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Hesitant q-rung orthopair fuzzy aggregation operators with their applications in multi-criteria decision making

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Abstract

The aim of this manuscript is to present a new concept of hesitant q-rung orthopair fuzzy sets (Hq-ROFSs) by combining the concept of the q-ROFSs as well as Hesitant fuzzy sets. The proposed concept is the generalization of the fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets, and Pythagorean fuzzy sets as well as intuitionistic hesitant fuzzy sets (IHFSs) and hesitant Pythagorean fuzzy sets (HPFSs).Furthermore some basic operational laws of hesitant q-rung orthopair fuzzy have been investigated. The score and accuracy functions are defined which play a vital role in decision making process for making comparison between the hesitant q-rung orthopair fuzzy numbers (Hq-ROFNs). Under the Hq-ROF environment, Hq-ROF weighted averaging (Hq-ROFWA) and Hq-ROF weighted geometric (Hq-ROFWG) operators are introduced and various properties of these aggregation operators are studied. Additionaly, a numerical application shows that how the proposed operators are utilized to solve multi-criteria decision making (MCDM) problems in which experts added their optimistic and pessimistic preferences. Finally the analysis of proposed method with other methods is presented which show that the method presented in this paper is more flexible and superior than existing methods.

Keywords: Hesitant fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, Hesitant q-rung orthopair fuzzy sets, Hesitant q-rung orthopair fuzzy weighted averaging operators, Hesitant q-rung orthopair fuzzy weighted geometric operators, multi-criteria decision making.

1 Introduction

The notion of fuzzy sets, originated by Zadeh [40], is a vigorous field which brought a revolution not only in the field of mathematics but also in various areas of the science and technology. This theory has various applications where imprecision and uncertainty is involved. In fuzzy sets it is required that each element of the universal set has a membership grade from a unit interval $[0,1]$. So in many situations of a real life not only the grade of membership, the grade of non-membership are also required. To overcome such situation Atanassove [2], presented the notion of intuitionistic fuzzy sets (IFSs) and this could be considered as a successful and important generalized structure of fuzzy sets. The approach of IFSs has the capability to comprehensively deal imprecision, and having broad applications in various fields such as cluster analysis [28, 34], pattern recognition [4, 14] and medical diagnoses [21, 25] etc. Atanassov [3] presented the applications of IF interpretations in decision making and for details see [24, 26, 30, 32]. Feng et al. [6] reformulated the generalized intuitionistic fuzzy soft sets and have studied several new operations on it and for details see [7, 20]. In daily life usually peoples are hesitant and irresolute when deciding about something, which makes it difficult for the decision maker to reach the final decision. To tackle such situation the theory of hesitant fuzzy set (HFS) was originated by Torra [27], in which for each element of the reference set the grade of membership consists of the set of several discrete values from a unit interval $[0,1]$ rather than a single number. Hu et al. [11] presented the concept of

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distance measure, similarity measure and entropy measure in hesitant fuzzy information and for details see [5]. Further Zhu et al. [42], mentioned the lack of non-membership in HFSs and originated the concept of dual hesitant fuzzy sets (DHFSs) that contain both of the grades of membership and non-membership respectively. On the same concept Yu et al. [39] presented the idea of DHF weighted averaging (DHFWA) and DHF weighted geometric (DHFWG) operators in DHFSs. Zhao et al. [41] defined some new operations and studied the concept of Einstein t-norm and t-conorm on DHFSs. The concept of DHFSs is more suitable to tackle uncertainty and fuzziness as compared to fuzzy sets, IFSs and HFSs.

Yager [35, 36], presented the notion of Pythagorean fuzzy sets (PFSs), which could be considered as an important generalization of IFSs. This concept of PFSs gives more freedom to the decision makers for assigning the values. The major change in IFSs and PFSs is that, in the PFSs the sum of membership and non-membership grades are greater than 1, but sum of their squares belongs to the unit interval $[0, 1]$. Many researchers showed their interest to MCDM problems by using PFSs. Some new aggregation operators are studied by Yager [36], to aggregate the PF numbers (PFNs) and their applications are studied MCDM problems. Hussain et al. [13] presented the rough structure of Pythagorean fuzzy ideals in semigroups. The operations of division and subtraction for PFNs was originated by Peng and Yang [23], and they also presented a superiority and inferiority ranking methods in Pythagorean fuzzy environment to solve MCDM problems. Garg [8, 9], studied various operators in Pythagorean fuzzy environment such as interval-valued PF weighted averaging and geometric operators and Einstein operators.

With the passage of time and the development of new theories once again decision makers faced complications to assign values to membership grade and non-membership grade. So to overcome these deficiencies Yager [37], presented the new generalization of PFSs, known as q-rung orthopair fuzzy sets (q-ROFS in short). In this notion the sum of qth power of membership grade and qth power of non-membership grade belongs to a unit interval $[0, 1]$ for $q \geq 1$. Ali [1], presented the notion of orbits in q-ROFSs. Hussain et al. [12] investigated a new approach is adopted to hybrid q-rung orthopair fuzzy sets with notions of covering rough set and TOPSIS. This means that the q-ROFS provides more space to the decision makers for the selection of membership and non-membership grades. Liu and Wang [19], proposed aggregation operators and studied their applications to MADM by using q-ROFSs information. Wei et al. [29], originated the notion of Heronian mean operators in MADM by using q-ROF information.

Peng et al. [22] originated the theory of intuitionistic HFSs (IHFSs) and originated their applications in group decision-making problems through a fuzzy cross-entropy. In IHFSs for every object of the given set the grades of membership and non-membership consists of sets of some discrete numbers from a unit interval $[0, 1]$ rather than a single value from a unit interval $[0, 1]$. Kang et al. [15] presented the combine study of IFSs and HFSs by defining distance and similarity measures. Furthermore Khan et al. [17] extended the notion of IHFSs to Pythagorean hesitant fuzzy sets (PHFSs). In PHFSs the sum of square of its membership grade and non-membership grade belongs to $[0, 1]$. Garg [10] investigated the aggregation operators in hesitant Pythagorean fuzzy sets (HPFSs). Liang and Xu [18] presented the evaluation of MCDM through a hesitant Pythagorean fuzzy information. Many researchers showed their keen interest in the generalizations of fuzzy sets such as IFSs, HFSs, PFSs, IHFSs and PHFSs due to their strong points of view to tackle the vagueness and uncertainty. Since these theories under the hesitant fuzzy environment have gained some attraction by the researchers and are valid only for those cases in which their corresponding sum of membership degree and non-membership degree is less than or equal to 1. However, if the decision maker may give their preference toward the object, in the form of discrete set, which the square sum of membership degree and non-membership degree exceed 1. So the ordinary IHFSs and PHFSs failed to handle such situations and is unable to classify the decision-making approaches. Therefore some more comprehensive model is required for such situations. To cope this situation, in this manuscript we introduced the concept of hesitant q-rung orthopair fuzzy sets (Hq-ROFSs), which is the generalized form of IHFSs and PHFSs. In Hq-ROFSs the sum of qth power of q-rung orthopair membership grade and qth power of q-rung orthopair non-membership grade belongs to $[0, 1]$ for $q \geq 1$. The remaining portions of the paper is designed as.

The Section 2, consists of some fundamental and primary notions related to IFSs and their generalizations. In Section 3, the concept of Hq-ROFSs is originated which is more effective and flexible for the decision makers because it gives more space and freedom in constraints as compared to IHFSs and PHFSs. The Section 4, consists of the study of Hq-ROF aggregation operators, that is Hq-ROF weighted averaging (Hq-ROFWA), Hq-ROF weighted geometric (Hq-ROFWG) operators and their related properties. In section 5, the proposed operators are utilized to MCDM in which the experts added their preferences in Hq-ROF environment. In Section 6, through a numerical example, it is demonstrated how the proposed operators works in decision making phenomena. A Subsection 6.1, consists of a comparative study of proposed method with the existing literature, which shows that the proposed method is more flexible and superior than the existing methods.The final Section 7, consists of the conclusion of the manuscript.

2 Preliminaries

This section is concerned with a brief review related to PFSs, q-ROFSs, HFSs, IHFSs and HPFSs.

Definition 2.1. [35, 36] *Let $\mathcal{S}$ be a universal set. A PFS $\mathcal{P}$ defined on $\mathcal{S}$ is an object having the form:*$\mathcal{P} = \{< t, h_{\mathcal{P}}(t), g_{\mathcal{P}}(t) > | t \in \mathcal{S}\}$, *such that* $h_{\mathcal{P}}(t) : \mathcal{S} \to [0,1]$ *denotes the membership grade and* $g_{\mathcal{P}}(t) : \mathcal{S} \to [0,1]$ *represents the non-membership grade of an element* $t \in \mathcal{S}$ *to* $\mathcal{P}$ *respectively. For any* $t \in \mathcal{S}$, *it must hold the condition* $0 \leq (h_{\mathcal{P}}(t))^2 + (g_{\mathcal{P}}(t))^2 \leq 1$ *and the degree of hesitancy/indeterminacy is given as*$\pi_{\mathcal{P}}(t) = \sqrt{1 - (h_{\mathcal{P}}(t))^2 - (g_{\mathcal{P}}(t))^2}$.

Definition 2.2. [37, 38] *Let $\mathcal{S}$ be a universal set. A q-ROFS $\mathcal{F}$ defined on $\mathcal{S}$ is an object represented by the following form:*

$$\mathcal{F} = \{< t, h_{\mathcal{F}}(t), g_{\mathcal{F}}(t) >_q | t \in \mathcal{S}, q \geq 1\},$$

where $h_{\mathcal{F}}(t) : \mathcal{S} \to [0,1]$ *represents the membership grade and* $g_{\mathcal{F}}(t) : \mathcal{S} \to [0,1]$ *represents the non-membership grade of an element* $t \in \mathcal{S}$ *to* $\mathcal{F}$. *For any* $t \in \mathcal{S}$, *it must hold the condition* $0 \leq (h_{\mathcal{F}}(t))^q + (g_{\mathcal{F}}(t))^q \leq 1$. *Moreover the indeterminacy for an element* $t \in \mathcal{S}$ *to* $\mathcal{F}$ *is defined as*

$$\pi_{\mathcal{F}}(t) = \sqrt[q]{1 - ((h_{\mathcal{F}}(t))^q + (g_{\mathcal{F}}(t))^q)}, \quad q \geq 1$$

For simplicity, $\mathcal{F}(t) = (h_{\mathcal{F}}(t), g_{\mathcal{F}}(t))$ *is known as q-rung orthopair fuzzy element (q-ROFE) if there is no confusion.*

Definition 2.3. [27] *Consider a set $\mathcal{S}$, then the HFS $\mathcal{H}$ on $\mathcal{S}$ is an object having the form:*

$$\mathcal{H} = \{< t, h_{\mathcal{H}}(t) > | t \in \mathcal{S}\}$$

where $h_{\mathcal{H}}(t)$ *represents the membership grade, which consists of set of some discrete numbers in* $[0,1]$, *of an element* $t \in \mathcal{S}$ *to the set* $\mathcal{H}$. *For simplicity, the hesitant fuzzy element (HFE) is written as* $h_{\mathcal{H}}(t)$. *Furthermore the cardinality of* $h_{\mathcal{H}}(t)$ *(number of the elements in* $h_{\mathcal{H}}(t)$*) is denoted as* $\#h_{\mathcal{H}}(t)$.

Definition 2.4. [42] *Let $\mathcal{S}$ be a universal set. A DHFS $\mathcal{D}$ on $\mathcal{S}$ is an object having the form:*

$$\mathcal{D} = \{< t, h_{\mathcal{D}}(t), g_{\mathcal{D}}(t) > | t \in \mathcal{S}\},$$

where $h_{\mathcal{D}}(t)$ *and* $g_{\mathcal{D}}(t)$ *are two finite subsets of unit interval* $[0,1]$, *which represent the possible dual hesitant membership degrees and dual hesitant non-membership degrees of the element* $t \in \mathcal{S}$ *to* $\mathcal{D}$ *respectively. For any* $t \in \mathcal{S}$, *it must hold the condition:*

$$0 \leq \Im, \lambda \leq 1, 0 \leq \Im^+ + \lambda^+ \leq 1$$

where $\Im \in h_{\mathcal{D}}(t), \lambda \in g_{\mathcal{D}}(t)$, $\Im^+ = \max_{\Im \in h_{\mathcal{D}}(t)} \{\Im\}$ *and* $\lambda^+ = \max_{\lambda \in g_{\mathcal{D}}(t)} \{\lambda\}$ *for all* $t \in \mathcal{S}$.

