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The role of states in triangle algebras

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Abstract

In this paper, we enlarge the language of triangle algebra by adding a unary operation that describes properties of a state. These structure algebras are called state triangle algebra. The vital properties of these algebras are given. The notion of state interval-valued residuated lattice (IVRL)-filters are introduced and give some examples and properties of them are given. Using this concept, we define two types IVRL-extended σ-filters of a state triangle algebra.

Keywords: Triangle algebra, IVRL-filter, State operator, State IVRL-filter.

1 Introduction

It is well known that certain information processing is based on the classical two-valued logic. Naturally, it is necessary to establish some rational logic systems as the logical foundation for uncertain information processing. For this reason, various kinds of non-classical logic systems have been extensively proposed and researched. To formalize the many-valued logics induced by continuous t-norms on the real unit interval $[0, 1]$, in 1998. Various logical algebras have been proposed as the semantical systems of non-classical logic systems. Among these logical algebras, residuated lattices are important algebraic structures. The concept of a residuated lattice was firstly introduced by M. Ward and R.P. Dilworth [14] as generalization of ideal lattices of rings. The lattice of filters of a residuated lattice was investigated in [10].

Van Gasse et al. introduced the notion of triangle algebras as a variety of residuated lattices equipped with unary operators ν and μ together with a third angular point u which is different from 0 and 1. They showed that these algebras serve as an equational representation of interval-valued residuated lattices (IVRLs). The authors defined triangle logic (TL) and showed that this logic is sound and complete with respect to the variety of triangle algebras [11]. The theory of triangle algebras has been enriched with filter theory. The same authors introduced the notion of IVRL-filters in triangle algebras and defined Boolean and prime IVRL-filters and revealed interesting properties of them [13]. Triangle algebras are different from the other algebraic structures, so triangle algebras play an important role in studying fuzzy logics and the related algebraic structures.

Semi-divisible residuated lattices are related to probability theory in the following way. In 1986 Mundici extended probaability theory on MV-algebras by defining states on these algebras and investigated the advantages of such approach in quantum logic framework [6]. In, 2006 Mundici showed that his approach fits well to De Finetti's subjective probabilities, too. Next, the notion of state was extended to even more general residuated structures and references thereon. A new approach to states on MV-algebras was presented by Flaminio and Montagna, they added a unary operation σ to the language of MV-algebras, which preserves the usual properties of states. The notion of states has been extended to other logical algebras such as BL-algebras, MTL-algebras and etc [5, 9]. Ciungu et al. defined a state operator and a strong state operator for a BL-algebra. They introduced the concept of state filters on a state BL-algebra and gave some properties of them [2].

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We are going to study more properties of triangle algebras. To this scope, firstly we define the concept of state triangle algebras and prove some theorems that determine some properties of them. Since the filter theory plays an important role in studding these algebras, we introduce the notion of state IVRL-filters. These state IVRL-filters have different function Because of the existence of two operators ν and μ and using them in the definition of state IVRL-filters in triangle algebras. Based on these facts, we give a classification for triangle algebras. One of our aims is to introduce some special state IVRL-extended filters and specific sets in state triangle algebras and consider them in details. We prove that (A, σ) is local iff $M_\sigma(A) := \{x \in A \mid ord(\sigma(\nu x)) = \infty\}$ is a state IVRL-filter of (A, σ). In this case $M_\sigma(A)$ is the only IVRL-extended maximal σ-filter of (A, σ).

This paper is organized as follows: In section 2, we give some basic definitions and results of residuated lattices and triangle algebras. In section 3, we define the notion of state triangle algebras and study some properties of them. In section 4, we introduce the state IVRL-filters and some type of them as IVRL-extended (maximal, prime)σ-filter and study them in details. Also, we discuss the relationship between these state IVRL-filters.

2 Preliminaries

In this section, we summarize some definitions and results of residuated lattices and triangle algebras, which will be used in the following sections. Firstly, we recall the definition of bounded commutative residuated latticed.

Definition 2.1. [11] *A bounded commutative residuated latticed is an algebra $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ with four binary operations and two constants 0,1 such that:*

- *$(L, \vee, \wedge, 0, 1)$ is a bounded lattice,*
- *operation $*$ is commutative and associative, with 1 as neutral element, and*
- *$x * y \leq z$ iff $x \leq y \rightarrow z$, for all x, y and z in L.*

The ordering $\leq$ and negation $\neg$ in a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ are defined as follows, for all x and y in L: $x \leq y$ iff $x \wedge y = x$ (or equivalently, iff $x \vee y = y$; or, also equivalently, iff $x \rightarrow y = 1$) and $\neg x = x \rightarrow 0$, $x^n = \underbrace{x * \cdots * x}_{n-times}$.

Recall that $a \in L$ is called a nilpotent element of L if $a^n = 0$, for some $n \in \mathbb{N}$.

Lemma 2.2. [10, 13] *Let $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ be a residuated lattice. Then the following properties are valid, for all x, y and z in L:*

(1) $x \vee y \leq (x \rightarrow y) \rightarrow y$ *(in particular $x \leq \neg\neg x$),*
(2) $x \rightarrow y \leq x * z \rightarrow y * z$,
(3) $(x \rightarrow y) * (y \rightarrow z) \leqslant (x \rightarrow z)$,
(4) *If $x \leqslant y$, then $x * z \leqslant y * z, z \rightarrow x \leqslant z \rightarrow y$ and $y \rightarrow z \leqslant x \rightarrow z$,*
(5) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x * y) \rightarrow z$,
(6) $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$,

Definition 2.3. [11] *Given a lattice $\mathcal{A} = (A, \vee, \wedge)$, its triangularization $\mathbb{T}(\mathcal{A})$ is the structure $\mathbb{T}(\mathcal{A}) = (Int(\mathcal{A}), \vee, \wedge)$ defined by*

- $Int(\mathcal{A}) = \{[x_1, x_2] \mid (x_1, x_2) \in A^2$ *and* $x_1 \leq x_2\}$,
- $[x_1, x_2] \wedge [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$,
- $[x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$.

The set $D_{\mathcal{A}} = \{[x, x] : x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{A})$.

Definition 2.4. [11] *An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathcal{A}), \vee, \wedge, \odot, \rightarrow_\odot, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{A})$ of a bounded lattice $\mathcal{A}$, in which the diagonal $D_{\mathcal{A}}$ is closed under $\odot$ and $\rightarrow_\odot$, i.e. $[x, x] \odot [y, y] \in D_{\mathcal{A}}$ and $[x, x] \rightarrow_\odot [y, y] \in D_{\mathcal{A}}$, for all $x, y \in A$.*

Definition 2.5. [11] *A triangle algebra is a structure $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ in which $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ is a* *residuated lattice, ν and μ are unary operations on A, u a constant, and satisfying the following conditions:*

$(T.1)$ $\nu x \leq x$, $(T.1^{'})$ $x \leq \mu x$,

$(T.2)$ $\nu x \leq \nu\nu x$, $(T.2^{'})$ $\mu\mu x \leq \mu x$,

$(T.3)$ $\nu(x \wedge y) = \nu x \wedge \nu y$, $(T.3^{'})$ $\mu(x \wedge y) = \mu x \wedge \mu y$,

$(T.4)$ $\nu(x \vee y) = \nu x \vee \nu y$, $(T.4^{'})$ $\mu(x \vee y) = \mu x \vee \mu y$,

$(T.5)$ $\nu u = 0$, $(T.5^{'})$ $\mu u = 1$,

$(T.6)$ $\nu\mu x = \mu x$, $(T.6^{'})$ $\mu\nu x = \nu x$,

$(T.7)$ $\nu(x \to y) \leq \nu x \to \nu y$,

$(T.8)$ $(\nu x \leftrightarrow \nu y) * (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y)$,

$(T.9)$ $\nu x \to \nu y \leq \nu(\nu x \to \nu y)$.

From now on $\mathcal{A} = (A, \vee, \wedge, \to, *, \nu, \mu, 0, u, 1)$ or simply A is a triangle algebra unless otherwise specified.

In triangle algebra $\mathcal{A}$, operator ν (necessity) and μ (possibility) are modal operators, and u (uncertainty, $u \neq 0, u \neq 1$) is a new constant. It turns out that triangle algebras are the equational representations of interval-valued residuated lattices (IVRLs).

