Iranian Journal of Fuzzy Systems
Volume 17, Number 5, (2020), pp. 13-22

Individual ergodic theorem for intuitionistic fuzzy observables using intuitionistic fuzzy state

K. Čunderlíková[1]

[1] *Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia*

cunderlikova.lendelova@gmail.com

Abstract

The classical ergodic theory has been built on σ-algebras. Later the Individual ergodic theorem was studied on more general structures like MV-algebras and quantum structures. The aim of this paper is to formulate the Individual ergodic theorem for intuitionistic fuzzy observables using **m**-almost everywhere convergence, where **m** is an intuitionistic fuzzy state. We show the Kolmogorov construction for intuitionistic fuzzy observables, too.

Keywords: The intuitionistic fuzzy event, the intuitionistic fuzzy observable, the intuitionistic fuzzy state, the product, the upper limit, the lower limit, the **m**-almost everywhere convergence, the **m**-preserving transformation, the individual ergodic theorem, the Kolmogorov construction.

1 Introduction

In [1, 2] K.T. Atanassov introduced the notion of intuitionistic fuzzy sets. Then in [8] B. Riečan defined the intuitionistic fuzzy state on the family of intuitionistic fuzzy events $\mathcal{F} = \{(\mu_A, \nu_A)\,; \mu_A + \nu_A \le 1_\Omega\}$, where μ_A, ν_A are $\mathcal{S}$-measurable functions, $\mu_A, \nu_A : \Omega \to [0,1]$, as a mapping **m** from the family $\mathcal{F}$ to the set R by the formula

$$\mathbf{m}\big((\mu_A, \nu_A)\big) = (1-\alpha)\int_\Omega \mu_A dP + \alpha\left(1 - \int_\Omega \nu_A dP\right),$$

where $P : \mathcal{S} \to [0,1]$ is a probability measure and $\alpha \in [0,1]$.

In paper [4] we defined the upper and the lower limits for sequence of intuitionistic fuzzy observables. We used an intuitionistic fuzzy state **m** to define the notion of almost everywhere convergence. We compared two concepts of **m**-almost everywhere convergence.

In paper [5] we studied the **m**-almost everywhere convergence of sequence of intuitionistic fuzzy observables $g_n(x_1,\dots,x_n) : \mathcal{B}(R) \to \mathcal{F}$ given by $g_n(x_1,\dots,x_n) = h_n \circ g_n^{-1}$, where $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ is the joint intuitionistic fuzzy observable of intuitionistic fuzzy observables $x_1,\dots,x_n$ and $g_n : R^n \to R$ is a Borel measurable function. We showed the connection between **m**-almost everywhere convergence of this sequence of intuitionistic fuzzy observables and P-almost everywhere convergence of random variables in classical probability space induced by Kolmogorov construction. This connection is a start point for proving the Individual ergodic theorem for intuitionistic fuzzy observables using **m**-almost everywhere convergence.

Recall that the formulation of the Individual ergodic theorem for intuitionistic fuzzy events with product first appeared in the paper [3]. There we used $\mathcal{P}$-almost everywhere convergence, where $\mathcal{P}$ was a separating intuitionistic fuzzy probability. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposited to two intuitionistic fuzzy states, it is usefull to study **m**-almost everywhere convergence, where **m** is an intuitionistic fuzzy state. In this paper we formulate the Individual ergodic theorem for intuitionistic fuzzy observables using **m**-almost everywhere convergence. We show the Kolmogorov construction for intuitionistic fuzzy observables, too.

Corresponding Author: K. Čunderlíková
Received: November 2019; Revised: February 2020; Accepted: April 2020.

Remark that in a whole text we use a notation IF as an abbreviation for intuitionistic fuzzy.

2 IF-events, IF-states, IF-observables and IF-mean value

First we start with definitions of basic notions (see [1, 2, 9]).

Definition 2.1. *Let Ω be a nonempty set. An IF-set $\mathbf{A}$ on Ω is a pair (μ_A, ν_A) of mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1_\Omega$.*

Definition 2.2. *Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a σ-algebra of subsets of Ω. An IF-event is called an IF-set $\mathbf{A} = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0,1]$ are $\mathcal{S}$-measurable.*

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \longrightarrow [0,1]$ will will called the membership function, $\nu_A : \Omega \longrightarrow [0,1]$ be called the non-membership function.

