

## The left (right) distributivity of semi-t-operators over 2-uninorms

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### Abstract

Recently the distributive equations involving various classes of aggregation operators have aroused widespread attention because of their importance in the theoretic and applied communities of fuzzy set theory. 2-uninorms and semi-t-operators are two special classes of aggregation operators and have been proved to be useful in many areas such as fuzzy decision making, approximate reasoning and so on. Therefore, the aim of this paper is to investigate the left (right) distributivity of semi-t-operators over 2-uninorms. We consider five subclasses of 2-uninorms and characterize the corresponding left (right) distributive equations of semi-t-operators over 2-uninorms.

*Keywords:* Fuzzy connective, distributivity, triangular norm, semi-t-operator, 2-uninorm.

## 1 Introduction

Aggregation operators have been widely investigated as the mathematical models of aggregating information (combining and integrating the given data into a representative value) [5]. T-norms and t-conorms, as two special classes of aggregation operators, are the fuzzy logical interpretations of logic connectives [18]. Since then, the extension of them has been an interesting topic. Nullnorms and t-operators were introduced in [23] and [7] respectively and it has been pointed out that they are equivalent [24]. As for the further extension, Drygaś introduced semi-t-operators [9] by eliminating the commutativity from the axioms of t-operators. T-operators generalized the notions of t-norms and t-conorms by using absorbing elements, while uninorms, introduced in [38], unified the two notions by using unit elements. Uninorms are extensively used in many fields like fuzzy logic [19, 15], neural network [12], fuzzy system modeling [37]. In the further study, Akella introduced 2-uninorms as a generalization of t-operators and uninorms in [2], which can be further extended to n-uninorms. A 2-uninorm is commutative, associative, increasing in each variable and has one absorbing element and two neutral elements respectively in two subintervals of the unit interval. The class of 2-uninorms can also be seen as a significant generalization of many other aggregation operators such as S-uninorms, T-uninorms and Bi-uninorms [25].

Distributivity problem was posed many years ago [1]. Due to its extensive applications in many fields such as pseudo-analysis, integration theory and fuzzy systems [29, 30, 31, 13, 16, 8, 34, 4], many researchers have studied this problem with aggregation operators [6, 17, 20, 22, 26, 28, 33] and fuzzy implications [3]. Xie and Liu discussed the distributivity of uninorms over nullnorms in [36]. Drygaś addressed the solutions of the distributivity in 2-uninorms in [11] and the distributivity of semi-t-operators over semi nullnorms in [9]. In [10], left distributivity and right distributivity of semi-t-operators over uninorms was studied by Drygaś and Qin et al. In [35], distributivity of 2-uninorms over semi-t-operators was studied by Wang and Liu. The proposal of 2-uninorms [2] uses a method similar to the ordinal sum of semi-groups, which is frequently used in the construction of aggregation operators. Many classes of aggregation operators such as uninorms and t-operators are just in subclasses of the class of 2-uninorms. The study of distributive equations involving 2-uninorms partly shows how the solutions change under the ordinal sum constructing method so it is correspondingly complicated and universal. For example, Drygaś's study in [11] can derive many related solutions of distributive equations between other aggregation operators. Besides, comparing with other aggregation operators,

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studies about distributive equation involving 2-uninorms are fewer and need to be enriched. Motivated by the facts above, we aim to characterize the left (right) distributive equations of semi-t-operators over 2-uninorms.

The rest of this paper is organized as follows. In section 2, we review some concepts and results about uninorms, (semi-)t-operators and 2-uninorms. In section 3, we discuss the left (right) distributivity of semi-t-operators over some main classes of 2-uninorms. The final section is the conclusion.

## 2 Preliminaries

We first recall some basic notions and assume that readers are familiar with the classical results of basic fuzzy connectives.

**Definition 2.1.** [18] A binary aggregation operator is a function  $F : [0, 1]^2 \rightarrow [0, 1]$  which is increasing in each variable and satisfies  $F(0, 0) = 0$  and  $F(1, 1) = 1$ .

**Definition 2.2.** [18] A semi-t-norm is an increasing operator  $T : [0, 1]^2 \rightarrow [0, 1]$  with neutral element 1. A semi-t-conorm is an increasing operator  $S : [0, 1]^2 \rightarrow [0, 1]$  with neutral element 0.

If a semi-t-norm(semi-t-conorm) satisfies the commutativity and the associativity, then it is a t-norm(t-conorm) [18]. The following introduced nullnorms and t-operators are famous extensions of t-norms and t-conorms.

**Definition 2.3.** [7] An operator  $V : [0, 1]^2 \rightarrow [0, 1]$  is called a nullnorm if it is commutative, associative, increasing in each variable, having a zero element  $z \in [0, 1]$  and satisfies  $V(0, x) = x$  for  $x \in [0, z]$ ,  $V(1, x) = x$  for  $x \in [z, 1]$ .

**Definition 2.4.** [23] An operator  $F : [0, 1]^2 \rightarrow [0, 1]$  is called a t-operator if it is commutative, associative, increasing in each variable and satisfies  $F(0, 0) = 0, F(1, 1) = 1$ , and the functions  $F_0(x) = F(0, x)$  and  $F_1(x) = F(1, x)$  are continuous on  $[0, 1]$ .

In [24] it was proved that t-operators and nullnorms have the same structures. However, when we omit the commutativity in the two definitions respectively, we obtain different operators. More specifically, for the class of t-operators we obtain the class of semi-t-operators introduced by Drygaś.

**Definition 2.5.** [9] An operator  $F : [0, 1]^2 \rightarrow [0, 1]$  is called a semi-t-operator if it is associative, increasing in each variable, fulfills  $F(0, 0) = 0, F(1, 1) = 1$ , and the functions  $F_0, F_1, F^0, F^1$  are continuous, where for  $x \in [0, 1]$ ,  $F_0(x) = F(0, x), F_1(x) = F(1, x), F^0(x) = F(x, 0), F^1(x) = F(x, 1)$ .

Denote the family of all semi-t-operators by  $\mathcal{F}_{a,b}$  where  $F(0, 1) = a, F(1, 0) = b$ .

**Theorem 2.6.** [9]  $F \in \mathcal{F}_{a,b}$  if and only if there exist an associative semi-t-norm  $T$  and an associative semi-t-conorm  $S$  such that

$$F(x, y) = \begin{cases} aS(\frac{x}{a}, \frac{y}{a}) & \text{if } (x, y) \in [0, a]^2, \\ b + (1 - b)T(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \text{if } (x, y) \in [b, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b & \text{if } (x, y) \in [b, 1] \times [0, b], \\ x & \text{otherwise,} \end{cases} \quad (1)$$

when  $a \leq b$  and

$$F(x, y) = \begin{cases} bS(\frac{x}{b}, \frac{y}{b}) & \text{if } (x, y) \in [0, b]^2, \\ a + (1 - a)T(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x, y) \in [a, 1]^2, \\ a & \text{if } (x, y) \in [0, a] \times [a, 1], \\ b & \text{if } (x, y) \in [b, 1] \times [0, b], \\ y & \text{otherwise,} \end{cases} \quad (2)$$

when  $a \geq b$ .