Theorem 2.6. [11] *There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.*

Proposition 2.7. [11] *Suppose $(A, \vee, \wedge, \to, 0, 1)$ is a residuated lattice such that $\neg$ is involutive. If there exists an element u in A such that $\neg u = u$, if ν is a unary operator on A that satisfies $T.1$- $T.6$, $T.8$, $T.9$ and if $(\nu x \leftrightarrow \nu y) * (\nu\neg x \leftrightarrow \nu\neg y) \leq x \leftrightarrow y$, then $(A, \vee, \wedge, \to, *, \nu, \mu, 0, u, 1)$ is a triangle algebra if we define $\mu x = \neg\nu\neg x$.*

Definition 2.8. [16] *A triangle algebra A is called an MTL-triangle algebra if $(x \to y) \vee (y \to x) = 1$. An MTL-triangle algebra A is called a BL-triangle algebra if $x \wedge y = x * (x \to y)$, for all $x, y \in A$.*

Definition 2.9. *A triangle algebra A is called RL-triangle algebra if $x \wedge y = x * (x \to y)$, for all $x, y \in A$.*

Definition 2.10. [13] *An IVRL-filter of A is a non-empty subset F of A satisfying:*
$(F.1)$ *if $x \in F, y \in A$ and $x \leq y$, then $y \in F$,*
$(F.2)$ *if $x, y \in F$, then $x * y \in F$,*
$(F.3)$ *if $x \in F$, then $\nu x \in F$.*

An alternative definition for an IVRL-filter F of a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is the following:
- $1 \in F$,
- for all x and y in A: if $x \in F$ and $x \to y \in F$, then $y \in F$.
- if $x \in F$, then $\nu x \in F$.

For all $x, y \in A$, we write $x \equiv_F y$ iff $x \to y$ and $y \to x$ are both in F.

$\equiv_F$ is always a congruence relation [13]. Note that $(F.3)$ is a necessary condition for this statement. Indeed, if $\equiv_F$ is a congruence relation on a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ and $x \in F$, then $x \equiv_F 1$ and therefore $\nu x \equiv_F \nu 1 = 1$, which is equivalent with $\nu x \in F$.

Proposition 2.11. [12] *In a triangle algebra $(A, \vee, \wedge, \to, *, \nu, \mu, 0, u, 1)$, the implication $\to$ and the product $*$ are completely determined by their action on $E(A)$ and the value of $u * u$,where $E(A) = \{x \in A \mid \nu x = x\}$. More specifically:*
- $\nu(x \to y) = (\nu x \to \nu y) \wedge (\mu x \to \mu y)$.
- $\mu(x \to y) = (\mu x \to (\mu(u * u) \to \mu y)) \wedge (\nu x \to \mu y)$.
- $\nu(x * y) = \nu x * \nu y$.
- $\mu(x * y) = (\nu x * \mu y) \vee (\mu x * \nu y) \vee (\mu x * \mu y * \mu(u * u))$.

Definition 2.12. [16] *The order of $x \in A$, denoted by $ord(x)$, is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If there is no such n, then $ord(x) = \infty$.*

Theorem 2.13. [16] *Let A be a triangle algebra. M is an IVRL-extended maximal filter of A iff for all $x \in A$, $x \notin M$, there exist $m \in M$, $n \geq 1$ such that $(m * \nu x^n)^k = 0$.*

Theorem 2.14. [16] *Let A be an MTL-triangle algebra. Then $Rad(A) = \{a \in A : \nu a \geq \neg(\nu a^n)$ for any $n \in \mathbb{N}\}$.*

Theorem 2.15. *Let F be an IVRL-filter of MTL-triangle algebra A. Then*

$$Rad(F) = \{a \in A : \neg(\nu a^n) \to \nu a \in F, \text{ for all } n \in \mathbb{N}\}$$

Definition 2.16. *A triangle algebra A is said to be local iff has exactly one IVRL-extended maximal filter.*

Definition 2.17. *Let $B(A)$ be the set of all complemented elements of the triangle algebra A (recall that an element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$; if such an element b exists, it is called a complement of a).*

3 State triangle algebras

In this section, we enlarge the language of triangle algebras by introducing a new operator, an internal state and study some related properties of such operators. The special set $Ker(\sigma)$ has been introduced and discuss relations between this set and B(A). Finally, we investigate the effect of σ on operators $\vee, \wedge, *$ and $\to$.

Definition 3.1. *Let A be a triangle algebra. A mapping $\sigma : A \longrightarrow A$ such that, for all $x, y \in A$, we have*
(ST_1) $\sigma(0) = 0$,
(ST_2) $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$,
(ST_3) $\sigma(x * y) = \sigma(x) * \sigma(x \to (x * y))$,
(ST_4) $\sigma(\sigma(x) * \sigma(y)) = \sigma(x) * \sigma(y)$,
(ST_5) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$,
(ST_6) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$,
(ST_7) $\sigma(\sigma(x) \vee \sigma(y)) = \sigma(x) \vee \sigma(y)$,
(ST_8) $\sigma(\nu x) = \nu(\sigma(x))$,
(ST_9) $\sigma(\mu x) = \mu(\sigma(x))$.
is said to be a state operator on A and the pair (A, σ) is said to be a state triangle algebra.

If σ is a state operator, then $Ker(\sigma) := \{x \in A \mid \sigma(\nu x) = 1\}$ is said to be the kernel of σ. A state operator σ is said to be faithful if $Ker(\sigma) = \{1\}$.

Example 3.2. (*i*) *Let A be a triangle algebra. Then (A, id_A) is a state triangle algebra.*
(*ii*) *Let $A = \{[0,0], [0,a], [0,b], [a,a], [a,b], [b,b], [0,1], [a,1], [b,1], [1,1]\}$. Define $\odot$ and $\Rightarrow$ as follows:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

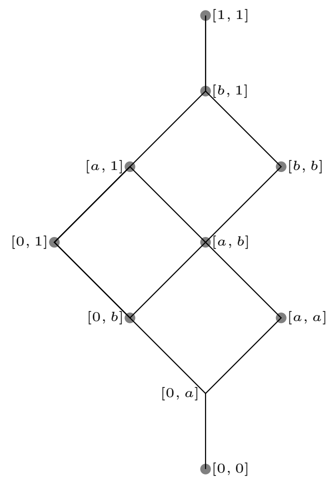
And we define ν, μ, $*$ and $\to$ as follows:

$$\nu[x_1, x_2] = [x_1, x_1], \quad \mu[x_1, x_2] = [x_2, x_2], \quad [x_1, x_2] * [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2],$$
$$[x_1, x_2] \to [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2].$$

Then $(A, \vee, \wedge, *, \to, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra with $[0,0]$ as the smallest and $[1,1]$ as the greatest element. We define the unary operation σ as follows:

$$\sigma(x) = \begin{cases} [0,0], & x = [0,0], [0,a], [0,b], [0,1] \\ [a,a], & x = [a,a], [a,b], [a,1] \\ [1,1], & x = [b,b], [b,1], [1,1]. \end{cases}$$

Then σ is a state operator on A. So (A, σ) is a state triangle algebra. Also, we have $\sigma(x * y) = \sigma(x) * \sigma(y)$ and $\sigma(x \to y) = \sigma(x) \to \sigma(y)$, for all $x, y \in A$. Thus σ is an endomorphism and $\sigma(A) = \{[0,0], [a,a], [1,1]\}$. Also, we have $Ker(\sigma) = \{[b,b], [b,1], [1,1]\}$ so σ is not faithful.



(iii) $\mathcal{L}^I = (L^I, \vee, \wedge, \mathcal{T}_{T,\alpha}, \mathcal{I}_{\mathcal{T}_{T,\alpha}}, \nu, \mu, [0,0], [0,1], [1,1])$ *is a triangle algebra if, for* $a = [a_1, a_2]$, $x = [x_1, x_2]$ *and* $y = [y_1, y_2]$ *in* L^I,

$$\mathcal{T}_{T,\alpha}(x, y) = [T(x_1, y_1), max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))],$$

induces a residuated lattice on $\mathcal{L}^I$*, with residual implicator*

$$\mathcal{I}_{\mathcal{T}_{T,\alpha}}(x, y) = [min(I_T(x_1, y_1), I_T(x_2, y_2)), min(I_T(T(x_2, \alpha), y_2), I_T(x_1, y_2))].$$

And $\nu x = [x_1, x_1]$ *and* $\mu x = [x_2, x_2]$*, for all* $\alpha \in I$ *and* $x = [x_1, x_2] \in L^I$ [11].
Let $T(x, y) = min(x, y)$ *and* $I_T(x, y) = \begin{cases} 1 & x \leq y \\ y & y < x \end{cases}$, $\alpha = 1$. *We define*

$$\sigma_a(x) = \begin{cases} x & x \leq a \\ 1 & otherwise. \end{cases}$$

It is clear that $\sigma_a(x)$ *is a state operator on* L^I*. Thus,* $(L^I, \sigma_a(x))$ *is a state triangle algebra.*

(iv) Let $A = \{[0,0], [0,v], [0,a], [0,b], [0,1], [v,v], [v,a], [v,b], [v,1], [a,a], [a,1], [b,b], [b,1], [1,1]\}$*. Define* $\odot$ *and* $\Rightarrow$ *as follows:*

$\odot$	0	v	a	b	1
0	0	0	0	0	0
v	0	v	v	v	v
a	0	v	a	v	a
b	0	v	v	b	b
1	0	v	a	b	1

$\Rightarrow$	0	v	a	b	1
0	1	1	1	1	1
v	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	v	a	b	1

