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Optimal control of linear fuzzy time-variant controlled systems

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Abstract

In this paper, we study linear fuzzy time-variant optimal control systems using the generalized differentiability concept and we present the general form of optimal controls and states. Some examples are provided to illustrate our results.

Keywords: Fuzzy differential equations, fuzzy optimal control, variation of constant formula.

1 Introduction

Fuzzy differential equations (FDEs) form a suitable setting for mathematical modeling of many real-world problems. An FDE may have different solutions, depending on the type of the derivative considered in the problem. The first and the most popular type of derivative is the Hukuhara derivative which is applied to FDEs in [15] . Under this setting, mainly the existence and uniqueness of the solutions to fuzzy differential equations are studied (see [10, 15] and references therein). Nevertheless, this approach has the disadvantage that it leads to solutions with increasing fuzziness as time goes by [6]. The strongly generalized differentiability was introduced in [5] and under this setting, the constant formulas for linear fuzzy differential equations are proposed in [7, 16, 17] . This concept allows us to resolve the above-mentioned shortcoming. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy number-valued functions than the Hukuhara derivative.

In the past few decades, fuzzy optimal control problems have attracted a great deal of attention, and the interest in the field of fuzzy optimal control theory has increased. Using the Hukuhara differentiability concept, necessary optimality conditions for fuzzy variational problems and for fuzzy optimal control problems are investigated in [13] and [14], respectively. In [21] and [22], the authors have used the generalized Hukuhara differentiability concept to study the time invariant dynamical systems with fuzzy initial conditions and time invariant dynamical systems with fuzzy parameters, respectively. The linear time varying fuzzy controlled systems with fuzzy boundary conditions are also studied in [23]. Under the granular differentiability concept, the sub-optimal control of a fuzzy linear dynamical system is studied in [19] and the fuzzy Bang-Bang control problem is revisited in [20]. A fuzzy time-optimal control problem is investigated in [2], where calculation of fuzzy optimal time is transformed to two classical time-optimal control problems with initial and final sets. Introducing a notion of a derivative of a fuzzy function in the new space of fuzzy numbers equipped with a scalar product, an optimal control problem with non-linear functional is formulated in [24], and an optimality condition is obtained in the form of a maximum principle. A fuzzy controller using fuzzy arithmetics and a new type of membership function is developed in [11] which is simple, fast and computationally efficient, compared to the classical techniques (Mamdani, Takagi Sugeno) and it can also adapt to the process dynamics. Some representative results on fuzzy fractional differential equations, controllability, approximate controllability, optimal control, and optimal feedback control for several different kinds of fractional evolution equations are surveyed in [1]. Fundamental system properties and specifications for control design such as robustness and stability are explained in [8], focusing on four intelligent control techniques, which are fuzzy logic control, neural networks, metaheuristics control tuning, and multi-agent systems. The optimal control problem with operating a fractional differential equations and partial differential equations at minimum quadratic objective function in the framework of neutrosophic environment

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and granular computing is studied in [25]. A discrete-time fuzzy interval optimal control problem via single-level constrained fuzzy arithmetic is presented in [9] and a fuzzy interval dynamic programming is developed to obtain the optimal fuzzy interval solution.

In this paper, we study the optimal control problem of the first order linear fuzzy controlled systems and, motivated by the results in [16], we propose the constant formulas for optimal controls and states under the strongly generalised differentiability concept. For this purpose, in next section, we first recall some fundamental concepts that are key to our discussion. In section 3, we present a necessary optimality condition and constant formulas for optimal controls and states in fuzzy time variant linear controlled systems, when boundary conditions are fuzzy numbers. The last section includes two examples illustrating the applicability of our results.

2 Preliminaries

In this section, we give some definitions and introduce the necessary notations which will be used throughout the paper, see for example, [4, 10] and references therein.

A fuzzy number is a mapping $\tilde{u} : \mathbb{R} \to [0,1]$ satisfying the properties: (i) $\tilde{u}$ is normal, i.e., there exists $s \in \mathbb{R}$ such that $\tilde{u}(s) = 1$, (ii) $\tilde{u}$ is fuzzy convex, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\}$, for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$, (iii) $\tilde{u}$ is upper semicontinuous on $\mathbb{R}$, (iv) $cl\{x \in \mathbb{R} : \tilde{u}(x) > 0\}$, denoted by $[\tilde{u}]^0$, is compact. Denote by $\mathcal{F}$ the set of all fuzzy numbers. For $0 < \alpha \leq 1$, let $[\tilde{u}]^\alpha = \{x \in \mathbb{R} : \tilde{u}(x) \geq \alpha\}$. It is well-known that for all $\alpha \in [0,1]$ and any $\tilde{u} \in \mathcal{F}$, the α-level set $[\tilde{u}]^\alpha$ is a non-empty compact interval $[\underline{u}_\alpha, \overline{u}_\alpha]$ where $\underline{u}_\alpha$ denotes the left-hand endpoint of $[\tilde{u}]^\alpha$ and $\overline{u}_\alpha$ denotes the right-hand endpoint of $[\tilde{u}]^\alpha$. Note that for any $\tilde{u} \in \mathcal{F}$, $\underline{u}_\alpha$ is a non-decreasing left continuous function, $\overline{u}_\alpha$ is a non-increasing left continuous function, and $\underline{u}_\alpha \leq \overline{u}_\alpha$ for $0 < \alpha \leq 1$. Moreover, if a parametric pair $(\underline{u}_\alpha, \overline{u}_\alpha), \alpha \in [0,1]$ satisfies these conditions, then it denotes the parametric form of a fuzzy number $\tilde{u} \in \mathcal{F}$ defined by $\tilde{u}(x) = \sup\{\alpha : \underline{u}_\alpha \leq x \leq \overline{u}_\alpha\}$. Therefore, a fuzzy number $\tilde{u} \in \mathcal{F}$ is completely characterized by its parametric form given by the pair $(\underline{u}_\alpha, \overline{u}_\alpha), \alpha \in [0,1]$.

In this paper, we consider the following LR-fuzzy numbers

$$\tilde{u}(x) = \begin{cases} L(\frac{\sigma - x}{\kappa}), & \text{if } x \in [\sigma - \kappa, \sigma], \\ R(\frac{x-\sigma}{\omega}), & \text{if } x \in [\sigma, \sigma + \omega], \end{cases} \tag{1}$$

where, σ, κ and ω are real numbers that $\kappa, \omega > 0$ and $L, R : [0,1] \to [0,1]$ are two non-increasing shape functions such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. An LR-fuzzy number $\tilde{u} \in \mathcal{F}$ characterized by (1) is denoted by $\tilde{u} = (\sigma, \kappa, \omega)_{LR}$. An LR-fuzzy number $\tilde{u}$ is called semi-symmetric if $L(x) = R(x), \forall x \in [0,1]$ and we use the notation $\tilde{u} = (\sigma, \kappa, \omega)_L$. Moreover, if the spreads of a semi-symmetric fuzzy number $\tilde{u}$ are equal, i.e., $\kappa = \omega$, the LR-fuzzy number $\tilde{u}$ is said to be symmetric and is denoted by $\tilde{u} = (\sigma, \kappa)_L$. A triangular fuzzy number $\tilde{u}$ is a semi-symmetric fuzzy number with $L(x) = 1 - x$ which we denote it by $\tilde{u} = (\sigma, \kappa, \omega)$. If $\kappa = \omega$ then $\tilde{u}$ is called a symmetric triangular fuzzy number and is denoted by $\tilde{u} = (\sigma, \kappa)$. Let $\mathcal{F}_T$ be the set of all triangular fuzzy numbers. We note that a real number σ can be considered as a symmetric triangular fuzzy number $\tilde{u} = (\sigma, 0)$; hence, $\mathbb{R} \subset \mathcal{F}_T \subset \mathcal{F}$. For $\tilde{u}, \tilde{v} \in \mathcal{F}$ and $\gamma \in \mathbb{R}$, the addition $\tilde{u} + \tilde{v}$, and the scalar multiplication $\gamma\tilde{u}$ are defined as $[\tilde{u} + \tilde{v}]^\alpha = [\tilde{u}]^\alpha + [\tilde{v}]^\alpha$ and $[\gamma\tilde{u}]^\alpha = \gamma[\tilde{u}]^\alpha$ for all $\alpha \in [0,1]$, where $[\tilde{u}]^\alpha + [\tilde{v}]^\alpha$ and $[\tilde{u}]^\alpha - [\tilde{v}]^\alpha$ mean the usual addition and subtraction of two intervals (subsets) of $\mathbb{R}$, respectively, and $\gamma[\tilde{u}]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$. Moreover the subtraction of $\tilde{u}$ and $\tilde{v}$ is defined as $\tilde{u} - \tilde{v} = \tilde{u} + (-\tilde{v})$. For two LR-fuzzy numbers $\tilde{u}_j = (\sigma_j, \kappa_j, \omega_j)_{LR} \in \mathcal{F}, j = 1,2$ and $\gamma \in \mathbb{R}$, we have $\tilde{u}_1 + \tilde{u}_2 = (\sigma_1 + \sigma_2, \kappa_1 + \kappa_2, \omega_1 + \omega_2)_{LR}$, and

