I}_Z}((g\circ f)_L^{\rightarrow}(D))\to M_{\mathcal{I}_Y}(g_L^{\leftarrow}((g\circ f)_L^{\rightarrow}(D)))\Big)\Big)\wedge\Big(\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_X}(A)\to M_{\mathcal{I}_Z}((g\circ f)_L^{\rightarrow}(A))\Big)\Big)\\
&= \Big(\bigwedge_{D\in L^X}\Big(M_{\mathcal{I}_Z}(g_L^{\rightarrow}(f_L^{\rightarrow}(D)))\to M_{\mathcal{I}_Y}(f_L^{\rightarrow}(D))\Big)\Big)\wedge\Big(\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_X}(A)\to M_{\mathcal{I}_Z}(g_L^{\rightarrow}(f_L^{\rightarrow}(A)))\Big)\Big)\\
&= \bigwedge_{A\in L^X}\Big(\Big(M_{\mathcal{I}_Z}(g_L^{\rightarrow}(f_L^{\rightarrow}(A)))\to M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\Big)\wedge\Big(M_{\mathcal{I}_X}(A)\to M_{\mathcal{I}_Z}(g_L^{\rightarrow}(f_L^{\rightarrow}(A)))\Big)\Big)\\
&\le \bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_X}(A)\to M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\Big)=II(f).
\end{aligned}
$$

□

Lemma 4.8. *Given two mappings $\mathcal{I}_X : L^X \longrightarrow M$ and $\mathcal{I}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then*

(1) $RI(f)=\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\to M_{\mathcal{I}_X}(A)\Big)$.

(2) $II(f)=\bigwedge_{D\in L^Y}\Big(M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\to M_{\mathcal{I}_Y}(B)\Big)$.

Proof. (1) On one hand,

$$\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\to M_{\mathcal{I}_X}(A)\Big)=\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\to M_{\mathcal{I}_X}(f_L^{\leftarrow}(f_L^{\rightarrow}(A)))\Big)\le\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B)\to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big)=RI(f).$$

On the other hand,

$$RI(f)=\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(f_L^{\rightarrow}(f_L^{\leftarrow}(B)))\to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big)\ge\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\to M_{\mathcal{I}_X}(A)\Big).$$

Then we know $RI(f)=\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\to M_{\mathcal{I}_X}(A)\Big)$.

(2) The proof is similar to (1). □

Theorem 4.9. *Given two mappings $\mathcal{I}_X : L^X \longrightarrow M$ and $\mathcal{I}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a bijective mapping. Then*

$$ISO(f)=\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_X}(A)\leftrightarrow M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\Big)=\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B)\leftrightarrow M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big).$$

5 Degree of quotient mappings

A surjection can be regarded as a quotient, for a surjection between $(X,\mathcal{I}_X)$ and $(Y,\mathcal{I}_Y)$ which are two (L,M)-fuzzy matroid-like spaces , we could consider the quotient degree with respect to $f : X \longrightarrow Y$. And in this section supposing X and Y are finite sets.

Definition 5.1. *Given two mappings $\mathcal{I}_X : L^X \longrightarrow M$ and $\mathcal{I}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ be a surjective mapping, Then*

$$QU(f)=\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B)\leftrightarrow M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big)$$

is called quotient degree of f with respect to $\mathcal{I}_X$ and $\mathcal{I}_Y$. It is easy to know $QU(f)\le RI(f)$.

Theorem 5.2. *Given two mappings $\mathcal{I}_X : L^X \longrightarrow M$ and $\mathcal{I}_Y : L^Y \longrightarrow M$, Let $D(\mathcal{I}_X)$ and $D(\mathcal{I}_Y)$ be the degrees of M-fuzzy family of independent L-fuzzy sets $\mathcal{I}_X$ and $\mathcal{I}_Y$, Let $f : X \longrightarrow Y$ ba a surjective mapping. Then*
$QU(f)=\bigvee\{a\in M \mid D(\mathcal{I}_Y)\wedge\mathcal{I}_Y(B)\wedge a\le D(\mathcal{I}_X)\wedge\mathcal{I}_X(f_L^{\leftarrow}(B))), D(\mathcal{I}_X)\wedge\mathcal{I}_X(f_L^{\leftarrow}B)\wedge a\le D(\mathcal{I}_Y)\wedge\mathcal{I}_Y(B)), \forall B\in L^X\}$.

The proof is similar to that of Theorem 4.2.

Theorem 5.3. *Given two mappings $\mathcal{I}_X : L^X \longrightarrow M$ and $\mathcal{I}_Y : L^Y \longrightarrow M$. Let $f : X \longrightarrow Y$ ba a surjective mapping. Then $RI(f)\wedge II(f)\le QU(f)$.*

Proof. f is surjective, so we have $f_L^{\rightarrow}(f_L^{\leftarrow}(D)) = D, \forall D \in L^Y$. Then

$$\begin{aligned}
RI(f) \wedge II(f) &= \left(\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big)\right) \wedge \left(\bigwedge_{A\in L^X}\Big(M_{\mathcal{I}_X}(A) \to M_{\mathcal{I}_Y}(f_L^{\rightarrow}(A))\Big)\right)\\
&\le \left(\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big)\right) \wedge \left(\bigwedge_{D\in L^Y}\Big(M_{\mathcal{I}_X}(f_L^{\leftarrow}(D)) \to M_{\mathcal{I}_Y}(f_L^{\rightarrow}(f_L^{\leftarrow}(D)))\Big)\right)\\
&= \left(\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big)\right) \wedge \left(\bigwedge_{D\in L^Y}\Big(M_{\mathcal{I}_X}(f_L^{\leftarrow}(D)) \to M_{\mathcal{I}_Y}(D)\Big)\right)\\
&= \bigwedge_{A\in L^X}\Big(\Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big) \wedge \Big(M_{\mathcal{I}_X}(f_L^{\leftarrow}(B)) \to M_{\mathcal{I}_Y}(B)\Big)\Big)\\
&\le \bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B) \leftrightarrow M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big) = QU(f).
\end{aligned}$$

□

Theorem 5.4. *Given three mappings $\mathcal{I}_X : L^X \longrightarrow M$, $\mathcal{I}_Y : L^Y \longrightarrow M$ and $\mathcal{I}_Z : L^Z \longrightarrow M$. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be surjective mappings. Then*

(1) $QU(f) \wedge QU(g) \le QU(g \circ f)$.