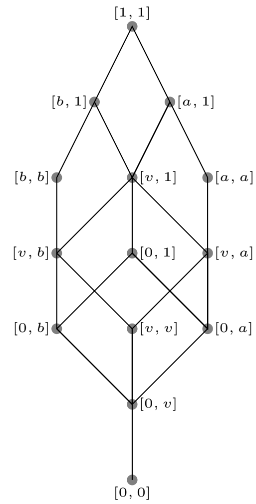
Define ν, μ, $*$ *and* $\to$ *one as follows:*

$$\nu[x_1, x_2] = [x_1, x_1],\ \ \mu[x_1, x_2] = [x_2, x_2] \text{ such that } [x_1, x_2] * [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2],$$
$$[x_1, x_2] \to [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2].$$

Then $(A, \vee, \wedge, *, \to, \nu, \mu, [0,0], [0,1], [1,1])$ *is a triangle algebra with* $[0,0]$ *as the smallest and* $[1,1]$ *as the greatest element. We define the unary operation* σ *as follows:*

$$\sigma(x) = \begin{cases} [0,0], & x = [0,0], [0,v], [v,v], [v,1] \\ [1,1], & otherwise. \end{cases}$$

Then σ *is a state operator on* A*. So* (A, σ) *is a state triangle algebra.*



Proposition 3.3. *Let (A, σ) be a state triangle algebra. Then the following statements hold:*
(i) *If $Ker(\sigma) \subseteq B(A)$, then $B(A/Ker(\sigma)) = B(A)/Ker(\sigma)$.*
(ii) *If $a \in A$ is a nilpotent element, then $a/Ker(\sigma) \in A/Ker(\sigma)$ is a nilpotent element.*

Proof. (i) Let $Ker(\sigma) \subseteq B(A)$. Then

$$\begin{aligned} B(A/Ker(\sigma)) &= \{[e] \in A/Ker(\sigma) \mid [e] \vee \neg[e] = [1]\} \\ &= \{[e] \in A/Ker(\sigma) \mid e/Ker(\sigma) \vee \neg e/Ker(\sigma) = 1/Ker(\sigma)\} \\ &= \{[e] \in A/Ker(\sigma) \mid e \vee \neg e = 1\} \\ &= B(A)/Ker(\sigma). \end{aligned}$$

(ii) It is clear. □

Lemma 3.4. *Let (A, σ) be a state triangle algebra. Then for all $x, y \in A$, we have:*
(1) $\sigma(1) = 1$.
(2) $\sigma(\neg x) = \neg(\sigma(x))$.
(3) *If $x \leq y$, then $\sigma(x) \leq \sigma(y)$.*
(4) *$\sigma(x * y) \geq \sigma(x) * \sigma(y)$ and if $x * y = 0$, then $\sigma(x * y) = \sigma(x) * \sigma(y)$.*
(5) *$\sigma(x \to y) \leq \sigma(x) \to \sigma(y)$ and if x, y are comparable, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$.*
(6) $\sigma(\sigma(x)) = \sigma(x)$.
(7) *$\sigma(A)$ is a triangle subalgebra of A.*
(8) $\sigma(A) = \{x \in A \mid x = \sigma(x)\}$.
(9) *If $ord(x) < \infty$, then $ord(\sigma(x)) \leq ord(x)$ and in MTL-triangle algebras we have $\sigma(x) \notin Rad(A)$.*
(10) *$\sigma(x \to y) = \sigma(x) \to \sigma(y)$ iff $\sigma(y \to x) = \sigma(y) \to \sigma(x)$.*
(11) *If $\sigma(A) = A$, then σ is the identity on A.*
(12) *If σ is faithful, then $x < y$ implies $\sigma(x) < \sigma(y)$.*
(13) *If σ is faithful, then either $\sigma(x) = x$ or $\sigma(x)$ and x are not comparable.*
(14) *If A is linear and σ faithful, then $\sigma(x) = x$, for all $x \in A$.*
(15) $\sigma(\nu(x * y)) = \sigma(\nu x) * \sigma(\nu x \to \nu(x * y))$.
(16) $u \notin Ker(\sigma)$.

Proof. The proof of parts $(1-6)$ and (8) is Similar to Proposition 3.5 of [8].

(7) From $(ST_1), (ST_4), (ST_5), (ST_6), (ST_7), (ST_8), (ST_9)$ and part (1), we have that $\sigma(A)$ is closed under all operators $*, \to, \vee, \wedge, \nu$ and μ. Thus $\sigma(A)$ is a triangle subalgebra of A.

(9) Let $ord(x) = n$. By part (4), $0 = \sigma(x^n) \geq \sigma(x)^n$. So $ord(\sigma(x)) \leq ord(x)$. From Theorem 2.14, we conclude any element of finite order cannot belong to $Rad(A)$.

(10) Let $\sigma(x \to y) = \sigma(x) \to \sigma(y)$. Then by part (5), $\sigma(y \to x) = \sigma(y) \to \sigma(y \wedge x) = \sigma(y) \to (\sigma(x) * (\sigma(x \to y)) = \sigma(y) \to (\sigma(x) * (\sigma(x) \to \sigma(y))) = \sigma(y) \to \sigma(x) \wedge \sigma(y) = \sigma(y) \to \sigma(x)$. The converse implication is proved by exchanging x and y in the previous formulas.

(11) For all $x \in A$, we have $x = \sigma(x_0)$, for some $x_0 \in A$. By part (7), we have $\sigma(x) = \sigma(\sigma(x_0)) = \sigma(x_0) = x$.

(12) Let $\sigma(x) = \sigma(y)$. Then $\sigma(y \to x) = \sigma(y) \to \sigma(x) = 1$. So $y \leq x$ which is a contradiction.

(13) Let $x \neq \sigma(x)$ and x and $\sigma(x)$ be comparable. Then $x < \sigma(x)$ or $\sigma(x) < x$ giving $\sigma(x) < \sigma(x)$, which is a contradiction.

(14) It is clear by (14).

(15) By Theorem 2.11 and (ST_3), we have $\sigma(\nu(x * y)) = \sigma(\nu x * \nu y) = \sigma(\nu x) * \sigma(\nu x \to (\nu x * \nu y)) = \sigma(\nu x) * \sigma(\nu x \to \nu(x * y))$.

(16) By $(T.5)$, $\sigma(\nu u) = \sigma(0) = 0$ so $u \notin Ker(\sigma)$. □

Proposition 3.5. *The following hold:*

(1) *If $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$, for all $x, y \in A$.*

(2) *If $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$, for all $x, y \in A$.*

(3) *If $\sigma(x \to y) = \sigma(x) \to \sigma(y)$, then $\sigma(x * y) = \sigma(x) * \sigma(y)$, for all $x, y \in A$.*

Proof. (1) We have $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y) = \sigma(x) \to (\sigma(x) \wedge \sigma(y)) = \sigma(x) \to \sigma(y)$.

(2) Since $(x \vee y) \to y = x \to y$, $\sigma(x \to y) = \sigma((x \vee y) \to y) = \sigma(x \vee y) \to \sigma((x \vee y) \wedge y) = (\sigma(x) \vee \sigma(y)) \to \sigma(y) = \sigma(x) \to \sigma(y)$.

(3) $\sigma(x * y) \to \sigma(z) = \sigma((x * y) \to z) = \sigma(x \to (y \to z)) = \sigma(x) \to (\sigma(y) \to \sigma(z)) = (\sigma(x) * \sigma(y)) \to \sigma(z)$. Let $z = \sigma(x) * \sigma(y)$. Then $\sigma(x * y) \to \sigma(\sigma * \sigma(y)) = \sigma(x) * \sigma(y) \to \sigma(\sigma(x) * \sigma(y)) = \sigma(x) * \sigma(y) \to \sigma(x) * \sigma(y) = 1$. Thus $\sigma(x * y) \to \sigma(\sigma(x) * \sigma(y)) = \sigma(x * y) \to \sigma(x) * \sigma(y) = 1$ and so $\sigma(x * y) \leq \sigma(x) * \sigma(y)$. By Lemma 3.4 (4), we have $\sigma(x * y) \geq \sigma(x) * \sigma(y)$ so $\sigma(x * y) = \sigma(x) * \sigma(y)$. □

In the following, we consider that under which conditions the converse of above proposition hold.

Proposition 3.6. *Let A be a RL-triangle algebra. Then we have*

(1) $\sigma(x \wedge y) = \sigma(x) * \sigma(x \to y)$.

(2) *If $\sigma(x \to y) = \sigma(x) \to \sigma(y)$, then $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$, for all $x, y \in A$.*

(3) *If $\sigma(x \to y) = \sigma(x) \to \sigma(y)$, then $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$.*

Proof. Similar to Proposition 3.6 of [8].

In the following example we show that the converse of part (3) of Proposition 3.5 is not generally true.

Example 3.7. *Let $A = \{0, u, 1\}$ be a chain. We define operations $\nu, \mu, *, \to$ as follows:*

x	νx
0	0
u	0
1	1

x	μx
0	0
u	1
1	1

$*$	0	u	1
0	0	0	0
u	0	u	u
1	0	u	1

$\to$	0	u	1
0	1	1	1
u	0	1	1
1	0	u	1

We define the unary operation σ as follows:

$$\sigma(x) = \begin{cases} 0 & x = 0, u \\ 1 & x = 1. \end{cases}$$

Then $(A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is a state triangle algebra. We have $\sigma(u * 0) = \sigma(u) * \sigma(0)$ but $\sigma(u \to 0) \neq \sigma(u) \to \sigma(0)$.

