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Uncertainty analysis of hierarchical granular structures for multi-granulation typical hesitant fuzzy approximation space

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Abstract

Hierarchical structures and uncertainty measures are two main aspects in granular computing, approximate reasoning and cognitive process. Typical hesitant fuzzy sets, as a prime extension of fuzzy sets, are more flexible to reflect the hesitance and ambiguity in knowledge representation and decision making. In this paper, we mainly investigate the hierarchical structures and uncertainty measures in typical hesitant fuzzy backgrounds. Firstly, we propose the parameterized scalar cardinalities of typical hesitant fuzzy elements, typical hesitant fuzzy sets and typical hesitant fuzzy relations based on a more reasonable partial orders with a disjunctive semantic meaning, respectively, where the parameters represent the decision makers' risk preferences. Secondly, we present four ordered relations for typical hesitant fuzzy space and four uncertainty measures to characterize the ambiguity in typical hesitant fuzzy approximation space and discuss their relationships. Thirdly, the hierarchical structures of a multi-granulation typical hesitant fuzzy space are analyzed by various multi-granulation typical hesitant fuzzy knowledge bases. In addition, we construct the framework of multi-granulation typical hesitant fuzzy rough sets in terms of optimistic and pessimistic attitudes. Finally, we study the uncertainty measures for the multi-granulation typical hesitant fuzzy approximation space based on the maximal and minimal knowledge bases, respectively.

Keywords: Typical hesitant fuzzy set, partial order, hierarchical structure, multi-granulation typical hesitant fuzzy approximation space, uncertainty measure.

1 Introduction

Granular computing (GrC), since presented by Zadeh [52, 53], has attracted much attention and been successfully applied in many fields such as fuzzy information systems and approximation space [3, 5, 11, 9, 8, 24, 40, 48], time series forecasting stock market [2, 5], fuzzy clustering [20, 21], cognitive computing [6, 14], and social networks-model [13]. Grc aims at finding some appropriate information granules, which can approximate the given concept to a certain extent. In the theory of GrC, an information granule, as the primary unit, is a collection of objects drawn together by similarity, indistinguishability and proximity of functionality [3, 24, 47]. Chen et al. [3] proposed the concept of granular fuzzy sets based on fuzzy similarity relations and then investigated the granular structures in the framework of GrC. Pedrycz et al. [23] proposed a characterization of numerical data by a collection of information granules to reveal the key structure of data, the topology and essential relationships in terms of information granules. Pedrycz and Homenda [22] presented and discussed a principle of justifiable granularity to support a coherent way of designing information granules in presence of experimental evidence.

The collection of granules induced by the information granulation of all the objects in the universe called the granular structure, in which all the objects are classified into a family of pairwise or overlapping granules. To reveal the interrelationships between granular structures, several hierarchical structures have been presented in the study of approximation spaces. Yao [48] studied hierarchical granulation and approximation structures for the stratified

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rough set approximations. Qian et al. [26, 25] defined four operations on a knowledge and uniformly expressed the finer or coarser relationships between the original and the generated granulation spaces. Xu et al. [44, 43] presented multi-granulation fuzzy rough sets in a fuzzy tolerance approximation space and generalized multi-granulation double-quantitative decision-theoretic rough set. Yang et al. [45] further proposed three various hierarchical structures to compare two different multi-granulation spaces from the viewpoints of partition and covering, respectively. Zhang and Yang [56] presented a new mechanism to make three-way group decisions with interval-valued decision-theoretic rough sets by calculating inclusion measures between two arbitrary interval-valued expected loss functions and the principle of justifiable granularity. Kong et al. [12] discussed three types of attribute reduction, including arbitrary union reduction, neighborhood union reduction and neighborhood intersection reduction. Zhang et al. [58] presented local multi-granulation decision-theoretic rough set in ordered information systems.

Information granularity is the average measure of information (knowledge) granules of a granular structure. The smaller the information granularity of a granular structure, the stronger its discernibility becomes. Uncertainty measure dominated by information granularity, including Shannon information entropy, information entropy and rough entropy, depicts the uncertainty of the actual structure of a granular structure efficiently [16, 25, 36, 35, 37, 39]. Shannon information entropy presented by Shannon [32], which measures the uncertainty of the actual structure, is a valuable mechanism for characterizing information contents in various models and applications in many diverse fields. Rough entropy, another new uncertainty measure in rough set theory, is employed to calculate the roughness degree of a fuzzy binary granular structure. Furthermore, the relationships among information entropy, rough entropy and knowledge granulation in complete information systems have been established by Liang et al. [15, 16]. Yu et al. [49] discussed uncertainty measures in multi-granulation with different graded rough set based on dominance relation.

Hesitant fuzzy sets (HFSs) [34], as a generalization of the classical fuzzy sets [50] and interval-valued fuzzy sets [51], are a fundamental and general tool to deal with hesitant and ambiguous problems. Most of the study [1, 4, 7, 31, 30, 42, 38, 46, 55] about HFSs are focused on the hypothesis that the hesitant fuzzy membership degree (HFE) is a finite and nonempty subsets of $[0, 1]$, which are called as typical hesitant fuzzy elements (THFEs) [1] and the relative HFSs are called as typical hesitant fuzzy sets (THFSs). Some basic operators [1, 42], information measures [4, 7, 30, 42], applications of HFSs to group decision making [31, 38, 42], and knowledge discovery with rough set theory [46, 55] have been studied in detail. Liang et al. [17] extended the range of applications of DTRSs to hesitant fuzzy information systems by considering the new evaluation format of hesitant fuzzy sets. Zhang et al. [54] developed the notion of multi-granulation decision-theoretic rough sets into the hesitant fuzzy linguistic background within the two-universe framework. Zhang et al. [57] presented multi-granulation hesitant fuzzy rough sets and corresponding applications via the extended definition [42] in the discussion of comparing two THFEs. Although there are magnificent discussions and conclusions about the knowledge acquisition and decision making in various hesitant approximation spaces, they have been studied via the extended definition [42] in the discussion of comparing two THFEs. So in this paper, under the framework of no extension corresponding to the discussion of comparing two THFEs, three vital suspending issues about the basic theory of THF approximation spaces will be discussed, including constructing the THF information granulation, investigating the hierarchy of information granular structure and measuring the degree of the information granularity. We will investigate the hierarchical structures and uncertainty measures of typical hesitant fuzzy approximation spaces on the basis of a scalar cardinality of THFSs.

The paper is organized as follows. In Section 2, we review some basic knowledge about THFEs and THFSs. We propose the scalar cardinalities of THFEs and THFSs in Section 3. The granulation and hierarchical structures of THF approximation space are presented in Section 4. Furthermore, four ordered relations will also be presented in THF approximation space. In what follows, we investigate the uncertainty measures for THF approximation space in Section 5. In Section 6, we investigate the hierarchical structures of the multi-granulation THF approximation space and the uncertainty measures for multi-granulation THF approximation space will be presented in Section 7. Finally, we end the paper with the conclusion in Section 8.

2 Preliminaries

In this section, we review some concepts and operations under typical hesitant fuzzy environment.

2.1 Typical hesitant fuzzy sets

THFSs are presented by Torra and Xu [34, 42] to deal with hesitant and incongruous problems.

Definition 2.1. [34] ***(THFEs)*** *Let U be a nonempty and finite universe of discourse, a typical hesitant fuzzy set on U is a function that when applied to U returns a finite and nonempty subset of $[0, 1]$, which can be presented as*

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follows, $A = \{\langle x, h_A(x)\rangle | x \in U\}$, *where* $h_A(x)$ *is called as typical hesitant fuzzy element (THFE for short) denoting the possible membership degrees of the element* $x \in U$ *to the THFS A and the number of elements in* $h_A(x)$ *is referred to be* $l(h_A(x))$. *The set of all the THFEs is denoted as* $\mathbb{H}$. *The set of all THFSs on U is denoted by THF(U). For all* $x \in U, a_i \in [0,\ 1]$, *and* $i = 1, 2, \cdots, m, h_A(x) = \{a_1, a_2, \cdots, a_m\}$ *denotes a constant THFS which notated by* $\widehat{a_{1\cdots m}}$ *and* $a_{1\cdots m} = h_{\widehat{a_{1\cdots m}}}(x) = \{a_1, a_2, \cdots, a_m\}$. *Particularly, for all* $x \in U$, $A = \emptyset$ *if* $h_A(x) = \{0\}$ *and* $A = U$ *if* $h_A(x) = \{1\}$. *Obviously, a constant THFS* $\widehat{a_{1\cdots m}}$ *will degenerate into a classical constant fuzzy set when* $m = 1$.

For simplicity, a THFE h will be noted by an increasing order: $h = \{h(1),\ h(2),\ \cdots,\ h(l)\}$ and $h(1) \le h(2) \le \cdots \le h(l)$. Let h be a THFE, the lowerbound and upperbound are defined as follows [34]:

- Lowerbound: $h^-(x) = \min h(x)$,
- Upperbound: $h^+(x) = \max h(x)$.