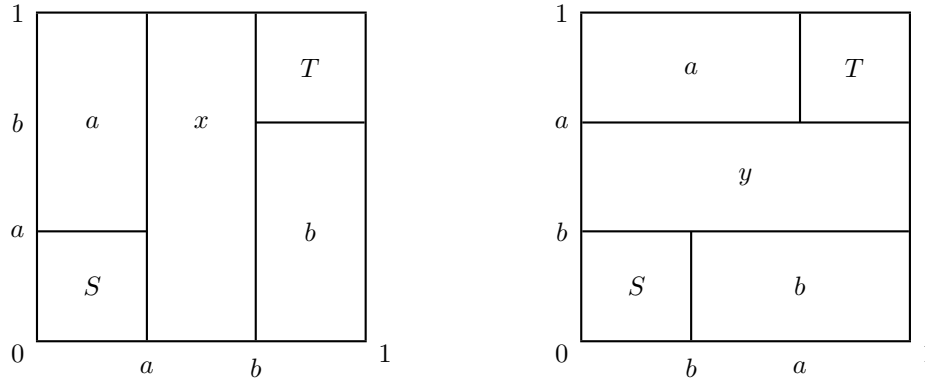


Fig.1 Structures of  $F$  in Theorem 2.6

**Definition 2.7.** [38] A uninorm is an operator  $U : [0, 1]^2 \rightarrow [0, 1]$ , which is commutative, associative, increasing in each variable and there exists a neutral element  $e \in [0, 1]$ .

Clearly, when  $e = 1$  it becomes a t-norm and when  $e = 0$  it becomes a t-conorm. From [21] we know  $U(0, 1) \in \{0, 1\}$ . If  $U(0, 1) = 0$ , the operator  $U$  is called conjunctive and if  $U(0, 1) = 1$  it is called disjunctive. Now we introduce two special subclasses of uninorms.

**Theorem 2.8.** [14] For an uninorm  $U$  with neutral element  $e \in (0, 1)$ , if  $f(x) = U(x, 1)$  and  $h(x) = U(x, 0)$  are continuous except at the point  $x = e$ , then  $U$  has one of the following forms

$$(1) \text{ If } U(0, 1) = 0, \text{ then } U(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [0, e]^2, \\ S(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{elsewhere.} \end{cases}$$

$$(2) \text{ If } U(0, 1) = 1, \text{ then } U(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [0, e]^2, \\ S(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ \max(x, y) & \text{elsewhere.} \end{cases}$$

Where operators  $T$  and  $S$  are isomorphic with a t-norm and a t-conorm respectively. Denote these two subclasses by  $\mathcal{U}_{min}$  and  $\mathcal{U}_{max}$  respectively.

It is clear that there is only one idempotent uninorm in the class of  $\mathcal{U}_{min}$  ( $\mathcal{U}_{max}$ ), which is denoted by  $U^{max}$  ( $U^{min}$ ). The concept of idempotent uninorms is useful. From [10] we have their characterization as follows.

**Theorem 2.9.** [10] A uninorm  $U$  with a neutral element  $e \in (0, 1)$  is idempotent if and only if there exists an Id-symmetrical (See [10]) decreasing function  $g : [0, 1] \rightarrow [0, 1]$  with fixed point  $e$ , such that  $U$  is given like

$$U(x, y) = \begin{cases} \min(x, y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g^2(x)), \\ \max(x, y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g^2(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g^2(x). \end{cases} \quad (3)$$

Motivated by ordinal sum of semi groups, Akella introduced 2-uninorms, the extension of uninorms.

**Definition 2.10.** [2] Let  $k \in (0, 1)$ ,  $e \in [0, k]$ ,  $f \in [k, 1]$ . An operator  $F : [0, 1]^2 \rightarrow [0, 1]$  is called a 2-uninorm if it is commutative, associative, increasing with respect to both variables and fulfills

- (1)  $F(e, x) = x$  for all  $x \leq k$ ,
- (2)  $F(f, x) = x$  for all  $k \leq x$ .

Let  $\mathcal{U}_{k(e,f)}$  denote the family of all 2-uninorms. From [39] we know some basic facts about them.

**Lemma 2.11.** [2] If  $F \in \mathcal{U}_{k(e,f)}$ , then

- (1)  $U_1(x, y) = \frac{F(kx, ky)}{k}$ ,  $U_2(x, y) = \frac{F(k+(1-k)x, k+(1-k)y)}{1-k}$  are uninorms with neutral element  $\frac{e}{k}$  and  $\frac{f-k}{1-k}$ , respectively.

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- (2)  $F(0, 1) \in \{0, k, 1\}$ ,  $F(1, k) \in \{k, 1\}$ ,  $F(0, k) \in \{0, k\}$ .
- (3)  $F(1, k) = F(1, e)$ ,  $F(0, k) = F(0, f)$ .

From [39], we know that there are five subclasses of  $\mathcal{U}_{k(e,f)}$  depending on the values of  $F(0, 1)$ ,  $F(0, k)$  and  $F(1, k)$ . For example, if  $F(0, 1) = 0$  and  $F(1, k) = k$  then we say  $F \in \mathcal{C}_{0,k}$ . So we have  $\mathcal{C}_{0,k}$ ,  $\mathcal{C}_{0,1}$ ,  $\mathcal{C}_{1,k}$ ,  $\mathcal{C}_{1,0}$  and  $\mathcal{C}_k$ . See [39] for more details about the structures of them. Furthermore, some more specific representations of these subclasses under discontinuity at the points  $e$  and  $f$  are also given.

**Theorem 2.12.** [39] For  $F \in \mathcal{U}_{k(e,f)}$ .

- (1) Let  $0 < e \leq k < f \leq 1$ ,  $F(x, 1)$  is continuous except perhaps at  $x = e$  and  $x = f$ ,  $F(0, 1) = 0$  and  $F(1, k) = k$  if and only if  $F$  has the structure

$$F = \begin{cases} U^{c_1} & \text{if } (x, y) \in [0, k]^2, \\ U^{c_2} & \text{if } (x, y) \in [k, 1]^2, \\ \min & \text{if } (x, y) \in (k, 1] \times [0, e] \cup [0, e] \times (k, 1], \\ k & \text{if } (x, y) \in [k, 1] \times [e, k] \cup [e, k] \times [k, 1]. \end{cases}$$

- (2) Let  $0 < e \leq k \leq f < 1$ ,  $F(x, 1)$  is continuous except perhaps at  $x = e$  and  $F(x, e)$  is continuous except perhaps at  $x = f$ ,  $F(0, 1) = 0$  and  $F(1, k) = 1$  if and only if  $F$  has the structure

$$F = \begin{cases} U^c & \text{if } (x, y) \in [0, k]^2, \\ U^d & \text{if } (x, y) \in [k, 1]^2, \\ \min & \text{if } (x, y) \in (k, 1] \times [0, e] \cup [0, e] \times (k, 1], \\ \max & \text{if } (x, y) \in (f, 1] \times [e, k] \cup [e, k] \times (f, 1], \\ k & \text{if } (x, y) \in [k, f] \times [e, k] \cup [e, k] \times [k, f]. \end{cases}$$