□

4 State IVRL-filters in state triangle algebras

From now on (A, σ) is a state triangle algebra unless otherwise specified.

Definition 4.1. *A non-empty subset $F \subseteq A$ is called a state IVRL-filter of (A, σ) if F is an IVRL-filter of A and $\sigma(\nu x) \in F$, for all $x \in F$.*

We will denote the set of σ-IVRL-filters of (A, σ) by $F_\sigma(A)$.

Example 4.2. *In Example 3.2 (iii), clearly $F = \{[1, 1]\}$ is a state IVRL-filter of (A, σ).*

Corollary 4.3. *Let (A,σ) be a state triangle algebras and F,G be two IVRL-filters of A such that $F \subseteq G$. If F is a state IVRL-filter of (A,σ), then for all $x \in A$, we have $\sigma(\nu x) \in F \subseteq G$, hence $\sigma(\nu x) \in G$. So G is a state IVRL-filter of (A,σ).*

Proposition 4.4. *Let (A,σ) be a state triangle algebra and F be a state IVRL-filter of (A,σ). Then $\sigma(A/F) = \sigma(A)/F$.*

Proof. By Lemma 3.4(8), we have

$$\sigma(A/F) = \{[x] \in A/F \mid [x] = \sigma([x])\} = \{[x] \in A/F \mid x = \sigma(x)\} = \sigma(A)/F.$$

□

Proposition 4.5. *Let (A,σ) be a state triangle algebra. Then $Ker(\sigma)$ is a state IVRL-filter of A.*

Proof. We have $\sigma(1) = 1$, so $1 \in Ker(\sigma)$. Let $x, y \in Ker(\sigma)$. Then $\sigma(\nu(x * y)) = \sigma((\nu x) * (\nu y)) \geq \sigma(\nu x) * \sigma(\nu y) = 1$, $\sigma(\nu(x * y)) = 1$. Thus $x * y \in Ker(\sigma)$. If $x \in Ker(\sigma)$ and $y \in A, x \leq y$, then $1 = \sigma(\nu x) \leq \sigma(\nu y)$. So $y \in Ker(\sigma)$. Also, let $x \in Ker(\sigma)$. Then we have $\sigma(\nu x) = 1$. Since $\nu\nu x = \nu x$, $\sigma(\nu\nu x) = 1$. Thus $\nu x \in Ker(\sigma)$. Therefore, $Ker(\sigma)$ is an IVRL-filter of A. □

Definition 4.6. *A proper σ-IVRL-filter of (A,σ) is called an IVRL-extended maximal filter of (A,σ) if it is not strictly contain in any proper σ-IVRL-filter of (A,σ).*

We denote the set of IVRL-extended maximal σ-filters of (A,σ) by $Max_\sigma(A)$ and $Rad_\sigma(A) = \cap_{F \in Max_\sigma(A)} F$.

Remark 4.7. *If $(F_i)_{i\in I}$ is a family of σ-IVRL-filter on a triangle algebra (A,σ), then $\cap_{i\in I} F_i$ is a σ-IVRL-filer of A.*

Proposition 4.8. *If F is an σ-IVRL-filter of (A,σ) and $x \in A$, then the σ-IVRL-filter generated by F and x is the set $[F,x)_\sigma = \{y \in A \mid \nu y \geq (f_1 * (\nu x * \sigma(\nu x))^{n_1}) * ... * (f_k * (\nu x * \sigma(\nu x))^{n_k})$, for some $f_i \in F_i, n_i \in \mathbb{N}, i \in \{1,...,k\}, k \geq 1\}$.*

Proof. Let $Y := \{y \in A \mid \nu y \geq (f_1 * v_1^{n_1}) * ... * (f_k * v_k^{n_k})$, for some $f_i \in F_i, n_i \in \mathbb{N}, i \in \{1,...,k\}, k \geq 1\}$. Clearly, Y is an IVRL-filter of A. Let $y \in Y$, $\nu y \geq (f_1 * (\nu x * \sigma(\nu x))^{n_1}) * ... * (f_k * (\nu x * \sigma(\nu x))^{n_k})$. Then by Lemma 2.2, and Lemma 3.4 we have

$$\begin{aligned}\sigma(\nu y) &\geq \sigma((f_1 * (\nu x * \sigma(\nu x))^{n_1}) * ... * (f_k * (\nu x * \sigma(\nu x))^{n_k}))\\ &\geq \sigma(f_1) * \sigma(\nu x * \sigma(\nu x))^{(n_1)} * ... * \sigma(f_k) * \sigma(\nu x * \sigma(\nu x))^{(n_k)}\\ &\geq \sigma(f_1) * \sigma(\sigma(\nu x))^{(2n_1)} * ... * \sigma(f_k) * \sigma(\sigma(\nu x))^{(2n_k)}\\ &\geq \sigma(f_1) * \sigma(\nu x * \sigma(\nu x))^{(2n_1)} * ... * \sigma(f_k) * \sigma(\nu x * \sigma(\nu x))^{(2n_k)}.\end{aligned}$$

Since $\sigma(f_i) \in F, i \in \{1,...,k\}$, it follows that $\sigma(\nu y) \in Y$. Thus $Y \in [A)_\sigma$. If $F' \in [A)_\sigma$ such that $F \subseteq F'$, $x \in F'$ and let $y \in Y$. Then $\nu y \geq (f_1 * (\nu x * \sigma(\nu x))^{n_1}) * ... * (f_k * (\nu x * \sigma(\nu x))^{n_k})$. But $(f_1 * (\nu x * \sigma(\nu x))^{n_1}) * ... * (f_k * (\nu x * \sigma(\nu x))^{n_k}) \in F'$. Thus $y \in F'$ and $Y = [F,x)_\sigma$. □

Proposition 4.9. *If $a, b \in A$, then the following hold:*

(1) *The σ-IVRL-filter generated by a is the set $[a)_\sigma = \{x \in A \mid \nu x \geq (\nu a * \sigma(\nu a))^n, n \geq 1\}$.*
(2) *If $a \leq b$, then $[b)_\sigma \subseteq [a)_\sigma$.*
(3) $[\sigma(a))_\sigma \subseteq [a)_\sigma$.
(4) $[a)_\sigma = [\nu a * \sigma(\nu a))_\sigma$
(5) $[a)_\sigma \vee [b)_\sigma = [(\nu a * \sigma(\nu a)) * (\nu b * \sigma(\nu b)))_\sigma$.
(6) $[a)_\sigma \cap [b)_\sigma = [(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)))_\sigma$.

Proof. (1) It follows from Proposition 4.8.

Properties $(2)-(4)$ are obvious.

(5) Let $u = \nu a * \sigma(\nu a), v = \nu b * \sigma(\nu b)$. Since $u * v \leq a, u * v \leq b$, By (2), we have $[a)_\sigma \subseteq [u * v)_\sigma, [b)_\sigma \subseteq [u * v)_\sigma$. Let $f \in [A)_\sigma$ so that $[a)_\sigma \subseteq F, [b)_\sigma \subseteq F$. Then $a, b \in F$ so $u * v \in F$. Thus $[u * v)_\sigma \subseteq F$ and $[a)_\sigma \vee [b)_\sigma = [u * v)_\sigma$.

(6) Since $\nu a * \sigma(\nu a) \leq (\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b))$, by (2) we have $[(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)))_\sigma \subseteq [\nu a * \sigma(\nu a))_\sigma = [a)_\sigma$. Similarly, $[(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)))_\sigma \subseteq [b)_\sigma$, so $[(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b))_\sigma \subseteq [a)_\sigma \cap [b)_\sigma$. Let $x \in [a)_\sigma \cap [b)_\sigma$. Then there are $m, n \geq 1$ such that $x \geq (\nu a * \sigma(\nu a))^m$ and $x \geq (\nu b * \sigma(\nu b))^n$. So $x \geq (\nu a * \sigma(\nu a))^m \vee (\nu b * \sigma(\nu b))^n \geq ((\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)))^{mn}$. So $x \in [(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)))_\sigma$. □

In the following example we show that the converse of parts $(2), (3)$ of the above proposition is not true.

Example 4.10. *In Example 3.2 (ii),*

- $[[b,1])_\sigma = \{[b,b],[b,1],[1,1]\} \subseteq [[a,a])_\sigma = A$. *But* $[a,a] \nleq [b,1]$.
- $[[b,1])_\sigma = \{[b,b],[b,1],[1,1]\}$ *and* $[\sigma([b,1]))_\sigma = \{[1,1]\}$. *So* $[[b,1])_\sigma \nsubseteq [\sigma([b,1]))_\sigma$ *and so* $[\sigma(a))_\sigma \nsupseteq [a)_\sigma$.