However, DHFSs are defined in terms of sets of values, as opposed to precise numbers, for the membership degrees and non-membership degrees of IFSs .Moreover, dual HFSs require the sum of the maximum membership degree and maximum non-membership degree to be no more than 1, which limits their application in certain cases. For example, decision-makers may deem that the possible membership degrees of an alternative against the criterion ‘excellent’ are 0.4 and 0.7, with its possible non-membership degree being 0.3 and 0.6, that is $\mathcal{D} = \{< t, \{0.4, 0.7\}, \{0.3, 0.6\} > | t \in \mathcal{S}\}$, implies $0.7+0.6 = 1.3 > 1$. In such circumstances, it is not possible to solve this problem by utilizing DHFSs. Therefore, in order to solve it, Peng et al [22], propose intuitionistic hesitant fuzzy sets (IHFSs), which are based on IFSs and HFSs.

Definition 2.5. [22] *Consider a universal set $\mathcal{S}$. An IHFS $\mathcal{I}$ on $\mathcal{S}$ is an object represented by the following:*

$$\mathcal{H} = \{< t, h_{\mathcal{H}}(t), g_{\mathcal{H}}(t) > | t \in \mathcal{S}\},$$

where $h_{\mathcal{H}}(t)$ *and* $g_{\mathcal{H}}(t)$ *are the subsets of* $[0,1]$, *which represents the intuitionistic hesitant membership and intuitionistic hesitant non-membership grades, of* $t \in \mathcal{S}$ *to* $\mathcal{H}$. *For every element* $t \in \mathcal{S}$, $\forall \Im(t) \in h_{\mathcal{H}}(t)$ *there exist* $\lambda(t) \in g_{\mathcal{H}}(t)$, *which holds the condition that* $0 \leq \Im(t) + \lambda(t) \leq 1$ *and similarly,* $\forall \lambda(t) \in g_{\mathcal{H}}(t)$, *there exist* $\Im(t) \in h_{\mathcal{H}}(t)$, *which holds the condition that* $0 \leq \Im(t) + \lambda(t) \leq 1$.

Definition 2.6. [18] *Let $\mathcal{S}$ be a universal set. A hesitant Pythagorean fuzzy set (HPFS) $\mathcal{P}_{\mathcal{H}}$ on $\mathcal{S}$ is an object denoted by the following:*

$$\mathcal{P}_{\mathcal{H}} = \{< t, h_{\mathcal{P}_{\mathcal{H}}}(t), g_{\mathcal{P}_{\mathcal{H}}}(t) > | t \in \mathcal{S}\},$$

such that $h_{\mathcal{P}_{\mathcal{H}}}(t)$ *and* $g_{\mathcal{P}_{\mathcal{H}}}(t)$ *are the subsets of* $[0,1]$, *represents the hesitant Pythagorean membership and hesitant Pythagorean non-membership grades. Moreover each element* $t \in \mathcal{S}$, $\forall \Im(t) \in h_{\mathcal{H}}(t)$ *there exist* $\lambda(t) \in g_{\mathcal{H}}(t)$, *which holds the condition that* $0 \leq (\Im(t))^2 + (\lambda(t))^2 \leq 1$ *and similarly,* $\forall \lambda(t) \in g_{\mathcal{H}}(t)$, *there exist* $\Im(t) \in h_{\mathcal{H}}(t)$, *which holds the condition that* $0 \leq (\Im(t))^2 + (\lambda(t))^2 \leq 1$.

3 Hesitant q-rung orthopair fuzzy set (Hq-ROFS)

This section is devoted to the study of Hq-ROFSs. The concept of Hq-ROFSs is more effective and flexible for the decision makers because this concept provides more space for the decision makers in assigning values as compared to IHFSs and PHFSs.

Definition 3.1. *Let $\mathcal{S}$ be a universal set. Then a Hq-ROFS $\mathcal{G}$ defined on $\mathcal{S}$ is an object given by the following:*

$$\mathcal{G} = \{< t, h_{\mathcal{G}}(t), g_{\mathcal{G}}(t)) >_q |t \in \mathcal{S}, q \geq 1\},$$

where $h_{\mathcal{G}}(t)$ and $g_{\mathcal{G}}(t)$ are two subsets of $[0,1]$, which represents the hesitant q-rung orthopair membership and hesitant q-rung orthopair non-membership grades of an object $t \in \mathcal{S}$ to the set $\mathcal{G}$. Moreover for each element $t \in \mathcal{S}$, $\forall \Im(t) \in h_{\mathcal{H}}(t)$ there exist $\lambda(t) \in g_{\mathcal{H}}(t)$, which holds the condition that $0 \leq (\Im(t))^q + (\lambda(t))^q \leq 1$ and similarly, $\forall \lambda(t) \in g_{\mathcal{H}}(t)$, there exist $\Im(t) \in h_{\mathcal{H}}(t)$, which holds the condition that $0 \leq (\Im(t))^q + (\lambda(t))^q \leq 1$.

In perspective of Definition 3.1, for each t belongs to $\mathcal{S}$ there are two sets, that is hesitant q-rung orthopair membership grade $h_{\mathcal{G}}(t)$ and hesitant q-rung orthopair non-membership grade $g_{\mathcal{G}}(t)$. The cardinality of $h_{\mathcal{G}}(t)$ and $g_{\mathcal{G}}(t)$ (number of the elements in $h_{\mathcal{G}}(t)$ and $g_{\mathcal{G}}(t)$) are represented by $\#h_{\mathcal{G}}(t)$ and $\#g_{\mathcal{G}}(t)$. Similar to the concepts of IHFSs and HPFSs, the Hq-ROFS also deal with two types of grades that is hesitant q-rung orthopair membership and hesitant q-rung orthopair non-membership grades of an object $t \in \mathcal{S}$ to the set $\mathcal{G}$, which provides more flexibility in assigning the values for each object in domain. The comparative study of Hq-ROFS with HFSs, PFSs, IHFSs and PHFSs are mentioned in Table 1.

Table 1, The comparative study of different fuzzy sets.

The evaluation format	The no. of elements in membership grade		The no. of elements in non-membership grade	
	one	Many	one	Many
HFSs [27]	√	√		
PFSs [35]	√		√	
IHFSs [22]	√	√	√	√
HPFSs [18]	√	√	√	√
Hq-ROFS	√	√	√	√

From *Table* 1, it is clear that Hq-ROFSs is the generalization of HFSs, PFSs, IHFSs and PHFSs but the main difference of Hq-ROFSs with the rest of these sets are their different constraints, which is clear from Definitions 2.5, 2.6 and 3.1. Actually Hq-ROFSs provides a huge space for decision makers to select membership and non-membership grades than the rest of the sets Define in 2.5, 2.6 and 3.1. For simplicity the pair $\mathcal{G} =< t, h_{\mathcal{G}}(t), g_{\mathcal{G}}(t)) >_q$ represent a hesitant q-rung fuzzy number (Hq-ROFN), which is represented by $\mathcal{E} = \mathcal{G}(h, g)$. For example, if the possible values of membership grades which the decision maker assigned to an object t of set $\mathcal{G}$ are $0.95, 0.9$ and 0.85, and for the possible values of non-membership grades are $0.75, 0.7$ and 0.65, then it can be represented as $\mathcal{E} = \mathcal{G}(h = \{0.95, 0.9, 0.85\}, g = \{0.75, 0.7, 0.65\})$. Here for all $\Im \in h$, there exists $\lambda \in g$, such that $\Im^q + \lambda^q \leq 1$, and for all $\lambda \in g$ there exists $\Im \in h$, such that $\Im^q + \lambda^q \leq 1$, for $q \geq 6$. So, in this case the ordinary IHFSs and PHFSs failed to tackle this situation. Therefore, to handle such situation a comprehensive model of Hq-ROFSs is defined in this paper.

Definition 3.2. *Let $\mathcal{E}_1 = \mathcal{G}(\hat{h}_1(t), \hat{g}_1(t))$ and $\mathcal{E}_2 = \mathcal{G}(\hat{h}_2(t), \hat{g}_2(t))$ be two Hq-ROFNs. Then $\mathcal{E}_1 \succcurlyeq \mathcal{E}_2$ holds for each $t \in \mathcal{S}$ such that $\hat{h}_1(t) \succcurlyeq \hat{h}_2(t) \Leftrightarrow \Im_1^{\delta(k)}(t) \geq \Im_2^{\delta(k)}(t)$ and $\hat{g}_1(t) \preccurlyeq \hat{g}_2(t) \Leftrightarrow \lambda_1^{\delta(k)}(t) \leq \lambda_2^{\delta(k)}(t)$, where $\Im_1^{\delta(k)}(t), \Im_2^{\delta(k)}(t)$ $\lambda_1^{\delta(k)}(t)$ and $\lambda_2^{\delta(k)}(t)$ represents the kth largest numbers of the membership grade and non-membership grade in $\hat{h}_1(t)$, $\hat{g}_1(t), \hat{h}_2(t)$ and $\hat{g}_2(t)$ respectively.*

where $[\delta(1), \delta(2), \delta(3), ..., \delta(n)]$ is a permutation of $(1, 2, ...n)$ which satisfies that $\mathcal{E}_{\delta(k\text{-}1)} \succcurlyeq \mathcal{E}_{\delta(k)}$ for $k = 2, 3, ..., n$.

Definition 3.3. *Consider two Hq-ROFNs $\mathcal{E}_1 = \mathcal{G}(h_1, g_1)$ and $\mathcal{E}_2 = \mathcal{G}(h_2, g_2)$. Then the union and intersection between them are defined as follows:*
(i) $\mathcal{E}_1 \cup \mathcal{E}_2 = \{h \in (h_1 \cup h_2)|h \geq \max(h_1^-, h_2^-), g \in (g_1 \cap g_2)|g \leq \min(g_1^+, g_2^+)\}$;
(ii) $\mathcal{E}_1 \cap \mathcal{E}_2 = \{h \in (h_1 \cap h_2)|h \leq \min(h_1^+, h_2^+), g \in (g_1 \cup g_2)|g \geq \max(g_1^-, g_2^-)\}$.
where h^-, h^+ and g^-, g^+ are the lower and upper bounds to h and g respectively, where $h^- = \underset{\Im \in h}{\cup} \min\{\Im\}$, $h^+ = \underset{\Im \in h}{\cup} \max\{\Im\}$, $g^- = \underset{\lambda \in g}{\cup} \min\{\lambda\}$ and $g^+ = \underset{\lambda \in g}{\cup} \max\{\lambda\}$.

Example 3.4. *Let* $\mathcal{E}_1 = \{\{0.1, 0.2, 0.4, 0.5\}, \{0.3, 0.6, 0.7\}\}$ *and* $\mathcal{E}_2 = \{\{0.3, 0.5, 0.7\}, \{0.2, 0.4, 0.6\}\}$, *then we have* $\mathcal{E}_1 \cup \mathcal{E}_2 = \{\{0.3, 0.4, 0.5, 0.7\}, \{0.2, 0.3, 0.4, 0.6\}\}$ *and* $\mathcal{E}_1 \cap \mathcal{E}_2 = \{\{0.1, 0.2, 0.3, 0.4, 0.5\}, \{0.3, 0.4, 0.6, 0.7\}\}$.

Definition 3.5. *For a Hq-ROFN $\mathcal{E} = \mathcal{G}(h, g)$ characterized by h and g, the score function of $\mathcal{E}$ is defined as*

$$\hat{S}(\mathcal{E}) = \frac{1}{\#h} \sum_{\Im \in h} \Im^q - \frac{1}{\#g} \sum_{\lambda \in g} \lambda^q,$$

where $\#h$ and $\#g$ represents the cardinality of h and g respectively.

Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2)$ be any two Hq-ROFNs. Then
i: If $\hat{S}(\mathcal{E}_1) > \hat{S}(\mathcal{E}_2)$, then $\mathcal{E}_1$ is superior than $\mathcal{E}_2$, represented by $\mathcal{E}_1 \succcurlyeq \mathcal{E}_2$,
ii: If $\hat{S}(\mathcal{E}_1) < \hat{S}(\mathcal{E}_2)$, then $\mathcal{E}_1$ is inferior than $\mathcal{E}_2$, represented by $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$,

Definition 3.6. *Let $\mathcal{E} = \mathcal{G}(h, g)$ be a Hq-ROFN. Then the accuracy function of $\mathcal{E}$ is $\mathcal{A}(\mathcal{E}) = \frac{1}{\#h} \sum_{\Im \in h} \Im^q + \frac{1}{\#g} \sum_{\lambda \in g} \lambda^q$.*

Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2)$ be any two Hq-ROFNs. Then
If $\hat{S}(\mathcal{E}_1) = \hat{S}(\mathcal{E}_2)$, then
(a) if $\mathcal{A}(\mathcal{E}_1) = \mathcal{A}(\mathcal{E}_2)$, then $\mathcal{E}_1$ is equivalent to $\mathcal{E}_2$, represented by $\mathcal{E}_1 \approx \mathcal{E}_2$,
(b) if $\mathcal{A}(\mathcal{E}_1) > \mathcal{A}(\mathcal{E}_2)$, then $\mathcal{E}_1$ is superior than $\mathcal{E}_2$, represented by $\mathcal{E}_1 \succcurlyeq \mathcal{E}_2$.
(c) If $\mathcal{A}(\mathcal{E}_1) < \mathcal{A}(\mathcal{E}_2)$, then $\mathcal{E}_1$ is inferior than $\mathcal{E}_2$, represented by $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$,

Example 3.7. *Let $\mathcal{E}_1 = \mathcal{G}(\{0.8, 0.7\}, \{0.75, 0.65\})$ and $\mathcal{E}_2 = \mathcal{G}(\{0.78, 0.67\}, \{0.82, 0.40299\})$ be two Hq-ROFNs. Then $\hat{S}(\mathcal{E}_1) = \hat{S}(\mathcal{E}_2) = 0.079$ when $q = 3$ and $\mathcal{A}(\mathcal{E}_1) = 0.776 > \mathcal{A}(\mathcal{E}_2) = 0.696$, so $\mathcal{E}_1$ is superior than $\mathcal{E}_2$, represented by $\mathcal{E}_1 \succcurlyeq \mathcal{E}_2$.*

Definition 3.8. *Consider a Hq-ROFN $\mathcal{E} = \mathcal{G}(h, g)$. Furthermore the hesitancy/indeterminacy degree of $\mathcal{E}$ can be defined as*

$$\pi_{\mathcal{E}} = \sqrt[q]{1 - \left(\frac{1}{\#h} \sum_{\Im \in h} \Im^q + \frac{1}{\#g} \sum_{\lambda \in g} \lambda^q \right)}$$

Definition 3.9. *Consider three Hq-ROFNs $\mathcal{E} = \mathcal{G}(h, g), \mathcal{E}_1 = \mathcal{G}(h_1, g_1)$ and $\mathcal{E}_2 = \mathcal{G}(h_2, g_2)$. Then some basic operations among them are defined as follows:*
(i) $\mathcal{E}_1 \oplus \mathcal{E}_2 = \bigcup_{\Im_1 \in h_1, \lambda_1 \in g_1, \Im_2 \in h_2, \lambda_2 \in g_2} \{\{\sqrt[q]{(\Im_1)^q + (\Im_2)^q - (\Im_1)^q (\Im_2)^q}\}, \{\lambda_1 \lambda_2\}\}$;
(ii) $\mathcal{E}_1 \otimes \mathcal{E}_2 = \bigcup_{\Im_1 \in h_1, \lambda_1 \in g_1, \Im_2 \in h_2, \lambda_2 \in g_2} \{\{\Im_1 \Im_2\}, \{\sqrt[q]{(\lambda_1)^q + (\lambda_2)^q - (\lambda_1)^q (\lambda_2)^q}\}\}$;
(iii) $\lambda \mathcal{E} = \bigcup_{\Im \in h, \lambda \in g} \{\{\sqrt[q]{1 - (1 - \Im^q)^\lambda}\}, \{\lambda^\lambda\}\}$, *where* $\lambda \in R$ *and* $\lambda > 0$;
(iv) $\mathcal{E}^\lambda = \bigcup_{\Im \in h, \lambda \in g} \{(\Im^\lambda), \sqrt[q]{1 - (1 - \lambda^q)^\lambda})\}$, *for* $\lambda > 0$;
(v) $\mathcal{E}^c = \mathcal{G}(g, h)$.

Theorem 3.10. *Let $\mathcal{E}_1 = \mathcal{G}(h_1, g_1)$ and $\mathcal{E}_2 = \mathcal{G}(h_2, g_2)$ be two Hq-ROFNs. Then we have:*
(i) $\mathcal{E}_1 \oplus \mathcal{E}_2 = \mathcal{E}_1 \oplus \mathcal{E}_1$;
(ii) $\mathcal{E}_1 \otimes \mathcal{E}_2 = \mathcal{E}_2 \otimes \mathcal{E}_1$;
(iii) $\lambda (\mathcal{E}_1 \oplus \mathcal{E}_2) = \lambda \mathcal{E}_1 \oplus \lambda \mathcal{E}_1; \ \ \lambda > 0$
(iv) $\lambda (\mathcal{E}_1 \otimes \mathcal{E}_2) = \lambda \mathcal{E}_2 \otimes \lambda \mathcal{E}_1, \ \ \lambda > 0$
(v) $(\mathcal{E}_1 \otimes \mathcal{E}_2)^\lambda = \mathcal{E}_2^\lambda \otimes \mathcal{E}_1^\lambda, \ \ \lambda > 0$
(vi) $(\lambda_1 + \lambda_2) \mathcal{E}_1 = \lambda_1 \mathcal{E}_1 \oplus \lambda_2 \mathcal{E}_1, \ \ \lambda_1, \lambda_2 > 0$
(vii) $\mathcal{E}_1^{\lambda_1 + \lambda_2} = \mathcal{E}_1^{\lambda_1} \otimes \mathcal{E}_1^{\lambda_2}, \ \ \lambda_1, \lambda_2 > 0.$

4 Hesitant q-rung orthopair fuzzy aggregation (Hq-ROFA) operators and their properties

This section is concerned with basic notions of related aggregation operators and their properties as follows:

Definition 4.1. *Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ be a collection of Hq-ROFNs. Furthermore consider the weight vector $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n)^T$ of $\mathcal{E}_i$ such that $\tilde{w}_i \geq 0$ $(i = 1, ..., n)$ where $\tilde{w}_i \in [0, 1]$ with $\sum_{i=1}^{n} \tilde{w}_i = 1$. Then the function for Hq-ROF weighted averaging (Hq-ROFWA) operator is defined as,*

$$Hq\text{-}ROFWA : \mathcal{H}^n \to \mathcal{H} \quad (\text{where } \mathcal{H} \text{ is the collection of all Hq-ROFNs})$$

such that

$$Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \overset{n}{\underset{i=1}{\oplus}} \tilde{w}_i \mathcal{E}_i = \tilde{w}_1 \mathcal{E}_1 \oplus \tilde{w}_2 \mathcal{E}_2 \oplus ... \oplus \tilde{w}_n \mathcal{E}_n$$

The aggregation result for Hq-ROFNs through operation rules is given as in Theorem 4.2.

Theorem 4.2. *Consider the collection* $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ *of Hq-ROFNs. Let* $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n)^T$ *be the weight vector of* $\mathcal{E}_i$ *such that* $\tilde{w}_i \geq 0 (i = 1, ..., n)$ *where* $_i \in [0, 1]$ *with* $\sum_{i=1}^{n} \tilde{w}_i = 1$. *Then*

$$Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}}\}, \{\prod_{i=1}^{n} \lambda_i^{\tilde{w} i}\}\}. \quad (1)$$

Proof. As we know

$$\tilde{w}_1 \mathcal{E}_1 = \underset{\Im_1 \in h_1, \lambda_1 \in g_1}{\cup} \{\{\sqrt[q]{1 - (1 - \Im_1^q)^{\tilde{w}_1}}\}, \{\lambda_1^{\tilde{w}_1}\}\} \quad \text{and} \quad \tilde{w}_2 \mathcal{E}_2 = \underset{\Im_2 \in h_2, \lambda_2 \in g_2}{\cup} \{\{\sqrt[q]{1 - (1 - \Im_2^q)^{\tilde{w}_2}}\}, \{\lambda_2^{\tilde{w}_2}\}\}$$

First we show that Eq. (1) is true for $n = 2$, we have

$$\begin{aligned} \tilde{w}_1 \mathcal{E}_1 \oplus \tilde{w}_2 \mathcal{E}_2 &= \underset{\substack{\Im_1 \in h_1, \lambda_1 \in g_1, \\ \Im_2 \in h_2, \lambda_2 \in g_2}}{\cup} \{\{\sqrt[q]{\begin{matrix} \left(1 - (1 - \Im_1^q)^{\tilde{w}_1}\right) + \left(1 - (1 - \Im_2^q)^{\tilde{w}_2}\right) \\ - \left(1 - (1 - \Im_1^q)^{\tilde{w}_1}\right) \left(1 - (1 - \Im_2^q)^{\tilde{w}_2}\right) \end{matrix}}\}, \{\lambda_1^{\tilde{w}_1} \lambda_2^{\tilde{w}_2}\}\} \\ &= \underset{\substack{\Im_1 \in h_1, \lambda_1 \in g_1, \\ \Im_2 \in h_2, \lambda_2 \in g_2}}{\cup} \{\{\sqrt[q]{1 - (1 - \Im_1^q)^{\tilde{w}_1} (1 - \Im_2^q)^{\tilde{w}_2}}\}, \{\lambda_1^{\tilde{w}_1} \lambda_2^{\tilde{w}_2}\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{2} (1 - \Im_i^q)^{\tilde{w}_i}}\}, \{\prod_{i=1}^{2} \lambda_i^{\tilde{w}_i}\}\} \end{aligned}$$

Now suppose Eq. (1) is true for $n = k$, then

$$Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_k) = \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{k} (1 - \Im_i^q)^{\tilde{w}_i}}\}, \{\prod_{i=1}^{k} \lambda_i^{\tilde{w}_i}\}\}$$

Next when n is increased single unit, we have

$$Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_k, \mathcal{E}_{k+1}) = \tilde{w}_1 \mathcal{E}_1 \oplus \tilde{w}_2 \mathcal{E}_2 \oplus ... \oplus \tilde{w}_k \mathcal{E}_k \oplus \tilde{w}_{k+1} \mathcal{E}_{k+1} = (\tilde{w}_1 \mathcal{E}_1 \oplus \tilde{w}_2 \mathcal{E}_2 \oplus ... \oplus \tilde{w}_k \mathcal{E}_k) \oplus \tilde{w}_{k+1} \mathcal{E}_{k+1}$$

$$\begin{aligned} &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{k} (1 - \Im_i^q)^{\tilde{w}_i}}\}, \{\prod_{i=1}^{k} \lambda_i^{\tilde{w}_i}\}\} \oplus \underset{\Im_{k+1} \in h_{k+1}, \lambda_{k+1} \in g_{k+1}}{\cup} \{\{\sqrt[q]{1 - (1 - \Im_{k+1}^q)^{\tilde{w}_{k+1}}}\}, \{\lambda_{k+1}^{\tilde{w}_{k+1}}\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{\begin{matrix} (1 - \prod_{i=1}^{k} (1 - \Im_i^q)^{\tilde{w}_i}) + ((1 - (1 - \Im_{k+1}^q)^{\tilde{w}_{k+1}}) \\ -(1 - \prod_{i=1}^{k} (1 - \Im_i^q)^{\tilde{w}_i})(1 - (1 - \Im_{k+1}^q)^{\tilde{w}_{k+1}}) \end{matrix}}\}, \{\prod_{i=1}^{k} \lambda_i^{\tilde{w}_i} \lambda_{k+1}^{\tilde{w}_{k+1}}\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{k} (1 - \Im_i^q)^{\tilde{w}_i} (1 - \Im_{k+1}^q)^{\tilde{w}_{k+1}}}\}, \{\prod_{i=1}^{k} \lambda_i^{\tilde{w}_i} \lambda_{k+1}^{\tilde{w}_{k+1}}\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{k+1} (1 - \Im_i^q)^{\tilde{w}_i}}\}, \{\prod_{i=1}^{k+1} \lambda_i^{\tilde{w}_i}\}\}. \end{aligned}$$

Hence Eq. (1) is true for $n = k + 1$. Therefore Eq. (1) is true for any value of n, which complete the proof. □

Proposition 4.3. *In the following, it is shown that* $Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$ *is also a* $Hq\text{-}ROFN$.