To propose partially ordered relations, unions and intersections for $\mathbb{H}$ and THFSs, we first define two operators of β-normalization and γ-normalization of h_1 with respect to h_2 as follows:

Let $\beta : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ be a function defined by:

$$\beta(h_1, h_2) = \begin{cases} h_1, & \text{if } l(h_1) \le l(h_2), \\ \{h_1(i) : i \in \{l(h_1) - l(h_2) + 1, \cdots, l(h_1)\}\}, & \text{otherwise.} \end{cases}$$

And $\gamma : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ be a function defined by, $\gamma(h_1, h_2) = \big\{h_1(i) : i \in \{1, \cdots, \min(l(h_1), l(h_2))\}\big\}$.

Note. For two THFEs $h_1, h_2 \in \mathbb{H}^{(m)} = \{h \in \mathbb{H},\ l(h) = m\}$, we define

$$h_1 \le_{(m)} h_2 \Longleftrightarrow h_1(i) \le h_2(i), \forall i = 1, \cdots, m.$$

Zhang and Yang [55] proposed the following partial order from the point of disjunctive semantic for THFSs and further defined the operators of intersection and union for THFS and THFEs.

Definition 2.2. [55] ***(Patial order, union, intersection and complement)*** *Let* $h,\ h_1,\ h_2$ *be two THFEs, then*

(1) $h_1 \le_h h_2$ *iff* $\begin{cases} h_1 \le_{(l_1)} \gamma(h_2, h_1), & \text{if } l_1 \le l_2, \\ \beta(h_1, h_2) \le_{(l_2)} h_2, & \text{else.} \end{cases}$

(2) $h_1 \cap_h h_2 = \begin{cases} h_2, & \text{if } l_1 \le l_2, \beta(h_2,\ h_1) \le_{(l_1)} h_1, \\ \{h_1(i) \wedge \gamma(h_2,\ h_1)(i),\ i = 1, \cdots, l_1\}, & \text{if } l_1 \le l_2, \beta(h_2,\ h_1) \nleq_{(l_1)} h_1, \\ h_2 \cap_h h_1, & \text{otherwise.} \end{cases}$

(3) $h_1 \cup_h h_2 = \begin{cases} h_2, & \text{if } l_1 \le l_2, h_1 \le_{(l_1)} \gamma(h_2,\ h_1), \\ \Big\{h_1(i) \vee \beta(h_2,\ h_1)(i),\ i = 1, \cdots, l_1\Big\}, & \text{if } l_1 \le l_2, h_1 \nleq_{(l_1)} \gamma(h_2,\ h_1), \\ h_2 \cup_h h_1, & \text{otherwise.} \end{cases}$

(4) $h^c = \Big\{1 - h(l - i + 1)\Big\}$

Zhang and Yang [55] have proved $\le_h$ is a partial order which satisfies reflexivity, antisymmetry and transitivity. While $\cap_h$ and $\cup_h$ satisfy the following fundamental properties of the sets theory except for the distributivity and associativity.

Proposition 2.3. [55] *For all* $h_1, h_2 \in \mathbb{H}$, $\cap_h$ *and* $\cup_h$ *satisfy the following properties*

1. *Idempotent:* $h_1 \cap_h h_1 = h_1, \quad h_1 \cup_h h_1 = h_1,$
2. *Commutativity:* $h_1 \cap_h h_2 = h_2 \cap_h h_1, \quad h_1 \cup_h h_2 = h_2 \cup_h h_1,$
3. *Absorption law:* $h_1 \cap_h (h_1 \cup_h h_2) = h_1, \quad h_1 \cup_h (h_1 \cap_h h_2) = h_1,$
4. *De'Morgan law:* $h_1 \cap_h h_2 = (h_1^c \cup_h h_2^c)^c, \quad h_1 \cup_h h_2 = (h_1^c \cap_h h_2^c)^c,$
5. $h_1 \cap_h h_2 \le_h h_1 \wedge h_2 \le_h h_1 \cup_h h_2,$
6. $h_1 \le_h h_2 \Longleftrightarrow h_1 \cap_h h_2 = h_1 \Longleftrightarrow h_1 \cup_h h_2 = h_2,$
7. *Double negation law:* $(h_1^c)^c = h_1.$

Theorem 2.4. [55] *The* $\le_h$ *defined in Definition 2.2 is a partial order between THFEs* h_1 *and* h_2.

Some operators between THFSs can be defined via those on THFEs.

Definition 2.5. [55] *Let* $A,\ B \in$ *THFS(U), then*

(1) $A \le_H B \Longleftrightarrow h_A(x) \le_h h_B(x),\ \forall x \in U,\ \ A <_H B \Longleftrightarrow h_A(x) <_h h_B(x),\ \forall x \in U,$
(2) *the intersection of A and B is denoted by* $A \cap_H B$, $h_{A\cap_H B}(x) = h_A(x) \cap_h h_B(x),\ \forall x \in U,$
(3) *the union of A and B is denoted by* $A \cup_H B$ *such that* $\forall x \in U$, $h_{A\cup_H B}(x) = h_A(x) \cup_h h_B(x),\ \forall x \in U,$
(4) $A - B = A \cap_H B^c$, *where* $B^c = \{\langle x, h_B^c(x)\rangle,\ \forall x \in U\}$.

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Definition 2.6. ***(THFR)*** *Let $U=\{x_1, x_2, \cdots, x_n\}$ be the universe of discourse, a typical hesitant fuzzy relation (THFR) R on U can be denoted by the following matrix:*

$$M(R) = \begin{bmatrix} h_R(x_1,x_1) & h_R(x_1,x_2) & \cdots & h_R(x_1,x_n) \\ h_R(x_2,x_1) & h_R(x_2,x_2) & \cdots & h_R(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_R(x_n,x_1) & h_R(x_n,x_2) & \cdots & h_R(x_n,x_n) \end{bmatrix}$$

where $h_R(x_i, x_j) \subseteq [0,1]$ $(1 \leq i,\ j \leq n)$ is the hesitant fuzzy similarity degree of x_i similar to x_j. For all $x \in U$, $R(x)$ is a THFS such that $h_R(x)(y) = h_R(x,y),\ \forall y \in U$.

The collection of all THFRs on U is denoted by THF($U\times U$)called the THF approximation space on U.

For any $R_1, R_2 \in$ THF($U\times U$), the partial order between them is defined as follows:

$R_1 \leq_R R_2 \Leftrightarrow h_{R_1}(x_i,x_j) \leq_h h_{R_2}(x_i,x_j),\ \forall x_i, x_j \in U$.

Definition 2.7. *Let $R_1, R_2 \in$ THF($U\times U$) be two THFRs, the union and intersection between them are defined as follows:*

(1) $R = R_1 \cup_R R_2 \Leftrightarrow h_R(x_i,x_j) = h_{R_1}(x_i,x_j) \cup_h h_{R_2}(x_i,x_j),\ \forall x_i, x_j \in U,$

(2) $R = R_1 \cap_R R_2 \Leftrightarrow h_R(x_i,x_j) = h_{R_1}(x_i,x_j) \cap_h h_{R_2}(x_i,x_j),\ \forall x_i, x_j \in U.$

3 The parameterized scalar cardinalities of typical hesitant fuzzy sets

The cardinality for fuzzy sets was initiated by De Luca and Termini [19] in terms of σ- count. Raleseu [29] first proposed two concepts of fuzzy cardinality: one of which gives the answer as a fuzzy set (or fuzzy number) and the other one gives an ordinary integer, furthermore, the interrelationships and properties between the two concepts are investigated. Wygralak proposed the notation of scalar cardinality for fuzzy sets from an axiomatic point of view [41]. The theory constitutes not only a powerful basis but also a useful tool for modelling and processing vague and imprecise quantitative information. Recently, Quirós et al. defined an axiomatic scalar cardinality of finite interval-valued hesitant fuzzy sets [28]. In this section, we will propose the parameterized scalar cardinalities of THFEs, THFSs and THFRs where the parameters reflect the decision makers' risk preference, respectively.

Definition 3.1. [41] *Let $U=\{x_1, x_2, \cdots, x_n\}$ be a reference set, $\forall A,\ B \in \mathrm{FS(U)},\ x, y \in \mathrm{U}$ and $a, b \in [0,1]$, the mapping $|\cdot| : \mathrm{FS(U)} \to [0,\infty)$ is a scalar cardinality measure for fuzzy sets if it satisfies the following properties:*

(1) *(coincidence)* $|\frac{1}{x}| = 1,$

(2) *(coincidence)* $a \leq b \Rightarrow |\frac{a}{x}| \leq |\frac{b}{y}|,$

(3) *(additivity)* $|A \cup B| = |A| + |B|$ *if* $A \cap B = \emptyset$ *where* $A \cup B(x) = \max\{A(x), B(x)\}, A \cap B(x) = \min\{A(x), B(x)\}, \forall x \in U.$

In terms of the decision making problems with THFSs, Xu and Xia [42] pointed out that the selection principle mainly relies on the decision makers' risk preferences. To the pessimistic principle, the smallest element of the THFE will be chosen, while the maximal value of THFE will be selected by the optimists. Due to decision makers' different risk preferences, the decision results may be different. Thus for $h = \{h(1), \cdots, h(l)\}$, in which the values of the elements are arranged by increasing order, i.e., $h^- = h(1)$ and $h^+ = h(l)$. A parameter η $(\eta \in [0,1])$ is added to evaluate the capability of the risk tolerance, which satisfies $\overline{h} = \eta h^+ + (1-\eta)h^-$. Obviously, the decision maker is pessimistic when $\eta = 0$. Whereas the optimistic when $\eta = 1$ and if $\eta = \frac{1}{2}$, the decision maker is neutral.