Proposition 4.11. *Let* (A,σ) *be a state triangle algebra and* $F_1, F_2 \in F_\sigma(A)$. *Then* $[F_1 \cup F_2)_\sigma = F_1 \vee F_2 = \{x \in A \mid \nu x \geq (f_1 * f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\}$.

Proof. Let $Y = \{x \in A \mid \nu x \geq (f_1 * f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1\}$. If $x, y \in Y$, then $\nu x \geq (f_1 * f_2)^k, \nu y \geq (g_1 * g_2)^l, f_1, g_1 \in F_1, f_2, g_2 \in F_2, k, l \geq 1$. By Lemma 2.2, we have $\nu(x*y) = \nu x * \nu y \geq (f_1 * f_2)^k * (g_1 * g_2)^l \geq ((f_1 * g_1) * (f_2 * g_2))^{k+l}$. Since $f_1 * g_1 \in F_1, f_2 * g_2 \in F_2$, $x * y \in Y$. If $x \in Y$ and $x \leq y, y \in A$, then $y \in Y$. Let $x \in Y$. Then $\nu x \geq (f_1 * f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1$ and so $\sigma(\nu x) \geq (\sigma(f_1) * \sigma(f_2))^k$. But $\sigma(f_1) \in F_1, \sigma(f_2) \in F_2$, so $\sigma(\nu x) \in Y$. Therefore, $Y \in [A)_\sigma$. Let $F \in [A)_\sigma$ such that $F_1 \cup F_2 \subseteq F$. If $x \in Y$, then $\nu x \geq (f_1 * f_2)^k, f_1 \in F_1, f_2 \in F_2, k \geq 1$. But $(f_1 * f_2)^k \in F$, thus $x \in F$. So $B \subseteq F$, $Y = [F_1 \cup F_2)_\sigma$. Similarly, $Y = F_1 \vee F_2$. ☐

Theorem 4.12. *Let* F *be a proper* σ*-IVRL-filter of* (A,σ). F *is an IVRL-extended maximal* σ*-filter iff for all* $x \in A \setminus F$, *there are* $f \in F$ *and* $m, n \geq 1$ *such that* $(f * \sigma(\nu x)^n)^m = 0$.

Proof. Let F be an IVRL-extended maximal σ-filter and $x \in A \setminus F$. Then $[f,x)_\sigma = A$, where $[F,x)_\sigma = \{y \in A \mid \nu y \geq f_1 * (\nu x * \sigma(\nu x))^{n_1} * ... * f_k * (\nu x * \sigma(\nu x))^{n_k}$ for some $f_i \in F, n_i \in \mathbb{N}, i \in \{1,...,k\}, k \geq 1\}$. Thus there are $f_i \in F, i \in \{1,...,k\}$ such that $f_1 * (\nu x * \sigma(\nu x))^{n_1} * ... * f_k * (x * \sigma(\nu x))^{n_k} \leq f_1 * (\nu x * \sigma(\nu x))^{n_1} * ... * f_k * (\nu x * \sigma(\nu x))^{n_k} = 0$. Let $g = f_1 * ... * f_k \in F, b = max\{n_1,...,n_k\}$ so we have $(g * (\nu x * \sigma(\nu x))^b)^k = 0$. Then $\sigma((g * (\nu x * \sigma(\nu x))^b)^k) = \sigma(0) = 0$. Since

$$\sigma((g * (\nu x * \sigma(\nu x))^b)^k) \geq \sigma(g * (\nu x * \sigma(\nu x))^b))^k \geq (\sigma(g) * \sigma(\nu x * \sigma(\nu x))^b)^k \geq (\sigma(g) * \sigma(\nu x)^{2b})^k.$$

So we have $(\sigma(g) * \sigma(\nu x)^{2b})^k = 0$. If $f = \sigma(g) \in F, n = 2b, m = k$, we have $(f * \sigma(\nu x)^n)^m = 0$.

Conversely, let $F' \in [A)_\sigma$ such that $F \subseteq F'$ and $F' \neq F$. Then there is $x \in F' \setminus F$. Then there are $f \in F$ and $m, n \geq 1$ such that $(f * \sigma(\nu x)^n)^m = 0$. Since $x \in F'$, $\sigma(\nu x) \in F'$ and so $(f * \sigma(\nu x)^n)^m \in F'$, thus $0 \in F'$. Thus $F' = A$, so and F is an IVRL-extended maximal σ-filter. ☐

Proposition 4.13. *Let* (A,σ) *be a state triangle algebra. Then we have*

(*i*) *If* F *is an IVRL-extended maximal filter of* $\sigma(A)$, *then* $\sigma^{-1}(F)$ *is an IVRL-extended maximal* σ*-filter of* (A,σ).
(*ii*) *If* F *is an IVRL-extended maximal* σ*-filter of* (A,σ), *then* $\sigma(F)$ *is an IVRL-extended maximal filter of* (A,σ).

Proof. (*i*) If $x, y \in \sigma^{-1}(F)$, then $\sigma(\nu x) * \sigma(\nu y) \in F$. But $\sigma(\nu x * \nu y) \geq \sigma(\nu x) * \sigma(\nu y)$, since $\sigma(\nu x * \nu y) \in \sigma(A)$ and F is an IVRL-filter of $\sigma(A)$, it follows that $\sigma(\nu x * \nu y) \in F$, $\nu x * \nu y \in \sigma^{-1}(F)$. Let $x, y \in A$ such that $\nu x \in \sigma^{-1}(F)$ and $x \leq y$. Then $\sigma(\nu x) \leq \sigma(\nu y)$ and since $\sigma(\nu x) \in F, \sigma(\nu y) \in \sigma(A)$. So $\sigma(\nu y) \in F$, $\nu y \in \sigma^{-1}(F)$. Hence $\sigma^{-1}(F)$ is an IVRL-filter of A. If $x \in \sigma^{-1}(F)$, then $\sigma(\nu x) \in F$ and $\sigma(\sigma(\nu x)) = \sigma(\nu x) \in F$. Thus $\sigma(\nu x) \in \sigma^{-1}(F), \sigma^{-1}(F) \in F_\sigma(A)$. Let $x \in A \setminus \sigma^{-1}(F)$. Then $\sigma(x) \in A \setminus F$. From Theorem 2.13, there are $f \in F$ and $m, n \geq 1$, such that $(f * \sigma(\nu x)^n)^m = 0$. Since $\sigma(f) = f \in F$, $f \in \neg\sigma(F)$. By Theorem 4.12, we have that $\neg\sigma(F)$ is an IVRL-extended maximal σ-filter of (A,σ).

(*ii*) Since $\sigma(F) = F \cap \sigma(A)$ and $\sigma(F)$ is an IVRL-filter of $\sigma(A)$. Let $x \in \sigma(A) \setminus \sigma(F)$. Then $x \in \sigma(A) \setminus F$. By Theorem 4.12, there are $f \in F$ and $m, n \geq 1$ such that $(f * \sigma(\nu x)^n)^m = 0$. Then $\sigma((f * \sigma(\nu x)^n)^m) = 0$, so $(\sigma(f) * \sigma(\sigma(\nu x))^n)^m = 0$. Since $\sigma(x) = x$, $(\sigma(f) * (\nu x)^n)^m = 0$. We have $\sigma(f) \in \sigma(F)$, from Theorem 2.13, $\sigma(F)$ is an IVRL-extended maximal filter of $\sigma(A)$. ☐

Definition 4.14. *A proper IVRL-*σ*-filter* P *of* (A,σ) *is called an IVRL-extended prime* σ*-filter if* $a, b \in A$ *such that* $(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)) \in P$, *then* $\nu a \in P$ *or* $\nu b \in P$.

We will denote the set of IVRL-extended prime σ-filters of (A,σ) by $Spec_\sigma(A)$.

Proposition 4.15. *Let* P *be a proper IVRL-*σ*-filter of* (A,σ). *Then the following are equivalent:*

(*i*) *If* $P_1, P_2 \in F_\sigma(A)$ *and* $P = P_1 \cap P_2$, *then* $P = P_1$ *or* $P = P_2$,
(*ii*) P *is an IVRL-extended prime* σ*-filter of* (A,σ).