$$\gamma\tilde{u} = \begin{cases} (\gamma\sigma, \gamma\kappa, \gamma\omega)_{LR}, & \text{if } \gamma \geq 0, \\ (\gamma\sigma, -\gamma\omega, -\gamma\kappa)_{RL}, & \text{if } \gamma < 0. \end{cases} \tag{2}$$

Therefore, for a semi-symmetric fuzzy number $\tilde{u} = (\sigma, \kappa, \omega)_L$, we have $\gamma\tilde{u} = (\gamma\sigma, \gamma^+\kappa - \gamma^-\omega, \gamma^+\omega - \gamma^-\kappa)_L$, where $\gamma^+ = \max\{0, \gamma\}$ and $\gamma^- = \min\{0, \gamma\}$. Let $\tilde{u}, \tilde{v} \in \mathcal{F}$. If there exists $\tilde{z} \in \mathcal{F}$ such that $\tilde{u} = \tilde{v} + \tilde{z}$, then $\tilde{z}$ is called the H-difference of $\tilde{u}$ and $\tilde{v}$ and it is denoted by $\tilde{z} = \tilde{u} \ominus \tilde{v}$. If $\tilde{u}_j = (\sigma_j, \kappa_j, \omega_j)_{LR} \in \mathcal{F}, j = 1, 2$, are two LR-fuzzy numbers with $\kappa_1 > \kappa_2$, $\omega_1 > \omega_2$, then $\tilde{z} = (\sigma_1 - \sigma_2, \kappa_1 - \kappa_2, \omega_1 - \omega_2)_{LR} \in \mathcal{F}$ is the H-difference of $\tilde{u}_1$ and $\tilde{u}_2$.

We also write $\tilde{u} \preceq \tilde{v}$, if $\underline{u}_\alpha \leq \underline{v}_\alpha$ and $\overline{u}_\alpha \leq \overline{v}_\alpha$ for all $\alpha \in [0,1]$. Moreover, $\tilde{u} = \tilde{v}$, if $\tilde{u} \preceq \tilde{v}$ and $\tilde{v} \preceq \tilde{u}$. The maximum of $\tilde{u}$ and $\tilde{v}$ denoted by $\widetilde{\max}\{\tilde{u}, \tilde{v}\}$ is a fuzzy number $\tilde{w} \in \mathcal{F}$ where $\overline{w}_\alpha = \max\{\overline{u}_\alpha, \overline{v}_\alpha\}$ and $\underline{w}_\alpha = \max\{\underline{u}_\alpha, \underline{v}_\alpha\}$ for all $\alpha \in [0,1]$. Moreover, we write $\tilde{w} = \widetilde{\min}\{\tilde{u}, \tilde{v}\}$ if $\overline{w}_\alpha = \min\{\overline{u}_\alpha, \overline{v}_\alpha\}$ and $\underline{w}_\alpha = \min\{\underline{u}_\alpha, \underline{v}_\alpha\}$ for all $\alpha \in [0,1]$.

We say that a fuzzy number-valued function $\tilde{x} : [a,b] \to \mathcal{F}$ satisfies the condition $(H1)$ at $t_0 \in (a,b)$ if $\tilde{x}(t_0+h) \ominus \tilde{x}(t_0)$ and $\tilde{x}(t_0) \ominus \tilde{x}(t_0 - h)$ exist for $h > 0$ sufficiently small and we say that $\tilde{x}$ satisfies the condition $(H2)$ at $t_0 \in (a,b)$ if $\tilde{x}(t_0) \ominus \tilde{x}(t_0 + h)$ and $\tilde{x}(t_0 - h) \ominus \tilde{x}(t_0)$ exist for $h > 0$ sufficiently small. Moreover, $\tilde{x}$ is said to be (i)-differentiable at

$t_0 \in (a,b)$ if satisfies condition $(H1)$ at t_0 and there exists an element $\tilde{x}'(t_0) \in \mathcal{F}$ such that

$$\lim_{h\to 0^+} D\big(\frac{\tilde{x}(t_0+h)\ominus\tilde{x}(t_0)}{h}, \tilde{x}'(t_0)\big) = \lim_{h\to 0^+} D\big(\frac{\tilde{x}(t_0)\ominus\tilde{x}(t_0-h)}{h}, \tilde{x}'(t_0)\big) = 0,$$

and said to be (ii)-differentiable at $t_0 \in (a,b)$ if satisfies condition $(H2)$ and there exists an element $\tilde{x}'(t_0) \in \mathcal{F}$ such that

$$\lim_{h\to 0^+} D\big(\frac{\tilde{x}(t_0-h)\ominus\tilde{x}(t_0)}{-h}, \tilde{x}'(t_0)\big) = \lim_{h\to 0^+} D\big(\frac{\tilde{x}(t_0)\ominus\tilde{x}(t_0+h)}{-h}, \tilde{x}'(t_0)\big) = 0,$$

where the metric $D : \mathcal{F}\times\mathcal{F} \to \mathbb{R}$ is defined as

$$D(\tilde{u},\tilde{v}) = \sup_{\alpha\in[0,1]} \max\{|\overline{u}_\alpha - \overline{v}_\alpha|, |\underline{u}_\alpha - \underline{v}_\alpha|\}, \quad u,v \in \mathcal{F}.$$

We say that $\tilde{x}$ is strongly generalized differentiable at $t_0 \in (a,b)$ if it is (i)-differentiable or (ii)-differentiable at t_0. If $\tilde{x}$ is (i)-differentiable at $t \in (a,b)$, then $[\tilde{x}'(t)]^\alpha = [\dot{\underline{x}}_\alpha(t), \dot{\overline{x}}_\alpha(t)]$ and if it is (ii)-differentiable at $t \in (a,b)$, then $[\tilde{x}'(t)]^\alpha = [\dot{\overline{x}}_\alpha(t), \dot{\underline{x}}_\alpha(t)]$, for all $\alpha \in [0,1]$. Moreover, the integral of $\tilde{x}$ on $[a,b]$ is a fuzzy number $\int_a^b \tilde{x}(t)dt \in \mathcal{F}$ with the parametric form $\left(\int_a^b \underline{x}_\alpha(t)dt\ ,\ \int_a^b \overline{x}_\alpha(t)dt\right), \alpha \in [0,1]$.