Proof. Let $h = \bigcup_{\Im_i \in h_i} \{ \sqrt[q]{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}} \}$ and $g = \bigcup_{\lambda_i \in g_i} \{ \prod_{i=1}^{n} \lambda_i^{\tilde{w}_i} \}$. Because $0 \leq \Im_i, \lambda_i \leq 1$, where $\Im_i \in h_i, \lambda_i \in g_i$, $\Im = \bigcup_{\Im_i \in h_i} \{ \sqrt[q]{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}} \}$ and $\lambda = \bigcup_{\lambda_i \in g_i} \{ \prod_{i=1}^{n} \lambda_i^{\tilde{w}_i} \}$, we have

$$\begin{aligned} 0 &\leq \bigcup_{\Im_i \in h_i} \{1 - \Im_i^q\} \leq 1 \Rightarrow 0 \leq \bigcup_{\Im_i \in h_i} \{(1 - \Im_i^q)^{\tilde{w}_i}\} \leq 1 \Rightarrow 0 \leq \bigcup_{\Im_i \in h_i} \{\prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}\} \leq 1 \\ &\Rightarrow \ 0 \leq \bigcup_{\Im_i \in h_i} \{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}\} \leq 1 \Rightarrow 0 \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}\}} \leq 1 \Rightarrow 0 \leq \Im \leq 1 \end{aligned}$$

and

$$0 \leq \bigcup_{\lambda_i \in g_i} \{\lambda_i^{\tilde{w}_i}\} \leq 1 \Rightarrow 0 \leq \bigcup_{\lambda_i \in g_i} \{\prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\} \leq 1 \Rightarrow 0 \leq \lambda \leq 1$$

so $0 \leq \Im, \lambda \leq 1$. Now, $\Im^q + \lambda^q \leq 1$, we have

$$\lambda^q \leq 1 - \Im^q \Rightarrow (\bigcup_{\lambda_i \in g_i} \{\prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\})^q \leq 1 - (\bigcup_{\Im_i \in h_i} \sqrt[q]{\{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}\}})^q,$$

then

$$\begin{aligned} 0 \leq \Im^q + \lambda^q &= (\bigcup_{\Im_i \in h_i} \sqrt[q]{\{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}\}})^q + (\bigcup_{\lambda_i \in g_i} \{\prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\})^q \\ &\leq (\bigcup_{\Im_i \in h_i} \sqrt[q]{\{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}\}})^q + 1 - (\bigcup_{\Im_i \in h_i} \sqrt[q]{\{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}\}})^q = 1 \Rightarrow 0 \leq \Im^q + \lambda^q \leq 1 \end{aligned}$$

Therefore, it is proved that $Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$ is a $Hq\text{-}ROFN$. □

Theorem 4.4. ***(Idempotency)*** *Consider the collection of $Hq\text{-}ROFNs$ $\mathcal{E} = \mathcal{G}(h, g) = \bigcup_{\Im \in h, \lambda \in g} \{\{\Im\}, \{\lambda\}\}$ and $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$. If for all i, $\Im_i = \Im, \lambda_i = \lambda$, where $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i and $\Im, \lambda$ are the objects of HFSs h, g, then*

$$Hq\text{-}ROFWA\,(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \mathcal{E}$$

Proof. From Theorem 4.2, we have

$$\begin{aligned} Hq - ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) &= \bigcup_{\Im_i \in h_i, \lambda_i \in g_i} \{\{ \sqrt[q]{1 - \prod_{i=1}^{n} (1 - \Im_i^q)^{\tilde{w}_i}} \}, \{\prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\}\} = \bigcup_{\Im \in h, \lambda \in g} \{\{ \sqrt[q]{1 - \prod_{i=1}^{n} (1 - \Im^q)^{\tilde{w}_i}} \}, \{\prod_{i=1}^{n} \lambda^{\tilde{w}_i}\}\} \\ &= \bigcup_{\Im \in h, \lambda \in g} \{\{ \sqrt[q]{1 - (1 - \Im^q)^{\sum_{i=1}^{n} \tilde{w}_i}} \}, \{\lambda^{\sum_{i=1}^{n} \tilde{w}_i}\}\} = \bigcup_{\Im \in h, \lambda \in g} \{\{ \sqrt[q]{1 - (1 - \Im^q)} \}, \{\lambda\}\} \\ &= \bigcup_{\Im \in h, \lambda \in g} \{\{\Im\}, \{\lambda\}\}, \end{aligned}$$

which implies $Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \mathcal{E}$, and so we get the required proof of the theorem. □

Theorem 4.5. ***(Boundedness)*** *Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ be the collection of $Hq\text{-}ROFNs$. Furthermore consider the weight vector $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n)^T$ of $\mathcal{E}_i$ such that $\tilde{w}_i \geq 0 (i = 1, ..., n)$ where $\tilde{w}_i \in [0, 1]$ with $\sum_{i=1}^{n} \tilde{w}_i = 1$. Then*

$$\begin{aligned} \mathcal{E}^- &\leq Hq\text{-}ROFWA\,(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \leq \mathcal{E}^+ \qquad \textbf{(i)} \\ \textit{where } \mathcal{E}^- &= \mathcal{G}(h^-, g^+),\ \mathcal{E}^+ = \mathcal{G}(h^+, g^-) \textit{ such that } h^- = \bigcup_{\Im_i \in h_i} \min\{\Im_i\}, \\ g^+ &= \bigcup_{\lambda_i \in g_i} \max\{\lambda_i\}, h^+ = \bigcup_{\Im_i \in h_i} \max\{\Im_i\}, g^- = \bigcup_{\lambda_i \in g_i} \min\{\lambda_i\} \end{aligned}$$

Proof. As

$$\bigcup_{\Im_i \in h_i} \min\{\Im_i\} \leq \bigcup_{\Im_i \in h_i} \{\Im_i\} \leq \bigcup_{\Im_i \in h_i} \max\{\Im_i\} \qquad (\mathbf{ii})$$

$$\text{and} \bigcup_{\lambda_i \in g_i} \min\{\lambda_i\} \leq \bigcup_{\lambda_i \in g_i} \{\lambda_i\} \leq \bigcup_{\lambda_i \in g_i} \max\{\lambda_i\} \qquad (\mathbf{iii})$$

Now from eq. (**ii**) we have

$$\bigcup_{\Im_i \in h_i} \min\{\Im_i\} \leq \bigcup_{\Im_i \in h_i} \{\Im_i\} \leq \bigcup_{\Im_i \in h_i} \max\{\Im_i\}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{\min\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\max\{(\Im_i)^q\}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\max\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\min\{(\Im_i)^q\}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{(1-\max\{(\Im_i)^q\})^{\tilde{w}_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{(1-\{(\Im_i)^q\})^{\tilde{w}_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{(1-\min\{(\Im_i)^q\})^{\tilde{w}_i}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{\prod_{i=1}^{n}(1-\max\{(\Im_i)^q\})^{\tilde{w}_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\prod_{i=1}^{n}(1-\{(\Im_i)^q\})^{\tilde{w}_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\prod_{i=1}^{n}(1-\min\{(\Im_i)^q\})^{\tilde{w}_i}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{(1-\max\{(\Im_i)^q\})^{\sum_{i=1}^{n}\tilde{w}_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\prod_{i=1}^{n}(1-\{(\Im_i)^q\})^{\tilde{w}_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{(1-\min\{(\Im_i)^q\})^{\sum_{i=1}^{n}\tilde{w}_i}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\max\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\prod_{i=1}^{n}(1-\{(\Im_i)^q\})^{w_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\min\{(\Im_i)^q\}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{-1+\min\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{-\prod_{i=1}^{n}(1-\{(\Im_i)^q\})^{w_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{-1+\max\{(\Im_i)^q\}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{1-1+\min\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\prod_{i=1}^{n}(1-\{(\Im_i)^q\})^{w_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{1-1+\max\{(\Im_i)^q\}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \sqrt[q]{\min\{(\Im_i)^q\}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\prod_{i=1}^{n}(1-\{(\Im_i)^q\})^{w_i}} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{\max\{(\Im_i)^q\}}$$

$$\Leftrightarrow \bigcup_{\Im_i \in h_i} \min\{\Im_i\} \leq \bigcup_{\Im_i \in h_i} \sqrt[q]{1-\prod_{i=1}^{n}(1-\{(\Im_i)^q\})^{w_i}} \leq \bigcup_{\Im_i \in h_i} \max\{\Im_i\} \qquad (\mathbf{vi})$$

Next from eq. (**iii**), we have $\bigcup_{\lambda_i \in g_i} \min\{\lambda_i\} \leq \bigcup_{\lambda_i \in g_i} \{\lambda_i\} \leq \bigcup_{\lambda_i \in g_i} \max\{\lambda_i\}$

$$\Leftrightarrow \bigcup_{\lambda_i \in g_i} \min\{(\lambda_i)^{\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \{(\lambda_i)^{\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \max\{(\lambda_i)^{\tilde{w}_i}\}$$

$$\Leftrightarrow \bigcup_{\lambda_i \in g_i} \min\{(\lambda_i)^{\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \{(\lambda_i)^{\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \max\{(\lambda_i)^{\tilde{w}_i}\}$$

$$\Leftrightarrow \bigcup_{\lambda_i \in g_i} \prod_{i=1}^{n} \min\{(\lambda_i)^{\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \prod_{i=1}^{n} \{(\lambda_i)^{\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \prod_{i=1}^{n} \max\{(\lambda_i)^{\tilde{w}_i}\}$$

$$\Leftrightarrow \bigcup_{\lambda_i \in g_i} \min\{(\lambda_i)^{\sum_{i=1}^{n}\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \prod_{i=1}^{n} \{(\lambda_i)^{\tilde{w}_i}\} \leq \bigcup_{\lambda_i \in g_i} \max\{(\lambda_i)^{\sum_{i=1}^{n}\tilde{w}_i}\}$$

$$\Leftrightarrow \underset{\lambda_i \in g_i}{\cup} \min\{\lambda_i\} \leq \underset{\lambda_i \in g_i}{\cup} \prod_{i=1}^{n}\{(\lambda_i)^{\tilde{w}_i}\} \leq \underset{\lambda_i \in g_i}{\cup} \max\{\lambda_i\} \qquad (\mathbf{v})$$

Thus we have

$\Leftrightarrow \underset{\Im_i \in h_i}{\cup} \min\{\Im_i\} - \underset{\lambda_i \in g_i}{\cup} \max\{\lambda_i\} \leq \underset{\Im_i \in h_i}{\cup} \sqrt[q]{1 - \prod_{i=1}^{n}(1 - \{(\Im_i)^q\})^{\tilde{w}_i}} - \underset{\lambda_i \in g_i}{\cup} \prod_{i=1}^{n}\{(\lambda_i)^{\tilde{w}_i}\} \leq \underset{\Im_i \in h_i}{\cup} \max\{\Im_i\} - \underset{\lambda_i \in g_i}{\cup} \min\{\lambda_i\}$
$\Rightarrow \hat{S}(\mathcal{E}^-) \leq \hat{S}(\mathcal{E}) \leq \hat{S}(\mathcal{E}^+)$
Therefore from eq. (**vi**) and (**v**) and Definition 3.5, we have $\mathcal{E}^- \leq Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \leq \mathcal{E}^+$. □

Theorem 4.6. ***(Monotonicity)*** *Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ and $\mathcal{L}_i = \mathcal{Q}(\varphi_i, \phi_i)(i = 1, 2, ..., n)$ be the collection of two families of Hq-ROFNs such that $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i and $\hat{\theta}_i, \hat{\eta}_i$ are the objects of HFSs φ_i, ϕ_i. If for all i, $\Im_i \geq \hat{\theta}_i$ and $\lambda_i \leq \hat{\eta}_i$, then:*
$Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \geq Hq\text{-}ROFWA(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n)$.