We propose the parameterized scalar cardinality measure for THFEs as follows:

Definition 3.2. *(Parameterized scalar cardinality of THFEs) Let $h_1, h_2 \in \mathbb{H}$ and $\eta \in [0,1]$, the mapping $|\cdot|_\eta : \mathbb{H} \to [0,\infty)$ is a parameterized scalar cardinality measure for THFEs if it satisfies the following properties:*

SCE1. *(coincidence)* $|\{1\}|_\eta = 1,$

SCE2. *(monotonicity)* $h_1 \leq_h h_2 \Rightarrow |h_1|_\eta \leq |h_2|_\eta,$

SCE3. *(additivity)* $|h_1 \cup_h h_2|_\eta = |h_1|_\eta + |h_2|_\eta$ *if* $h_1 = \{0\}$ *or* $h_2 = \{0\}.$

Definition 3.3. *Let $h \in \mathbb{H}$ and $\eta \in [0,1]$, we define $|h|_\eta$ as follows, $|h|_\eta = \eta h^+ + (1-\eta)h^-$.*

Theorem 3.4. *Let $h \in \mathbb{H}$ and $\eta \in [0,1]$, then $|h|_\eta$ defined in Definition 3.3 is a parametrized scalar cardinality of THFEs.*

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Proof. For any $h \in \mathbb{H}$ and $\eta \in [0,1]$, we need to prove $|h|_\eta$ satisfies three axiom definitions in Definition 3.2

SCE1. $h = \{1\} \Rightarrow h^- = h^+ = \{1\} \Rightarrow |h|_\eta = \eta h^+(1-\eta)h^- = 1$.

SCE2. If $h_1 \leq_h h_2$, we conclude $h_1^- \leq h_2^-$ and $h_1^+ \leq h_2^+$, thus for $\forall \eta \in [0,1]$, $|h_1|_\eta = \eta h_1^+ + (1-\eta)h_1^- \leq \eta h_2^+ + (1-\eta)h_2^- = |h_2|_\eta$.

SCE3. It is obvious. □

A single-valued special THFS is defined as follows:

Definition 3.5. *Let $a \in \mathbb{H}$ be a THFE, then a special THFS a/x, where $x \in \mathrm{U}$, is given by the THFE $h_{a/x}$ as follows:*

$$h_{a/x}(y) = \begin{cases} a, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

Remark 3.6. *If $a \in [0,1]$, then $\forall x \in \mathrm{U}$, the $a/x \in \mathrm{THFS(U)}$ degenerates to one type of FSs.*

Definition 3.7. *(Parametrized scalar cardinality of THFSs) Let $A, B \in \mathrm{THFS(U)}$, then for all $x_i, x_j \in \mathrm{U}, h_1, h_2 \in \mathbb{H}$, the mapping $|\cdot|_\eta : \mathrm{THFS(U)} \to [0, \infty)$ is a parametrized scalar cardinality measure for THFSs if it satisfies the following properties:*

SCS1. *(coincidence)* $|\frac{\{1\}}{x}|_\eta = 1$,

SCS2. *(monotonicity)* $h_1 \leq_h h_2 \Rightarrow |\frac{h_1}{x_i}|_\eta \leq |\frac{h_2}{x_j}|_\eta$,

SCS3. *(additivity)* $|A \cup_H B|_\eta = |A|_\eta + |B|_\eta$ *if* $A = \emptyset$ *or* $B = \emptyset$.

Definition 3.8. *Let $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in U\} \in \mathrm{THFS(U)}$. For $\forall x_i \in \mathrm{U}, \eta \in [0,1]$, $|A|_\eta$ is defined as $|A|_\eta = \sum\limits_{i=1}^{n} |h_A(x_i)|_\eta$.*

Theorem 3.9. *Let $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in U\} \in \mathrm{THFS(U)}$. For all $\eta \in [0,1]$, $|A|_\eta$ in Definition 3.8 is a parametrized scalar cardinality of A.*

Proof. We need to prove $|A|_\eta$ satisfies the three axioms in Definition 3.7.

SC1. We conclude from $A = \frac{\{1\}}{x}$ that there exists $x_i \in U$, such that $h_A(x_i) = \{1\}$ and $h_A(x_j) = \{0\}, \forall x_j \neq x_i$. Then $h_A^-(x_i) = h_A^+(x_i) = 1 \Rightarrow |h_A(x_i)|_\eta = \eta \times 1 + (1-\eta) \times 1 = 1$, thus $|A|_\eta = \sum\limits_{j=1}^{n} |h_A(x_j)|_\eta = |h_A(x_i)|_\eta + \sum\limits_{j \neq i} |h_A(x_j)|_\eta = 1$.

SC2. If $h_1 \leq_h h_2$, from Definition 2.2, we conclude $h_1^- \leq h_2^-$ and $h_1^+ \leq h_2^+$, thus for $\forall \eta \in [0,1]$, $|h_1|_\eta = \eta h_1^+ + (1-\eta)h_1^- \leq \eta h_2^+ + (1-\eta)h_2^- = |h_2|_\eta$. $\forall x_i, x_j \in \mathrm{U}$, we have $|\frac{h_1}{x_i}|_\eta = \sum\limits_{j=1}^{n} |h_A(x_j)|_\eta = |h_1|_\eta + \sum\limits_{j \neq i} |h_A(x_j)|_\eta = |h_1|_\eta \leq |h_2|_\eta = |\frac{h_2}{x_j}|_\eta$.

SC3. It is obvious.

□

Remark 3.10. *We conclude from above that η reflects the decision makers' risk preference. $\forall A = \{\langle x_i, h_A(x_i)\rangle | x_i \in \mathrm{U}\}$, the minimal scalar cardinality of A will be obtained by letting $\eta = 0$ and the maximal scalar cardinality of A will be gotten when $\eta = 1$, analogously, the decision maker will be regard as a neutral one if $\eta = \frac{1}{2}$. And we have*

$$|A|_0 = \sum_{i=1}^{n} |h_A(x_i)|_0 = \sum_{i=1}^{n} h_A^-(x_i),$$
$$|A|_1 = \sum_{i=1}^{n} |h_A(x_i)|_1 = \sum_{i=1}^{n} h_A^+(x_i),$$
$$|A|_{\frac{1}{2}} = \sum_{i=1}^{n} |h_A(x_i)|_{\frac{1}{2}} = \sum_{i=1}^{n} \frac{h_A^-(x_i) + h_A^+(x_i)}{2}.$$

Furthermore, we define the parametrized scalar cardinality of THFRS.

Definition 3.11. *(Parametrized scalar cardinality of THFRs) Let R be a THFR on $U = \{x_1, \cdots, x_n\}$ where $S_R(x_i) = \{\frac{h_R(x_i,x_j)}{x_j} | 1 \leq j \leq n\}$, $1 \leq i \leq n$. The mapping $|\cdot| : \mathrm{THF(U \times U)} \to [0, \infty)$ is a parametrized scalar cardinality measure for THFRs if it satisfies the following properties:*

SCR1. *(coincidence)* $|\frac{\{1\}}{x}|_\eta = 1$,

SCR2. *(monotonicity)* $R_1 \leq_R R_2 \Rightarrow |\frac{h_{R_1}(x_m,x_n)}{x_i}|_\eta \leq |\frac{h_{R_2}(x_m,x_n)}{x_j}|_\eta, x_m, x_n \in U$,

SCR3. *(additivity)* $|R_1 \cup_R R_2|_\eta = |R_1|_\eta + |R_2|_\eta$ *if* $h_{R_1}(x_i) = \emptyset$ *or* $h_{R_2}(x_i) = \emptyset$.

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Definition 3.12. *Let R be a THFR on* $\mathrm{U} = \{x_1, \cdots, x_n\}$ *where* $S_R(x_i) = \{\frac{h_R(x_i,x_j)}{x_j} | 1 \le j \le n\},\ 1 \le i \le n$. *The parametrized scalar cardinalities of R can be defined as follows:*

$$|R|_\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_R(x_i,x_j)|_\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(\eta h_R^+(x_i,x_j) + (1-\eta)h_R^-(x_i,x_j)\Big).$$

Theorem 3.13. *Let R be a THFR on* $\mathrm{U} = \{x_1, \cdots, x_n\}$. *For all* $\eta \in [0,1]$, $|R|_\eta$ *in Definition 3.12 is a parametrized scalar cardinality of R.*

Proof. It can be proved similarly to the proofs of Theorems 3.4 and 3.9.

□

Remark 3.14. *With respect to decision makers' risk preference, we can conclude from Definition 3.12. that the maximal, minimal and neutral parametrized scalar cardinality can be derived as follows:*

$$|R|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_R(x_i,x_j)|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} h_R^+(x_i,x_j),$$
$$|R|_0 = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_R(x_i,x_j)|_0 = \sum_{i=1}^{n}\sum_{j=1}^{n} h_R^-(x_i,x_j),$$
$$|R|_{\frac{1}{2}} = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_R(x_i,x_j)|_{\frac{1}{2}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{h_R^+(x_i,x_j)+h_R^-(x_i,x_j)}{2},$$

for all $x_i, x_j \in \mathrm{U}$ *and* $R \in$ *THF(U× U).*

4 Granulation and hierarchical structures of the typical hesitant fuzzy approximation space

In this section, we will present the hierarchical structures of typical hesitant fuzzy approximation space by four ordered relations and further discuss the relative properties.