3 Linear control systems with fuzzy boundary conditions

Let $a, b : [t_0, t_f] \to \mathbb{R}$ be continuous functions and $\tilde{x}_0 \in \mathcal{F}$. Consider the following fuzzy control system

$$\begin{cases} \tilde{x}'(t) = a(t)\tilde{x}(t) + b(t)\tilde{u}(t), \\ \tilde{x}(t_0) = \tilde{x}_0, \end{cases} \tag{3}$$

where $\tilde{x} : [t_0, t_f] \to \mathcal{F}$ and $\tilde{u} : [t_0, t_f] \to \mathcal{F}$ are the fuzzy state and the fuzzy control functions, respectively. Depending on the type of differentiability of $\tilde{x}$, we propose two types of solutions for control system (3); (i)-differentiable solution and (ii)-differentiable solution. In each type of the solutions, we may have (i)-differentiability or (ii)-differentiability for control $\tilde{u}$. In the sequel, we introduce the appropriate classes of states $\tilde{x}$ and controls $\tilde{u}$, in which the type of differentiability of $\tilde{u}$ depends on the function $b(t)$ and the type of the differentiability of $\tilde{x}$.

The pair $(\tilde{x}, \tilde{u})$ is said to be a (i)-solution of the fuzzy control system (3), if $\tilde{x}$ is a (i)-differentiable function satisfying the fuzzy control system (3), corresponding to $\tilde{u}$. A (ii)-solution of the fuzzy control system (3) is defined, similarly. Using the results in [16], we have the following theorem.

Theorem 3.1. *(a) Assume that* $a(t) \geq 0$, $\forall t \in [t_0, t_f]$. *For a (i)-solution* $(\tilde{x}, \tilde{u})$ *of the fuzzy control system* (3), *we have*

$$\tilde{x}(t) = e^{\int_{t_0}^t a(\tau)d\tau}\left(\tilde{x}_0 + \int_{t_0}^t b(s)\tilde{u}(s)e^{-\int_{t_0}^s a(\tau)d\tau}ds\right), \tag{4}$$

and for a (ii)-solution $(\tilde{x}, \tilde{u})$ *of the fuzzy control system* (3), *we have*

$$\begin{aligned} \tilde{x}(t) = \cosh\left(\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 \ominus \int_{t_0}^t \left[b(s)\tilde{u}(s)\sinh\left(\int_{t_0}^s a(\tau)d\tau\right) - b(s)\tilde{u}(s)\cosh\left(\int_0^s a(\tau)d\tau\right)\right]ds\right) \\ \ominus -\sinh\left(\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 \ominus \int_{t_0}^t \left[b(s)\tilde{u}(s)\sinh\left(\int_{t_0}^s a(\tau)d\tau\right) - b(s)\tilde{u}(s)\cosh\left(\int_0^s a(\tau)d\tau\right)\right]ds\right), \end{aligned} \tag{5}$$

provided that the H-differences in (5) *exist.*
(b) Assume that $a(t) \leq 0$, $\forall t \in [t_0, t_f]$. *For a (i)-solution* $(\tilde{x}, \tilde{u})$ *of the fuzzy control system* (3), *we have*

$$\begin{aligned} \tilde{x}(t) = \cosh\left(\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 + \int_{t_0}^t \left[b(s)\tilde{u}(s)\cosh\left(\int_{t_0}^s a(\tau)d\tau\right) \ominus b(s)\tilde{u}(s)\sinh\left(\int_0^s a(\tau)d\tau\right)\right]ds\right) \\ +\sinh\left(\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 + \int_{t_0}^t \left[b(s)\tilde{u}(s)\cosh\left(\int_{t_0}^s a(\tau)d\tau\right) \ominus b(s)\tilde{u}(s)\sinh\left(\int_0^s a(\tau)d\tau\right)\right]ds\right), \end{aligned} \tag{6}$$

and for a (ii)-solution $(\tilde{x}, \tilde{u})$ *of the fuzzy control system* (3), *we have*

$$\tilde{x}(t) = e^{\int_{t_0}^t a(\tau)d\tau}\left(\tilde{x}_0 \ominus \int_{t_0}^t (-b(s)\tilde{u}(s))e^{-\int_{t_0}^s a(\tau)d\tau}ds\right), \tag{7}$$

provided that the H-difference in (7) *exists.*

A (i)-solution $(\tilde{x},\tilde{u})$ of the fuzzy control system (3) is called a (i)-admissible pair if $\tilde{x}$ satisfies the following end condition

$$\tilde{x}(t_f)=\tilde{x}_f, \tag{8}$$

where $\tilde{x}_f\in\mathcal{F}$. A (ii)-admissible pair of the fuzzy control system (3) is defined, similarly. Consider a fuzzy functional given by

$$J(\tilde{x},\tilde{u})=-\frac{1}{2}\int_{t_0}^{t_f}\tilde{u}^2(t)dt. \tag{9}$$

Denote by $\mathcal{U}_{(i)}$, the set of all (i)-admissible pairs $(\tilde{x},\tilde{u})$ of the fuzzy control system (3) that

$$\tilde{u}^{\min}\preceq\tilde{u}(t)\preceq\tilde{u}^{\max}, \tag{10}$$

where $\tilde{u}^{\min}, \quad \tilde{u}^{\max}\in\mathcal{F}$. A pair $(\tilde{x}^*,\tilde{u}^*)\in\mathcal{U}_{(i)}$ is called a (i)-optimal pair of the fuzzy optimal control problem (FOCP), if

$$J(\tilde{x},\tilde{u})\preceq J(\tilde{x}^*,\tilde{u}^*),\ \forall(\tilde{x},\tilde{u})\in\mathcal{U}_{(i)}. \tag{11}$$

Similarly, the set of all (ii)-admissible paries of the fuzzy control system (3) is denoted by $\mathcal{U}_{(ii)}$ and a pair $(\tilde{x}^*,\tilde{u}^*)\in\mathcal{U}_{(ii)}$ is called a (ii)-optimal pair of the FOCP, if

$$J(\tilde{x},\tilde{u})\preceq J(\tilde{x}^*,\tilde{u}^*),\ \forall(\tilde{x},\tilde{u})\in\mathcal{U}_{(ii)}. \tag{12}$$

In this paper, similar to the classic optimal control problems, the fuzzy quantity $\int_{t_0}^{t_f}\tilde{u}^2(\tau)d\tau$ is used to measure the total input energy to steer the state $\tilde{x}$ from $\tilde{x}_0$ to $\tilde{x}_f$ through the control function $\tilde{u}$; hence, our FOCP can be named as a minimum energy problem.

For a $\tilde{x}\in\mathcal{F}$, we use the notation $x_\alpha=(\underline{x}_\alpha,\overline{x}_\alpha),\alpha\in[0,1]$. Following the references [21, 23, 12, 14], we seek a (i)-optimal pair $(\tilde{x}^*,\tilde{u}^*)\in\mathcal{U}_{(i)}$ of the FOCP by maximizing the functional

$$J(x_\alpha,u_\alpha)=-\frac{1}{2}\int_{t_0}^{t_f}(\underline{u}_\alpha^2(t)+\overline{u}_\alpha^2(t))dt, \tag{13}$$

subject to

$$\begin{cases}\dot{\underline{x}}_\alpha(t)=a^-(t)\overline{x}_\alpha(t)+a^+(t)\underline{x}_\alpha(t)+b^-(t)\overline{u}_\alpha(t)+b^+(t)\underline{u}_\alpha(t),\\ \dot{\overline{x}}_\alpha(t)=a^+(t)\overline{x}_\alpha(t)+a^-(t)\underline{x}_\alpha(t)+b^+(t)\overline{u}_\alpha(t)+b^-(t)\underline{u}_\alpha(t),\end{cases} \tag{14}$$

and

$$\underline{u}_\alpha^{\min}\leq\underline{u}_\alpha(t)\leq\underline{u}_\alpha^{\max},\ \overline{u}_\alpha^{\min}\leq\overline{u}_\alpha(t)\leq\overline{u}_\alpha^{\max},\quad\forall t\in[t_0,t_f]. \tag{15}$$