If $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$, $\mathbf{B} = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Lukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$\mathbf{A} \oplus \mathbf{B} = ((\mu_A + \mu_B) \wedge 1_\Omega, (\nu_A + \nu_B - 1_\Omega) \vee 0_\Omega)), \quad \text{and} \quad \mathbf{A} \odot \mathbf{B} = ((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega))$$

and the partial ordering is given by $\mathbf{A} \leq \mathbf{B}$ if and only if $\mu_A \leq \mu_B, \nu_A \geq \nu_B$. In paper we use max-min connectives defined by

$$\mathbf{A} \vee \mathbf{B} = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \quad \text{and} \quad \mathbf{A} \wedge \mathbf{B} = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$$

and the de Morgan rules $(a \vee b)^* = a^* \wedge b^*$, and $(a \wedge b)^* = a^* \vee b^*$, where $a^* = 1 - a$.

Example 2.3. *Fuzzy set $f : \Omega \longrightarrow [0,1]$ can be regarded as IF-set, if we put $\mathbf{A} = (f, 1_\Omega - f)$. If $f = \chi_A$, then the corresponding IF-set has the form $\mathbf{A} = (\chi_A, 1_\Omega - \chi_A) = (\chi_A, \chi_{A'})$. In this case $\mathbf{A} \oplus \mathbf{B}$ corresponds to the union of sets, $\mathbf{A} \odot \mathbf{B}$ to the intersection of sets and $\leq$ to the set inclusion.*

In the IF-probability theory [10] instead of the notion of probability we use the notion of state.

Definition 2.4. *Let $\mathcal{F}$ be the family of all IF-events in Ω. A mapping $\mathbf{m} : \mathcal{F} \to [0,1]$ is called an IF-state, if the following conditions are satisfied:*

(i) $\mathbf{m}((1_\Omega, 0_\Omega)) = 1$, $\mathbf{m}((0_\Omega, 1_\Omega)) = 0$;

(ii) *if* $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$ *and* $\mathbf{A}, \mathbf{B} \in \mathcal{F}$, *then* $\mathbf{m}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{m}(\mathbf{A}) + \mathbf{m}(\mathbf{B})$;

(iii) *if* $\mathbf{A}_n \nearrow \mathbf{A}$ *(i.e.* $\mu_{A_n} \nearrow \mu_A$, $\nu_{A_n} \searrow \nu_A$*), then* $\mathbf{m}(\mathbf{A}_n) \nearrow \mathbf{m}(\mathbf{A})$.

Probably the most useful result in the IF-state theory is the following representation theorem [8]:

Theorem 2.5. *To each IF-state $\mathbf{m} : \mathcal{F} \to [0,1]$ there exists exactly one probability measure $P : \mathcal{S} \to [0,1]$ and exactly one $\alpha \in [0,1]$ such that for each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$,*

$$\mathbf{m}(\mathbf{A}) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha \bigg(1 - \int_\Omega \nu_A dP \bigg).$$

Proof. In [8] *Theorem.* □

The third basic notion in the probability theory is the notion of an observable. Let $\mathcal{J}$ be the family of all intervals in R of the form $[a, b) = \{x \in R : a \leq x < b\}$. Then the σ-algebra $\sigma(\mathcal{J})$ is denoted $\mathcal{B}(R)$ and it is called the σ-algebra of Borel sets, its elements are called Borel sets.

Definition 2.6. *By an IF-observable on $\mathcal{F}$ we understand each mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A \cap B = \emptyset$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) *if* $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$.

If we denote $x(A) = \big(x^\flat(A), 1_\Omega - x^\sharp(A)\big)$ for each $A \in \mathcal{B}(R)$, then $x^\flat, x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables, where $\mathcal{T} = \{f : \Omega \to [0,1]; f$ is $\mathcal{S} -$ measurable$\}$.

Remark 2.7. *Sometimes we need to work with n-dimensional IF-observable $x : \mathcal{B}(R^n) \to \mathcal{F}$ defined as a mapping with the following conditions:*

(i) $x(R^n) = (1_\Omega, 0_\Omega)$, $x(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A \cap B = \emptyset$, $A, B \in \mathcal{B}(R^n)$, *then* $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$ *and* $x(A \cup B) = x(A) \oplus x(B)$;

(iii) *if* $A_n \nearrow A$, *then* $x(A_n) \nearrow x(A)$ *for each* $A, A_n \in \mathcal{B}(R^n)$.

If $n = 1$, then we simply say that x is an IF-observable.

Similarly as in the classical case the following theorem can be proved [7, 10].