Proof. From Definition 3.2, it is clear that, if $\mathcal{E}_i \geq \mathcal{L}_i$, then $h_i \geq \varphi_i$ and $\phi_i \geq g_i$. Now if
$h_i \geq \varphi_i \Rightarrow \underset{\Im_i \in h_i}{\cup} \{\Im_i\} \geq \underset{\hat{\theta}_i \in \varphi_i}{\cup} \left\{\hat{\theta}_i\right\} \Leftrightarrow \underset{\Im_i \in h_i}{\cup} \{(\Im_i)^q\} \geq \underset{\hat{\theta}_i \in \varphi_i}{\cup} \left\{\left(\hat{\theta}_i\right)^q\right\}$, where $q \geq 1$,

$$\Leftrightarrow \underset{\Im_i \in h_i}{\cup} \sqrt[q]{\{(\Im_i)^q\}} \geq \underset{\theta_i \in \varphi_i}{\cup} \sqrt[q]{\left\{\left(\hat{\theta}_i\right)^q\right\}} \Leftrightarrow \underset{\hat{\theta}_i \in \varphi_i}{\cup} \sqrt[q]{1 - \left\{\left(\hat{\theta}_i\right)^q\right\}} \geq \underset{\Im_i \in h_i}{\cup} \sqrt[q]{1 - \{(\Im_i)^q\}}$$

$$\Leftrightarrow \underset{\hat{\theta}_i \in \varphi_i}{\cup} \sqrt[q]{\left(1 - \left\{\left(\hat{\theta}_i\right)^q\right\}\right)^{\tilde{w}_i}} \geq \underset{\Im_i \in h_i}{\cup} \sqrt[q]{(1 - \{(\Im_i)^q\})^{\tilde{w}_i}}$$

$$\Leftrightarrow \underset{\hat{\theta}_i \in \varphi_i}{\cup} \sqrt[q]{\prod_{i=1}^{n}\left(1 - \left\{\left(\hat{\theta}_i\right)^q\right\}\right)^{\tilde{w}_i}} \geq \underset{\Im_i \in h_i}{\cup} \sqrt[q]{\prod_{i=1}^{n}(1 - \{(\Im_i)^q\})^{\tilde{w}_i}}$$

$$\Leftrightarrow \underset{\Im_i \in h_i}{\cup} \sqrt[q]{1 - \prod_{i=1}^{n}(1 - \{(\Im_i)^q\})^{\tilde{w}_i}} \geq \underset{\hat{\theta}_i \in \varphi_i}{\cup} \sqrt[q]{1 - \prod_{i=1}^{n}\left(1 - \left\{\left(\hat{\theta}_i\right)^q\right\}\right)^{\tilde{w}_i}} \qquad \text{(a)}$$

Next if

$$\phi_i \geq g_i, \text{ then } \underset{\hat{\eta}_i \in \phi_i}{\cup} \{\hat{\eta}_i\} \geq \underset{\lambda_i \in g_i}{\cup} \{\lambda_i\} \Leftrightarrow \underset{\hat{\eta}_i \in \phi_i}{\cup} \{(\hat{\eta}_i)^{\tilde{w}_i}\} \geq \underset{\lambda_i \in g_i}{\cup} \{(\lambda_i)^{\tilde{w}_i}\} \Leftrightarrow \underset{\hat{\eta}_i \in \phi_i}{\cup} \{\prod_{i=1}^{n}(\hat{\eta}_i)^{\tilde{w}_i}\} \geq \underset{\lambda_i \in g_i}{\cup} \{\prod_{i=1}^{n}(\lambda_i)^{\tilde{w}_i}\} \qquad \text{(b)}$$

Let $\mathcal{E} = Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) =$ and $\mathcal{L} = Hq\text{-}ROFWA(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n)$. Then from *eq.* (a) and (b), we have $\hat{S}(\mathcal{E}) \geq \hat{S}(\mathcal{L})$
If $\hat{S}(\mathcal{E}) > \hat{S}(\mathcal{L})$, then $Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) > Hq\text{-}ROFWA(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n)$. If $\hat{S}(\mathcal{E}) = \hat{S}(\mathcal{L})$, then $Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = Hq\text{-}ROFWA(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n)$, then,

$$\frac{1}{\#h}\sum_{\Im \in h}\Im^q - \frac{1}{\#g}\sum_{\lambda \in g}\lambda^q = \frac{1}{\#\varphi}\sum_{\hat{\theta} \in \varphi}^{q}\hat{\theta}^q - \frac{1}{\#\phi}\sum_{\lambda \in \phi}^{q}\hat{\eta}^q$$

$$\text{implies } \frac{1}{\#h}\sum_{\Im \in h}\Im^q = \frac{1}{\#\varphi}\sum_{\hat{\theta} \in \varphi}^{q}\hat{\theta}^q \text{ and } \frac{1}{\#g}\sum_{\lambda \in g}\lambda^q = -\frac{1}{\#\phi}\sum_{\hat{\eta} \in \phi}^{q}\hat{\eta}^q$$

Furthermore

$$\mathcal{A}(\mathcal{E}) = \frac{1}{\#h}\sum_{\Im \in h}\Im^q + \frac{1}{\#g}\sum_{\lambda \in g}\lambda^q = \frac{1}{\#\varphi}\sum_{\hat{\theta} \in \varphi}^{q}\hat{\theta}^q + \frac{1}{\#\phi}\sum_{\lambda \in \phi}^{q}\hat{\eta}^q \mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{L})$$

Therefore $Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = Hq\text{-}ROFWA(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n)$. □

Theorem 4.7. *Let $\mathcal{E} = \mathcal{G}(h, g) = \underset{\Im \in h, \lambda \in g}{\cup} \{\{\Im\}, \{\lambda\}\}$ and $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ be the collection of Hq-ROFNs. If $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i and $\Im, \lambda$ are the objects of HFSs h, g. Then*

$$Hq\text{-}ROFWA(\mathcal{E}_1 \oplus \mathcal{E}, \mathcal{E}_2 \oplus \mathcal{E}, ..., \mathcal{E}_n \oplus \mathcal{E}) = Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \oplus \mathcal{E}$$

Proof. For any i, we have

$$\begin{aligned}\mathcal{E}_i \oplus \mathcal{E} &= \underset{\Im_i \in h_i, \lambda_i \in g_i, \Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{(\Im_i)^q + (\Im)^q - (\Im_i)^q (\Im)^q}\}, \{\lambda_i \lambda\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i, \Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{1 - (1-(\Im_i)^q)(1-(\Im)^q)}\}, \{\lambda_i \lambda\}\}\end{aligned}$$

Now from Theorem 4.2, we have

$$\begin{aligned}Hq-ROFWA(\mathcal{E}_1 \oplus \mathcal{E}, \mathcal{E}_2 \oplus \mathcal{E}, ..., \mathcal{E}_n \oplus \mathcal{E}) &= \underset{\Im_i \in h_i, \lambda_i \in g_i, \Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{n} [(1-(\Im_i)^q)(1-(\Im)^q)]^{\tilde{w}_i}}\}, \{\prod_{i=1}^{n} (\lambda_i \lambda)^{\tilde{w}_i}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i, \Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{1 - [(1-(\Im)^q)]^{\sum_{i=1}^{n} \tilde{w}_i} \prod_{i=1}^{n} [1-(\Im_i)^q]^{\tilde{w}_i}}\}, \{\lambda^{\sum_{i=1}^{n} \tilde{w}_i} \prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i, \Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{1 - (1-(\Im)^q) \prod_{i=1}^{n} [1-(\Im_i)^q]^{\tilde{w}_i}}\}, \{\lambda \prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\}\} \quad (2)\end{aligned}$$

Furthermore

$$\begin{aligned}Hq-ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \oplus \mathcal{E} &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{n} (1-\Im_i^q)^{\tilde{w}_i}}\}, \{\prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\}\} \oplus \underset{\Im \in h, \lambda \in g}{\cup} \{\{\Im\}, \{\lambda\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i, \Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{1 - (1-(\Im)^q)(1-[1-\prod_{i=1}^{n} [1-(\Im_i)^q]^{\tilde{w}_i}])}\}, \{\lambda \prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i, \Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{1 - (1-(\Im)^q) \prod_{i=1}^{n} [1-(\Im_i)^q]^{\tilde{w}_i}}\}, \{\lambda \prod_{i=1}^{n} \lambda_i^{\tilde{w}_i}\}\} \quad (3)\end{aligned}$$

Therefore, from Eqs. (2) and (3), $Hq\text{-}ROFWA(\mathcal{E}_1 \oplus \mathcal{E}, \mathcal{E}_2 \oplus \mathcal{E}, ..., \mathcal{E}_n \oplus \mathcal{E}) = Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \oplus \mathcal{E}$. □

Theorem 4.8. *Consider the collection $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ of Hq-ROFNs such that $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i. If $\lambda > 0$, then:*

$$Hq\text{-}ROFWA(\lambda\mathcal{E}_1, \lambda\mathcal{E}_2, ..., \lambda\mathcal{E}_n) = \lambda Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$$

Proof. Since$\lambda\mathcal{E} = \underset{\Im \in h, \lambda \in g}{\cup} \{\{\sqrt[q]{1-(1-\Im^q)^\lambda}\}, \{\lambda^\lambda\}\}$, where $\lambda \in R$ and $\lambda > 0$ Then according to Theorem 4.2, we have

$$\begin{aligned}Hq-ROFWA(\lambda\mathcal{E}_1, \lambda\mathcal{E}_2, ..., \lambda\mathcal{E}_n) &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{n} [[1-(\Im_i)^q]^\lambda]^{\tilde{w}_i}}\}, \{\prod_{i=1i}^{n} [\lambda_i^\lambda]^{\tilde{w}_i}\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{n} [1-(\Im_i)^q]^{\lambda\tilde{w}_i}}\}, \{\prod_{i=1i}^{n} \lambda_i^{\lambda\tilde{w}_i}\}\} \quad (4)\end{aligned}$$

Furthermore

$$\begin{aligned}\lambda Hq-ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) &= \lambda(\underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{n} [1-(\Im_i)^q]^{\tilde{w}_i}}\}, \{\prod_{i=1i}^{n} \lambda_i^{\tilde{w}_i}\}\}) \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - [1-(1-\prod_{i=1}^{n} [1-(\Im_i)^q]^{\tilde{w}_i}]^\lambda}\}, \{[\prod_{i=1i}^{n} \lambda_i^{\tilde{w}_i}]^\lambda\}\} \\ &= \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\sqrt[q]{1 - \prod_{i=1}^{n} [1-(\Im_i)^q]^{\lambda\tilde{w}_i}}\}, \{\prod_{i=1i}^{n} \lambda_i^{\lambda\tilde{w}_i}\}\} \quad (5)\end{aligned}$$

Therefore, from Eqs. (4) and (5) $Hq\text{-}ROFWA(\lambda\mathcal{E}_1, \lambda\mathcal{E}_2, ..., \lambda\mathcal{E}_n) = \lambda Hq\text{-}ROFWA(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)$. □

Theorem 4.9. *Let* $\mathcal{E} = \mathcal{G}(h,g) = \bigcup_{\Im\in h,\lambda\in g}\{\{\Im\},\{\lambda\}\}$ *and* $\mathcal{E}_i = \mathcal{G}(h_i,g_i)(i=1,2,...,n)$ *be the collection of Hq-ROFNs such that* $\Im,\lambda$ *are the objects of HFSs* h,g *and* $\Im_i,\lambda_i$ *are the objects of HFSs* h_i,g_i. *If for* $\lambda>0$, *then:*

$$Hq\text{-}ROFWA\left(\lambda\mathcal{E}_1\oplus\mathcal{E},\lambda\mathcal{E}_2\oplus\mathcal{E},...,\lambda\mathcal{E}_n\oplus\mathcal{E}\right)=\lambda Hq\text{-}ROFWA\left(\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_n\right)\oplus\mathcal{E}$$

Proof. The proof is similar to Theorems 4.7 and 4.8. □

Theorem 4.10. *Let* $\mathcal{E}_i=\mathcal{G}(h_i,g_i)(i=1,2,...,n)$ *and* $\mathcal{L}_i=\mathcal{Q}(\varphi_i,\phi_i)(i=1,2,...,n)$ *be the collection of two families of Hq-ROFNs such that* $\Im_i,\lambda_i$ *are the objects of HFSs* h_i,g_i *and* $\hat{\theta}_i,\hat{\eta}_i$ *are the objects of HFSs* φ_i,ϕ_i. *Then* $Hq\text{-}ROFWA\left(\mathcal{E}_1\oplus\mathcal{L}_1,\mathcal{E}_2\oplus\mathcal{L}_2,...,\mathcal{E}_n\oplus\mathcal{L}_n\right)=Hq\text{-}ROFWA\left(\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_n\right)\oplus Hq\text{-}ROFWA\left(\mathcal{L}_1,\mathcal{L}_2,...,\mathcal{L}_n\right).$