Using the Pontryagin's maximum principle [18], the necessary conditions for optimal controls are obtained as

$$\underline{u}_\alpha^*(t)=\max\left\{\underline{u}_\alpha^{\min},\min\left\{\underline{u}_\alpha^{\max},b^+(t)\underline{\lambda}_\alpha^*(t)+b^-(t)\overline{\lambda}^*(t)\right\}\right\}, \tag{16}$$

$$\overline{u}_\alpha^*(t)=\max\left\{\overline{u}_\alpha^{\min},\min\left\{\overline{u}_\alpha^{\max},b^+(t)\overline{\lambda}_\alpha^*(t)+b^-(t)\underline{\lambda}^*(t)\right\}\right\}, \tag{17}$$

where $\underline{\lambda}_\alpha^*$ and $\overline{\lambda}_\alpha^*$ are the costate variables satisfying

$$\begin{cases}\dot{\underline{\lambda}}_\alpha(t)=-a^-(t)\overline{\lambda}_\alpha(t)-a^+(t)\underline{\lambda}_\alpha(t),\\ \dot{\overline{\lambda}}_\alpha(t)=-a^+(t)\overline{\lambda}_\alpha(t)-a^-(t)\underline{\lambda}_\alpha(t).\end{cases} \tag{18}$$

Obviously, if for all $t\in[0,t_f]$, the pairs $(\underline{\lambda}_\alpha^*(t),\overline{\lambda}_\alpha^*(t))$, $\alpha\in[0,1]$, denote the parametric form of the fuzzy numbers $\tilde{\lambda}^*(t)\in\mathcal{F}$, then $t\to\tilde{\lambda}^*(t)$ is a (ii)-solution of

$$\dot{\tilde{\lambda}}(t)=-a(t)\tilde{\lambda}(t),\qquad t\in[0,t_f]. \tag{19}$$

Moreover, the equations (16) and (17) can be written as

$$\tilde{u}^*(t)=\widetilde{\max}\left\{\tilde{u}^{\min},\widetilde{\min}\left\{\tilde{u}^{\max},b(t)\tilde{\lambda}^*(t)\right\}\right\}. \tag{20}$$

Similarly, if $(\tilde{x}^*, \tilde{u}^*)$ is a (ii)-optimal pair of the FOCP, then the pair (x^*_α, u^*_α), $\alpha \in [0,1]$, maximizes the functional (13) subject to

$$\begin{cases} \dot{\underline{x}}_\alpha(t) = a^+(t)\overline{x}_\alpha(t) + a^-(t)\underline{x}_\alpha(t) + b^+(t)\overline{u}_\alpha(t) + b^-(t)\underline{u}_\alpha(t), \\ \dot{\overline{x}}_\alpha(t) = a^-(t)\overline{x}_\alpha(t) + a^+(t)\underline{x}_\alpha(t) + b^-(t)\overline{u}_\alpha(t) + b^+(t)\underline{u}_\alpha(t), \end{cases} \tag{21}$$

and (15). In this case, the necessary optimality conditions are given by (16) and (17), where $\underline{\lambda}^*_\alpha$ and $\overline{\lambda}^*_\alpha$ are costate variables satisfying

$$\begin{cases} \dot{\underline{\lambda}}_\alpha(t) = -a^+(t)\overline{\lambda}_\alpha(t) - a^-(t)\underline{\lambda}_\alpha(t), \\ \dot{\overline{\lambda}}_\alpha(t) = -a^-(t)\overline{\lambda}_\alpha(t) - a^+(t)\underline{\lambda}_\alpha(t). \end{cases} \tag{22}$$

Obviously, if for all $t \in [0, t_f]$, the pairs $(\underline{\lambda}^*_\alpha(t), \overline{\lambda}^*_\alpha(t))$, $\alpha \in [0,1]$, denote the parametric form of the fuzzy numbers $\tilde{\lambda}^*(t) \in \mathcal{F}$, then $t \to \tilde{\lambda}^*(t)$ is a (i)-solution of (19). We summarize our results in the following theorem.

Theorem 3.2. *If $(\tilde{x}^*, \tilde{u}^*) \in \mathcal{U}_{(i)}$ is a (i)-optimal pair to the fuzzy optimal control problem (FOCP), then $\tilde{u}^*$ is given by* (20)*, where $\tilde{\lambda}^*$ is a (ii)-solution of* (19)*. Moreover, if $(\tilde{x}^*, \tilde{u}^*) \in \mathcal{U}_{(ii)}$ is a (ii)-optimal pair of the FOCP, then $\tilde{u}^*$ is given by* (20)*, where $\tilde{\lambda}^*$ is a (i)-solution to* (19)*.*

Remark 3.3. *Theorem* 2.9 *in [17] and Theorem* 3.2 *enable us to discuss the type of the differentiability for $\tilde{u}^*$, when there is no constraint* (10) *on control function. For example, for a (ii)-optimal pair $(\tilde{x}^*, \tilde{u}^*) \in \mathcal{U}_{(ii)}$, the control function $\tilde{u}^*$ is given by $\tilde{u}^*(t) = b(t)\tilde{\lambda}^*(t)$, where $\tilde{\lambda}^*(t)$ is a (i)-differentiable, according to the Theorem* 3.2*. Therefore, by Theorem* 2.9 *in [17], if $b(t)\dot{b}(t) > 0$, then $\tilde{u}^*$ is (i)-differentiable at t and if $b(t)\dot{b}(t) < 0$, then $\tilde{u}^*$ may be (i)-differentiable or (ii)-differentiable at t, depending on the satisfying $\tilde{u}^*$ in the conditions (H_1) or (H_2) at t. The case that $(\tilde{x}^*, \tilde{u}^*) \in \mathcal{U}_{(i)}$ is (i)-optimal pair, is similar.*

In the absence of the constraint (10), we can derive explicit formulas for optimal pairs of the FOCP as following lemma.

Lemma 3.4. *(a) Let $a(t) \geq 0, \forall t \in [t_0, t_f]$. Then, for a (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ of the FOCP, we have*

$$\tilde{u}^*(t) = b(t)e^{-\int_{t_0}^t a(\tau)d\tau}\tilde{\lambda}_0, \tag{23}$$

$$\tilde{x}^*(t) = e^{\int_{t_0}^t a(\tau)d\tau}\left(\tilde{x}_0 + \tilde{\lambda}_0 \int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds\right), \tag{24}$$

and for a (ii)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ of the FOCP, we have

$$\tilde{u}^*(t) = b(t)\cosh\left(\int_{t_0}^t a(\tau)d\tau\right)\tilde{\lambda}_0 - b(t)\sinh\left(\int_{t_0}^t a(\tau)d\tau\right)\tilde{\lambda}_0, \tag{25}$$

$$\begin{aligned} \tilde{x}^*(t) = \cosh\left(\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 \ominus \int_{t_0}^t \left[b^2(s)\sinh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0 - b^2(s)\cosh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0\right]ds\right) \\ \ominus \sinh\left(-\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 \ominus \int_{t_0}^t \left[b^2(s)\sinh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0 - b^2(s)\cosh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0\right]ds\right). \end{aligned} \tag{26}$$

provided that the H-differences in (26) *exist.*
(b) If $a(t) \leq 0, \forall t \in [t_0, t_f]$, then the fuzzy control function $\tilde{u}^*$ and the fuzzy state function $\tilde{x}^*$ given by

$$\tilde{u}^*(t) = b(t)\cosh(\int_{t_0}^t a(\tau)d\tau)\tilde{\lambda}_0 \ominus b(t)\sinh(\int_{t_0}^t a(\tau)d\tau)\tilde{\lambda}_0, \tag{27}$$

$$\begin{aligned} \tilde{x}^*(t) = \cosh\left(\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 + \int_{t_0}^t \left[b^2(s)\cosh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0 \ominus b^2(s)\sinh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0\right]ds\right), \\ + \sinh\left(\int_{t_0}^t a(\tau)d\tau\right)&\left(\tilde{x}_0 + \int_{t_0}^t \left[b^2(s)\cosh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0 \ominus b^2(s)\sinh\left(2\int_{t_0}^s a(\tau)d\tau\right)\tilde{\lambda}_0\right]ds\right) \end{aligned} \tag{28}$$

form a (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ of the FOCP, provided that the H-differences in (27) *and* (28) *exist. Moreover, the fuzzy control function $\tilde{u}^*$ and the fuzzy state function $\tilde{x}^*$ given by*

$$\tilde{u}^*(t) = b(t)e^{-\int_{t_0}^t a(\tau)d\tau}\tilde{\lambda}_0, \tag{29}$$

$$\tilde{x}^*(t) = e^{\int_{t_0}^t a(\tau)d\tau}\left(\tilde{x}_0 \ominus \tilde{\lambda}_0 \int_{t_0}^t -b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds\right) \tag{30}$$

constitute a (ii)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ *of the FOCP, provided that the H-difference in* (30) *exists.*