- (3) Let  $0 \leq e < k \leq f < 1$ ,  $F(x, 0)$  is continuous except perhaps at  $x = e$  and  $x = f$ ,  $F(0, 1) = 1$  and  $F(0, k) = k$  if and only if  $F$  has the structure

$$F = \begin{cases} U^{d_1} & \text{if } (x, y) \in [0, k]^2, \\ U^{d_2} & \text{if } (x, y) \in [k, 1]^2, \\ \max & \text{if } (x, y) \in (f, 1] \times [0, k] \cup [0, k] \times (f, 1], \\ k & \text{if } (x, y) \in [k, f] \times [0, k] \cup [0, k] \times [k, f]. \end{cases}$$

- (4) Let  $0 < e \leq k \leq f < 1$ ,  $F(x, 0)$  is continuous except perhaps at  $x = f$  and  $F(x, f)$  is continuous except perhaps at  $x = e$ ,  $F(0, 1) = 1$  and  $F(0, k) = 0$  if and only if  $F$  has the structure

$$F = \begin{cases} U^c & \text{if } (x, y) \in [0, k]^2, \\ U^d & \text{if } (x, y) \in [k, 1]^2, \\ \min & \text{if } (x, y) \in (k, f] \times [0, e] \cup [0, e] \times (k, f], \\ \max & \text{if } (x, y) \in (f, 1] \times [0, k] \cup [0, k] \times (f, 1], \\ k & \text{if } (x, y) \in [k, f] \times [e, k] \cup [e, k] \times [k, f]. \end{cases}$$

- (5)  $F(x, 0)$  and  $F(x, 1)$  are continuous except perhaps at  $x = e$  and  $x = f$ ,  $F(0, 1) = k$  if and only if  $F$  has the structure

$$F = \begin{cases} U^d & \text{if } (x, y) \in [0, k]^2, \\ U^c & \text{if } (x, y) \in [k, 1]^2, \\ k & \text{if } (x, y) \in [k, 1] \times [0, k] \cup [0, k] \times [k, 1]. \end{cases}$$

Where,  $U^c$ ,  $U^{c_1}$ ,  $U^{c_2}$  are operators isomorphic with uninorms in  $\mathcal{U}_{\min}$ , and  $U^d$ ,  $U^{d_1}$ ,  $U^{d_2}$  are operators isomorphic to uninorms in  $\mathcal{U}_{\max}$ . For the convenience of the discussion in next section, we denote these five subclasses of 2-uninorms by  $\mathcal{C}_{0,k}^*$ ,  $\mathcal{C}_{0,1}^*$ ,  $\mathcal{C}_{1,k}^*$ ,  $\mathcal{C}_{1,0}^*$ ,  $\mathcal{C}_k^*$ .

### 3 The left (right) distributivity of semi-t-operators over 2-uninorms

**Definition 3.1.** [10] Consider two aggregation operators  $F, G : [0, 1]^2 \rightarrow [0, 1]$ . We claim that  $F$  is distributive over  $G$  if the left and the right distributive equations hold

$$F(x, G(y, z)) = G(F(x, y), F(x, z)) \quad (4)$$

$$F(G(x, y), z) = G(F(x, z), F(y, z)) \quad (5)$$

for  $x, y, z \in [0, 1]$ .

**Lemma 3.2.** [27] Every increasing operator  $F : [0, 1]^2 \rightarrow [0, 1]$  is distributive over  $\max$  and  $\min$ .

**Lemma 3.3.** Let  $F$  be an aggregation operator,  $U$  be an idempotent uninorm with neutral element  $e \in (0, 1)$  and  $F$  is left distributive over  $U$ , then the following statements hold.

- (1) If there exists  $a \in (e, 1]$  fulfilling that  $F(x, y) \geq \max(x, y)$  and  $F(x, 0) = x$  for  $(x, y) \in [0, a]^2$ , then  $F(x, y) = \max(x, y)$  for  $(x, y) \in [e, a] \times [0, e]$  and  $U$  is  $U^{\max}$ .
- (2) If there exists  $a \in [0, e)$  fulfilling that  $F(x, y) \leq \min(x, y)$  and  $F(x, 1) = x$  for  $(x, y) \in [a, 1]^2$ , then  $F(x, y) = \min(x, y)$  for  $(x, y) \in [a, e] \times [e, 1]$  and  $U$  is  $U^{\min}$ .

*Proof.* We prove (1), the proof of (2) is quite similar.

$$\text{First we know } U(x, y) = \begin{cases} \min(x, y) & \text{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \text{if } (x, y) \in [e, 1]^2. \end{cases}$$

For  $x \in [e, a]$ , we have  $x = F(x, 0) = F(x, U(0, e)) = U(F(x, 0), F(x, e)) = U(x, F(x, e))$ . If we assume  $F(x, e) \neq x$ , then  $F(x, e) > x \geq e$ . We obtain  $x = U(x, F(x, e)) = \max(x, F(x, e)) = F(x, e)$ , which is a contradiction. So  $F(x, e) = x$  and  $F(x, 0) = x$ , i.e.  $F(x, y) = \max(x, y)$  in  $[e, a] \times [0, e]$ .

According to Theorem 2.9, there exists a function  $g$  that  $U$  and  $g$  satisfy (3). Now we assume that there exists a  $x_0 \in [0, e)$  such that  $g(x_0) > e$ . Then there exists such  $y_0 \in (e, \min(a, g(x_0)))$  such that  $U(x_0, y_0) = \min(x_0, y_0)$ . Naturally we obtain  $e = \max(e, x) = F(e, x) = F(e, \min(x, y)) = F(e, U(x, y)) = U(F(e, x), F(e, y)) = U(e, F(e, y)) = F(e, y) \geq y > e$ , which is a contradiction. Hence such  $x_0$  do not exist, i.e.  $U$  is  $U^{\max}$ .  $\square$

We consider the left distributivity of  $F \in \mathcal{F}_{a,b}$  over  $G \in \mathcal{U}_{k(e,f)}$  and start with the case of  $a \leq b$ . Firstly, it is easy to see the idempotency of  $G$ .

**Lemma 3.4.** Let  $a \leq b$ . If  $F \in \mathcal{F}_{a,b}$  is left distributive over  $G \in \mathcal{U}_{k(e,f)}$ , then  $G$  is idempotent.

*Proof.* For  $x \in [a, 1]$ , we have  $x = F(x, 1) = F(x, G(1, 1)) = G(F(x, 1), F(x, 1)) = G(x, x)$ . For  $x \in [0, b]$ , we have  $x = F(x, 0) = F(x, G(0, 0)) = G(F(x, 0), F(x, 0)) = G(x, x)$ . So  $G$  is idempotent.  $\square$

Since there are five subclasses of 2-uninorms, we study the distributivity conditions subclass by subclass. From [39], we know that when  $G$  is in a certain subclass, the order relationship for parameters of  $G$  is also restrained. For example, the claim  $G \in \mathcal{C}_{1,0}$  simultaneously means that  $0 < e \leq k \leq f < 1$ .

**Theorem 3.5.** Let  $0 = a \leq b$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_{1,0}$  or  $\mathcal{C}_{1,k}$ .  $F$  is left distributive over  $G$  if and only if  $b \geq f$ ,  $G$  is idempotent and  $G(x, y) = \max(x, y)$  for  $(x, y) \in (b, 1] \times [0, 1] \cup [0, 1] \times (b, 1]$ .

*Proof.* From Lemma 3.4,  $G$  is idempotent.