Theorem 2.8. *Let $x : \mathcal{B}(R) \longrightarrow \mathcal{F}$ be an IF-observable, $\mathbf{m} : \mathcal{F} \longrightarrow [0, 1]$ be an IF-state. Define the mapping $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0, 1]$ by the formula $\mathbf{m}_x(C) = \mathbf{m}(x(C))$. Then $\mathbf{m}_x : \mathcal{B}(R) \longrightarrow [0, 1]$ is a probability measure.*

Proof. In [7] *Proposition 3.1.* □

Since now $\mathbf{m}_x : \mathcal{B}(R) \to [0, 1]$ plays an analogous role as $P_\xi : \mathcal{B}(R) \to [0, 1]$, we can define **IF-expected value** $\mathbf{E}(x)$ by the same formula (see [7]).

Definition 2.9. *We say that an IF-observable x is an integrable IF-observable, if the integral $\int_R t \; d\mathbf{m}_x(t)$ exists. In this case we define IF-expected value $\mathbf{E}(x) = \int_R t \; d\mathbf{m}_x(t)$. If the integral $\int_R t^2 \; d\mathbf{m}_x(t)$ exists, then we define IF-dispersion $\mathbf{D}^2(x)$ by the formula*

$$\mathbf{D}^2(x) = \int_R t^2 \; d\mathbf{m}_x(t) - \big(\mathbf{E}(x)\big)^2 = \int_R (t - \mathbf{E}(x))^2 \; d\mathbf{m}_x(t).$$

3 Product operation, joint IF-observable and function of several IF-observables

In [6] we introduced the notion of product operation on the family of IF-events $\mathcal{F}$ and showed an example of this operation.

Definition 3.1. *We say that a binary operation $\cdot$ on $\mathcal{F}$ is product if it satisfies the following conditions:*

(i) $(1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2)$ *for each* $(a_1, a_2) \in \mathcal{F}$;

(ii) *the operation $\cdot$ is commutative and associative;*

(iii) *if* $(a_1, a_2) \odot (b_1, b_2) = (0_\Omega, 1_\Omega)$ *and* $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$, *then*

$$(c_1, c_2) \cdot \big((a_1, a_2) \oplus (b_1, b_2)\big) = \big((c_1, c_2) \cdot (a_1, a_2)\big) \oplus \big((c_1, c_2) \cdot (b_1, b_2)\big)$$

and $\big((c_1, c_2) \cdot (a_1, a_2)\big) \odot \big((c_1, c_2) \cdot (b_1, b_2)\big) = (0_\Omega, 1_\Omega)$, *for each* $(c_1, c_2) \in \mathcal{F}$;

(iv) *if* $(a_{1n}, a_{2n}) \searrow (0_\Omega, 1_\Omega)$, $(b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$ *and* $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$, *then* $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \searrow (0_\Omega, 1_\Omega)$.

In the following theorem is the example of product operation for IF-events.

Theorem 3.2. *The operation $\cdot$ defined by $(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$ for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ is product operation on $\mathcal{F}$.*

Proof. In [6] *Theorem 1.* □

In [9] B. Riečan defined the notion of a joint IF-observable and he proved its existence.

Definition 3.3. *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be two IF-observables. The joint IF-observable of the IF-observables x, y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) *if* $A, B \in \mathcal{B}(R^2)$ *and* $A \cap B = \emptyset$, *then* $h(A \cup B) = h(A) \oplus h(B)$ *and* $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$;

(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^2)$ *and* $A_n \nearrow A$, *then* $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ *for each* $C, D \in \mathcal{B}(R)$.

Theorem 3.4. *For each two IF-observables* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *there exists their joint IF-observable.*

Proof. In [9] *Theorem 3.3.* □

Remark 3.5. *The joint IF-observable of IF-observables* x, y *from Definition 3.3 is two-dimensional IF-observable.*

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. About this says the following definition.

Definition 3.6. *Let* $x_1, \ldots, x_n : \mathcal{B}(R) \to \mathcal{F}$ *be IF-observables,* h_n *their joint IF-observable and* $g_n : R^n \to R$ *a Borel measurable function. Then we define the IF-observable* $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ *by the formula* $g_n(x_1, \ldots, x_n)(A) = h_n\big(g_n^{-1}(A)\big)$, *for each* $A \in \mathcal{B}(R)$.

4 Kolmogorov construction

In this section we introduce the notion of compatibility of intuitionistic fuzzy observables as follows.

Definition 4.1. *Let* $(\mathcal{F}, \cdot)$ *be a family of IF-events with product and* $x, y : \mathcal{B}(R) \to \mathcal{F}$ *be the IF-observables on* $\mathcal{F}$. *We say that IF-observables* x, y *are compatible, if there exists their joint IF-observable* h.

We can generalize the notion of compatibility for k IF-observables $x_{i_1}, \ldots, x_{i_k}$.