Proof. Since

$$\mathcal{E}_i\oplus\mathcal{L}_i=\bigcup_{\substack{\Im_i\in h_i,\lambda_i\in g_i,\\ \hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}}\{\{\sqrt[q]{(\Im_i)^q+\left(\hat{\theta}_i\right)^q-(\Im_i)^q\left(\hat{\theta}_i\right)^q}\},\{\lambda_i\hat{\eta}_i\}\}=\bigcup_{\substack{\Im_i\in h_i,\lambda_i\in g_i,\\ \hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}}\{\{\sqrt[q]{1-[1-(\Im_i)^q][1-\left(\hat{\theta}_i\right)^q]}\},\{\lambda_i\hat{\eta}_i\}\}$$

Then according to Theorem 4.2, we have

$$\begin{aligned}Hq-ROFWA\left(\mathcal{E}_1\oplus\mathcal{L}_1,\mathcal{E}_2\oplus\mathcal{L}_2,...,\mathcal{E}_n\oplus\mathcal{L}_n\right)&=\bigcup_{\substack{\Im_i\in h_i,\lambda_i\in g_i,\\ \hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}}\{\{\sqrt[q]{1-\prod_{i=1}^{n}\left([1-(\Im_i)^q][1-\left(\hat{\theta}_i\right)^q]\right)^{\tilde{w}_i}}\},\{\prod_{i=1}^{n}[\lambda_i\hat{\eta}_i]^{\tilde{w}_i}\}\}\\
&=\bigcup_{\substack{\Im_i\in h_i,\lambda_i\in g_i,\\ \hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}}\{\{\sqrt[q]{1-\prod_{i=1}^{n}[1-(\Im_i)^q]^{\tilde{w}_i}\prod_{i=1}^{n}[1-\left(\hat{\theta}_i\right)^q]^{\tilde{w}_i}}\},\{\prod_{i=1}^{n}[\lambda_i\hat{\eta}_i]^{\tilde{w}_i}\}\}\\
&=\bigcup_{\substack{\Im_i\in h_i,\lambda_i\in g_i,\\ \hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}}\{\{\sqrt[q]{1-\prod_{i=1}^{n}(1-(\xi_i)^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\prod_{i=1}^{n}(1-(\Im_i)^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\},\\
&\qquad\{\prod_{i=1}^{n}[\lambda_i]^{\tilde{w}_i}\prod_{i=1}^{n}[\hat{\eta}_i]^{\tilde{w}_i}\}\}\end{aligned}\tag{6}$$

Next

$$\begin{aligned}&Hq-ROFWA\left(\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_n\right)\oplus Hq-ROFWA\left(\mathcal{L}_1,\mathcal{L}_2,...,\mathcal{L}_n\right)\\
&=\bigcup_{\Im_i\in h_i,\lambda_i\in g_i}\{\{\sqrt[q]{1-\prod_{i=1}^{n}[1-(\Im_i)^q]^{\tilde{w}_i}}\},\{\prod_{i=1}^{n}\lambda_i^{\tilde{w}_i}\}\}\oplus\bigcup_{\hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}\{\{\sqrt[q]{1-\prod_{i=1}^{n}[1-\left(\hat{\theta}_i\right)^q]^{\tilde{w}_i}}\},\{\prod_{i=1}^{n}{}_i\hat{\eta}_i^{\tilde{w}_i}\}\}\\
&=\bigcup_{\substack{\Im_i\in h_i,\lambda_i\in g_i,\\ \hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}}\{\{\sqrt[q]{\begin{array}{c}(1-\prod_{i=1}^{n}[1-(\Im_i)^q]^{\tilde{w}_i})+(1-\prod_{i=1}^{n}[1-\left(\hat{\theta}_i\right)^q]^{\tilde{w}_i})-\\(1-\prod_{i=1}^{n}[1-(\Im_i)^q]^{\tilde{w}_i})(1-\prod_{i=1}^{n}[1-\left(\hat{\theta}_i\right)^q]^{\tilde{w}_i})\end{array}}\},\\
&\qquad\{\prod_{i=1}^{n}\lambda_i^{\tilde{w}_i}\prod_{i=1}^{n}{}_i\hat{\eta}_i^{\tilde{w}_i}\}\}\\
&=\bigcup_{\substack{\Im_i\in h_i,\lambda_i\in g_i,\\ \hat{\theta}_i\in\varphi_i,\hat{\eta}_i\in\phi_i}}\{\{\sqrt[q]{1-\prod_{i=1}^{n}(1-(\xi_i)^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\prod_{i=1}^{n}(1-(\Im_i)^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\},\\
&\qquad\{\prod_{i=1}^{n}[\lambda_i]^{\tilde{w}_i}\prod_{i=1}^{n}[\hat{\eta}_i]^{\tilde{w}_i}\}\}\end{aligned}\tag{7}$$

Therefore, from Eqs. (6) and (7),

$$Hq-ROFWA\left(\mathcal{E}_1\oplus\mathcal{L}_1,\mathcal{E}_2\oplus\mathcal{L}_2,...,\mathcal{E}_n\oplus\mathcal{L}_n\right)=Hq-ROFWA\left(\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_n\right)\oplus Hq-ROFWA\left(\mathcal{L}_1,\mathcal{L}_2,...,\mathcal{L}_n\right).$$

□

Definition 4.11. *Consider the collection $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ of Hq-ROFNs. Moreover let $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n)^T$ be the weight vector of $\mathcal{E}_i$ where $\tilde{w}_i \geq 0 (i = 1, ..., n)$ such that $\tilde{w}_i \in [0, 1]$ with $\sum_{i=1}^{n} \tilde{w}_i = 1$. Then the mapping for Hq-ROF weighted geometric (Hq-ROFWG) operator is defined as*

$$Hq\text{-}ROFWG : \mathcal{H}^n \to \mathcal{H} \quad (\text{where } \mathcal{H} \text{ is the collection of all Hq-ROFNs})$$

such that

$$Hq\text{-}ROFWG(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \overset{n}{\underset{i=1}{\otimes}} \mathcal{E}_i^{\tilde{w}_i} = \mathcal{E}_1^{\tilde{w}_1} \otimes \mathcal{E}_2^{\tilde{w}_2} \otimes ... \otimes \mathcal{E}_n^{\tilde{w}_n}$$

The aggregation result for Hq-ROFNs through operation rule is given as in Theorem 4.12.

Theorem 4.12. *Suppose that $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ be the collection of Hq-ROFNs. Let us consider the weigh vector $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n)^T$ of $\mathcal{E}_i$ such that $\tilde{w}_i \geq 0 (i = 1, ..., n)$ where $\tilde{w}_i \in [0, 1]$ with $\sum_{i=1}^{n} \tilde{w}_i = 1$. Then*

$$Hq\text{-}ROFWG(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \underset{\Im_i \in h_i, \lambda_i \in g_i}{\cup} \{\{\prod_{i=1}^{n} \Im_i^{\tilde{w}_i}\}, \{\sqrt[q]{1 - \prod_{i=1}^{n} [1 - \lambda_i]^{\tilde{w}_i}}\}\}.$$

Proof. Proof follows from Theorem 4.2. □

Theorem 4.13. ***(Idempotency)*** *Consider the collection $\mathcal{E} = \mathcal{G}(h, g) = \underset{\Im \in h, \lambda \in g}{\cup} \{\{\Im\}, \{\lambda\}\}$ and $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ of Hq-ROFNs. Now for all i, $\Im_i = \Im, \lambda_i = \lambda$, where $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i and $\Im, \lambda$ are the objects of HFSs h, g, then*

$$Hq\text{-}ROFWG\,(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) = \mathcal{E}$$

Proof. Proof follows from Theorem 4.4. □

Theorem 4.14. ***(Boundedness)*** *Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ be the collection of $Hq\text{-}ROFNs$. Now consider the weight vector $\tilde{w} = (\tilde{w}_1, \tilde{w}_2, ..., \tilde{w}_n)^T$ of $\mathcal{E}_i$ such that $\tilde{w}_i \geq 0 (i = 1, ..., n)$ where $\tilde{w}_i \in [0, 1]$ with $\sum_{i=1}^{n} \tilde{w}_i = 1$. Then*

$$\mathcal{E}^- \leq Hq\text{-}ROFWG\,(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \leq \mathcal{E}^+ \qquad \textbf{(i)}$$

where $\mathcal{E}^- = \mathcal{G}(h^-, g^+)$, $\mathcal{E}^+ = \mathcal{G}(h^+, g^-)$ such that $h^- = \underset{\Im_i \in h_i}{\cup} \min\{\Im_i\}$,

$$g^+ = \underset{\lambda_i \in g_i}{\cup} \max\{\lambda_i\}, h^+ = \underset{\Im_i \in h_i}{\cup} \max\{\Im_i\}, g^- = \underset{\lambda_i \in g_i}{\cup} \min\{\lambda_i\}$$

Proof. Proof directly follows from Theorem 4.5. □

Theorem 4.15. ***(Monotonicity)*** *Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ and $\mathcal{L}_i = \mathcal{Q}(\varphi_i, \phi_i)(i = 1, 2, ..., n)$ be the collection of two families of Hq-ROFNs such that $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i and $\hat{\theta}_i, \hat{\eta}_i$ are the objects of HFSs φ_i, ϕ_i. If for all i, $\Im_i \geq \hat{\theta}_i$ and $\lambda_i \leq \hat{\eta}_i$, then:*
$Hq\text{-}ROFWG\,(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \geq Hq\text{-}ROFWG\,(\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n)$.

Proof. Proof is straightforward as Theorem 4.6. □

Theorem 4.16. *Suppose that the collection $\mathcal{E} = \mathcal{G}(h, g) = \underset{\Im \in h, \lambda \in g}{\cup} \{\{\Im\}, \{\lambda\}\}$ and $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ of Hq-ROFNs. Let $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i and $\Im, \lambda$ are the objects of HFSs h, g. Then*

$$Hq\text{-}ROFWG\,(\mathcal{E}_1 \otimes \mathcal{E}, \mathcal{E}_2 \otimes \mathcal{E}, ..., \mathcal{E}_n \otimes \mathcal{E}) = Hq\text{-}ROFWG\,(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n) \otimes \mathcal{E}$$

Proof. Straightforward as Theorem 4.7. □

Theorem 4.17. *Let $\mathcal{E}_i = \mathcal{G}(h_i, g_i)(i = 1, 2, ..., n)$ be the collection of Hq-ROFNs such that $\Im_i, \lambda_i$ are the objects of HFSs h_i, g_i. If $\lambda > 0$, then:*

$$Hq\text{-}ROFWG\,(\lambda\mathcal{E}_1, \lambda\mathcal{E}_2, ..., \lambda\mathcal{E}_n) = [Hq\text{-}ROFWG\,(\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n)]^{\lambda}$$

Proof. The proof is straightforward as in Theorem 4.8. □

Theorem 4.18. *Let* $\mathcal{E} = \mathcal{G}(h,g) = \underset{\Im\in h,\lambda\in g}{\cup}\{\{\Im\},\{\lambda\}\}$ *and* $\mathcal{E}_i = \mathcal{G}(h_i,g_i)(i = 1,2,...,n)$ *be the collection of Hq-ROFNs such that* $\Im,\lambda$ *are the objects of HFSs* h,g *and* $\Im_i,\lambda_i$ *are the objects of HFSs* h_i,g_i. *If for* $\lambda > 0$, *then:*

$$Hq\text{-}ROFWG\left(\mathcal{E}_1^\lambda \otimes \mathcal{E}, \mathcal{E}_2^\lambda \otimes \mathcal{E}, ..., \mathcal{E}_n^\lambda \otimes \mathcal{E}\right) = \left[Hq\text{-}ROFWG\left(\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_n\right)\right]^\lambda \otimes \mathcal{E}$$

Proof. By using the proofs of Theorems 4.16 and 4.17, we can get required proof. □

Theorem 4.19. *Let* $\mathcal{E}_i = \mathcal{G}(h_i,g_i)(i = 1,2,...,n)$ *and* $\mathcal{L}_i = \mathcal{Q}(\varphi_i,\phi_i)(i = 1,2,...,n)$ *be the collection of two families of Hq-ROFNs such that* $\Im_i,\lambda_i$ *are the objects of HFSs* h_i,g_i *and* $\hat{\theta}_i,\hat{\eta}_i$ *are the objects of HFSs* φ_i,ϕ_i. *Then* $Hq\text{-}ROFWG\left(\mathcal{E}_1 \otimes \mathcal{L}_1, \mathcal{E}_2 \otimes \mathcal{L}_2, ..., \mathcal{E}_n \otimes \mathcal{L}_n\right) = Hq\text{-}ROFWG\left(\mathcal{E}_1,\mathcal{E}_2,...,\mathcal{E}_n\right) \otimes Hq\text{-}ROFWG\left(\mathcal{L}_1,\mathcal{L}_2,...,\mathcal{L}_n\right).$

Proof. Proof directly follows from Theorem 4.10. □

5 An approach to multi-criteria decision making with hesitant q-rung orthopair fuzzy information

This section consists of a technique for multi-criteria decision making (MCDM). Here the concepts of Hq-ROFA operators will be employed, which is stated in Section 4. Major steps for this decision making method and its associated process are presented in the following.