For equation  $F(x, G(0, 1)) = G(F(x, 0), F(x, 1))$ , taking arbitrary fixed  $x \geq b$  we obtain  $x = F(x, 1) = F(x, G(0, 1)) = G(F(x, 0), F(x, 1)) = G(x, b)$ . So  $b \geq f$ .

For equation  $F(1, G(0, x)) = G(F(1, 0), F(1, x))$ , taking arbitrary fixed  $x > b$  we have the equation  $F(1, G(0, x)) = G(F(1, 0), F(1, x)) = G(b, x) = x > b$ . Hence  $G(0, x) = F(1, G(0, x)) = x$ . By  $G(0, x) = x$  and  $G(f, x) = x$  we know  $G(x, y) = \max(x, y)$  for  $x \in (b, 1] \times [0, 1] \cup [0, 1] \times (b, 1]$ .

Conversely, let  $F$  and  $G$  satisfy these conditions. We to check if the left distributivity equation holds. Without loss of generality, we set  $y \geq z$ .

1° Case  $x \in [0, b]$ .  $F(x, G(y, z)) = x = G(x, x) = G(F(x, y), F(x, z))$ .

2° Case  $x \in (b, 1]$ . First it is easy to see the facts that  $G(y, z) \leq b$  for  $y < b$  and  $F(x, y), F(x, z) \in [b, 1]$ . Then, we have  $F(x, G(y, z)) = \begin{cases} F(x, y) & \text{if } y \geq b \\ b & \text{if } y < b \end{cases} = F(x, y) = G(F(x, y), F(x, z))$ .

So  $F$  is left distributive over  $G$ . □

**Lemma 3.6.** *Let  $0 < a \leq b$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_{1,0}$ . If  $F$  is left distributive over  $G$ , then  $0 < e = k \leq f < a \leq b \leq 1$ .*

*Proof.* From Lemma 3.4 we know  $G$  is idempotent.

Consider the equation  $F(x, G(0, 1)) = G(F(x, 0), F(x, 1))$ . Taking  $x \in [0, a]$  we have  $a = G(x, a)$ , so  $a > f$ .

Consider the equation  $F(x, G(0, f)) = G(F(x, 0), F(x, f))$ . Taking  $x \in [f, a]$  we have  $x = G(x, F(x, f)) = F(x, f)$ , so  $F(x, f) = \max(x, f)$  for  $x \in [0, a]$ . Taking  $x = e$  we have  $e = F(e, G(0, f)) = G(F(e, 0), F(e, f)) = G(e, f) = k$ . So  $0 < e = k \leq f < a \leq b \leq 1$ . □

**Lemma 3.7.** *Let  $0 < a \leq b$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_{1,0}$ . If  $F$  is left distributive over  $G$ , then the upper underlying uninorm of  $G$  is in  $\mathcal{U}_{max}$  and  $F$  has the following form*

$$F(x, y) = \begin{cases} S(x, y) & \text{if } (x, y) \in [0, f]^2, \\ \max(x, y) & \text{if } (x, y) \in [f, a] \times [0, f], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ T(x, y) & \text{if } (x, y) \in [b, 1]^2, \\ A(x, y) & \text{elsewhere.} \end{cases} \tag{6}$$

where  $T$  is isomorphic to an associative semi-t-norm,  $S$  is isomorphic to an associative semi-t-conorm,  $A$  is an increasing operator that preserves the associativity of  $F$ .

*Proof.* From Lemma 3.4 and Lemma 3.6, we know  $G$  that is idempotent and  $0 < e = k \leq f < a \leq b < 1$ . From the proof of Lemma 3.6 we have  $F(x, f) = \max(x, f)$  for  $x \in [0, a]$ . So  $F$  is of the form (6). Since  $F(f, x) = f$  for  $x \in [0, f]$ , using the method as in the proof of Lemma 3.3 we know that the upper underlying uninorm of  $G$  is in  $\mathcal{U}_{max}$ . □

Now we have obtained several necessary conditions. With the continuity of the border of the involved 2-uninorms known, we can give the accurate solutions of the distributive equations.

**Theorem 3.8.** *Let  $0 < a \leq b$ .  $F \in \mathcal{F}_{a,b}$  is left distributive over  $G \in \mathcal{C}_{1,0}^*$  if and only if  $F$  is as in Lemma 3.7 and  $G$  is given by*

$$G(x, y) = \begin{cases} \min & \text{if } (x, y) \in [0, f]^2, \\ \max & \text{otherwise.} \end{cases} \tag{7}$$

*Proof.* Let  $F$  is left distributive over  $G$ . Directly from Lemma 3.4, 3.6 and 3.7,  $F$  has the form of (6) and  $G$  has the form (7).

Conversely, let  $F$  and  $G$  be given by (6) and (7). We start to check whether the left distributivity equation holds. Without loss of generality we set  $y \geq z$ .

1° Case  $(y, z) \in (f, 1] \times [0, 1]$ . If  $F(x, y) = f$ , then  $f < y = F(0, y) \leq F(x, y) = f$ . So in this case  $F(x, y) > f$ , so  $F(x, G(y, z)) = F(x, y) = \max(F(x, y), F(x, z)) = G(F(x, y), F(x, z))$ .

2° Case  $(y, z) \in [0, f]^2$ .

For  $x > f$ , we have  $F(x, z) = F(x, y) > f$  so  $F(x, G(y, z)) = F(x, z) = F(x, y) = G(F(x, y), F(x, z))$ .

For  $x \leq f$ , we have  $F(x, y), F(x, z) \leq f$  so  $F(x, G(y, z)) = F(x, z) = G(F(x, y), F(x, z))$ .

So  $F$  is left distributive over  $G$ . □

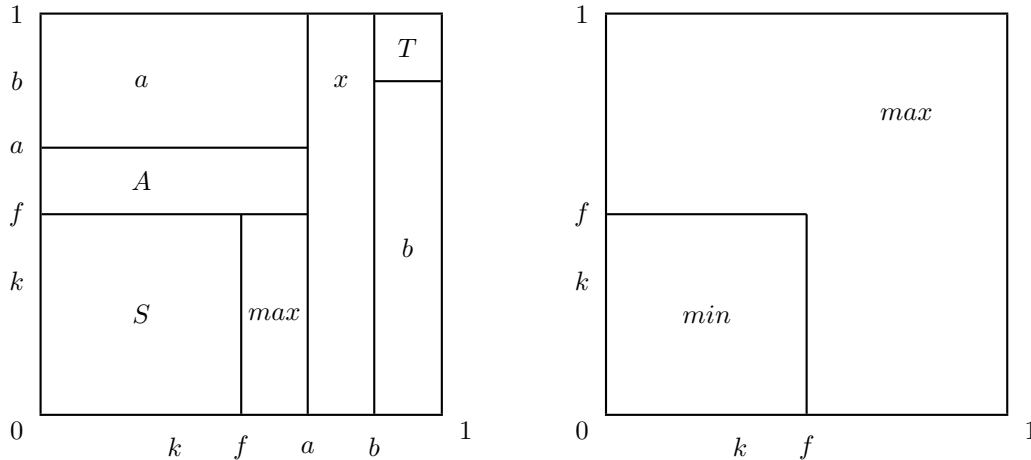


Fig.2 Structures of  $F$  and  $G$  in Theorem 3.8

**Lemma 3.9.** Let  $0 < a \leq b$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_{1,k}$ . If  $F$  is left distributive over  $G$ , then  $0 \leq e < a \leq k \leq f \leq b \leq 1$  or  $0 \leq e < k \leq f < a \leq b \leq 1$ .