Definition 4.2. *Let* $J \subset N$, $J = \{i_1, \ldots, i_k\}$ *and* $x_{i_1}, \ldots, x_{i_k}$ *be the IF-observables on* $\mathcal{F}$. *We say that the IF-observables* $x_{i_1}, \ldots, x_{i_k}$ *are compatible, if there exists a mapping* $h_J : \mathcal{B}(R^{|J|}) \to \mathcal{F}$ *satisfying the following conditions:*

(i) $h_J(R^{|J|}) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$

(ii) *if* $A, B \in \mathcal{B}(R^{|J|})$ *and* $A \cap B = \emptyset$, *then* $h_J(A \cup B) = h_J(A) \oplus h_J(B)$ *and* $h_J(A) \odot h_J(B) = (0_\Omega, 1_\Omega)$;

(iii) *if* $A, A_1, \ldots \in \mathcal{B}(R^{|J|})$ *and* $A_n \nearrow A$, *then* $h_J(A_n) \nearrow h_J(A)$;

(iv) $h_J(A_{i_1} \times \ldots \times A_{i_k}) = x_{i_1}(A_{i_1}) \cdot \ldots \cdot x_{i_k}(A_{i_k})$ *for each* $A_{i_1}, \ldots, A_{i_k} \in \mathcal{B}(R)$.

By *Definition 4.2* to every compatible IF-observables $x_1, \ldots, x_n$ there exists a morphism $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ (i.e. $h_n(R^n) = (1_\Omega, 0_\Omega)$, h_n is additive and continuous) such that $h_n(A_1 \times \ldots \times A_n) = x_1(A_1) \cdot \ldots \cdot x_n(A_n)$, for each $A_1, \ldots, A_n \in \mathcal{B}(R)$.

Lemma 4.3. *The mapping* $h_J : \mathcal{B}(R^{|J|}) \to \mathcal{F}$ *from Definition 4.2 satisfies the following conditions:*

(v) *if* $A \in \mathcal{B}(R)$, *then* $h_J(\{(t_1, \ldots, t_i, \ldots, t_k) \mid (t_1, \ldots, t_i, \ldots, t_k) \in R^{|J|}, t_i \in A\}) = x_i(A)$;

(vi) *if* $J_1 \subset J_2 \subset N$, *then* $h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = h_{J_1}(A)$ *for each* $A \in \mathcal{B}(R^{|J_1|})$, *where* $\pi_{J_2,J_1} : \mathcal{B}(R^{|J_2|}) \to \mathcal{B}(R^{|J_1|})$ *is the projection.*

Proof. (v)

$$\begin{aligned} h_J(\{(t_1, \ldots, t_i, \ldots, t_k) | (t_1, \ldots, t_i, \ldots, t_k) \in R^{|J|}, t_i \in A\}) &= h_J(R \times \ldots \times R \times A \times R \times \ldots \times R) \\ &= x_1(R) \cdot \ldots \cdot x_{i-1}(R) \cdot x_i(A) \cdot x_{i+1}(R) \cdot \ldots \cdot x_k(R) \\ &= (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \cdot x_i(A) \cdot (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \\ &= (1_\Omega, 0_\Omega) \cdot x_i(A) \cdot (1_\Omega, 0_\Omega) = x_i(A) \end{aligned}$$

(vi) Let $J_1 \subset J_2 \subset N$; $A = A_{t_1} \times \ldots \times A_{t_k} \in \mathcal{B}(R^{|J_1|})$,

$$\pi^{-1}_{J_1,J_2}(A) = R \times \ldots \times R \times A_{t_1} \times \ldots \times A_{t_k} \times R \times \ldots \times R \in \mathcal{B}(R^{|J_2|}).$$

Then

$$\begin{aligned} h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) &= x_{s_1}(R) \cdot \ldots \cdot x_{s_i}(R) \cdot x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \cdot x_{s_j}(R) \cdot \ldots \cdot x_{s_n}(R) \\ &= (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \cdot x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \cdot (1_\Omega, 0_\Omega) \cdot \ldots \cdot (1_\Omega, 0_\Omega) \end{aligned}$$

$$\begin{aligned} &= (1_\Omega, 0_\Omega) \cdot x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \cdot (1_\Omega, 0_\Omega) \\ &= x_{t_1}(A_{t_1}) \cdot \ldots \cdot x_{t_k}(A_{t_k}) \\ &= h_{J_1}(A). \end{aligned}$$

Let we consider the set $\mathcal{L} = \{A \in \mathcal{B}(R^{|J|}),\ h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = h_{J_1}(A)\}$ and denote $\mathcal{D}$ the set of all rectangles $A_{t_1} \times \ldots \times A_{t_k}$; $A_{t_1}, \ldots, A_{t_k} \in \mathcal{B}(R)$.