4.1 Typical hesitant fuzzy granulation

Definition 4.1. ***(THF granular structure)*** *For any* $R \in$ *THF(U× U), a THF binary granular structure determined by it is defined as*

$$K(R) = \big(R(x_1), R(x_2), \cdots, R(x_n)\big)$$

where $R(x_i) = \{\frac{h_R(x_i,x_j)}{x_j} | 1 \le j \le n\}$ *is called the THF information granule by* x_i $(1 \le i \le n)$ *and R.*

The collection of all THF granular structures on U is denoted by $\mathfrak{K}(U)$.

In what follows, we denote $\cup_P$ and $\cap_P$ to represent the union and intersection between typical hesitant fuzzy granular structures and we denote the union and intersection between two typical hesitant fuzzy relations by $\cup_R$ and $\cap_R$, respectively.

Definition 4.2. *Let* $K(P) = (P(x_1),\ P(x_2),\ \cdots,\ P(x_n)\ ),\ K(Q) = (Q(x_1),\ Q(x_2),\ \cdots, Q(x_n)\ ) \in \mathfrak{K}(U)$ *where* $P, Q \in$ *THF(U× U). Four operators (i.e., intersection* $\cap_P$*, union* $\cup_P$*, munis* $-$ *and complement c) on* $\mathfrak{K}(U)$*, can be defined as follows:*

$$K(P) \cap_P K(Q) = \{P \cap_R Q(x_i) | P \cap_R Q(x_i) = P(x_i) \cap_H Q(x_i), 1 \le i \le n\},$$
$$K(P) \cup_P K(Q) = \{P \cup_R Q(x_i) | P \cup_R Q = P(x_i) \cup_H Q(x_i), 1 \le i \le n\},$$
$$K(P) - K(Q) = \{P - Q(x_i) | P - Q(x_i) = P(x_i) \cap_H Q^c(x_i), 1 \le i \le n\},$$
$$K^c(P) = \{P^c(x_i) | 1 \le i \le n\},$$

where

$$P(x_i) \cap_H Q(x_i) = \left\{\frac{h_P(x_i,x_j) \cap_h h_Q(x_i,x_j)}{x_j} | 1 \le j \le n\right\},\ 1 \le i \le n,$$
$$P(x_i) \cup_H Q(x_i) = \left\{\frac{h_P(x_i,x_j) \cup_h h_Q(x_i,x_j)}{x_j} | 1 \le j \le n\right\},\ (1 \le i \le n),$$
$$P^c(x_i) = \left\{\frac{\{1\}}{x_i}, \frac{h_P^c(x_i,x_j)}{x_j} | 1 \le j \le n, j \ne i\right\},\ 1 \le i \le n.$$

It can be proved that $\cap_P, \cup_P$ *and c possess the following properties derived directly from Proposition 2.3.*

Proposition 4.3. *Let* $K(I) = (S_I(x_1), S_I(x_2), \cdots, S_I(x_n)\ )$ *and* $S_I(x_i) = \{\frac{\{1\}}{x_j} | 1 \le j \le n\}$*, then for any* $K(P), K(Q) \in \mathfrak{K}(U)$*the intersection* $\cap_P$*, union* $\cup_P$ *and complement c on* $\mathfrak{K}(U)$ *satisfy the following properties respectively*

(1) *identity law:* $K(P) \cap_P K(I) = K(P)$,

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(2) *zero law:* $K(P) \cup_P K(I) = K(I)$,
(3) *commutativity:* $K(P) \cap_P K(Q) = K(Q) \cap_P K(P)$, $K(P) \cup_P K(Q) = K(Q) \cup_P K(P)$,
(4) *double negation law:* $(K^c(P))^c = K(P)$,
(5) *de'Morgan law:* $K(P) \cap_P K(Q) = (K^c(P) \cup_P K^c(Q))^c$, $K(P) \cup_P K(Q) = (K^c(P) \cap_P K^c(Q))^c$,
(6) *absorption law:* $K(P) \cap_P (K(P) \cup_P K(Q)) = K(P)$, $K(P) \cup_P (K(P) \cap_P K(Q)) = K(P)$.

4.2 Four order relations in the typical hesitant fuzzy approximation space

Definition 4.4. *Let* $K(R) = \big(R(x_1), R(x_2), \cdots, R(x_n)\big) \in \mathfrak{K}(U)$*. The parametrized scalar cardinalities of* $K(R)$ *can be defined by:* $|K(R)|_\eta = \sum_{i=1}^{n} |R(x_i)|_\eta$.

Remark 4.5. *With respect to decision makers' risk preference, we conclude from Definition 4.4 that the maximal, minimal and neutral THF information granule derived from* $x_i \in \mathrm{U}$ *and THFR* R *can be calculated as follows:*

$$|K(R)|_1 = \sum_{i=1}^{n} |R(x_i)|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_R(x_i,x_j)|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} h_R^+(x_i,x_j),$$

$$|K(R)|_0 = \sum_{i=1}^{n} |R(x_i)|_0 = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_R(x_i,x_j)|_0 = \sum_{i=1}^{n}\sum_{j=1}^{n} h_R^-(x_i,x_j),$$

$$|K(R)|_{\frac{1}{2}} = \sum_{i=1}^{n} |R(x_i)|_{\frac{1}{2}} = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_R(x_i,x_j)|_{\frac{1}{2}} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{h_R^+(x_i,x_j)+h_R^-(x_i,x_j)}{2}.$$

In what follows, we present four ordered relations between two THF granular structures and further investigate the relative properties.

Definition 4.6. *Let* $K(P) = (P(x_1),\ P(x_2),\ \cdots,\ P(x_n)\ )$, $K(Q) = (Q(x_1),\ Q(x_2),\ \cdots,\ Q(x_n)\ ) \in \mathfrak{K}(U)$*. Four ordered relations on* $\mathfrak{K}(U)$ *are defined as, respectively*

(1) $K(P) \leq_{P1} K(Q) \Longleftrightarrow P(x_i) \leq_H Q(x_i), 1 \leq i \leq n \Longleftrightarrow h_P(x_i,x_j) \leq_h h_Q(x_i,x_j), 1 \leq i,j \leq n$,
$K(P) = K(Q) \Longleftrightarrow P(x_i) = Q(x_i), 1 \leq i \leq n \Longleftrightarrow h_P(x_i,x_j) = h_Q(x_i,x_j),\ 1 \leq i,j \leq n$,
$K(P) <_{P1} K(Q) \Longleftrightarrow K(P) \leq_{P1} K(Q) \wedge K(P) \neq K(Q)$,
(2) $K(P) \leq_{P2} K(Q) \Longleftrightarrow |P(x_i)|_\eta \leq |Q(x_i)|_\eta,\ 1 \leq i \leq n$,
$K(P) \approx_{P2} K(Q) \Longleftrightarrow |P(x_i)|_\eta = |Q(x_i)|_\eta,\ 1 \leq i \leq n$,
$K(P) <_{P2} K(Q) \Longleftrightarrow K(P) \leq_{P2} K(Q) \wedge K(P) \not\approx_{P2} K(Q)$,
(3) $K(P) \leq_{P3} K(Q) \Longleftrightarrow$ *for* $K(P)$*, there exists a sequence* $K(\mathscr{Q}')$ *of* $K(Q)$ *such that* $|P(x_i)|_\eta \leq |S_{\mathscr{Q}'}(x_i)|_\eta, 1 \leq i \leq n$*, where* $K(\mathscr{Q}') = (Q(x_1'), Q(x_2'), \cdots, Q(x_n'))$ *and* $x_1', x_2', \cdots, x_n'$ *is a permutation of* $x_1, x_2, \cdots, x_n$,
$K(P) \approx_{P3} K(Q) \Longleftrightarrow$ *for* $K(P)$*, there exists a sequence* $K(\mathscr{Q}')$ *of* $K(Q)$ *such that* $|P(x_i)|_\eta = |S_{\mathscr{Q}'}(x_i)|_\eta, 1 \leq i \leq n$,
$K(P) <_{P3} K(Q) \Longleftrightarrow K(P) \leq_{P3} K(Q) \wedge K(P) \not\approx_{P3} K(Q)$,
(4) $K(P) \leq_{P4} K(Q) \Longleftrightarrow |K_P|_\eta \leq |K_Q|_\eta$,
$K(P) \approx_{P4} K(Q) \Longleftrightarrow |K_P|_\eta = |K_Q|_\eta$,
$K(P) <_{P4} K(Q) \Longleftrightarrow K(P) \leq_{P4} K(Q) \wedge K(P) \not\approx_{P4} K(Q)$.

The following propositions will be obtained directly by Definition 4.6

Proposition 4.7. *Let* $K(P), K(Q) \in \mathfrak{K}(U)$*, then,*
(1) $K(P) \leq_{Pi} K(Q) \Longleftrightarrow K^c(Q) \leq_{Pi} K^c(P),\ 1 \leq i \leq 4$,
(2) $K(P) \cap_P K(Q) \leq_{Pi} K(P),\ K(P) \cap_P K(Q) \leq_{Pi} K(Q),\ 1 \leq i \leq 4$,
(3) $K(P) \leq_{Pi} K(P) \cup_P K(Q),\ K(Q) \leq_{Pi} K(P) \cup_P K(Q),\ 1 \leq i \leq 4$.