*Proof.* Directly from Lemma 3.4  $G$  is idempotent. So the underlying uninorms of  $G$  is idempotent.

Consider the equation  $F(x, G(0, 1)) = G(F(x, 0), F(x, 1))$ . Let  $x \geq b$ , then  $x = F(x, 1) = F(x, G(0, 1)) = G(F(x, 0), F(x, 1)) = G(b, x)$  and consequently  $b \geq f$ ; let  $x \leq a$ , then  $a = G(x, a)$  and consequently  $e < a \leq k$  or  $a > f$ .  $\square$

**Lemma 3.10.** Let  $0 < a \leq b$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_{1,k}$ . If  $F$  is left distributive over  $G$ , then the lower underlying uninorm of  $G$  is in  $\mathcal{U}_{max}$  and

(1) In the case  $0 \leq e < a \leq k \leq f \leq b \leq 1$ ,  $F$  has the following form

$$F(x, y) = \begin{cases} S(x, y) & \text{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \text{if } (x, y) \in [e, a] \times [0, e], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ T(x, y) & \text{if } (x, y) \in [b, 1]^2, \\ A(x, y) & \text{elsewhere.} \end{cases} \quad (8)$$

(2) In the case  $0 \leq e < k \leq f < a \leq b \leq 1$ , let  $g(x) = \inf\{y \in [0, a] | F(x, y) \geq f\}$  and  $F$  has the following form

$$F(x, y) = \begin{cases} S(x, y) & \text{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \text{if } (x, y) \in ([e, k] \cup [f, a]) \times [0, e] \cup [k, \inf\{x | g(x) \leq e\}] \times [0, \min(k, g(x))], \\ B(x, y) & \text{if } (x, y) \in [\sup\{x | g(x) \geq k\}, a] \times [\max(k, g(x), f)], \\ C(x, y) & \text{if } (x, y) \in [\sup\{x | g(x) \geq e\}, f] \times [g(x), e], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ T(x, y) & \text{if } (x, y) \in [b, 1]^2, \\ A(x, y) & \text{elsewhere.} \end{cases} \quad (9)$$

Where  $T$  is isomorphic to an associative semi-t-norm,  $S$  are isomorphic with an associative semi-t-conorm.  $A, B, C$  are increasing operators that preserve the associativity of  $F$  and moreover, the value of  $B(x, y)$  and  $C(x, y)$  only depends on  $x$  respectively.



*Proof.* From Lemma 3.4 and Lemma 3.9, we know  $G$  is idempotent and either  $0 \leq e < a \leq k \leq f \leq b \leq 1$  or  $0 \leq e < k \leq f < a \leq b \leq 1$ .

(1) In the case  $0 \leq e < a \leq k \leq f \leq b \leq 1$ .

From Lemma 3.3 we know  $F(x, y) = \max(x, y)$  in  $[e, a] \times [0, e]$ , so  $F$  is of the form (8).

(2) In the case  $0 \leq e < k \leq f < a \leq b \leq 1$ .

Consider the equation  $F(x, G(0, e)) = G(F(x, 0), F(x, e))$ . Taking  $x \in [f, a]$  we have  $x = F(x, 0) = G(x, F(x, e)) = F(x, e)$ . Taking  $x \in [e, k)$  we have  $x = G(x, F(x, e)) < k \Rightarrow x = \max(x, F(x, e)) = F(x, e)$ . Therefore,  $F(x, y) = \max(x, y)$  for  $(x, y) \in ([e, k) \cup [f, a]) \times [0, e]$ .

For  $x \in [0, a]$ ,  $(y, z) \in [0, e]^2 \cup [k, f]^2$  and  $F(x, y), F(x, z) \in [f, 1]$ , we have  $F(x, \min(y, z)) = F(x, G(y, z)) = G(F(x, y), F(x, z)) = F(x, \max(y, z)) \Rightarrow F(x, y) = F(x, z)$ . Therefore, if  $F(x, y) \geq f$ , then its value only depends on  $x$  when  $(x, y)$  is in  $[0, a] \times [k, f]$  or  $[0, a] \times [0, e]$ .

For  $x \in [k, f]$ ,  $y \in (e, k]$ , when  $F(x, y) \in [0, f)$  we have  $F(x, y) = F(x, G(y, 0)) = G(F(x, y), x) = x$ . Therefore,  $F(x, y) = \max(x, y)$  for  $(x, y) \in [k, f] \times (e, k]$  and  $F(x, y) < f$ .

So we obtain the two possible structures of  $F$ . In both cases by using Lemma 3.3 we know that the lower underlying uninorm of  $G$  is  $U^{\max}$ . □

Lemma 3.10 reveals that  $g(x)$  matters the structure of  $F$ . It is obvious that  $g(x)$  is decreasing and  $g(0) = f$ ,  $g(k) \geq e$ ,  $g(f) = 0$ . In the following discussion we will see that with more specific structure of  $G$  given, structure of  $g$  becomes more clear.

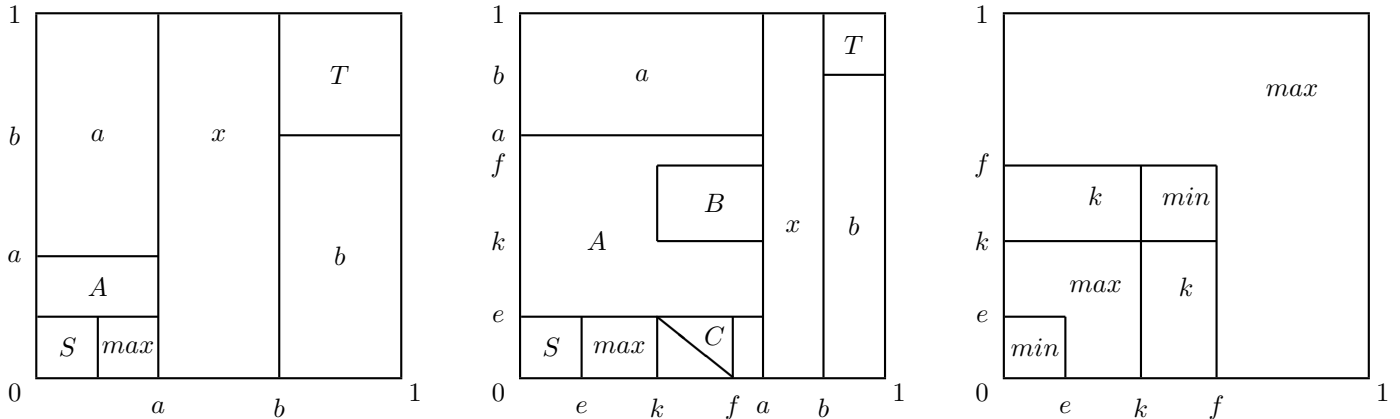


Fig.3 Two structures of  $F$  and structure of  $G$  in Theorem 3.11

**Theorem 3.11.** Let  $0 < a \leq b$ .  $F \in \mathcal{F}_{a,b}$  is left distributive over  $G \in \mathcal{C}_{1,k}^*$  if and only if  $F$  is the in first case in Lemma 3.10 or in the second case with  $g(x) \notin (k, f)$  for all  $x \in [0, f]$  and  $g(x) \notin (e, k]$  for all  $x \in [k, f]$  and  $G$  is given by

$$G(x, y) = \begin{cases} U^{\max} & \text{if } (x, y) \in [0, k]^2 \cup [k, 1]^2, \\ \max & \text{if } (x, y) \in (f, 1] \times [0, k) \cup [0, k) \times (f, 1], \\ k & \text{otherwise,} \end{cases} \tag{10}$$

*Proof.* Let  $F$  is left distributive over  $G$ . Directly from Lemmas 3.4, 3.9 and 3.10,  $F$  has the form of (8) or (9) and  $G$  has the form (10).