On the basic concepts of this proposed method for MCDM are given. Let $\mathcal{X} = \{t_1,t_2,...,t_n\}$ be any set of n feasible alternatives and $\mathcal{C}=\{\mathcal{C}_1,\mathcal{C}_2,\ ...,\mathcal{C}_m\}$ be the finite set of criteria. Next consider the weight vector $\tilde{w} = (\tilde{w}_1,\tilde{w}_2,...,\tilde{w}_m)^T$ of all criteria such that $0 \le \tilde{w}_i \le 1$ with $\sum_{i=1}^{m}\tilde{w}_i = 1$ for $i = 1,....,m$. Decision makers $\mathcal{D}_{mem}$ and $\mathcal{D}_{non-mem}$ put forward the assessment values of all the alternatives $t_i(i = 1,...n)$ corresponding to the set of criteria $\mathcal{C}_j(j = 1,...,m)$ by h_{ij} and g_{ij} respectively. So combining these two values as a Hq-ROFN we have $\mathcal{E}_{ij} = (h_{ij},g_{ij})$. This means that the decision maker $\mathcal{D}_{mem}$ provides membership grade h_{ij} to the alternative t_i according to the criteria $\mathcal{C}_j$. Whereas decision maker $\mathcal{D}_{non-mem}$ provides non-membership grade g_{ij} to the alternative t_i according to the criteria $\mathcal{C}_j$. So on the analysis of $\mathcal{D}_{mem}$ and $\mathcal{D}_{non-mem}$ the Hq-ROF decision matrix $\mathcal{D} = (\mathcal{E}_{ij})_{m\times n}$ can be constructed.

Furthermore, on the base of Hq-ROFWA (or Hq-ROFWG) operator to the MCDM problems, the decision making process using Hq-ROF information is as follows:

Step 1. By using Hq-ROFWA (or Hq-ROFWG) operator, aggregate the Hq-ROFWEs $\mathcal{E}_{ij}$ for each criteria $\mathcal{C}_j$, that is to calculate all the preference values $\mathcal{E}_i$ of the alternatives t_i.

$$\begin{aligned}
\mathcal{E}_i &= Hq\text{-}ROFWA(\mathcal{E}_{i1},\mathcal{E}_{i2},...,\mathcal{E}_{in})\\
&= \overset{n}{\underset{j=1}{\oplus}}\tilde{w}_j\mathcal{E}_{ij}\\
&= \underset{\Im_{ij}\in h_{ij},\lambda_{ij}\in g_{ij}}{\cup}\{\{\sqrt[q]{1-\prod_{j=1}^{n}(1-\Im_{ij}^q)^{\tilde{w}_j}}\},\{\prod_{j=1}^{n}\lambda_{ij}^{\tilde{w}_j}\}\}
\end{aligned}$$

or the hesitant q-rung orthopair fuzzy weighted geometric operator:

$$\begin{aligned}
\mathcal{E}_i &= Hq\text{-}ROFWG(\mathcal{E}_{i1},\mathcal{E}_{i2},...,\mathcal{E}_{in})\\
&= \overset{n}{\underset{j=1}{\otimes}}\mathcal{E}_{ij}^{\tilde{w}_j}\\
&= \underset{\Im_{ij}\in h_{ij},\lambda_{ij}\in g_{ij}}{\cup}\{\{\prod_{j=1}^{n}\Im_{ij}^{\tilde{w}_j}\},\{\sqrt[q]{1-\prod_{j=1}^{n}[1-\lambda_{ij}^q]^{\tilde{w}_j}}\}\}
\end{aligned}$$

Step 2. With the help of Definition 3.5, calculate the score function $\hat{S}\left(\mathcal{E}_i\right)(i = 1,...,m)$ of all the hesitant q-rung orthopair fuzzy preference values $\mathcal{E}_i$. If there exist equality among the score functions, then with the help of Definition 3.6, calculate the accuracy functions of all hesitant q-rung orthopair fuzzy preference values $\mathcal{E}_i$.

Step 3. Finally calculate the rank of alternatives t_i through the score function and get the best alternative. Greater the score function better the alternative is.

6 An illustrative example

Recently improvement the strategy of energy among the countries is becoming a common debate in the world. For the energy development the selection criteria of energy project is a critical step due to the unstable and uncertain environment because of the incorrect energy policy effects economic development and environment. Therefore it is significant to pick the most applicable energy policy.

In this section the proposed operators is presented for the supporting and selection process of energy projects in decision making based on Hq-ROFS which relates to the assessment and rank of standard energy projects for conventional energy event. Consider that there are five energy projects (alternatives) $\mathcal{X} = \{t_1, t_2, t_3, t_4, t_5\}$ and upon the four criteria to be considered: (1) $\mathcal{C}_1$: technological; (2) $\mathcal{C}_2$: environmental; (3) $\mathcal{C}_3$: socio-political and (4) $\mathcal{C}_4$: economic, that is, $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ and the assessment of energy projects summarized by Xu and Xia [33]. Moreover, to selecet the weight vector $\tilde{w} = (0.15, 0.3, 0.2, 0.35)^T$ for the proposed criteria. Under the circumstances of hesitant q-rung orthopair fuzzy environment, the invited experts evaluate these alternatives with Hq-ROFNs. However, some of the values can be repeated here but this does not means the repeated values has more significance than the values which is repeated less times. Therefore from the experts opinion the Hq-ROF decision matrix $\mathcal{D} = (\mathcal{E}_{ij})_{m\times n}$ can be constructed and is given in Table 2.

Table 2, Hq-ROF decision matrix for $q = 3$

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
t_1	$\mathcal{E}(\{0.7, 0.3\}, \{0.3, 0.2\})$	$\mathcal{E}(\{0.6, 0.4\}, \{0.1, 0.3\})$	$\mathcal{E}(\{0.6, 0.1\}, \{0.2, 0.4\})$	$\mathcal{E}(\{0.6, 0.4\}, \{0.3, 0.4\})$
t_2	$\mathcal{E}(\{0.3, 0.6\}, \{0.2, 0.4\})$	$\mathcal{E}(\{0.7, 0.2\}, \{0.2, 0.3\})$	$\mathcal{E}(\{0.8, 0.4\}, \{0.1, 0.2\})$	$\mathcal{E}(\{0.7, 0.5\}, \{0.2, 0.3\})$
t_3	$\mathcal{E}(\{0.8, 0.4\}, \{0.1, 0.2\})$	$\mathcal{E}(\{0.8, 0.5\}, \{0.2, 0.1\})$	$\mathcal{E}(\{0.7, 0.3\}, \{0.1, 0.3\})$	$\mathcal{E}(\{0.8, 0.6\}, \{0.1, 0.2\})$
t_4	$\mathcal{E}(\{0.5, 0.6\}, \{0.3, 0.4\})$	$\mathcal{E}(\{0.6, 0.3\}, \{0.4, 0.2\})$	$\mathcal{E}(\{0.6, 0.4\}, \{0.3, 0.2\})$	$\mathcal{E}(\{0.1, 0.7\}, \{0.2, 0.3\})$
t_5	$\mathcal{E}(\{0.6, 0.2\}, \{0.2, 0.4\})$	$\mathcal{E}(\{0.7, 0.1\}, \{0.1, 0.2\})$	$\mathcal{E}(\{0.4, 0.2\}, \{0.2.0.5\})$	$\mathcal{E}(\{0.2, 0.3\}, \{0.7, 0.1\})$

Step 1. By using Hq-ROFWA operator, aggregate the Hq-ROFWEs $\mathcal{E}_{ij}$ for each criteria $\mathcal{C}_j$, that is to find all the preference values $\mathcal{E}_i$ of the alternatives $t_i\,(i = 1, ..., 5)\,(j = 1, ..., 4)$, that is:

$\mathcal{E}_1 = \{\{0.359, 0.450, 0.457, 0.465, 0.470, 0.517, 0.523, 0.526, 0.528, 0.531, 0.537, 0.572, 0.575, 0.580, 0.583, 0.618\}, \{0.187, 0.199, 0.207, 0.215, 0.220, 0.228, 0.238, 0.253, 0.260, 0.277, 0.288, 0.299, 0.306, 0.318, 0.331, 0.351\}$

$\mathcal{E}_2 = \{\{0.401, 0.455, 0.536, 0.557, 0.563, 0.565, 0.584, 0.589, 0.632, 0.636, 0.650, 0.651, 0.655, 0.667, 0.700, 0.714\}, \{0.174, 0.193, 0.197, 0.2, 0.201, 0.218, 0.222, 0.223, 0.226, 0.227, 0.230, 0.251, 0.251, 0.256, 0.260, 0.289\}\}$

$\mathcal{E}_3 = \{\{0.508, 0.582, 0.596, 0.642, 0.647, 0.648, 0.684, 0.688, 0.693, 0.697, 0.726, 0.726, 0.730, 0.754, 0.760, 0.784\}, \{0.1, 0.111, 0.123, 0.125, 0.128, 0.137, 0.138, 0.141, 0.153, 0.157, 0.159, 0.170, 0.174, 0.176, 0.196, 0.217\}\}$

$\mathcal{E}_4 = \{\{0.345, 0.384, 0.420, 0.447, 0.466, 0.487, 0.510, 0.527, 0.556, 0.570, 0.585, 0.598, 0.607, 0.618, 0.631, 0.641\}, \{0.212, 0.222, 0.230, 0.241, 0.245, 0.256, 0.262, 0.266, 0.273, 0.277, 0.284, 0.296, 0.302, 0.315, 0.327, 0.342\}\}$

$\mathcal{E}_5 = \{\{0.181, 0.233, 0.259, 0.288, 0.343, 0.360, 0.372, 0.387, 0.498, 0.506, 0.511, 0.518, 0.536, 0.542, 0.547, 0.553\}, \{0.128, 0.141, 0.153, 0.157, 0.170, 0.174, 0.188, 0.209, 0.252, 0.279, 0.302, 0.310, 0.336, 0.344, 0.372, 0.413\}\}$

Step 2. Now to calculate the score function $\hat{S}\,(\mathcal{E}_i)\,(i = 1, ..., 5)$ of all the hesitant q-rung orthopair fuzzy preference values $\mathcal{E}_i$, that is:

$\hat{S}\,(\mathcal{E}_1) = 0.126, \hat{S}\,(\mathcal{E}_2) = 0.197, \hat{S}\,(\mathcal{E}_3) = 0.319, \hat{S}\,(\mathcal{E}_4) = 0.159, \hat{S}\,(\mathcal{E}_5) = 0.0684$

Step 3. Since $\hat{S}\,(\mathcal{E}_3) > \hat{S}\,(\mathcal{E}_2) > \hat{S}\,(\mathcal{E}_4) > \hat{S}\,(\mathcal{E}_1) > \hat{S}\,(\mathcal{E}_5)$. Therefore, we have $t_3 > t_2 > t_4 > t_1 > t_5$. Thus the best option is t_3.