Proposition 4.8. *From the definition of* $\leq_{Pi}, 1 \leq i \leq 4$*, we conclude the following implications:*
$\leq_{P1} \Longrightarrow \leq_{P2} \Longrightarrow \leq_{P3} \Longrightarrow \leq_{P4}$.

Proof. Let $K(P),\ K(Q) \in \mathfrak{K}(\mathrm{U})$, then $K(P) \leq_{P1} K(Q) \Longleftrightarrow P(x_i) \leq_H Q(x_i), 1 \leq i \leq n \Longleftrightarrow h_P(x_i,x_j) \leq_h h_Q(x_i,x_j), 1 \leq i,j \leq n \Rightarrow h_P^-(x_i,x_j) \leq h_Q^-(x_i,x_j) \wedge h_P^+(x_i,x_j) \leq h_Q^+(x_i,x_j), 1 \leq i,j \leq n \Rightarrow |h_P(x_i,x_j)|_\eta = \eta h_P^+(x_i,x_j) + (1-\eta)h_P^-(x_i,x_j) \leq \eta h_Q^+(x_i,x_j) + (1-\eta)h_Q^-(x_i,x_j) = |h_Q(x_i,x_j)|_\eta \Rightarrow |P(x_i)|_\eta = \sum_{i=1}^{n} |h_P(x_i,x_j)|_\eta \leq \sum_{i=1}^{n} |h_Q(x_i,x_j)|_\eta = |Q(x_i)|_\eta, 1 \leq i \leq n$. i.e., $\leq_{P1} \Longrightarrow \leq_{P2}$ can be obtained.
Similarly, we can prove the other implication relations .

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Concluding above, the ordered relation $\leq_{P1}$ is the strongest order between the four orders defined in Definition 4.6 and $(\mathfrak{K}(\mathrm{U}), \leq_{P1})$ composes a poset.

In this subsection, we construct the THF granular structures and discuss the basic operators. Some ordered relations have been presented to describe the hierarch structure of the THF granules.

5 Uncertainty measures for the typical hesitant fuzzy approximation space

In this section, the THF granularity, THF information entropy, THF rough entropy and THF information Shannon entropy will be introduced and their properties will be discussed, respectively. We further study their relationships.

5.1 Granularity of typical hesitant fuzzy granular structures

Definition 5.1. ***(THF granulation)*** *Let $K(R) = (S_R(x_1), S_R(x_2), \cdots, S_R(x_n)) \in \mathfrak{K}(U)$, where $P(x_i) = \{\frac{h_P(x_i,x_j)}{x_j} | 1 \leq j \leq n\}$, $1 \leq i \leq n$. The THF granulation of $K(R)$ can be defined as follows:*

$$G_\eta K(R) = \frac{1}{n}\sum_{i=1}^{n}\frac{|R(x_i)|_\eta}{n} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|h_R(x_i,x_j)|_\eta}{n} = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\eta h_R^+(x_i,x_j) + (1-\eta)h_R^-(x_i,x_j)}{n}.$$

From the definition, the following properties can be held directly.

Proposition 5.2. *Let $K(R) \in \mathfrak{K}(U)$, then*

(1) $G_\eta K(R) + G_{1-\eta}K^c(R) = 1$.

(2) *If $K(R) = (R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(U)$ degenerates to a fuzzy granular structure, then the THF granularity of $K(R)$ degenerates to a fuzzy information granularity. That is, $G_\eta K(R) = \frac{1}{n}\sum\limits_{i=1}^{n}\frac{|\widetilde{R}(x_i)|}{n}$.*

Proof. (1) and (2) can be proved straightforwardly from Definition 5.1. □

We conclude from Definition 4.1 that the THF granularity is an extension of the fuzzy granularity.

Theorem 5.3. *Let $K(P),\ K(Q) \in \mathfrak{K}(U)$. Then,*

(1) $K(P) \leq_{P1} K(Q) \Longrightarrow G_\eta K(P) \leq G_\eta K(Q)$,

(2) $K(P) \leq_{P2} K(Q) \Longrightarrow G_\eta K(P) \leq G_\eta K(Q)$, $K(P) \approx_{P2} K(Q) \Longrightarrow G_\eta K(P) = G_\eta K(Q)$,

(3) $K(P) \leq_{P3} K(Q) \Longrightarrow G_\eta K(P) \leq G_\eta K(Q)$, $K(P) \approx_{P3} K(Q) \Longrightarrow G_\eta K(P) = G_\eta K(Q)$,

(4) $K(P) \leq_{P4} K(Q) \Longleftrightarrow G_\eta K(P) \leq G_\eta K(Q)$, $K(P) \approx_{P4} K(Q) \Longleftrightarrow G_\eta K(P) = G_\eta K(Q)$.

Proof. The theorem can be obtained from Definitions 4.6 and 5.1. □

5.2 Information entropy of typical hesitant fuzzy granular structures

Fuzzy granular structure characterizes the size of information content of a fuzzy granule and information entropy is used to measure the uncertainty of a fuzzy granule [26]. In what follows, we will study the information entropy of typical hesitant fuzzy granular structures.

Definition 5.4. ***(THF information entropy)*** *Let $K(R) = (S_R(x_1), S_R(x_2), \cdots, S_R(x_n)) \in \mathfrak{K}(U)$. The THF information entropy of $K(R)$ can be defined as follows:*

$$E_\eta K(R) = \frac{1}{n}\sum_{i=1}^{n}(1 - \frac{|R(x_i)|_\eta}{n}) = 1 - \frac{1}{n^2}\sum_{i=1}^{n}|R(x_i)|_\eta.$$

Proposition 5.5. *Let $K(R) = (S_R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(U)$, then*

(1) $E_\eta K(R) = 1 - E_{1-\eta}K^c(R)$,

(2) *If $K(R)$ degenerates to a fuzzy granule, then the THF information entropy of $K(R)$ degenerates to a fuzzy information entropy,*

$$E_\eta K(R) = EK(\widetilde{R}) = \frac{1}{n}\sum_{i=1}^{n}\Big(1 - \frac{|\widetilde{R}(x_i)|}{n}\Big).$$

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Proof. For $K(R) = (R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(\mathrm{U})$, we conclude from the definition of THF granulation and THF information entropy that $E_\eta K(R) = 1 - G_\eta K(R)$, thus they are straightforward from Proposition 5.2. □

In conclusion, from Definitions 4.6 and 5.1, we can get the following theorem.

Theorem 5.6. *Let* $K(P),\ K(Q) \in \mathfrak{K}(U)$. *Then,*
(1) $K(P) \leq_{P1} K(Q) \Longrightarrow E_\eta K(P) \geq E_\eta K(Q)$,
(2) $K(P) \leq_{P2} K(Q) \Longrightarrow E_\eta K(P) \geq E_\eta K(Q),\ \ K(P) \approx_{P2} K(Q) \Longrightarrow E_\eta K(P) = E_\eta K(Q)$,
(3) $K(P) \leq_{P3} K(Q) \Longrightarrow E_\eta K(P) \geq E_\eta K(Q),\ \ K(P) \approx_{P3} K(Q) \Longrightarrow E_\eta K(P) = E_\eta K(Q)$,
(4) $K(P) \leq_{P4} K(Q) \Longleftrightarrow E_\eta K(P) \geq E_\eta K(Q),\ \ K(P) \approx_{P4} K(Q) \Longleftrightarrow E_\eta K(P) = E_\eta K(Q)$.

5.3 Rough entropy of typical hesitant fuzzy granular structures

In this subsection, we will propose the typical hesitant fuzzy rough entropy to measure the ambiguity and roughness of typical hesitant fuzzy granular structures.

Definition 5.7. ***(THF rough entropy)*** *Let* $K(R) = (R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(U)$. *The THF rough entropy of* $K(R)$ *can be defined as follows:*

$$Er_\eta K(R) = \frac{1}{n}\sum_{i=1}^{n} \log_2 |R(x_i)|_\eta.$$

Proposition 5.8. *If* $K(R) = (R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(U)$ *degenerates to a fuzzy granular structure, then the THF rough entropy of* $K(R)$ *degenerates to a fuzzy rough entropy*

$$Er_\eta K(R) = \frac{1}{n}\sum_{i=1}^{n} \log_2 |\widetilde{R}(x_i)|.$$

We can also obtain the following Theorem directly.

Theorem 5.9. *Let* $K(P),\ K(Q) \in \mathfrak{K}(U)$. *Then,*
(1) $K(P) \leq_{P1} K(Q) \Longrightarrow Er_\eta K(P) \leq Er_\eta K(Q)$,
(2) $K(P) \leq_{P2} K(Q) \Longrightarrow Er_\eta K(P) \leq Er_\eta K(Q),\ \ K(P) \approx_{P2} K(Q) \Longrightarrow Er_\eta K(P) = Er_\eta K(Q)$,
(3) $K(P) \leq_{P3} K(Q) \Longrightarrow Er_\eta K(P) \leq Er_\eta K(Q),\ \ K(P) \approx_{P3} K(Q) \Longrightarrow Er_\eta K(P) = Er_\eta K(Q)$.

5.4 Information Shannon entropy of typical hesitant fuzzy granular structures

Shannon's entropy was extended by Hu et al. [10] to the fuzzy granular structure, which characterizes the uncertainty of the fuzzy binary granular structures. We in this section will propose the information Shannon entropy in the environment of THF granular structures.