We first try to figure out what  $g(x)$  should be like if  $F$  is in the form (9). Let  $x \in [0, f)$  and consider the equation  $F(x, G(0, f)) = G(F(x, 0), F(x, f))$ . If  $F(x, f) > f$ , then  $F(x, k) = G(x, F(x, f)) = \max(x, F(x, f)) = F(x, f) > f$ . Otherwise,  $F(x, f) = f$ , i.e.  $g(x) = f$  for  $x \in (0, f)$ . Hence  $g(x) \notin (k, f)$  for all  $x \in [0, f]$ . Assume that there exists a point  $(x, y) \in [k, f] \times (e, k]$  with  $y = g(x)$ . Considering the equation  $f = F(x, y) = F(x, G(y, 0)) = G(F(x, y), x) = x$  we obtain a contradiction. Hence  $g(x) \notin (e, k]$  for all  $x \in [k, f]$ .

Conversely, we start to check whether the left distributive equation holds. Let  $F$  and  $G$  be given by (8) and (10) first. Without loss of generality, we set  $y \geq z$  in the following discussion.



1° Case  $x \in [0, a]$ . For  $(y, z) \in [0, a]^2$ , the proof is the same as Theorem 3.8.

For  $(y, z) \in [a, 1] \times [0, 1]$ , we have  $F(x, y) \leq a$ . Hence  $F(x, G(y, z)) = F(x, y) = a = \max(a, F(x, z)) = G(F(x, y), F(x, z))$ .

2° Case  $x \in [a, b]$ .  $F(x, G(y, z)) = x = G(x, x) = G(F(x, y), F(x, z))$ .

3° Case  $x \in [b, 1]$ .

For  $(y, z) \in [b, 1]^2$ , it is obvious.

For  $(y, z) \in [b, 1] \times [0, b]$ ,  $F(x, G(y, z)) = F(x, y) = b = \min(b, F(x, z)) = G(F(x, y), F(x, z))$ .

For  $(y, z) \in [0, b]^2$ , we have  $G(y, z) \leq b$ . Hence  $F(x, G(y, z)) = b = G(b, b) = G(F(x, y), F(x, z))$ .

Then let  $F$  and  $G$  be given by (9) and (10) and  $g(x) \notin (k, f)$  for all  $x \in [0, f]$  and  $g(x) \notin (e, k]$  for all  $x \in [k, f]$ .

1° Case  $(x, y, z) \notin [0, f]^3$ .

If  $y > f$ , then  $F(x, y) > f$  is valid. So we know that  $F(x, G(y, z)) = F(x, y) = \max(F(x, y), F(x, z)) = G(F(x, y), F(x, z))$ .

If  $y \leq f$ , then  $x > f$  is valid. We have  $F(x, y), F(x, z) > f$  so  $G(F(x, y), F(x, z)) = F(x, y)$ . Meanwhile, for  $(y, z) \in [0, e]^2 \cup [k, f]^2$ ,  $F(x, G(y, z)) = F(x, z) = F(x, y)$ . For  $(y, z) \in [k, f] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, k) = F(x, y)$ . For  $(y, z) \in (e, k] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, \max(y, z)) = F(x, y)$ . So the equation always holds.

2° Case  $(x, y, z) \in [0, f]^3$  with  $F(x, y) > f$ .

First we see  $G(F(x, y), F(x, z)) = F(x, y)$ . Meanwhile, for  $(y, z) \in [0, e]^2 \cup [k, f]^2$ ,  $F(x, G(y, z)) = F(x, z) = F(x, y)$ . For  $(y, z) \in [k, f] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, k) = F(x, y)$ . For  $(y, z) \in (e, k] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, \max(y, z)) = F(x, y)$ . So the equation still holds.

3° Case  $(x, y, z) \in [k, f] \times [0, f]^2$  with  $F(x, y) \leq f$ . First we see  $F(x, y), F(x, z) \in [k, f]$ , then  $G(F(x, y), F(x, z)) = F(x, \min(y, z)) = F(x, z)$ . So we only need to consider  $(y, z) \in [e, f] \times [0, k]$ . Notice that  $G(y, z) \leq k$  and we have  $F(x, G(y, z)) = x = F(x, z) = G(F(x, y), F(x, z))$ .

4° Case  $(x, y, z) \in [0, k] \times [0, f]^2$  with  $F(x, y) \leq f$ .

For  $(y, z) \in [0, k]^2$ , the situation coincides with Theorem 3.8.

For  $(y, z) \in [k, f] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, k) = k = G(F(x, y), F(x, z))$ .

For  $(y, z) \in [k, f]^2$ , we have  $G(y, z) \leq b$ ,  $F(x, G(y, z)) = F(x, \min(y, z)) = G(F(x, y), F(x, z))$ .

□

**Lemma 3.12.** Let  $a \leq b$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_k$ . If  $F$  is left distributive over  $G$ , then  $0 \leq e \leq k \leq a \leq b \leq f \leq 1$  or  $0 \leq e \leq a \leq k \leq b \leq f \leq 1$  or  $0 \leq e \leq a \leq b \leq k \leq f \leq 1$ .

*Proof.* From Lemma 3.4  $G$  is idempotent. If  $a < e$ , then for  $x < a$ , we have  $a = F(x, k) = F(x, G(0, 1)) = G(F(x, 0), F(x, 1)) = G(x, a) = \min(x, a)$ , which is a contradiction. So  $a \geq e$  and similarly,  $b \leq f$ . So we obtain three possible cases  $0 \leq e \leq k \leq a \leq b \leq f \leq 1$ ,  $0 \leq e \leq a \leq k \leq b \leq f \leq 1$  and  $0 \leq e \leq a \leq b \leq k \leq f \leq 1$ . □

**Lemma 3.13.** Let  $a \leq b$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_k$ . If  $F$  is left distributive over  $G$ , then the upper and lower underlying uninorm of  $G$  are in  $\mathcal{U}_{\min}$  and  $\mathcal{U}_{\max}$  respectively and

(1) In the case  $0 \leq e \leq k \leq a \leq b \leq f \leq 1$ ,  $F$  has the following form

$$F(x, y) = \begin{cases} S(x, y) & \text{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \text{if } (x, y) \in [e, a] \times [0, e] \cup [k, a] \times [e, k], \\ A(x, y) & \text{if } (x, y) \in [0, k] \times (e, a] \cup [k, a] \times (k, a], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ T(x, y) & \text{if } (x, y) \in [f, 1]^2, \\ B(x, y) & \text{if } (x, y) \in [b, 1] \times [b, f], \\ \min(x, y) & \text{if } (x, y) \in [b, f] \times [f, 1]. \end{cases} \quad (11)$$