Proof. ($i \Rightarrow ii$) Let $(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)) \in P$, for all $a, b \in A$. Let $P_1 = [P,a)_\sigma$ and $P_2 = [P,b)_\sigma$. Clearly, $P \subseteq P_1 \cap P_2$. Let $x \in P_1 \cap P_2$ and $u = \nu a * \sigma(\nu a)$ and $v = \nu b * \sigma(\nu b)$. By Proposition 4.8, there are $p_i, q_j \in P$ and

$m_i, n_j \geq 1, i \in \{1, ..., k\}, j \in \{1, ..., l\}, k, l \geq 1$, such that $\nu x \geq p_1 * u^{m_1} * ... * p_k * u^{m_k}$ and $\nu x \geq q_1 * v^{n_1} * ... * q_1 * v^{n_1}$. Then

$$\begin{aligned}\nu x &\geq (p_1 * u^{m_1} * ... * p_k * u^{m_k}) \vee (q_1 * v^{n_1} * ... * q_1 * v^{n_1}) \\ &\geq *_{1\leq i\leq k} *_{1\leq j\leq l} (*((p_i * u^{m_i}) \vee (q_j * v^{n_j}))) \\ &\geq *_{1\leq i\leq k} *_{1\leq j\leq l} (*((p_i \vee q_j) * (p_i \vee v^{n_j}) * (u^{m_i} \vee q_j) * (u^{m_i} \vee q_j) * (u^{m_i} \vee v^{n_j}))) \\ &\geq *_{1\leq i\leq k} *_{1\leq j\leq l} (*((p_i \vee q_j) * (p_i \vee v^{n_j}) * (u^{m_i} \vee q_j) * (u \vee v)^{m_i n_j})).\end{aligned}$$

But $p_i \vee q_j, p_i \vee v^{n_j}, u^{m_i} \vee q_j, (u \vee v)^{m_i n_j} \in P, i \in \{1, ..., k\}, j \in \{1, ..., l\}$. So $x \in P$ and $P = P_1 \cap P_2$. Thus $P = P_1$ or $P = P_2$. that is $\nu a \in P$ or $\nu b \in P$.

$(ii \Rightarrow i)$ Let $P_1, P_2 \in F_\sigma(A)$ such that $P = P_1 \cap P_2$. If $P \neq P_1$ and $P \neq P_2$ and $a \in P_1 \setminus P$ and $b \in P_2 \setminus P$, then $\nu a * \sigma(\nu a) \in P_1, \nu b * \sigma(\nu b) \in P_2$. So $(\nu a * \sigma(\nu a)) \vee (\nu b * \sigma(\nu b)) \in P_1 \cap P_2 = P$. Thus $\nu a \in P$ or $\nu b \in P$, a contradiction. Thus $P = P_1$ or $P = P_2$. □

Proposition 4.16. *Let F be a proper σ-IVRL-filter of (A, σ). Then there exists an IVRL-extended maximal σ-filter F_0 of (A, σ) such that $F \subseteq F_0$.*

Proof. Clearly, the set $A_F = \{F' \mid F'$ is a proper σ-IVRL-filter and $F \subseteq F'\}$ is nonvoid and inductively ordered by inclusion. So by Zorn lemma, A_F has a maximal element F_0. We are going to prove that F_0 is an IVRL-extended maximal σ-filter of (A, σ). If F_1 is a proper σ-IVRL-filter of (A, σ) such that $F_0 \subseteq F_1$, then $F_1 \in A_F$ and the maximality of F_0 implies that $F_1 = F_0$. □

Proposition 4.17. *Let $a \in A, a < 1$. Then there is an IVRL-extended prime σ-filter P of (A, σ), such that $a \notin P$.*

Proof. Let $F_a = \{F \mid F$ is an IVRL-σ-filter and $a \notin F\}$. Then $\{1\} \in F_a$, so $F_a \neq \emptyset$. We show that F_a is inductively ordered and by zorn lemma, there is a maximal element P of F_a. We will prove that P is an IVRL-extended prime σ-filter. Let $x, y \in A$ such that $(\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y)) \in P$ and $\nu x, \nu y \notin P$. Considering the sets $[P, x)_\sigma$ and $[P, y)_\sigma$. Then P is strictly contained in $[P, x)_\sigma$ and $[P, y)_\sigma$. The maximality of P implies that $a \in [P, x)_\sigma \cap [P, y)_\sigma$. Then there are $a_i, b_j \in P$ and $m_i, n_j \geq 1, i \in \{1, ..., k\}, j \in \{1, ..., l\}, k, l \geq 1$ such that $\nu a \geq a_1 * (\nu x * \sigma(\nu x))^{m_1} * ... * a_k * (\nu x * \sigma(\nu x))^{m_k}$ and $\nu a \geq b_1 * (\nu y * \sigma(\nu y))^{n_1} * ... * b_k * (\nu y * \sigma(\nu y))^{n_k}$. Similar to proposition 4.15, we have $\nu a \in P$, a contradiction. So P is an IVRL-extended prime σ-filter. □

In the following example we show that the concepts of IVRL-extended maximal σ-filters and IVRL-extended prime σ-filters on state triangle algebras are different from the concepts of IVRL-extended maximal filters and IVRL-extended prime filters on triangle algebras.

Example 4.18. *Let $A = \{[0, 0], [0, a], [0, b], [0, c], [0, d], [0, 1], [a, b], [a, d], [c, d], [a, a], [b, b], [c, c], [d, d], [a, 1], [c, 1], [d, 1], [b, 1], [1, 1]\}$. Define $\odot$ and $\Rightarrow$ as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\Rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

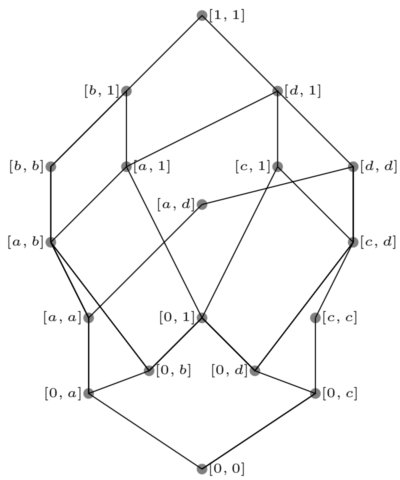
And we define ν, μ, $*$ and $\rightarrow$ as follows:

$$\nu[x_1, x_2] = [x_1, x_1],\ \mu[x_1, x_2] = [x_2, x_2] \text{ such that } [x_1, x_2] * [y_1, y_2] = [x_1 \odot y_1, x_2 \odot y_2],$$

$$[x_1, x_2] \rightarrow [y_1, y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2].$$

Then $(A, \vee, \wedge, *, \rightarrow, \nu, \mu, [0, 0], [0, 1], [1, 1])$ is a triangle algebra with $[0, 0]$ as smallest and $[1, 1]$ as greatest element. We define the unary operation σ as follows:

$$\sigma(x) = \begin{cases} [1, 1], & x = [c, c], [d, d], [c, d], [c, 1], [d, 1], [1, 1] \\ [0, 0], & \textit{otherwise.} \end{cases}$$



Then σ is a state operator on A. So (A,σ) is a state triangle algebra. The IVRL-filters of A are $\{[1,1]\}, \{[b,b],[b,1],[1,1]\}$, $\{[c,c],[c,d],[c,1],[d,1],[1,1]\}, A$. The IVRL-$\sigma$-filters of (A,σ) are $\{[1,1]\}, \{[c,c],[c,d],[c,1],[d,1],[1,1]\}, A$. Since $\{[1,1]\} = \{[b,b],[b,1],[1,1]\} \cap \{[c,c],[c,d],[c,1],[d,1],[1,1]\}$, the IVRL-filter $\{[1,1]\}$ is not an IVRL-extended prime filter of A. But $\{[1,1]\}$ is an IVRL-extended prime σ-filter of (A,σ). Thus the notion of IVRL-extended prime filter and IVRL-extended prime σ-filter of (A,σ) are not the same.

Proposition 4.19. *Let (A,σ) be a state triangle algebra, $F \in F_\sigma(A)$ and $a \in A \setminus F$. Then the following hold:*
(i) *There is $P \in Spec_\sigma(A)$ such that $F \subseteq P$ and $a \notin P$.*
(ii) *$F = \cap_{P \in Spec_{\sigma(A)}, F \subseteq P} P$.*
(iii) *$F = \cap_{P \in Spec_{\sigma(A)}} P = \{1\}$.*

Proof. (i) It follows from Proposition 4.17.
(ii) Clearly, $F \subseteq \cap_{P \in Spec_{\sigma(A)}, F \subseteq P} P$. Let $a \in \cap_{P \in Spec_{\sigma(A)}, F \subseteq P} P$ and $a \notin F$. Then from (i) it follows that there is $P \in Spec_\sigma(A)$ such that $F \subseteq P$ and $a \notin P$, a contradiction.
(iii) It follows from (ii), for $F = \{1\}$. □

Proposition 4.20. *Let (A,σ) be a state triangle algebra and P be a proper IVRL-σ-filter of (A,σ). Then the following are equivalent:*
(i) *$P \in Spec_\sigma(A)$,*
(ii) *for every $x,y \in A$ for which $\nu x * \sigma(\nu x), \nu y * \sigma(\nu y) \in A \setminus P$, there is $z \in A \setminus P$ such that $\nu x * \sigma(\nu x) \leq \nu z$ and $\nu y * \sigma(\nu y) \leq \nu z$,*
(iii) *if $F_\sigma(x) \cap F_\sigma(y) \subseteq P$, then $x \in P$ or $y \in P$.*

Proof. ($i \Leftrightarrow ii$) Let $x,y \in A$ such that $\nu x * \sigma(\nu x), \nu y * \sigma(\nu y) \in A \setminus P$ and $z = (\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y))$. If $z \in P$, then $\nu x \in P$ or $\nu y \in P$. So $\nu x * \sigma(\nu x) \in P$ or $\nu y * \sigma(\nu y) \in P$, a contradiction. Thus $x \in A \setminus P$ and $\nu x * \sigma(\nu x) \leq \nu z$ and $\nu y * \sigma(\nu y) \leq \nu z$.