Proof. Obviously, in the absence of the constraint (10), we have $\tilde{u}^*(t) = b(t)\tilde{\lambda}^*(t),\ \forall t \in [t_0, t_f]$. Assume that $a(t) \geq 0, \forall t \in [t_0, t_f]$. From (6), a (i)-solution to (19) is given by

$$\tilde{\lambda}^*(t) = \cosh(\int_{t_0}^{t} a(\tau)d\tau)\ \tilde{\lambda}_0 - \sinh(\int_{t_0}^{t} a(\tau)d\tau)\tilde{\lambda}_0.$$

Therefore, according to Theorem 3.2, the control function $\tilde{u}^*$ in (ii)-optimal pair is given by (25). Moreover, the state function $\tilde{x}^*$ in (ii)-optimal pair, given by (26), is obtained by substituting the control function $\tilde{u}^*$ into (5); since, it is easy to check that

$$b(s)\tilde{u}^*(s)\sinh(\int_{t_0}^{s} a(\tau)d\tau) - b(s)\tilde{u}^*(s)\cosh(\int_{t_0}^{s} a(\tau)d\tau) = b^2(s)\sinh(2\int_{t_0}^{s} a(\tau)d\tau)\tilde{\lambda}_0 - b^2(s)\cosh(2\int_{t_0}^{s} a(\tau)d\tau)\tilde{\lambda}_0.$$

On the other hand, from (7), a (ii)-solution to (19) is given by $\tilde{\lambda}^*(t) = e^{-\int_{t_0}^{t} a(\tau)d\tau}\tilde{\lambda}_0$. Hence, the fuzzy control function (23) and the fuzzy state function (24), obtained by substituting the control function (23) into (4), constitute a (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$.

Now, let $a(t) \leq 0,\ \forall t \in [t_0, t_f]$. According to (4), the (i)-solution to (19) is given by $\tilde{\lambda}^*(t) = e^{-\int_{t_0}^{t} a(\tau)d\tau}\tilde{\lambda}_0$. Therefore, the fuzzy control function $\tilde{u}^*$ in (ii)-optimal pair is given by (29). Substituting $\tilde{u}^*$ into (7) gives the fuzzy state function $\tilde{x}^*$ in (ii)-optimal pair as (30). Moreover, according to (5), we see that

$$\tilde{\lambda}^*(t) = \cosh(\int_{t_0}^{t} a(\tau)d\tau)\ \tilde{\lambda}_0 \ominus \sinh(\int_{t_0}^{t} a(\tau)d\tau)\tilde{\lambda}_0,$$

is a (ii)-solution to (19). Therefore, (27) is the fuzzy control function $\tilde{u}^*$ in (i)-optimal pair. Beside, since we have

$$b(s)\tilde{u}^*(s)\cosh(\int_{t_0}^{s} a(\tau)d\tau) \ominus b(s)\tilde{u}^*(s)\sinh(\int_{t_0}^{s} a(\tau)d\tau) = b^2(s)\cosh(2\int_{t_0}^{s} a(\tau)d\tau)\tilde{\lambda}_0 \ominus b^2(s)\sinh(2\int_{t_0}^{s} a(\tau)d\tau)\tilde{\lambda}_0,$$

therefore, substituting $\tilde{u}^*$ into (6) gives (28). □

We note that, the general formula for optimal pairs of the FOCP can be considered as a combination of the formulas given in Lemma 3.4, depending on the sign of $a(t)$ on each subintervals of $[t_0, t_f]$. Moreover, the fuzzy constant $\tilde{\lambda}_0$ in Lemma 3.4 is used to adjust the final condition $\tilde{x}^*(t_f) = \tilde{x}_f$. In the other word, $\tilde{\lambda}_0$ is obtained by imposing the condition $\tilde{x}^*(t_f) = \tilde{x}_f$ and solving a simple fuzzy linear equation. For example, from (24), we obtain

$$\tilde{\lambda}_0 = \frac{e^{-\int_{t_0}^{t_f} a(\tau)d\tau}\tilde{x}_f \ominus \tilde{x}_0}{\int_{t_0}^{t_f} b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds}, \tag{31}$$

and, from (30), we obtain

$$\tilde{\lambda}_0 = \frac{\tilde{x}_0 \ominus e^{-\int_{t_0}^{t_f} a(\tau)d\tau}\tilde{x}_f}{-\int_{t_0}^{t_f} b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds}. \tag{32}$$

Remark 3.5. *Let* $\tilde{y}(t) = \tilde{x}_0 \ominus \tilde{\lambda}_0 \int_{t_0}^{t} -b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds$. *We have*

$$\overline{y}_\alpha(t) = \overline{x}_{0\alpha} + \underline{\lambda}_{0\alpha}\int_{t_0}^{t} b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds. \tag{33}$$

Substituting $\underline{\lambda}_{0\alpha}$ *from* (32) *in* (33) *gives* $\overline{y}_\alpha(t) = (1-\gamma(t))\overline{x}_{0\alpha} + \gamma(t)e^{-\int_{t_0}^{t_f} a(\tau)d\tau}\overline{x}_{f\alpha}$, *where* $\gamma(t) = \frac{\int_{t_0}^{t} b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds}{\int_{t_0}^{t_f} b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds}$. *In a similar way, we obtain* $\underline{y}_\alpha(t) = (1-\gamma(t))\underline{x}_{0\alpha} + \gamma(t)e^{-\int_{t_0}^{t_f} a(\tau)d\tau}\underline{x}_{f\alpha}$. *Moreover,* $0 \leq \gamma(t) \leq 1,\ t \in [t_0, t_f]$*; therefore*

$$\tilde{y}(t) = (1-\gamma(t))\tilde{x}_0 + \gamma(t)e^{-\int_{t_0}^{t_f} a(\tau)d\tau}\tilde{x}_f.$$

Indeed, the existence of the H-difference in (32) *guarantees the existence of the H-difference in* (30).

According to the following lemma, if the boundary conditions in (3) are semi-symmetric fuzzy numbers, then the values of state functions and control functions in optimal pairs are semi-symmetric fuzzy numbers too.

Lemma 3.6. *Let* $\tilde{x}_0 = (x_{0c}, x_{0l}, x_{0r})_L,\ \tilde{x}_f = (x_{fc}, x_{fl}, x_{fr})_L \in \mathcal{F}$.
(a) If $a(t) \geq 0,\ \forall t \in [t_0, t_f]$*, then a (ii)-optimal pair* $(\tilde{x}^*, \tilde{u}^*)$ *of the FOCP is given by*

$$\begin{aligned}\tilde{u}^*(t) = \Big(& b(t)e^{-\int_{t_0}^t a(\tau)d\tau}\lambda_{0c}, \big[b^+(t)\cosh(\int_{t_0}^t a(\tau)d\tau) - b^-(t)\sinh(\int_{t_0}^t a(\tau)d\tau)\big]\lambda_{0l}\\ &+\big[b^+(t)\sinh(\int_{t_0}^t a(\tau)d\tau) - b^-(t)\cosh(\int_{t_0}^t a(\tau)d\tau)\big]\lambda_{0r}, \big[b^+(t)\cosh(\int_{t_0}^t a(\tau)d\tau) - b^-(t)\sinh(\int_{t_0}^t a(\tau)d\tau)\big]\lambda_{0r}\\ &+\big[b^+(t)\sinh(\int_{t_0}^t a(\tau)d\tau) - b^-(t)\cosh(\int_{t_0}^t a(\tau)d\tau)\big]\lambda_{0l}\Big)_L,\end{aligned} \tag{34}$$