(2) In the case  $0 \leq e \leq a \leq k \leq b \leq f \leq 1$ ,  $F$  has the following form

$$F(x, y) = \begin{cases} S(x, y) & \text{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \text{if } (x, y) \in [e, a] \times [0, e], \\ A(x, y) & \text{if } (x, y) \in [0, a] \times [e, a], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ T(x, y) & \text{if } (x, y) \in [f, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in [b, f] \times [f, 1], \\ B(x, y) & \text{if } (x, y) \in [b, 1] \times [b, f]. \end{cases} \tag{12}$$

(3) In the case  $0 \leq e \leq a \leq b \leq k \leq f \leq 1$ ,  $F$  has the following form

$$F(x, y) = \begin{cases} S(x, y) & \text{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \text{if } (x, y) \in [e, a] \times [0, e], \\ A(x, y) & \text{if } (x, y) \in [0, a] \times [e, a], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ T(x, y) & \text{if } (x, y) \in [f, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in [b, f] \times [f, 1] \cup [b, k] \times [k, f], \\ B(x, y) & \text{if } (x, y) \in (k, 1] \times [b, f] \cup [b, k] \times [b, k]. \end{cases} \tag{13}$$

Where  $T$  is isomorphic to an associative semi-t-norm,  $S$  is isomorphic to an associative semi-t-conorm,  $A, B$  are increasing operators that preserves the associativity of  $F$ .

*Proof.* From Lemma 3.4  $G$  is idempotent. From Lemma 3.12, there are three possibilities to be discussed.

(1) In the case  $0 \leq e \leq k \leq a \leq b \leq f \leq 1$ .

For equation  $F(x, G(0, 1)) = G(F(x, 0), F(x, 1))$ , taking  $x \in [0, a]$  we have  $F(x, k) = G(x, a) = \begin{cases} k & \text{if } x \in [0, k] \\ x & \text{if } x \in [k, a] \end{cases} = \max(x, k)$ , so  $F(x, y) = \max(x, y)$  in  $[k, a] \times [0, k]$  and  $F$  is isomorphic to a semi-conorm in  $[0, k]^2$ . By using Lemma 3.3 in  $[0, k]^2$  and  $[k, 1]^2$  we know  $F$  is in the form of (14).

(2) In the case  $0 \leq e < k \leq f < a \leq b < 1$ .

Use Lemma 3.3 in  $[0, k]^2$  and  $[k, 1]^2$  respectively and we obtain the structures of  $F$  and  $G$ .

(3) In the case  $0 \leq e \leq a \leq b \leq k \leq f \leq 1$ .

Notice that the proof just goes the same way as it in (1).

By Lemma 3.3, the upper and lower underlying uninorm of  $G$  are in  $U_{min}$  and  $U_{max}$  respectively. □

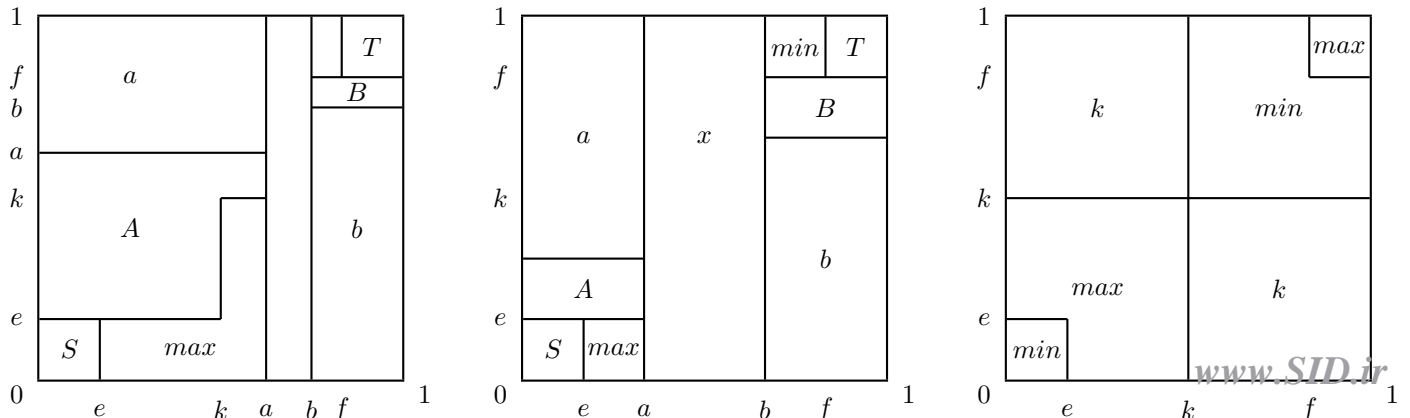


Fig.4 Two Structures of  $F$  and the structure of  $G$  in Theorem 3.14

**Theorem 3.14.** Let  $a \leq b$ .  $F \in \mathcal{F}_{a,b}$  is left distributive over  $G \in \mathcal{C}_k$  if and only if  $F$  is in one of the cases in Lemma 3.13 and  $G$  is idempotent, given by

$$G(x, y) = \begin{cases} U^{max} & \text{if } (x, y) \in [0, k]^2, \\ U^{min} & \text{if } (x, y) \in [k, 1]^2, \\ k & \text{otherwise.} \end{cases} \quad (14)$$

*Proof.* Let  $F$  is left distributive over  $G$ . Directly from Lemmas 3.4, 3.12 and 3.13,  $F$  has three possible form (11), (12) and (13).

Conversely, we start to check whether the left distributive equation holds. Let  $F$  and  $G$  be given by (11) and (14) first. Without loss of generality, we always set  $y \geq z$  in the following discussion.

1° Case  $x \in [0, k]$ .

For  $(y, z) \in [0, k]^2$ , it just coincides with the situation of the proof of Theorem 3.8.

For  $(y, z) \in [k, 1] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, k) = k = G(F(x, y), F(x, z))$ .

For  $(y, z) \in [k, 1] \times [k, 1]$ ,  $F(x, G(y, z)) = F(x, z) = \min(F(x, y), F(x, z)) = G(F(x, y), F(x, z))$ .

For  $(y, z) \in [f, 1]^2$ ,  $F(x, G(y, z)) = F(x, y) = a = G(a, a) = G(F(x, y), F(x, z))$ .

2° Case  $x \in [k, a]$ . Since  $F(x, y), F(x, z) \in [k, a]$ , we have  $G(F(x, y), F(x, z)) = F(x, z)$ .

For  $(y, z) \in [e, k] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, y) = \max(x, y) = x = \max(x, z) = F(x, z)$ .

For  $(y, z) \in [k, 1] \times [0, k]$ ,  $F(x, G(y, z)) = F(x, k) = x = \max(x, z) = F(x, z)$ .

For  $(y, z) \in [f, 1]^2$ ,  $F(x, G(y, z)) = F(x, y) = a = F(x, z)$ .

For  $(y, z)$  elsewhere,  $G(x, y) = \min(x, y)$  so it is obvious.

3° Case  $x \in [a, b]$ .  $F(x, G(y, z)) = x = G(F(x, y), F(x, z))$ .

4° Case  $x \in [b, 1]$ . Still, the method is similar to the proof of Theorem 3.8.

Since when  $F$  is in (13) the proof remains almost the same, we have only one case left. Let  $F$  and  $G$  be given by (12) and (14).