(2) $QU(g \circ f) \wedge RI(f) \wedge RI(g) \le QU(g)$.

Proof. (1)

$$\begin{aligned}
QU(f) \wedge QU(g) &= \left(\bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B) \leftrightarrow M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big)\right) \wedge \left(\bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C) \leftrightarrow M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\Big)\right)\\
&\le \left(\bigwedge_{D\in L^Z}\Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(D)) \leftrightarrow M_{\mathcal{I}_X}(f_L^{\leftarrow}(g_L^{\leftarrow}(D)))\Big)\right) \wedge \left(\bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C) \leftrightarrow M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\Big)\right)\\
&= \left(\bigwedge_{D\in L^Z}\Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(D)) \leftrightarrow M_{\mathcal{I}_X}((g\circ f)_L^{\leftarrow}(D))\Big)\right) \wedge \left(\bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C) \leftrightarrow M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\Big)\right)\\
&= \bigwedge_{C\in L^Z}\Big(\big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C)) \leftrightarrow M_{\mathcal{I}_X}((g\circ f)^{-1}(C))\big) \wedge \big(M_{\mathcal{I}_Z}(C) \leftrightarrow M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\big)\Big)\\
&\le \bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C) \leftrightarrow M_{\mathcal{I}_X}((g\circ f)_L^{\leftarrow}(C))\Big) = QU(g\circ f).
\end{aligned}$$

(2) Consider $QU(g \circ f)$,

$$QU(g\circ f) = \bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C) \leftrightarrow M_{\mathcal{I}_X}(g\circ f)_L^{\leftarrow}(C)\Big) \le \bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_X}((g\circ f)_L^{\leftarrow}C) \to M_{\mathcal{I}_Z}(C)\Big)$$

and

$$RI(f) = \bigwedge_{B\in L^Y}\Big(M_{\mathcal{I}_Y}(B) \to M_{\mathcal{I}_X}(f_L^{\leftarrow}(B))\Big) \le \bigwedge_{D\in L^Z}\Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(D)) \to M_{\mathcal{I}_X}((g\circ f)_L^{\leftarrow}(D))\Big).$$

Then, we have $QU(g \circ f) \wedge RI(f) \le \bigwedge_{D\in L^Z}\Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(D)) \to M_{\mathcal{I}_Z}(D)\Big)$.

Hence

$$\begin{aligned}
QU(g\circ f) \wedge RI(f) \wedge RI(g) &\le \left(\bigwedge_{D\in L^X}\Big(M_{\mathcal{I}_X}(g_L^{\leftarrow}(D)) \to M_{\mathcal{I}_Z}(D)\Big)\right) \wedge \left(\bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C) \to M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\Big)\right)\\
&= \bigwedge_{C\in L^Z}\Big(\Big(M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C)) \to M_{\mathcal{I}_Z}(C)\Big) \wedge \Big(M_{\mathcal{I}_Z}(C) \to M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\Big)\Big)\\
&= \bigwedge_{C\in L^Z}\Big(M_{\mathcal{I}_Z}(C) \leftrightarrow M_{\mathcal{I}_Y}(g_L^{\leftarrow}(C))\Big)\\
&= QU(g).
\end{aligned}$$

□

6 Conclusions

In this paper, the degree of M-fuzzy family of independent L-fuzzy sets with respect to a mapping from L^X to M is introduced to generalize the notion of (L, M)-fuzzy independent structure. Besides, weak degree and independent-to-independent degree of a function between two matroids are defined. Based on these, some special degrees (e.g. quotient degrees and isomorphism degrees) with respect to mappings between two (L, M)-fuzzy matroid-like spaces are defined and studied in details. Also, we give characterizations of these degrees and investigate relationships among them. In the view of degree, many notions of matroids (such as circuit, base, and so on) can be generalized. It would be agreeable that those degrees with respect to the same (L, M)-fuzzy matroid become equal. And we can generalise L^X to a lattice or consider greedoids which is closer connected with convex structures.

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Degrees of *M*-fuzzy families of independent *L*-fuzzy sets

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درجه خانواده‌های M-فازی، مجموعه‌های L-فازی مستقل

چکیده. این مقاله ماترویدهای فازی را از نظر درجه مورد مطالعه قرار می‌دهد. در ابتدا، با معرفی درجه خانواده M-فازی مجموعه‌های L-فازی مستقل نسبت به یک نگاشت از L^X به M، مفهوم ساختار مستقل (L, M)- فازی را تعمیم می‌دهیم. ثابت می‌شود که این نوع از درجات در برخی از اصول موضوعه مشابه با ساختار مستقل (L, M)- فازی صدق می‌کنند. سپس برخی از درجات خاص (به عنوان مثال، درجات خارج‌قسمتی و درجات یکریختی) نسبت به نگاشت‌های بین دو فضای ماتروید-مانند (L, M)- فازی را با جزئیات تعریف و بررسی می‌کنیم. بالاخره، مشخصه‌هایی از این درجات را ارائه و رابطه بین آن‌ها را مورد بررسی قرار می‌دهیم.