$$\begin{aligned}\tilde{x}^*(t) = \Big(& e^{\int_{t_0}^t a(\tau)d\tau}(x_{0c} + \lambda_{0c}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds), x_{0l}\cosh(\int_{t_0}^t a(\tau)d\tau) - x_{0r}\sinh(\int_{t_0}^t a(\tau)d\tau) -\\ &\lambda_{0l}\int_{t_0}^t b^2(s)\sinh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds - \lambda_{0r}\int_{t_0}^t b^2(s)\cosh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds,\\ &x_{0r}\cosh(\int_{t_0}^t a(\tau)d\tau) - x_{0l}\sinh(\int_{t_0}^t a(\tau)d\tau) - \lambda_{0r}\int_{t_0}^t b^2(s)\sinh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds -\\ &\lambda_{0l}\int_{t_0}^t b^2(s)\cosh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds\Big)_L\end{aligned} \tag{35}$$

and a (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ *of the FOCP is given by*

$$\tilde{u}^*(t) = e^{-\int_{t_0}^t a(\tau)d\tau}\Big(b(t)\lambda_{0c}, b^+(t)\lambda_{0l} - b^-(t)\lambda_{0r}, b^+(t)\lambda_{0r} - b^-(t)\lambda_{0l}\Big)_L, \tag{36}$$

$$\begin{aligned}\tilde{x}^*(t) = e^{\int_{t_0}^t a(\tau)d\tau}\Big(& x_{0c} + \lambda_{0c}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds, x_{0l} + \lambda_{0l}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds,\\ &x_{0r} + \lambda_{0r}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds\Big)_L.\end{aligned} \tag{37}$$

(b) If $a(t) \leq 0,\ \forall t \in [t_0, t_f]$*, then the fuzzy control function* $\tilde{u}^*$ *given by* (36) *and the fuzzy state function*

$$\tilde{x}^*(t) = e^{\int_{t_0}^t a(\tau)d\tau}\Big(x_{0c} + \lambda_{0c}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds, x_{0l} - \lambda_{0r}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds, x_{0r} - \lambda_{0l}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds\Big)_L \tag{38}$$

constitute a (ii)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ *of the FOCP and the fuzzy control function* $\tilde{u}^*$*, given by* (34) *and the fuzzy state function* $\tilde{x}^*$ *given by*

$$\begin{aligned}\tilde{x}^*(t) = \Big(& e^{\int_{t_0}^t a(\tau)d\tau}(x_{0c} + \lambda_{0c}\int_{t_0}^t b^2(s)e^{-2\int_{t_0}^s a(\tau)d\tau}ds), x_{0l}\cosh(\int_{t_0}^t a(\tau)d\tau) - x_{0r}\sinh(\int_{t_0}^t a(\tau)d\tau)+\\ &\lambda_{0l}\int_{t_0}^t b^2(s)\cosh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds + \lambda_{0r}\int_{t_0}^t b^2(s)\sinh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds,\\ &x_{0r}\cosh(\int_{t_0}^t a(\tau)d\tau) - x_{0l}\sinh(\int_{t_0}^t a(\tau)d\tau) + \lambda_{0r}\int_{t_0}^t b^2(s)\cosh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds+\\ &\lambda_{0l}\int_{t_0}^t b^2(s)\sinh\big[\int_{t_0}^t a(\tau)d\tau - 2\int_{t_0}^s a(\tau)d\tau\big]ds\Big)_L\end{aligned} \tag{39}$$

form a (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ *to FOCP.*

Proof. The equations (34)-(39), are direct consequence of Lemma (3.4) which are obtained by inserting $\tilde{x}_0 = (x_{0c}, x_{0l}, x_{0r})_L$ and $\tilde{\lambda}_0 = (\lambda_{0c}, \lambda_{0l}, \lambda_{0r})_L \in \mathcal{F}$ into (23)-(30) and performing some simplifications. □

Remark 3.7. *Obviously, if* $\tilde{x}_0 = (x_0, 0)$ *and* $\tilde{x}_f = (x_f, 0)$*, then we have a crisp optimal control problem. Setting* $\tilde{\lambda}_0 = (\lambda_0, 0)$ *and using the equations* (34)-(39)*, it is concluded that the (i)-optimal and (ii)-optimal pairs are equal on I* *characterized by* $\tilde{x}^*(t) = (x^*(t), 0)$, $\tilde{u}^*(t) = (u^*(t), 0)$ *with*

$$x^*(t) = e^{\int_{t_0}^{t} a(\tau)d\tau}(x_0 + \lambda_0 \int_{t_0}^{t} b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds), \; u^*(t) = b(t)e^{-\int_{t_0}^{t} a(\tau)d\tau}\lambda_0,$$

where $\lambda_0 = \frac{e^{-\int_{t_0}^{t_f} a(\tau)d\tau}x_f - x_0}{\int_{t_0}^{t_f} b^2(s)e^{-2\int_{t_0}^{s} a(\tau)d\tau}ds}$, *obtained from* (31) *satisfies the condition* $x^*(t_f) = x_f$. *These results are exactly consistent with those obtained by applying the Pontryagin's maximum principle to maximize the functional* $J(x, u) = -\frac{1}{2}\int_{t_0}^{t_f} u^2(t)dt$, *subject to the constraints* $\dot{x}(t) = a(t)x(t) + b(t)u(t)$, $x(t_0) = x_0$, $x(t_f) = x_f$ [18].

4 Examples

In this section, we present two examples to illustrate the new results.

Example 4.1. *Consider a FOCP maximizing the functional* $J(\tilde{x}, \tilde{u}) = -\frac{1}{2}\int_0^1 \tilde{u}^2(t)dt$ *subject to*

$$\tilde{x}'(t) = (t - 0.5)\tilde{x}(t) + \cos(\pi t)\tilde{u}(t), \tilde{x}(0) = \tilde{x}_0, \tilde{x}(1) = \tilde{x}_f.$$

We first set $\tilde{x}_0 = (0, 0.5, 1)$ *and* $\tilde{x}_f = (2, 0.05, 0.1)$ *and propose the (ii)-optimal pair for this problem. From Lemma 3.6, for* $t \in [0, 0.5]$, *we have*

$$\tilde{x}^*(t) = e^{0.5t^2-0.5t}\Bigg(4.2123\int_0^t \cos^2(\pi s)e^{s-s^2}ds, 0.5 - 0.6679\int_0^t \cos^2(\pi s)e^{s-s^2}ds, 1 - 1.5104\int_0^t \cos^2(\pi s)e^{s-s^2}ds\Bigg),$$

$$\tilde{u}^*(t) = \cos(\pi t)e^{0.5t-0.5t^2}(4.2123, 1.5105, 0.6679),$$

and for $t \in [0.5, 1]$, *we have*

$$\tilde{x}^*(t) = \Bigg(e^{0.5t^2-0.5t+0.125}(1.6372 + 2.5549\int_{0.5}^t cos^2(\pi s)e^{s-s^2}ds), 0.2754\cosh(0.5t^2 - 0.5t + 0.125) - 0.5075\sinh(0.5t^2 - 0.5t + 0.125) - 1.5104\int_{0.5}^t \cos^2(\pi s)\sinh(0.5t^2 - 0.5t - s^2 + s - 0.125)ds - 0.6679\int_{0.5}^t \cos^2(\pi s)\cosh(0.5t^2 - 0.5t - s^2 + s - 0.125)ds, 0.5075\cosh(0.5t^2 - 0.5t + 0.125) - 0.2754\sinh(0.5t^2 - 0.5t + 0.125) - 0.6679\int_{0.5}^t \cos^2(\pi s)\sinh(0.5t^2 - 0.5t - s^2 + s - 0.125)ds - 1.5104\int_{0.5}^t \cos^2(\pi s)\cosh(0.5t^2 - 0.5t - s^2 + s - 0.125)ds\Bigg),$$

$$\tilde{u}^*(t) = -\cos(\pi t)\Bigg(-4.2123e^{0.5t-0.5t^2-0.125}, 1.5104\sinh(0.5t^2 - 0.5t + 0.125) + 0.6679\cosh(0.5t^2 - 0.5t + 0.125), 0.6679\sinh(0.5t^2 - 0.5t + 0.125) + 1.5104\cosh(0.5t^2 - 0.5t + 0.125)\Bigg).$$