1° Case  $x \in [0, a]$ .

For  $y \leq a$ , from [32] the equation holds.

For  $y > a$ , we see  $G(y, z) > a$ ,  $F(x, y) = a$  and  $F(x, z) \leq a$ . Hence  $F(x, G(y, z)) = a = G(a, F(x, z)) = G(F(x, y), F(x, z))$ .

2° Case  $x \in [a, b]$ .  $F(x, G(y, z)) = x = G(x, x) = G(F(x, y), F(x, z))$ .

3° Case  $x \in [b, 1]$ . The proof is similar to the case 1° above.

So  $F$  is distributive over  $G$ . □

Up to now, we have discussed the distributive conditions when  $G \in \mathcal{C}_{1,0}$ ,  $\mathcal{C}_{1,k}$  and  $\mathcal{C}_k$ . In fact, the discussions when  $G \in \mathcal{C}_{0,1}$  and  $\mathcal{C}_{0,k}$  are very similar. So we only show the results.

**Theorem 3.15.** Let  $a \leq b = 1$ ,  $F \in \mathcal{F}_{a,b}$  and  $G \in \mathcal{C}_{0,1}$  or  $\mathcal{C}_{0,k}$ .  $F$  is left distributive over  $G$  if and only if  $a \leq e$ ,  $G$  is idempotent and  $G(x, y) = \min(x, y)$  for  $x \in [0, a] \times [0, 1] \cup [0, 1] \times [0, a]$ .

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**Theorem 3.16.** Let  $a \leq b < 1$ .  $F \in \mathcal{F}_{a,b}$  is left distributive over  $G \in \mathcal{C}_{0,1}^*$  if and only if  $0 \leq a \leq b < e \leq k = f < 1$  and  $F$  is in the form

$$F(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in [b, e] \times [e, 1], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ S(x, y) & \text{if } (x, y) \in [0, a]^2, \\ A(x, y) & \text{elsewhere.} \end{cases} \quad (15)$$

and  $G$  is given by

$$G(x, y) = \begin{cases} \max & \text{if } (x, y) \in [e, 1]^2, \\ \min & \text{otherwise.} \end{cases} \quad (16)$$

where  $T$  is isomorphic to an associative semi- $t$ -norm,  $S$  is isomorphic to an associative semi- $t$ -conorm,  $A$  is an increasing operator that preserves the associativity of  $F$ .

**Theorem 3.17.** Let  $a \leq b < 1$ .  $F \in \mathcal{F}_{a,b}$  is left distributive over  $G \in \mathcal{C}_{0,k}^*$  if and only if the structure of  $F$  is in one of the following cases

(1)  $0 \leq a \leq e \leq k \leq b < f \leq 1$ ,  $F$  has the following form

$$F(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [f, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in [b, f] \times [f, 1], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ S(x, y) & \text{if } (x, y) \in [0, a]^2, \\ A(x, y) & \text{elsewhere.} \end{cases} \quad (17)$$

(2)  $0 \leq a \leq b < e \leq k < f \leq 1$ , let  $g(x) = \sup\{y \in [0, a] \mid F(x, y) \leq e\}$  then  $g(x) \notin (e, k)$  for all  $x \in [e, 1]$  and  $g(x) \notin (k, f]$  for all  $x \in [e, k]$  and  $F$  has the following form

$$F(x, y) = \begin{cases} T(x, y) & \text{if } (x, y) \in [f, 1]^2, \\ \min(x, y) & \text{if } (x, y) \in ((k, f] \cup [b, e]) \times [f, 1] \cup [\sup\{x \mid g(x) \geq f\}, k] \times [\max(k, g(x)), 1], \\ B(x, y) & \text{if } (x, y) \in [b, \inf\{x \mid g(x) \leq k\}] \times [e, \min(k, g(x))], \\ C(x, y) & \text{if } (x, y) \in [e, \inf\{x \mid g(x) \leq f\}] \times [f, g(x)], \\ a & \text{if } x \leq a \leq y, \\ b & \text{if } y \leq b \leq x, \\ x & \text{if } a \leq x \leq b, \\ S(x, y) & \text{if } (x, y) \in [0, a]^2, \\ A(x, y) & \text{elsewhere.} \end{cases} \quad (18)$$

and  $G$  is given by

$$G(x, y) = \begin{cases} U^{\min} & \text{if } (x, y) \in [0, k]^2 \cup [k, 1]^2, \\ \min & \text{if } (x, y) \in [0, e] \times (k, 1] \cup (k, 1] \times [0, e), \\ k & \text{otherwise.} \end{cases} \quad (19)$$

Where  $T$  is isomorphic to an associative semi- $t$ -norm,  $S$  are isomorphic with an associative semi- $t$ -conorm.  $A, B, C$  are increasing operators that preserve the associativity of  $F$  and moreover, the value of  $B(x, y)$  and  $C(x, y)$  only depends on  $x$  respectively.

If we consider the right distributivity of  $F \in \mathcal{F}_{a,b}$  over  $G \in \mathcal{U}_{k(e,f)}$  with the condition  $b \leq a$ , the results are quite symmetrical with all we have just discussed above.

## 4 Conclusions

In this paper, we have studied the left (right) distributivity of semi-t-operators  $F \in \mathcal{F}_{a,b}$  over 2-uninorms  $G$ . We have given a detailed study of the left (right) distributive equation with  $a \leq b$  ( $a \geq b$ ). For the rest cases, we find that even the idempotency of  $G$  on  $(a, b)$  cannot be promised so we believe the results will be too complicated to study. In fact, as suggested in [11], solutions of the distributivity involving 2-uninorms are closely related to the specific structures of the given operators, so to consider the most general situation without giving the investigated 2-uninorm in specific subclasses is not entirely possible.

In our discussion we find the case when  $G \in \mathcal{C}_{1,k}$  is very interesting. The structure of the semi-t-operator  $F$  in this case varies with its contour on a parameter of  $G$  and in some neighbour areas of this contour the distributive equation is transformed into  $F(x, y) = F(x, z)$ . Besides, Lemma 3.3 can be applied in other studies about distributivity too.

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## The left (right) distributivity of semi-t-operators over 2-uninorms

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## توزیع پذیری چپ (راست) نیم-t - عملگرها روی ۲-هم نرم

**چکیده.** اخیراً معادلات توزیع پذیر، آغازگر کلاس‌های متعددی از عملگرهای انباشتگی، توجه گسترده‌ای را به دلیل اهمیت آن‌ها در انجمن‌های نظری و عملی نظریه مجموعه‌های فازی برانگیخته است. ۲-تک نرم‌ها و نیم-t-عملگرها دو کلاس خاص از عملگرهای انباشتگی می‌باشند و ثابت کرده‌اند که در بسیاری از حوزه‌ها مانند تصمیم‌گیری فازی، استدلال تقریب و غیره مفید می‌باشند. از این رو، هدف این مقاله تحقیق و بررسی توزیع‌پذیری چپ (راست) نیم-t-عملگرها روی ۲-تک نرم‌ها می‌باشد. ما پنج زیرکلاس از ۲-تک نرم‌ها را در نظر می‌گیریم و معادلات توزیع‌پذیر چپ (راست) متناظر نیم-t-عملگرها روی ۲-نرم‌ها را مشخص می‌سازیم.