Definition 5.10. ***(THF information Shannon entropy)*** *Let* $K(R) = (R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(U)$, *the THF information Shannon entropy of* $K(R)$ *is then defined as follows:*

$$H_\eta K(R) = -\frac{1}{n}\sum_{i=1}^{n} \log_2 \frac{|R(x_i)|_\eta}{n}.$$

Proposition 5.11. *If* $K(R) = (R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(U)$ *degenerates to a fuzzy granular structure, then the THF information Shannon entropy of* $K(R)$ *degenerates to a fuzzy information Shannon entropy*

$$H_\eta K(R) = -\frac{1}{n}\sum_{i=1}^{n} \log_2 |\frac{\widetilde{R}(x_i)|}{n}.$$

Theorem 5.12. *Let* $K(P),\ K(Q) \in \mathfrak{K}(U)$. *Then,*
(1) $K(P) \leq_{P1} K(Q) \Longrightarrow H_\eta K(P) \geq H_\eta K(Q)$,
(2) $K(P) \leq_{P2} K(Q) \Longrightarrow H_\eta K(P) \geq H_\eta K(Q),\ \ K(P) \approx_{P2} K(Q) \Longrightarrow H_\eta K(P) = H_\eta K(Q)$,
(3) $K(P) \leq_{P3} K(Q) \Longrightarrow H_\eta K(P) \geq H_\eta K(Q),\ \ K(P) \approx_{P3} K(Q) \Longrightarrow H_\eta K(P) = H_\eta K(Q)$.

Having investigated the notations and relative properties of THF rough entropy and THF information Shannon entropy, the relationships of above two uncertainty measures can be built as follows:

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Theorem 5.13. *Let $K(R) = (R(x_1), R(x_2), \cdots, R(x_n)) \in \mathfrak{K}(U)$, then $Er_\eta K(R) + H_\eta K(R) = \log_2 n$.*

Proof. It can be obtained from Definitions 5.7 and 5.10. □

6 Hierarchical structure of the multi-granulation typical hesitant fuzzy approximation space

The method of multi-granulation presents a promising direction in rough sets by approximating the concepts with multiple granular structures represented by binary relations [24, 27]. Yang et al. [45] presented three different hierarchical structures on the multi-granulation spaces based on partition and covering. Moreover, the interrelationship between these hierarchical structures and multi-granulation rough sets were investigated thoroughly. In this section, we present three hierarchical structures in the multi-granulation typical hesitant fuzzy approximation spaces and the framework of optimistic and pessimistic multi-granulation typical hesitant fuzzy rough sets will be proposed.

6.1 Three ordered relations in multi-granulation typical hesitant fuzzy approximation space

Definition 6.1. *Let $\mathfrak{K}(U)$ be a THF approximation space, the pair $(\mathrm{U}, 2^{THF(U\times U)})$ is called a multi-granulation THF approximation space (MG-THFAS), where $2^{THF(U\times U)}$ represents the power set of THF(U× U).*

For all $\Omega \in 2^{THF(U\times U)}, (\mathrm{U}, \Omega)$ is recognized as a multi-granulation THF knowledge base of $(\mathrm{U}, 2^{THF(U\times U)})$.

In what follows, we present three ordered relations to describe the difference between arbitrary two hierarchical structures of multi-granulation THF approximation space.

Definition 6.2. *Let $(\mathrm{U}, 2^{THF(U\times U)})$ be a MG-THFAS on U and $\Omega_1, \Omega_2 \in 2^{THF(U\times U)}$. If $\forall Q \in \Omega_2$, there must exist $P \in \Omega_1$ such that $P(x) \leq_H Q(x), \forall x \in \mathrm{U}$, we say Ω_1 is finer than Ω_2 and represent it by $\Omega_1 \trianglelefteq_1 \Omega_2$. Moreover, $\Omega_1 \triangleleft_1 \Omega_2$ when $\Omega_1 \trianglelefteq_1 \Omega_2$ and $\Omega_1 \neq \Omega_2$, which means Ω_1 is finer than Ω_2 strictly, in which $\Omega_1 = \Omega_2$ denotes $\forall Q \in \Omega_2$, there exists $P \in \Omega_1$, such that $P = Q$ and vice verse.*

Definition 6.3. *Let $(\mathrm{U}, 2^{THF(U\times U)})$ be a MG-THFAS on U and $\Omega_1, \Omega_2 \in 2^{THF(U\times U)}$. If $\forall P \in \Omega_1$, there must exist $Q \in \Omega_2$ such that $P(x) \leq_H Q(x), \forall x \in U$, we say Ω_1 is finer than Ω_2 or Ω_2 is coarser than Ω_1 and denote it by $\Omega_1 \trianglelefteq_2 \Omega_2$. Ω_1 is finer than Ω_2 strictly when $\Omega_1 \trianglelefteq_2 \Omega_2$ and $\Omega_1 \neq \Omega_2$, which is denoted by $\Omega_1 \triangleleft_2 \Omega_2$.*

Definition 6.4. *Let $(\mathrm{U}, 2^{THF(U\times U)})$ be a MG-THFAS on U and $\Omega_1, \Omega_2 \in 2^{THF(U\times U)}$. If $\forall P \in \Omega_1,\ Q \in \Omega_2$, $P(x) \leq_H Q(x), \forall x \in U$, we say Ω_1 is finer than Ω_2 or Ω_2 is coarser than Ω_1 and represent it by $\Omega_1 \trianglelefteq_3 \Omega_2$. Moreover, $\Omega_1 \triangleleft_3 \Omega_2$ means Ω_1 is strictly finer than Ω_2 when $\Omega_1 \trianglelefteq_3 \Omega_2$ and $\Omega_1 \neq \Omega_2$.*

Let $\trianglelefteq_1$, $\trianglelefteq_2$ and $\trianglelefteq_3$ are three hierarchical structures on a MG-THFAS $(\mathrm{U}, 2^{THF(U\times U)})$, we conclude from above definition that Ω_3 is finer than the other two ones, and the interrelationship between three hierarchical structures are presented as follows:

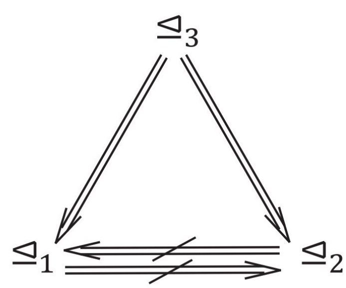

Figure1: Relationships between three hierarchical structures

We in what follows present the minimum and maximum description of an object w.r.t. a multi-granulation THF knowledge base.

Definition 6.5. *Let $(\mathrm{U}, 2^{THF(U\times U)})$ be a MG-THFAS on U, in which a multi-granulation THF knowledge base $\Omega \in 2^{THF(U\times U)}$. $\forall x \in U$, a minimum and maximum granule of x w.r.t. Ω can be defined as follows:*

(1) *$P(x)$ is a minimal granule of x w.r.t. Ω if and only if $\nexists Q \in \Omega$ such that $Q(x) <_H P(x)$,*

(2) *$Q(x)$ is a maximal granule of x w.r.t. Ω if and only if $\nexists P \in \Omega$ such that $Q(x) <_H P(x)$,*

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(3) *Let $S^{\min}_{\Omega}(x)$ be the minimal granule of x w.r.t. Ω, which satisfies $S^{\min}_{\Omega}(x) = \{P(x)|P \in \Omega \wedge \nexists Q(Q(x) <_H P(x))\}$, then the minimal description of x w.r.t. Ω can be represented as $\Omega^{\min}(x) = \{P|P(x) \in S^{\min}_{\Omega}(x)\}$.*

(4) *Let $S^{\max}_{\Omega}(x)$ the maximal granule of x w.r.t. Ω, which satisfies $S^{\max}_{\Omega}(x) = \{Q(x)|Q \in \Omega \wedge \nexists P(Q(x) <_H P(x))\}$, then the maximal description of x w.r.t. Ω can be denoted by $\Omega^{\max}(x) = \{Q|S_Q(x) \in S^{\max}_{\Omega}(x)\}$.*

6.2 Multi-granulation typical hesitant fuzzy rough sets

In this section, we propose two types of multi-granulation THF rough sets (MG-THFRS) in terms of optimistic and pessimistic views.

Definition 6.6. *Let $(\mathrm{U}, 2^{THF(U\times U)})$ be a MG-THFAS on U and $\Omega \in 2^{THF(U\times U)}$. The optimistic upper and lower approximations of THFS A w.r.t. Ω, denoted by $\overline{\Omega}^O(A)$ and $\underline{\Omega}^O(A)$, respectively, are defined as follows:*

$$\overline{\Omega}^O A = \Big\{\langle x, h_{\overline{\Omega}^O A}(x)\rangle | x \in U\Big\},\ \underline{\Omega}^O A = \Big\{\langle x, h_{\underline{\Omega}^O A}(x)\rangle | x \in U\Big\}.$$

Where

$$h_{\overline{\Omega}^O A}(x) = \bigcap_{\substack{h\\ R\in\Omega}} \bigcup_{\substack{h\\ y\in U}} [h_R(x,y) \cap_h h_A(y)],\ h_{\underline{\Omega}^O A}(x) = \bigcup_{\substack{h\\ R\in\Omega}} \bigcap_{\substack{h\\ y\in \mathrm{U}}} [h_{R^c}(x,y) \cup_h h_A(y)].$$

The pair $(\underline{\Omega}^O(A), \overline{\Omega}^O(A))$ are recognized as the optimistic multi-granulation THF rough approximation operators of A w.r.t. the multi-granulation THF knowledge base Ω.