Next to determine the Hq-ROFWG operator, their main steps are described as follows:

Step 1$^{'}$. By using Hq-ROFWG operator, aggregate the Hq-ROFWEs $\mathcal{E}_{ij}$ for each criteria $\mathcal{C}_j$, that is to determine all the preference values $\mathcal{E}_i$ of the alternatives $t_i\,(i = 1, ..., 5)\,(j = 1, ..., 4)$, that is:

$$\begin{aligned}\mathcal{E}_1 \;=\; & \{\{0.290, 0.300, 0.328, 0.330, 0.335, 0.372, 0.378, 0.380, 0.416, 0.469, 0.472,\\ & 0.479, 0.533, 0.541, 0.544, 0.614\}, \{0.233, 0.249, 0.273, 0.285, 0.288, 0.296,\\ & 0.299, 0.306, 0.316, 0.323, 0.326, 0.331, 0.334, 0.342, 0.355, 0.362\}\}\end{aligned}$$

$$\begin{aligned}\mathcal{E}_2 \;=\; & \{\{0.336, 0.373, 0.378, 0.386, 0.420, 0.429, 0.435, 0.482, 0.490, 0.544, 0.551,\\ & 0.563, 0.612, 0.624, 0.633, 0.702\}, \{0.188, 0.2, 0.231, 0.237, 0.240, 0.245, 0.248,\\ & 0.255, 0.267, 0.273, 0.276, 0.277, 0.281, 0.285, 0.302, 0.307\}\}\end{aligned}$$

$$\mathcal{E}_3 = \{\{0.465, 0.515, 0.516, 0.536, 0.551, 0.571, 0.592, 0.594, 0.610, 0.612, 0.635, 0.658, 0.676, 0.702, 0.704, 0.779\}, \{0.1, 0.127, 0.146, 0.151, 0.161, 0.165, 0.177, 0.184, 0.188, 0.194, 0.203, 0.206, 0.211, 0.214, 0.221, 0.228\}\}$$
$$\mathcal{E}_4 = \{\{0.234, 0.240, 0.253, 0.260, 0.288, 0.296, 0.312, 0.320, 0.461, 0.474, 0.500, 0.514, 0.568, 0.584, 0.616, 0.633\}, \{0.222, 0.245, 0.255, 0.260, 0.273, 0.277, 0.285, 0.300, 0.304, 0.3176, 0.323, 0.326, 0.334, 0.337, 0.342, 0.353\}\}$$
$$\mathcal{E}_5 = \{\{0.162, 0.187, 0.187, 0.192, 0.215, 0.220, 0.221, 0.254, 0.291, 0.335, 0.336, 0.343, 0.386, 0.394, 0.396, 0.455\}, \{0.151, 0.177, 0.230, 0.242, 0.304, 0.311, 0.332, 0.338, 0.518, 0.521, 0.528, 0.530, 0.544, 0.546, 0.552, 0.554\}\}$$

Step 2$'$. Now to calculate the score function $\hat{S}(\mathcal{E}_i)$ $(i = 1, ..., 5)$ of all the hesitant q-rung orthopair fuzzy preference values $\mathcal{E}_i$, that is:
$\hat{S}(\mathcal{E}_1) = 0.058, \hat{S}(\mathcal{E}_2) = 0.122, \hat{S}(\mathcal{E}_3) = 0.229, \hat{S}(\mathcal{E}_4) = 0.067, \hat{S}(\mathcal{E}_5) = -0.058$

Step 3$'$. Since $\hat{S}(\mathcal{E}_3) > \hat{S}(\mathcal{E}_2) > \hat{S}(\mathcal{E}_4) > \hat{S}(\mathcal{E}_1) > \hat{S}(\mathcal{E}_5)$. Therefore, we have $t_3 > t_1 > t_2 > t_5 > t_4$. Thus the desirable option is t_3.

From the above analysis, it is clear that the ranking results for both Hq-ROFWA and Hq-ROFWG are the same and the most desirable energy project is for t_3.

6.1 Comparison of proposed method with existing methods

To demonstrate the effectiveness and prominent character of the proposed method, the analysis of comparative study is organized to utilized the illustrative example 6, by using the MCDM methods consist of hesitant fuzzy weighted average (HFWA) and hesitant fuzzy weighted geometric (HFWG) by Xia and Xu [31], intuitionistic HFWA (IHFWA) and intuitionistic HFWG (IHFWG) by Peng et al. [22] or DHFWA and DHFWG by Yu et al. [39], and Pythagorean HFWA (PHFWA) and Pythagorean HFWG (PHFWG) by Sajjad et al. [16] which are the special cases of Hq-ROFWA and Hq-ROFWG operators (suppose $q = 3$). The comparative study of these methods are shown as Table 3.

Table3. Ranking results from different methods for example 6,

Operator	Score Functions					Ranking
	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$	
HFWA[31]	0.490	0.581	0.66	0.483	0.357	$t_3 > t_2 > t_1 > t_4 > t_5$
HFWG[31]	0.424	0.497	0.607	0.410	0.286	$t_3 > t_2 > t_1 > t_4 > t_5$
IHFWA[22, 39]	0.229	0.355	0.510	0.212	0.111	$t_3 > t_2 > t_1 > t_4 > t_5$
IHFWG[22, 39]	0.138	0.256	0.445	0.126	−0.047	$t_3 > t_2 > t_1 > t_4 > t_5$
PHFWA[16]	0.184	0.288	0.426	0.182	0.089	$t_3 > t_2 > t_1 > t_4 > t_5$
PHFWG[16]	0.092	0.185	0.340	0.084	−0.054	$t_3 > t_2 > t_1 > t_4 > t_5$
Hq-ROFWA	0.126	0.197	0.319	0.159	0.0684	$t_3 > t_2 > t_4 > t_1 > t_5$
Hq-ROFWG	0.058	0.122	0.229	0.067	−0.058	$t_3 > t_2 > t_4 > t_1 > t_5$

From Table 3, it is clear that the ranking results are slightly different (that is t_1 and t_4) under the same evaluation data. However the optimal ranking results the same as is t_3. In HFSs the calculation technique is simple and narrow because of only tackle membership grade and ignore non-membership grade. In different scenario of decision making apart from the grades of membership, the grades of non-memberships is also required, so to handle this situation Peng et al. [22], introduced IHFSs, and Zhu et al. [42] originated the notion of DHFS but the decision making applications of IHFSs and DHFSs are also restricted because of sum of membership grade and non-membership grade belongs to $[0, 1]$. After that Sajjad [16], developed this concept to PHFSs. The main difference between IHFSs and PHFSs is that, in the later case sum of membership and non-membership grades is greater than 1, but sum of their squares belongs to the unit interval $[0, 1]$. Here if we assign the membership grade 0.9 and non-membership grade 0.8. In this case $(0.9)^2 + (0.8)^2 > 1$ and all these notions are failed. The method we developed in this paper handle this situation, that is $(0.9)^q + (0.8)^q < 1$ for $q \geq 5$. From the above discussion it is clear that both IHFSs and PHFSs are special form of Hq-ROFSs. If $q = 1$ then Hq-ROFSs is degenerated to IHFSs and if $q = 2$ then Hq-ROFSs reduced to PHFSs. So from the analysis it is clear that method developed in this paper is more suitable to tackle a diverse situation by adapting its own parameters q. Therefore the proposed method is most superior because it provides more flexibility and freedom in the process of aggregation information due to parameter q. Therefore the method presented in this paper is more suitable than the other methods for decision making problems.

The value of parameter q play a vital role the flexibility of Hq-ROFWA and Hq-ROFWG operators. Different ranking results are derived by assigning different input to parameter q. The detail of ranking outcomes are shown in Table 4 and Table 5. From Table 4, it is clear that the final ranking results of Hq-ROFWA are slightly different by increasing the values of parameter q but the optimal results is the same overall that is t_3. Furthermore the value of score function become relatively larger when the parameter q is relatively smaller (from 1 to 5). On other hand the value of score function become smaller in general by increasing the value of parameter q. From the analysis it is clear that when parameter q lie between interval 1 to 5, then the behavior of decision maker become optimistic, while by increasing the input of parameter q, the behavior of decision maker become more pessimistic. Similarly if we see the ranking result from Table 5, then the derived optimal result are the same overall and in addition the rest of the rules are the same as we find in Table 4. Thus different decision maker can set different input to the parameter q on the bases of their preference.

7 Conclusion

The notion of q-ROFS is originally initiated by Yager. Actually the generalized q-ROFSs inherits the virtues of IFS and PFS in relaxing the restriction on the membership grade and on the non-membership grade. The aim of this manuscript is to propose the concept of Hq-ROFSs, which is the hybridization of IHFSs [22] and HPFSs [18]. In IHFSs and HPFSs a decision maker is bound to the constraints but our proposed notion of Hq-ROFS provide more flexibility and freedom to the decision maker. The comparative study of Hq-ROFS with HFSs, PFSs, DHFSs and HPFSs are summarized in Table 1. Some hesitant q-rung orthopair fuzzy operational laws have been presented. The score and accuracy functions are defined which play a vital role on making comparison between the Hq-ROFNs. Under the hesitant q-rung orthopair fuzzy environment, we introduced Hq-ROFWA operator, Hq-ROFWG operator and studied various properties of these operators. Furthermore the proposed aggregation operators are utalized to solve decision making problems where the experts added their favorable preference. In Table 3, we presented the comparative study of proposed method with other methods which show that the proposed method is more flexible and superior than others.

Table 4, Ranking results based on Hq-ROFWA operators by using the different q

q	Score Functions					Ranking
	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$	
$q = 1$	0.490	0.581	0.66	0.483	0.357	$t_3 > t_2 > t_1 > t_4 > t_5$
$q = 2$	0.092	0.185	0.340	0.084	-0.054	$t_3 > t_2 > t_1 > t_4 > t_5$
$q = 3$	0.058	0.122	0.229	0.067	-0.058	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 5$	0.050	0.104	0.176	0.058	0.032	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 8$	0.025	0.039	0.079	0.016	0.010	$t_3 > t_2 > t_1 > t_4 > t_5$
$q = 10$	0.00474	0.021	0.048	0.007	0.00473	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 15$	0.0006	0.0052	0.015	0.00099	0.00075	$t_3 > t_2 > t_4 > t_5 > t_1$
$q = 20$	0.00008	0.0014	0.0047	0.000152	0.000122	$t_3 > t_2 > t_4 > t_5 > t_1$

Table 5, Ranking results based on Hq-ROFWG operators by using the different q

q	Score Functions					Ranking
	$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$	$\mathcal{E}_4$	$\mathcal{E}_5$	
$q = 1$	0.138	0.256	0.445	0.126	-0.047	$t_3 > t_2 > t_1 > t_4 > t_5$
$q = 2$	0.184	0.288	0.426	0.182	0.089	$t_3 > t_2 > t_1 > t_4 > t_5$
$q = 3$	0.126	0.197	0.319	0.159	0.0684	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 5$	0.015	0.044	0.098	0.024	-0.031	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 8$	0.0030	0.0102	0.0292	0.0051	-0.0105	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 10$	0.00096	0.00412	0.01398	0.00182	-0.00505	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 15$	0.000061	0.00050	0.00263	0.00015	-0.00084	$t_3 > t_2 > t_4 > t_1 > t_5$
$q = 20$	0.0000045	0.000069	0.00059	0.0000128	-0.00014	$t_3 > t_2 > t_4 > t_1 > t_5$

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Hesitant q-rung orthopair fuzzy aggregation operators with their applications in multi-criteria decision making

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عملگرهای تجمع فازی q orthopair – پله مردد به همراه کاربرد آن‌ها در تصمیم‌گیری چند منظوره

چکیده. هدف از این مقاله، ارائه یک مفهوم جدید از مجموعه‌های فازی q orthopair – پله مردد (Hq-ROFSs) به وسیله ترکیب مفهوم q-ROFSs و مجموعه‌های فازی مردد است. مفهوم پیشنهادی، تعمیمی از مجموعه‌های فازی، مجموعه‌های فازی شهودی، مجموعه‌های فازی مردد، و مجموعه‌های فازی فیثاغورسی بعلاوه‌ی، مجموعه‌های فازی مردد شهودی (IHFSs) و مجموعه‌های فازی فیثاغورسی مردد (HPFSs) می‌باشد. علاوه‌براین، برخی از قوانین عملی پایه‌ای فازی q orthopair – پله مردد مورد بررسی قرار گرفته‌اند. توابع امتیاز و دقت که نقش حیاتی در پروسه تصمیم‌گیری برای مقایسه بین اعداد فازی q orthopair – پله (Hq-ROFNs) مردد دارند، تعریف شده‌اند.

تحت شرایط Hq-ROF ، عملگرهای برآورد میانگین وزین (Hq-ROFWA) و هندسه وزین Hq-ROF (Hq-ROFWG) معرفی شده‌اند و ویژگی‌های متعدد این عملگرهای تجمع مورد بررسی قرار گرفته‌اند. بعلاوه، یک کاربرد عددی نشان می‌دهد که عملگرهای پیشنهادی چگونه بکار گرفته شده‌اند تا مسائل تصمیم‌گیری چندمنظوره را حل کنند که در آن‌ها متخصصین، اولویت‌های خوش‌بینانه و بدبینانه را اضافه کرده‌اند. در پایان، آنالیز روش پیشنهادی با روش‌های دیگر ارائه گردیده که نشان می دهد روش ارائه شده در این مقاله انعطاف پذیرتر و نسبت به روش‌های موجود برتری دارد.