Conversely, let $x,y \in A$ such that $(\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y)) \in P$, $x \notin P$ and $y \notin P$. Then $\nu x * \sigma(\nu x), \nu y * \sigma(\nu y) \in A \setminus P$. So there is $z \in A \setminus P$ such that $\nu x * \sigma(\nu x) \leq z$ and $\nu y * \sigma(\nu y) \leq z$. But $(\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y)) \leq z$, that is $z \in P$, a contradiction.

($i \Leftrightarrow iii$) By Proposition 4.9(6), we have $[x)_\sigma \cap [y)_\sigma = [(\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y)))_\sigma$. Thus $(\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y)) \in P$ and $\nu x \in P$ or $\nu y \in P$.

Conversely, if $(\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y)) \in P$, $x,y \in A$, then $[(\nu x * \sigma(\nu x)) \vee (\nu y * \sigma(\nu y))))_\sigma \subseteq P$. Hence $[x)_\sigma \cap [y)_\sigma \subseteq P$ and $\nu x \in P$ or $\nu y \in P$. □

Definition 4.21. *Let (A,σ) be a state triangle algebra. A subalgebra B of A is called closed relative to σ if $x \in B$ implies $\sigma(x) \in B$.*

Example 4.22. *In Example 3.2 (ii), $B = \{[a,a],[1,1]\}$ is closed relative to σ.*

Proposition 4.23. *Let (A,σ) be a state triangle algebra and B be a subalgebra of A closed relative to σ and $F \in F(A)$. Then $B \cap F \in [B)_\sigma$.*

Proof. Let $x,y \in B \cap F$. Then $\nu x * \nu y \in B \cap F$. If $x \in B \cap F$ and $x \leq y, y \in B$, then $\nu y \in F$. Thus $\nu y \in B$. Thus $B \cap F$ is an IVRL-filter of B. Let $x \in B \cap F$. Then $\sigma(x) \in B \cap F$ and $B \cap F \in [B)_\sigma$. □

It is clear that $|F(B)| \leq |F(A)|$, $|Spec(B)| \leq |Spec(A)|$ and $|Max(B)| \leq |Max(A)|$. So we have $|[B)_\sigma| \leq |[A)_\sigma|$, $|Spec_\sigma(B)| \leq |Spec_\sigma(A)|$ and $|Max_\sigma(B)| \leq |Max_\sigma(A)|$.

Proposition 4.24. *Let (A,σ) be a state triangle algebra. Then $|Max_\sigma(A)| = |Max(\sigma(A))|$.*

Proof. Clearly, if B is a subalgebra of A closed relative to σ, then $|Max_\sigma(B)| \leq |Max(\sigma(A))|$. If $B = \sigma(A)$, we have $|Max_\sigma(\sigma(A))| \leq |Max_\sigma(A)|$. Since on $\sigma(A)$ the concepts of IVRL-filter and σ-IVRL-filter are the same, $Max_\sigma(\sigma(A)) = Max(\sigma(A))$. Thus $|Max(\sigma(A))| \leq |Max_\sigma(A)|$.

Conversely, let $F \in Max_\sigma(A)$. We must prove that $F \cap \sigma(A) \in Max(\sigma(A))$. If $F \cap \sigma(A) = \sigma(A)$, then $0 \in F$, a contradiction. Thus $F \cap \sigma(A)$ is a proper σ-IVRL-filter. There is $x \in \sigma(A) \setminus (F \cap \sigma(A)) = \sigma(A) \setminus F$, since $F \in Max_\sigma(A)$, by Theorem 4.12, there are $f \in F$ and $m, n \geq 1$ such that $(f * \sigma(\nu x)^n)^m = 0$. Thus $\sigma((f * \sigma(\nu x)^n)^m) = 0$ and so $(\sigma(f) * \sigma(\nu x)^n)^m = 0$. Since $\sigma(x) = x$, $(\sigma(f) * (\nu x)^n)^m = 0$. We know that $\sigma(f) \in F \cap \sigma(A)$ so by Theorem 2.13, $F \cap \sigma(A) \in Max(\sigma(A))$. We define function $Q : Max_\sigma(A) \longrightarrow Max(\sigma(A))$ that $Q(F) = F \cap \sigma(A)$. If $F_1, F_2 \in Max_\sigma(A)$, then $Q(F_1) = Q(F_2)$. Suppose there is $x \in F_1 \setminus F_2$. Since $x \in F_1$, $\sigma(x) \in F_1$. So $\sigma(x) \in F_1 \cap \sigma(A) = F_2 \cap \sigma(A)$. Thus $\sigma(x) \in F_2$. Since $x \notin F_2$, from the maximality of F_2, there are $m, n \geq 1$ and $f \in F_2$ such that $(f * \sigma(\nu x)^n)^m = 0$. But $(f * \sigma(\nu x)^n)^m \in F_2$ thus $0 \in F_2$ and $F_2 = A$, a contradiction. Therefore $F_1 \subseteq F_2$, similarly $F_2 \subseteq F_1$ so $F_1 = F_2$. Hence Q is injective and $|Max_\sigma(A)| \leq |Max(\sigma(A))|$. □

Definition 4.25. *A triangle algebra A is called simple if it has only two IVRL-filters $\{1\}$ and A. A state triangle algebra (A,σ) is called simple if $\sigma(A)$ is a simple triangle algebra. A state triangle algebra (A,σ) is called simple relative to $F_\sigma(A)$ if it has only two state IVRL-filter $\{1\}$ and A.*

Example 4.26. *In Example 3.7, clearly (A,σ) is a simple state triangle algebra.*

Proposition 4.27. *If (A,σ) is a state triangle algebra where A is simple, then $\sigma(A)$ and (A,σ) are simple.*

Proof. Let F be an IVRL-filter of $\sigma(A), F \neq \{1\}$. By Proposition 4.13(i), the set $\sigma^{-1}(F)$ is an IVRL-filter of A. So $\sigma^{-1}(F) = \{1\}$ or $\sigma^{-1}(F) = A$. If $x \in F$, then $\sigma(x) = x$ and $x \in \sigma^{-1}(F)$. So $F \subseteq \sigma^{-1}(F)$ and $\sigma^{-1}(F) = A$. It follows that $0 \in \sigma^{-1}(F)$ and $\sigma(0) \in F$, thus $0 \in F$, and $F = \sigma(A)$. □

Proposition 4.28. *Let (A,σ) be a state triangle algebra and (A,σ) be a simple relative to $[A)_\sigma$. Then (A,σ) is simple.*

Proof. Let F be an IVRL-filter of $\sigma(A)$, $F \neq \{1\}$. By Proposition 4.13(i) the set $\neg\sigma(F)$ is an IVRL-σ-filter of (A,σ). Since $F \subseteq \neg\sigma(F)$, $\neg\sigma(F) \neq \{1\}$. So $\neg\sigma(F) = A$ and $0 \in \neg\sigma(F)$. Hence $0 = \sigma(0) \in F$ and $F = \sigma(A)$. □

Corollary 4.29. *Let (A,σ) be a simple state triangle algebra relative to $F_\sigma(A)$. We know that $Ker(\sigma) \in [A)_\sigma$ and $Ker(\sigma) \neq A$. Thus $Ker(\sigma) = \{1\}$ and so σ is faithful.*

Proposition 4.30. *Let (A,σ) be a state triangle algebra. Then $Rad(Ker(\sigma)) = \sigma^{-1}(Rad(\{1\}))$.*

Proof. By Lemma 3.4(6), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} a \in Rad(Ker(\sigma)) &\Leftrightarrow \neg(\nu a^n) \to \nu a \in Ker(\sigma) \Leftrightarrow \sigma(\nu(\neg(\nu a^n) \to \nu a)) = 1 \\ &\Leftrightarrow \sigma(\neg(\nu a^n) \to \nu a) = 1 \Leftrightarrow \sigma(\neg(\nu a^n)) \to \sigma(\nu a) = 1 \\ &\Leftrightarrow \sigma(a) \in Rad(\{1\}) \Leftrightarrow a \in \sigma^{-1}(Rad(\{1\}). \end{aligned}$$

□

Definition 4.31. *A triangle algebra A is called semisimple if $Rad(A) = \{1\}$. A state triangle algebra (A,σ) is called semisimple if $\sigma(A)$ is simple, that is $Rad(\sigma(A)) = \{1\}$. A state triangle algebra (A,σ) is called semisimple relative to $[A)_\sigma$ if $Rad_\sigma(A) = \{1\}$.*

Example 4.32. *In Example 4.18, $\sigma(A) = \{[0,0],[1,1]\}$ is simple, but $ker(\sigma) = \{[c,c],[c,1],[c,d],[d,d],$ $[d,1],[1,1]\} \neq \{[1,1]\}$.*

Theorem 4.33. *Let (A,σ) be a state triangle algebra. Then the following are equivalent:*
(i) *(A,σ) is simple relative to $[A)_\sigma$,*
(ii) *(A,σ) is simple and σ is faithful.*

Proof. ($i \Rightarrow ii$) It is clear by Proposition 4.28 and corollary 4.29.