Analogously, the pessimistic multi-granulation THF rough sets are presented as follows:

Definition 6.7. *Let $(\mathrm{U}, 2^{THF(U\times U)})$ be a MG-THFAS on U and $\Omega \in 2^{THF(U\times U)}$ be a multi-granulation THF knowledge base. The pessimistic upper and lower approximations of THFS A w.r.t. Ω, denoted by $\overline{\Omega}^P(A)$ and $\underline{\Omega}^P(A)$, respectively, are proposed as follows:*

$$\overline{\Omega}^P A = \Big\{\langle x, h_{\overline{\Omega}^P A}(x)\rangle | x \in \mathrm{U}\Big\},\ \underline{\Omega}^P A = \Big\{x, h_{\underline{\Omega}^P A}(x)\rangle | x \in \mathrm{U}\Big\}.$$

Where

$$h_{\overline{\Omega}^P A}(x) = \bigcup_{\substack{h\\ R\in\Omega}} \bigcup_{\substack{h\\ y\in \mathrm{U}}} [h_R(x,y) \cap_h h_A(y)],\ h_{\underline{\Omega}^P A}(x) = \bigcap_{\substack{h\\ R\in\Omega}} \bigcap_{\substack{h\\ y\in \mathrm{U}}} [h_{R^c}(x,y) \cup_h h_A(y)].$$

The pair $(\underline{\Omega}^P(A), \overline{\Omega}^P(A))$ are viewed as the pessimistic multi-granulation THF rough approximation operators of A w.r.t. the multi-granulation THF knowledge base Ω.

In terms of two views of multi-granulation THF rough set models, the optimistic one is more accurate to describe the THF A compared with the pessimistic model with the reason that for a THFS A and multi-granulation THF knowledge base Ω, $\underline{\Omega}^O A \geq_H \underline{\Omega}^P A$, while $\overline{\Omega}^O A \leq_H \overline{\Omega}^P A$.

Theorem 6.8. *Let $(\mathrm{U}, 2^{THF(U\times U)})$ be a MG-THFAS on U and Ω_1, $\Omega_2 \in 2^{THF(U\times U)}$ be two multi-granulation THF knowledge bases. Then for any THFS A on U, we obtain*

(1) $\Omega_1 \trianglelefteq_1 \Omega_2 \Longleftrightarrow \underline{\Omega_1}^O A \geq_H \underline{\Omega_2}^O A$,

(2) $\Omega_1 \trianglelefteq_1 \Omega_2 \Longleftrightarrow \overline{\Omega_1}^O A \leq_H \overline{\Omega_2}^O A$.

Proof. ($\Rightarrow$) It can be proved from Definition 6.6.

($\Leftarrow$) Without loss of generality, we suppose $\Omega_1 = \{P\} \ntrianglelefteq_1 \Omega_2 = \{Q\}$, then there must exist $x_i \in \mathrm{U}$, such that $P(x_i) \nleq_H Q(x_i)$. So there exists $x_j \in \mathrm{U}$ which satisfies $h_P(x_i,x_j) \nleq_h h_Q(x_i,x_j)$, specially, we assume $h_P(x_i,x_j) >_h h_Q(x_i,x_j)$. Suppose $A = \{\langle h_{P^c}(x_i,y), y\rangle | y \in \mathrm{U}\}$, then $h_{\underline{\Omega_1}^O A}(x_i) = \bigcup_{h, P\in\Omega_1} \bigcap_{h, y\in U} [h_{P^c}(x,y) \cup_h h_A(y)] = \bigcap_{h, y\in U} [h_{P^c}(x_i,y) \cup_h h_A(y)] = [h_{P^c}(x_i,x_j) \cup_h h_A(x_j)] \bigcap_{h, y\neq x_j} [h_{P^c}(x_i,y) \cup_h h_A(y)] = h_{P^c}(x_i,x_j) \bigcap_{h, y\neq x_j} [h_{P^c}(x_i,y)] \leq_h h_{Q^c}(x_i,x_j) \bigcap_{h, y\neq x_j} [h_{P^c}(x_i,y) \cup_h h_{Q^c}(x_i,y)] = \bigcup_{h, Q\in\Omega_2} \bigcap_{h, y\in U} [h_{Q^c}(x,y) \cup_h h_A(y)] = h_{\underline{\Omega_2}^O A}(x_i)$, which contradicts the condition that $\underline{\Omega_1}^O A \geq_H \underline{\Omega_2}^O A$ for any THF A on U. □

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Similarly, the relationships between pessimistic multi-granulation THF approximation operators based the multi-granulation THF knowledge bases can be obtained in the following theorem:

Theorem 6.9. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on U and* $\Omega_1,\ \Omega_2 \in 2^{THF(U\times U)}$ *be two multi-granulation THF knowledge bases. Then for any THFS A on U, we have*

(1) $\Omega_1 \trianglelefteq_2 \Omega_2 \Longleftrightarrow \underline{\Omega_1}^P A \geq_H \underline{\Omega_2}^P A$,

(2) $\Omega_1 \trianglelefteq_2 \Omega_2 \Longleftrightarrow \overline{\Omega_1}^P A \leq_H \overline{\Omega_2}^P A$.

Proof. It can be straightforward from Theorem 6.8. □

Corollary 6.10. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on U and* $\Omega_1,\ \Omega_2 \in 2^{THF(U\times U)}$ *be two multi-granulation THF knowledge bases. Then for any THFS A on U,* $\Omega_1 \trianglelefteq_3 \Omega_2 \Longleftrightarrow \underline{\Omega_1}^O A \geq_H \underline{\Omega_2}^O A \wedge \overline{\Omega_1}^O A \leq_H \overline{\Omega_2}^O A \wedge \underline{\Omega_1}^P A \geq_H \underline{\Omega_2}^P A \wedge \overline{\Omega_1}^P A \leq_H \overline{\Omega_2}^P A$.

Theorem 6.11. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on U and* $\Omega_1,\ \Omega_2 \in 2^{THF(U\times U)}$ *be two multi-granulation THF knowledge bases. If two THF relations* $R_1,\ R_2 \in \Omega$ *such that* $R_1 \leq_R R_2$*, then for any THFS A on U, we obtain:*

(1) $\underline{\Omega}^O A = \underline{\Omega \setminus R_2}^O A, \qquad \overline{\Omega}^O A = \overline{\Omega \setminus R_2}^O A$,

(2) $\underline{\Omega}^P A = \underline{\Omega \setminus R_1}^P A, \qquad \overline{\Omega}^P A = \overline{\Omega \setminus R_1}^P A$.

Specially, if there exists an object $x \in \mathrm{U}$ *such that* $S_{R_1}(x) \leq_H S_{R_2}(x)$*, then:*

(1)′ $\underline{\Omega}^O A(x) = \underline{\Omega \setminus R_2}^O A(x), \qquad \overline{\Omega}^O A(x) = \overline{\Omega \setminus R_2}^O A(x)$,

(2)′ $\underline{\Omega}^P A(x) = \underline{\Omega \setminus R_1}^P A(x), \qquad \overline{\Omega}^P A(x) = \overline{\Omega \setminus R_1}^P A(x)$.

Thus, if $R_1,\ R_2 \in \Omega$ and there exists an object $x \in \mathrm{U}$ such that $S_{R_1} \leq_H S_{R_2}(x_2)$, then the optimistic multi-granulation THF upper and lower approximations of $A(x)$ are irrelevant to R_2. While the pessimistic multi-granulation THF upper and lower approximations of $A(x)$ are irrelevant to R_1.

Combining with Definition 6.5, we in what follows shows that the minimum (maximum) granule of an object with respect to the given multigranulation THF knowledge base can be used to explore the intrinsic characteristic of optimistic (pessimistic) MTHFRSs.

Theorem 6.12. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on U, in which a multi-granulation THF knowledge base* $\Omega \in 2^{THF(U\times U)}$ *exists. For any THFS A on U, then*

(1) $\overline{\Omega}^O A(x) = \overline{\Omega^{\min}(x)}^O A(x), \quad \underline{\Omega}^O A(x) = \underline{\Omega^{\min}(x)}^O A(x)$,

(2) $\overline{\Omega}^P A(x) = \overline{\Omega^{\max}(x)}^P A(x), \quad \underline{\Omega}^P A(x) = \underline{\Omega^{\max}(x)}^P A(x)$.

Proof. It can be concluded from Definitions 6.6, 6.7 and Theorem 6.11. □

7 Uncertainty measures for the multi-granulation typical hesitant fuzzy approximation space

Liang and Shi [16] introduced the concepts of information entropy, rough entropy and knowledge granulation in rough set theory, and established the relationships among those concepts. To further reveal the essence of uncertainty measures in the framework of multi-granulation approximation space, several theories such as the structures, uncertainty measures and reduction have been presented in [16, 18, 33]. In what follows, we propose four types of uncertainty measures for THFSs based on the maximum and minimum granule of the objects and further discuss their properties with the multi-granulation THF knowledge bases.