The plots of the (ii)-optimal pair are depicted in Figure 1. We have $\tilde{u}^*(t) = f(t)\tilde{\lambda}_0$, $t \in [0, 0.5]$, *where* $\tilde{\lambda}_0 = (4.2123, 1.5105, 0.6679)$ *and* $f(t) = \cos(\pi t)e^{0.5t-0.5t^2}$. *It is easy to see that* $f(t) > 0$, $\dot{f}(t) > 0$, $t \in (0, 0.0457)$ *and* $f(t) > 0$, $\dot{f}(t) < 0$, $t \in (0.0457, 0.5)$*; hence, by Theorem 10 in [5], the control function* $\tilde{u}^*$ *is (i)-differentiable with increasing length of the support on* $(0, 0.0457)$ *and (ii)-differentiable with decreasing length of the support on* $(0.0457, 0.5)$, *as shown in the left plot of Figure 1. On the other hand, on the interval* $[0.5, 1]$, *we have* $b(t) = cos(\pi t)$ *and the corresponding fuzzy function* $\tilde{\lambda}^*$ *(the coefficient of b in* $\tilde{u}^*$*) is (i)-differentiable, according to Theorem* 3.2*. Obviously* $b(t)\dot{b}(t) > 0$, $t \in (0.5, 1)$*; therefore, by Remark* 3.3*, the fuzzy control function* $\tilde{u}^*$ *is (i)-differentiable on* $(0.5, 1)$ *with increasing length of the support, as shown in the left plot of Figure 1. The right plot in Figure 1 shows that the trajectory of the fuzzy state* $\tilde{x}^*$ *in (ii)-optimal pair has a decreasing length of the support and it is (ii)-differentiable.*

Now, we set $\tilde{x}_0 = (0, 0.1)$, $\tilde{x}_f = (-1, 0.25, 0.5)$ *and propose the (i)-optimal pair. From Lemma 3.6, the (i)-optimal pair* $(\tilde{x}^*, \tilde{u}^*)$ *for this problem is given by*

$$\tilde{u}^*(t) = cos(\pi t)\Bigg(-1.8882e^{0.5t-0.5t^2}, 0.6971\cosh(0.5t^2 - 0.5t) + 0.2248\sinh(0.5t^2 - 0.5t), 0.2248\cosh(0.5t^2 - 0.5t) + 0.6971\sinh(0.5t^2 - 0.5t)\Bigg),$$

$$\tilde{x}^*(t) = \Bigg(-1.8882e^{0.5t^2-0.5t}\int_0^t \cos^2(\pi s)e^{s-s^2}ds,$$

$$0.1e^{0.5t-0.5t^2} + 0.6971\int_0^t \cos^2(\pi s)\cosh(0.5t^2 - 0.5t - s^2 + s)ds + 0.2248\int_0^t cos^2(\pi s)\sinh(0.5t^2 - 0.5t - s^2 + s)ds,$$

$0.1e^{0.5t-0.5t^2} + 0.2248\int_0^t \cos^2(\pi s)\cosh(0.5t^2 - 0.5t - s^2 + s)ds + 0.6971\int_0^t \cos^2(\pi s)\sinh(0.5t^2 - 0.5t - s^2 + s)ds\Big),$
if $t \in [0, 0.5]$, *and*

$\tilde{u}^*(t) = -e^{0.5t-0.5t^2-0.125}\cos(\pi t)\Big(1.8882, 0.2248, 0.6971\Big),$

$\tilde{x}^*(t) = e^{0.5t^2-0.5t+0.125}\Big(-0.4689 - 1.8882\int_{0.5}^t \cos^2(\pi s)e^{s-s^2-0.25}ds, 0.2875 + +0.6971\int_{0.5}^t \cos^2(\pi s)e^{s-s^2-0.25}ds, 0.1681 + +0.2248\int_{0.5}^t \cos^2(\pi s)e^{s-s^2-0.25}ds\Big)$, *if* $t \in [0.5, 1]$.

Figure 2 depicts the (i)-optimal pair. The control function $\tilde{u}^*$ *in (i)-optimal pair can be written as* $\tilde{u}^*(t) = b(t)\tilde{\lambda}^*(t)$, $t \in [0, 0.5]$, *where* $b(t) = cos(\pi t)$ *and* $\tilde{\lambda}^*(t)$ *is the coefficient of* $b(t)$ *which is a (ii)-differentiable function, according to Theorem* 3.2. *Moreover* $b(t)\dot{b}(t) < 0$, $t \in (0, 0.5)$; *hence, by Remark* 3.3, *the fuzzy function* $\tilde{u}^*$ *is (ii)-differentiable with decreasing length of the support on* $[0, 0.5]$, *as shown in the left plot of Figure 2. Beside, we have* $\tilde{u}^*(t) = f(t)\tilde{\lambda}_0$, $t \in [0.5, 1]$, *where* $f(t) = -e^{0.5t-0.5t^2-0.125}\cos(\pi t)$ *and* $\tilde{\lambda}_0 = (1.8882, 0.2248, 0.6971)$. *Obviously,* $f(t) > 0$, $\dot{f}(t) > 0$, $t \in (0.5, 0.9543)$ *and* $f(t) > 0$, $\dot{f}(t) < 0$, $t \in (0.9543, 1)$; *hence, by Theorem 10 in [5], it is concluded that* $\tilde{u}^*$ *is a (i)-differentiable function and the length of its support is increasing on* $(0.5, 0.9543)$ *and it is a (ii)-differentiable function and the length of its support is decreasing on* $(0.9543, 1)$, *as shown in the left plot of Figure 2. The right plot in Figure 2 shows the trajectory of the fuzzy state* $\tilde{x}^*$ *in (i)-optimal pair, which is (i)-differentiable with increasing length of the support.*

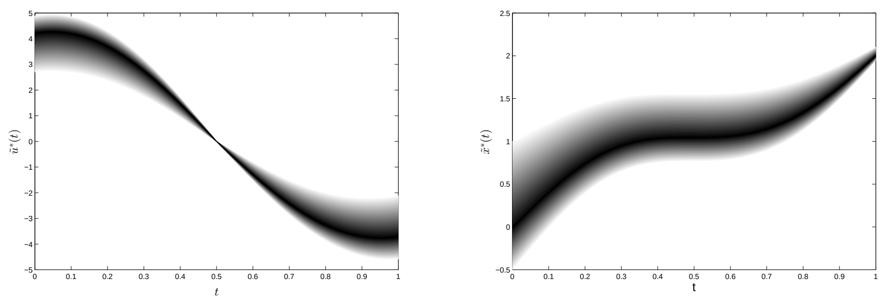

Figure 1: The (ii)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ of Example 4.1.

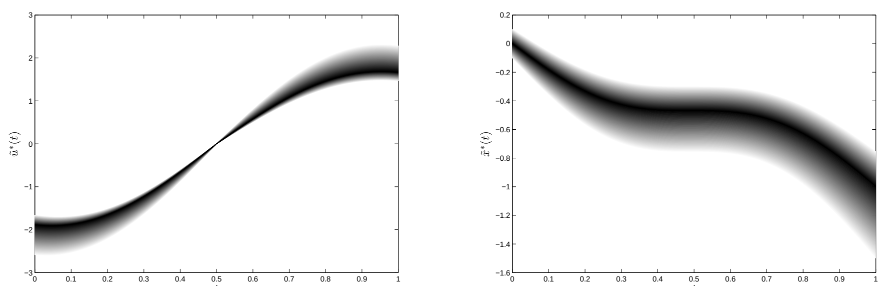

Figure 2: The (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ of Example 4.1.