($ii \Rightarrow i$) Let $F \in [A)_\sigma, F \neq \{1\}$. Then $\sigma(F) = F \cap \sigma(A)$ is an IVRL-filter of A. Since $\sigma(A)$ is simple, $\sigma(A) = \{1\}$ or $\sigma(F) = \sigma(A)$. But σ is faithful and $F \neq \{1\}$. So $\sigma(F) \neq \{1\}$ and $\sigma(F) = \sigma(A)$. Thus $0 \in \sigma(F)$ and $0 \in F$. Therefore $F = A$. □

Proposition 4.34. *Let (A, σ) be a state triangle algebra. Then $Rad(\sigma(A)) = \sigma(Rad_\sigma(A))$.*

Proof. Let $x \in Rad_\sigma(A)$. Then $x = \sigma(y), y \in Rad_\sigma(A)$. Let M be an IVRL-extended maximal filter of $\sigma(A)$. By Proposition 4.13 (i), $\neg\sigma(M)$ is an IVRL-extended maximal σ-filter of (A, σ). Then $y \in \neg\sigma(M)$, $\sigma(y) \in M$. So $x \in M$ and $x \in Rad(\sigma(A))$. Conversely, let $x \in Rad(\sigma(A))$ and M be an IVRL-extended maximal σ-filter of (A, σ). By Proposition 4.13 (ii), $\sigma(M)$ is an IVRL-extended maximal filter of $\sigma(A)$. So $x \in \sigma(M) = M \cap \sigma(A)$ and $x \in M$. Thus $x \in Rad_\sigma(A)$. Since $x = \sigma(x)$, $x \in \sigma(Rad_\sigma(A))$. □

Proposition 4.35. *Let (A, σ) be a state triangle algebra. Then (A, σ) is semisimple and σ is faithful iff (A, σ) is semisimple relative to $[A)_\sigma$.*

Proof. By Proposition 4.34, $\sigma(Rad_\sigma(A)) = Rad(\sigma(A)) = \{1\}$. So $Rad_\sigma(A) \subseteq ker(\sigma) = \{1\}$ and $Rad_\sigma(A) = \{1\}$.

Conversely, $Rad(\sigma(A)) = \sigma(Rad_\sigma(A)) = \sigma(\{1\}) = \{1\}$, so (A, σ) is semisimple. Let $x \in A$. Then $\sigma(x) = 1$. If $x \neq 1$, then $x \notin Rad_\sigma(A)$. there is an IVRL-extended maximal σ-filter M so that $x \notin M$. By Theorem 4.12, there are $f \in M$ and $m, n \geq 1$ such that $(f * \sigma(\nu x)^m)^n = 0$. Therefore $f^n = 0$ and $0 \in F$, a contradiction. So $\nu x = 1$, $x = 1$ and σ is faithful. □

Definition 4.36. *A state triangle algebra (A, σ) is called local if $\sigma(A)$ is local. A state triangle algebra (A, σ) is called local relative to $[A)_\sigma$ if it has only one IVRL-extended maximal σ-filter.*

Example 4.37. *In Example 4.18, (A, σ) is local.*

Theorem 4.38. *Let (A, σ) be a state triangle algebra. (A, σ) is local relative to $[A)_\sigma$ iff (A, σ) is local.*

Proof. Let F be the only IVRL-extended maximal σ-filter of (A, σ). We must show that $\sigma(F)$ is the only IVRL-extended maximal filter of $\sigma(A)$. If $\sigma(F) = \sigma(A)$, then $0 \in \sigma(F)$ and $0 \in F$, a contradiction. So $\sigma(F)$ is a proper IVRL-filter of $\sigma(A)$. Let G be an IVRL-filter of $\sigma(A), G \neq \sigma(A)$ and let $x \in G$. By Proposition 4.13(i), we have that $\neg\sigma(G)$ is an IVRL-σ-filter of (A, σ). Let $\neg\sigma(G) = A$. Then $0 \in \neg\sigma(G)$ and $0 \in G$, a contradiction. So $\neg\sigma(G)$ is a proper σ-filter of (A, σ). Thus $\neg\sigma(G) \subseteq F$. Since $\nu x = \sigma(\nu x) \in G$, $\nu x \in \neg\sigma(G)$ so $\nu x \in F$. We know that $x = \sigma(x)$, so $\nu x \in \sigma(F)$. Thus $G \subseteq \sigma(F)$ and so (A, σ) is local.

Conversely, let G be the only IVRL-extended maximal filter of $\sigma(A)$. By Proposition 4.13(i), $\neg\sigma(G)$ is an IVRL-extended maximal σ-filter of (A, σ). We prove that $\neg\sigma(G)$ is only IVRL-extended maximal σ-filter. Let $F \in [A)_\sigma, F \neq A$. Then $F \cap \sigma(A) = \sigma(F)$ is a proper IVRL-filter of $\sigma(A)$. Thus $F \cap \sigma(A) \subseteq G$. If $x \in F$, then $\sigma(\nu x) \in F \cap \sigma(A) \subseteq G$. So $\nu x \in \neg\sigma(G)$ and $F \subseteq \neg\sigma(G)$. □

Proposition 4.39. *Let (A, σ) be a state triangle algebra and F be a proper σ-IVRL-filter of (A, σ). Then $F \subseteq M_\sigma(A) := \{x \in A \mid ord(\sigma(\nu x)) = \infty\}$.*

Proof. Let $x \in F$. Then $(x * \sigma(\nu x))^n \in F$, for every $n \in \mathbb{N}$. If $x \notin M_\sigma(A)$, then there is $m \geq 1$ such that $(\sigma(\nu x))^m = 0$. We have $(x*\sigma(\nu x))^m \leq (\sigma(\nu x))^m$, so $(x*\sigma(\nu x))^m = 0$. Thus $0 \in F$, which is a contradiction. Therefore $F \subseteq M_\sigma(A)$. □

Theorem 4.40. *Let (A, σ) be a state triangle algebra. Then (A, σ) is local iff $M_\sigma(A)$ is a σ-IVRL-filter of (A, σ). In this case $M_\sigma(A)$ is the only IVRL-extended maximal σ-filter of (A, σ).*

Proof. Let (A, σ) be local. Then (A, σ) has only one IVRL-extended maximal σ-filter F. By Proposition 4.39, $F \subseteq M_\sigma(A)$. If $x \in M_\sigma(A)$ and $x \notin F$, then $[x)_\sigma \subsetneq F$. Since F is the only IVRL-extended maximal filter, $[x)_\sigma = A$ so $0 \in [x)_\sigma$ thus there is $n \geq 1$ such that $\sigma((x * \sigma(\nu x))^n) \geq (\sigma(x * \sigma(\nu x)))^n \geq (\sigma(x * \sigma(\nu x)))^n \geq (\sigma(x) * \sigma(\sigma(\nu x)))^n \geq \sigma^{2n}(\nu x)$. Thus $\sigma^{2n}(\nu x) = 0$, that is $x \notin M_\sigma(A)$, which is a contradiction. So $M_\sigma(A) \subseteq F$ and $M_\sigma(A) = F$. Therefore $M_\sigma(A)$ is an IVRL-σ-filter of (A, σ).

Conversely, let $M_\sigma(A)$ be an IVRL-σ-filter of (A, σ) and F be a proper IVRL-σ-filter of (A, σ). By Proposition 4.39, $F \subseteq M_\sigma(A)$, so $M_\sigma(A)$ is the only IVRL-extended maximal σ-filter of (A, σ). Hence (A, σ) is local relative to $[A)_\sigma$ and so (A, σ) is local. □

5 Conclusions

In this paper, motivated by the previous research of triangle algebras, we extended the concept of state triangle algebras. We introduce and study these algebras and their state IVRL-filters. Using the notion of state IVRL-filters, we characterize kinds of state IVRL-filters and focus on these and the relations between these IVRL-filters were determined. As an application of state triangle algebras, we show the relation between these algebras and some special sets of triangle algebras such as radical of an IVRL-filter and local triangle algebras and etc. We proved that (A, σ) is local relative to $[A)_\sigma$ iff (A, σ) is local.

The investigation of other such generalizations can be an interesting object for further work.

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The role of states in triangle algebras

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نقش حالت‌ها در جبرهای مثلثی

چکیده. در این مقاله، جبرهای مثلثی را با اضافه کردن عملگر یکتایی حالت، توسیع می‌دهیم. این ساختارهای جبری را جبرهای مثلثی حالت نامیم و برخی ویژگی‌های اساسی این جبرها را ارائه می‌دهیم. سپس مفهوم فیلترهای مشبکه مانده بازه‌ای مقدار (IVRL) حالت را معرفی کرده و چند مثال و ویژگی هایی از آن‌ها را مورد بررسی قرار می‌دهیم. در پایان، با استفاده از این مفهوم، دو نوع σ- فیلتر توسیع- یافته IVRL از جبرهای مثلثی حالت را معرفی می‌کنیم.