Definition 7.1. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on U, in which a multi-granulation THF knowledge base* $\Omega \in 2^{THF(U\times U)}$ *exists. Two types of THF granularity on* Ω *can then be defined as follows:*

$$G^{\cap}_{\eta}(\Omega) = \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcap_{R\in\Omega^{\min}(x)} R(x)|_{\eta} = \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcap_{R\in\Omega} R(x)|_{\eta}, \quad G^{\cup}_{\eta}(\Omega) = \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcap_{R\in\Omega^{\max}(x)} R(x)|_{\eta} = \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcup_{R\in\Omega} R(x)|_{\eta}.$$

Theorem 7.2. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on U, in which two multi-granulation THF knowledge bases* $\Omega_1,\ \Omega_2 \in 2^{THF(U\times U)}$ *exist. Then we conclude that:*

(1) $\Omega_1 \trianglelefteq_1 \Omega_2 \Longrightarrow G^{\cap}_{\eta}(\Omega_1) \leq G^{\cap}_{\eta}(\Omega_2)$,

(2) $\Omega_1 \trianglelefteq_2 \Omega_2 \Longrightarrow G^{\cup}_{\eta}(\Omega_1) \leq G^{\cup}_{\eta}(\Omega_2)$.

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Definition 7.3. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on* U*, in which a multi-granulation THF knowledge base* $\Omega \in 2^{THF(U\times U)}$ *exists. Two types of THF information entropy on* Ω *can then be defined as follows:*

$$E_\eta^{\cap}(\Omega) = 1 - \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcap_{R\in\Omega^{\min}(x)} R(x)|_\eta = 1 - \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcap_{R\in\Omega} R(x)|_\eta = 1 - G_\eta^{\cap}(\Omega),$$

$$E_\eta^{\cup}(\Omega) = 1 - \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcap_{R\in\Omega^{\max}(x)} R(x)|_\eta = 1 - \frac{1}{n^2}\sum_{x\in \mathrm{U}} |\bigcup_{R\in\Omega} R(x)|_\eta = 1 - G_\eta^{\cup}(\Omega).$$

Theorem 7.4. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on* U*, in which two multi-granulation THF knowledge bases* $\Omega_1,\ \Omega_2 \in 2^{THF(U\times U)}$ *exist. Then*

(1) $\Omega_1 \trianglelefteq_1 \Omega_2 \Longrightarrow E_\eta^{\cap}(\Omega_1) \geq E_\eta^{\cap}(\Omega_2)$,
(2) $\Omega_1 \trianglelefteq_2 \Omega_2 \Longrightarrow E_\eta^{\cup}(\Omega_1) \geq E_\eta^{\cup}(\Omega_2)$.

Definition 7.5. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on* U*, in which a multi-granulation THF knowledge base* $\Omega \in 2^{THF(U\times U)}$ *exists. Two types of THF rough entropy on* Ω *can then be defined as follows:*

$$Er_\eta^{\cap}(\Omega) = \frac{1}{n}\sum_{x\in \mathrm{U}} \log_2 |\bigcap_{R\in\Omega^{\min}(x)} R(x)|_\eta = \frac{1}{n}\sum_{x\in \mathrm{U}} \log_2 |\bigcap_{R\in\Omega} R(x)|_\eta,$$

$$Er_\eta^{\cup}(\Omega) = \frac{1}{n}\sum_{x\in \mathrm{U}} \log_2 |\bigcap_{R\in\Omega^{\max}(x)} R(x)|_\eta = \frac{1}{n}\sum_{x\in \mathrm{U}} \log_2 |\bigcup_{R\in\Omega} R(x)|_\eta.$$

Theorem 7.6. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on* U*, in which two multi-granulation THF knowledge bases* $\Omega_1,\ \Omega_2 \in 2^{THF(U\times U)}$ *exist. Thus*

(1) $\Omega_1 \trianglelefteq_1 \Omega_2 \Longrightarrow Er_\eta^{\cap}(\Omega_1) \leq Er_\eta^{\cap}(\Omega_2)$,
(2) $\Omega_1 \trianglelefteq_2 \Omega_2 \Longrightarrow Er_\eta^{\cup}(\Omega_1) \leq Er_\eta^{\cup}(\Omega_2)$.

Definition 7.7. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on* U*, in which a multi-granulation THF knowledge base* $\Omega \in 2^{THF(U\times U)}$ *exists. Two types of THF information Shannon entropy on* Ω *can then be defined as follows:*

$$H_\eta^{\cap}(\Omega) = \frac{1}{n}\sum_{x\in \mathrm{U}} \log_2 \frac{\left|\bigcap_{R\in\Omega^{\min}(x)} R(x)\right|_\eta}{n} = -\frac{1}{n}\sum_{x\in U} \log_2 \frac{\left|\bigcap_{R\in\Omega} R(x)\right|_\eta}{n} = \log_2 n - Er_\eta^{\cap}(\Omega),$$

$$H_\eta^{\cup}(\Omega) = \frac{1}{n}\sum_{x\in U} \log_2 \frac{\left|\bigcap_{R\in\Omega^{\max}(x)} R(x)\right|_\eta}{n} = -\frac{1}{n}\sum_{x\in \mathrm{U}} \log_2 \frac{\left|\bigcup_{R\in\Omega} R(x)\right|_\eta}{n} = \log_2 n - Er_\eta^{\cup}(\Omega).$$

Theorem 7.8. *Let* $(\mathrm{U}, 2^{THF(U\times U)})$ *be a MG-THFAS on* U*, in which two multi-granulation THF knowledge bases* $\Omega_1,\ \Omega_2 \in 2^{THF(U\times U)}$ *exist. Thus*

(1) $\Omega_1 \trianglelefteq_1 \Omega_2 \Longrightarrow H_\eta^{\cap}(\Omega_1) \geq H_\eta^{\cap}(\Omega_2)$,
(2) $\Omega_1 \trianglelefteq_2 \Omega_2 \Longrightarrow H_\eta^{\cup}(\Omega_1) \geq H_\eta^{\cup}(\Omega_2)$.

8 Conclusions

This paper has systematically studied the hierarchical structures and uncertainty measures for THF approximate space by the parameterized scalar cardinalities of THFs and THFEs. We have defined THF granularity, THF information entropy, THF rough entropy and THF information Shannon entropy to characterize the uncertainty of the THF approximation space by proposing four ordered relations of THF approximation space. In addition, the properties of the four uncertainty measures with the ordered relations have been investigated. Furthermore, we have analyzed the hierarchical structures of multi-granulation THF approximation space by presenting three various ordered relations on multi-granulation THF knowledge bases and we have proposed the framework of multi-granulation typical hesitant fuzzy rough sets in terms of pessimistic and optimistic attitudes and discussed the properties of approximation operators with the multi-granulation knowledge bases. The uncertainty measures for the multi-granulation typical hesitant fuzzy approximation space based on the maximal and minimal knowledge bases have been presented. In the future, the uncertainty measures and hierarchical structures will be applied to granule selection, granular computing and optimal hierarchical structure of multi-granulation hierarchical structures under the multi-granulation THF approximation space.

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Uncertainty analysis of hierarchical granular structures for multi-granulation typical hesitant fuzzy approximation space

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تحلیل عدم اطمینان ساختارهای دانه‌ای سلسله مراتبی برای فضای تقریب فازی مردد نوعی چند-دانه

چکیده. اندازه‌های عدم اطمینان و ساختارهای سلسله مراتبی دو دیدگاه اصلی در محاسبه دانه‌ای، استدلال تقریب، و روند آگاهی است. مجموعه‌های فازی مردد نوعی، به عنوان توسیع اولیه از مجموعه های فازی، در انعکاس تردید و ابهام در ارائه دانش و تصمیم‌گیری انعطاف‌پذیرتر می‌باشند. در این مقاله، بطوراساسی، ساختارهای سلسله مراتبی و اندازه‌های عدم اطمینان در زمینه‌های فازی مردد نوعی را بررسی می‌کنیم. ابتدا، اعداد اصلی عددی پارامتری شده عناصر فازی مردد نوعی، مجموعه‌های فازی مردد و روابط فازی مردد نوعی را به ترتیب براساس ترتیب‌های جزئی قابل قبول‌تر، با یک مفهوم گسسته، بطوری‌که پارامترها اولویت‌های ریسک، تصمیم‌گیرنده‌ها را بیان می‌کنند، پیشنهاد می‌کنیم. سپس، چهار رابطه ترتیب برای فضای فازی مردد نوعی و چهار اندازه‌ی عدم اطمینان جهت مشخص سازی ابهام در فضای تقریب فازی مردد ارائه می‌دهیم و روابط آن‌ها را مورد بررسی قرار می‌دهیم. بعد از آن، ساختارهای سلسله مراتبی فضای فازی مردد نوعی چند-دانه، براساس دانش فازی مردد نوعی چند-دانه مختلف مورد بحث قرار گرفته‌اند. بعلاوه، چارچوب مجموعه‌های طبیعی فازی مردد نوعی چند- دانه را برحسب رویکرد خوش‌بینانه و بدبینانه بنا می‌کنیم. بالاخره، برای فضای تقریب فازی مردد نوعی چنددانه، به ترتیب براساس پایگاه‌های دانش ماکسیمال و مینیمال اندازه‌های عدم اطمینان را مورد مطالعه قرار می‌دهیم.

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