Example 4.2. *We maximize functional* $J(\tilde{x}, \tilde{u}) = -\frac{1}{2}\int_0^{50} \tilde{u}^2(t)dt$ *subject to the Gomertz population growth model [3] given by* $\tilde{x}'(t) = ae^{-kt}\tilde{x}(t) + \tilde{u}(t)$, *where* $k > 0$ *and* $\tilde{u}$ *is a control function denoting the external forces injected to environment. We first set* $\tilde{x}_0 = (2, 2)_{1-z^2}$ *and* $\tilde{x}_f = 10$. *From Lemma 3.4, the (ii)-optimal pair* $(\tilde{x}^*, \tilde{u}^*)$ *for this FOCP is*

$$\tilde{x}^*(t) = \Big(e^{2.5(1-e^{-0.2t})}(2 - 0.7291\int_0^t e^{5(e^{-0.2s}-1)}ds), 2e^{2.5(e^{-0.2t}-1)} - 0.0003058\int_0^t e^{2.5e^{-0.2t}-5e^{-0.2s}+2.5}ds\Big)_{1-z^2},$$

$$\tilde{u}^*(t) = \Big(-0.7291e^{2.5(e^{-0.2t}-1)}, 0.0003058e^{2.5(1-e^{-0.2t})}\Big)_{1-z^2}.$$

The plot of the (ii)-optimal pair is depicted in Figure 3. Here, we have $b(t) = 1$; *hence,* $\tilde{u}^*(t) = \tilde{\lambda}^*(t)$. *From the left plot in Figure 3, the support of* $\tilde{u}^*$ *is increasing which is due to the fact that it is (i)-differentiable, according to Theorem*

3.2. Moreover, the fuzzy state $\tilde{x}^*$ is a (ii)-differentiable function with decreasing support, as shown in the right plot of Figure 3.
Now, we set $\tilde{x}_0 = (2,1,2)_{\sqrt{1-z},1-z}$ and $\tilde{x}_f = (100,3,7)_{\frac{1-e^{1-z^2}}{1-e}}$. According to Lemma 3.4, the (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ is

$$\tilde{x}^*(t) = e^{2.5(1-e^{-0.2t})}\Big((2,1,2)_{\sqrt{1-z},1-z} + \big[(5.08, 0.152, 0.355)_{\frac{1-e^{1-z^2}}{1-e}} \ominus (1.2368, 0.6184, 1.2368)_{\sqrt{1-z},1-z}\big] \int_0^t e^{5(e^{-0.2s}-1)} ds\Big),$$

$$\tilde{u}^*(t) = e^{2.5(e^{-0.2t}-1)}\Big((5.08, 0.1524, 0.3556)_{\frac{1-e^{1-z^2}}{1-e}} \ominus (1.2368, 0.6184, 1.2368)_{\sqrt{1-z},1-z}\Big).$$

Figure 4 shows the (i)-optimal pair. By a similar discussion to the (ii)-optimal pair case, the control function $\tilde{u}^*$ in (i)-optimal pair is (ii)-differentiable and the corresponding state $\tilde{x}^*$ is (i)-differentiable; hence, the supports of $\tilde{u}^*$ and $\tilde{x}^*$ have decreasing and increasing length, respectively, which is shown in Figure 4. Figures 1-4, confirm the good performance of the fuzzy control function $\tilde{u}^*$ in steering the state $\tilde{x}^*$ from the initial condition $\tilde{x}_0$ to the final condition $\tilde{x}_f$, in each example.

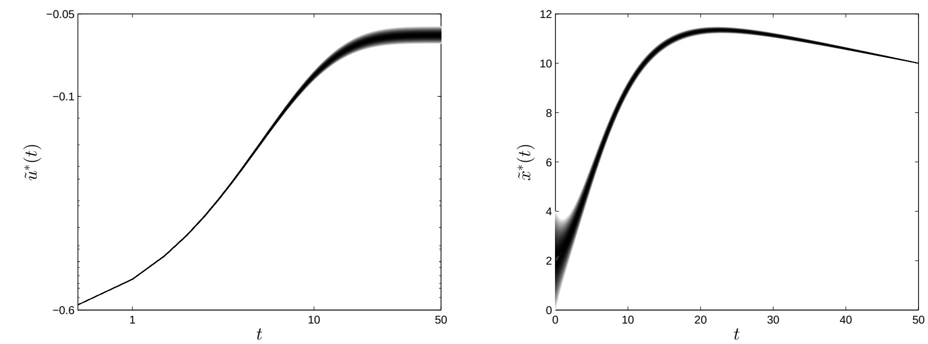

Figure 3: The (ii)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ of Example 4.2.

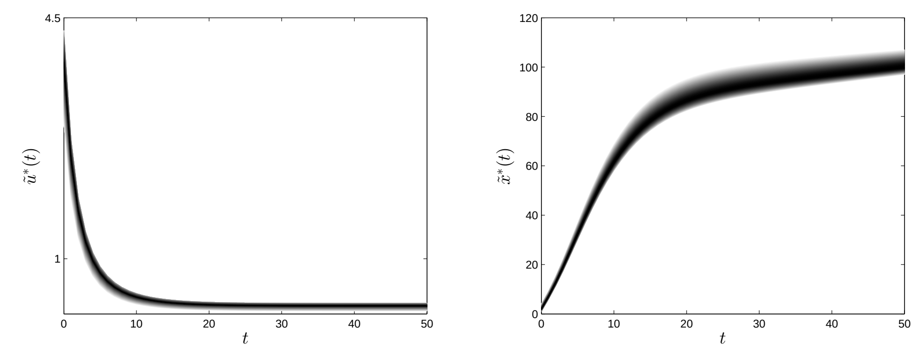

Figure 4: The (i)-optimal pair $(\tilde{x}^*, \tilde{u}^*)$ of Example 4.2.

5 Conclusion

In this paper, we have studied a fuzzy optimal control problem by minimizing the required energy to steer a fuzzy state satisfying a time varying linear control system with a fuzzy initial condition, to a given target state described by a fuzzy number. To this purpose, the concept of generalized differentiability has been considered. Applying the Pontryagin Maximum Principle to α-cuts of the fuzzy control system, we obtained the necessary optimality conditions for two types of control functions and corresponding state functions which constitute the (i)-optimal and the (ii)-optimal pairs, depending on the type of differentiability considered for the state. In particular, we proposed the constant formulas for the optimal pairs. Two illustrative examples are given to support the reliability and efficiency of the proposed method.

For further research, one can extend the presented approach to the other types of fuzzy optimal control problems. For example, setting $J(\tilde{x}, \tilde{u}) = -\int_{t_0}^{t_f} |\tilde{u}(\tau)| d\tau$ gives a fuzzy minimum fuel problem, setting $J(t_f, \tilde{x}, \tilde{u}) = -\int_{t_0}^{t_f} d\tau$ gives a fuzzy time-optimal control problem and $J(\tilde{x}, \tilde{u}) = -\int_{t_0}^{t_f} (\tilde{x}(\tau) \ominus \tilde{x}_f)^2 + \tilde{u}^2(\tau) d\tau$ gives a fuzzy tracing problem, where we ignore the condition (8). For more details on different types of optimal control problems, see [18].

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Optimal control of linear fuzzy time-variant controlled systems

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کنترل بهینه سیستم های کنترل خطی فازی زمان متغیر

چکیده. در این مقاله با استفاده از مفهوم مشتق‌پذیری تعمیم‌یافته، سیستم‌های کنترل خطی فازی زمان متغیر را مطالعه می‌کنیم و شکل کلی وضعیت و کنترل‌های بهینه را معرفی می‌کنیم. مثال‌هایی برای نشان دادن نتایج ارائه شده‌اند.