Evidently $\mathcal{L} \supset \mathcal{D}$. The properties of mapping h_J imply that $\mathcal{L}$ is $q-\sigma$-algebra over ring $s(\mathcal{D})$ generated by set $\mathcal{D}$. Therefore

$$\mathcal{L} \supset q - \sigma(s(\mathcal{D})) = \sigma(s(\mathcal{D})) = \mathcal{B}(R^{|J_1|})$$

that implies $h_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = h_{J_1}(A)$, where $A \in \mathcal{B}(R^{|J_1|})$. □

Proposition 4.4. *Let* $\mathbf{m}$ *be an IF-state on a family of IF-events with product* $(\mathcal{F}, \cdot)$*. Define* $P_n : \mathcal{B}(R^n) \to [0,1]$ *by the formula*

$$P_n(A) = \mathbf{m}(h_n(A)), \quad A \in \mathcal{B}(R).$$

Then P_n *is a probability measure such that*

$$P_n(\{(t_1, \ldots, t_i, \ldots, t_n) \mid (t_1, \ldots, t_i, \ldots, t_n) \in R^n,\ t_i \in A\}) = \mathbf{m}(x_i(A)) = \mathbf{m}_{x_i}(A).$$

Proof. The first assertion is clear. Further

$$\begin{aligned} P_n(\{(t_1, \ldots, t_i, \ldots, t_n)|(t_1, \ldots, t_i, \ldots, t_n) \in R^n, t_i \in A\}) &= \mathbf{m}(h_n(R \times \ldots \times R \times A \times R \times \ldots \times R)) \\ &= \mathbf{m}(x_1(R) \cdot \ldots \cdot x_{i-1}(R) \cdot x_i(A) \cdot x_{i+1}(R) \cdot \ldots \cdot x_n(R)) \\ &= \mathbf{m}((1_\Omega, 0_\Omega) \cdot x_i(A) \cdot (1_\Omega, 0_\Omega)) = \mathbf{m}(x_i(A)) = \mathbf{m}_{x_i}(A). \end{aligned}$$

□

Proposition 4.5. *Let* $\emptyset \neq J \subset N$*,* J *be finite,* $J = \{t_1, \ldots, t_k\}$*. Then there exists exactly one probability measure* $P_J : \mathcal{B}(R^k) \to [0,1]$ *such that*

$$P_J(A_1 \times \ldots \times A_k) = \mathbf{m}(x_{t_1}(A_1) \cdot \ldots \cdot x_{t_k}(A_k))$$

for each $A_1, \ldots, A_k \in \mathcal{B}(R)$.

Proof. Let $I = \{1, \ldots, t_k\} \supset J$, $\pi_{I,J}$ be the projection from R^{t_k} to R^k. Then

$$\pi^{-1}_{I,J}(A_1 \times \ldots \times A_k) = B_1 \times \ldots \times B_{t_k},$$

where $B_{t_i} = A_i$ $(i = 1, 2, \ldots, k)$; $B_j = R$ if $j \notin J$. Therefore

$$\begin{aligned} P_{t_k}(\pi^{-1}_{I,J}(A_1 \times \ldots \times A_k)) &= P_{t_k}(B_1 \times \ldots \times B_{t_k}) \\ &= \mathbf{m}(h_{t_k}(B_1 \times \ldots \times B_{t_k})) \\ &= \mathbf{m}(x_1(B_1) \cdot \ldots \cdot x_{t_k}(B_{t_k})) \\ &= \mathbf{m}(x_{t_1}(A_1) \cdot \ldots \cdot x_{t_k}(A_k)). \end{aligned}$$

Put $P_J = P_{t_k} \circ \pi^{-1}_{I,J} : \mathcal{B}(R^k) \to [0,1]$. Then P_J is a probability measure with the property stated in *Proposition 4.5.*

If μ is other measure with this property, then P_J coincides with μ on each rectangles and therefore they coincide on $\mathcal{B}(R^k)$. □

By property (vi) we obtained a family of probability measures $\{P_J \mid \emptyset \neq J \subset N, J \text{ finite}\}$ given by

$$P_J(A) = \mathbf{m}\big(h_J(A)\big),\ A \in \mathcal{B}(R).$$

The family satisfies the Kolmogorov consistency condition. E.g., if $J_2 = \{1,2,3\}$, $J_1 = \{1,3\}$ and $\pi_{J_2,J_1} : R^3 \to R^2$ is the projection (assigning to a triple (t_1, t_2, t_3) a pair (t_1, t_3)) then

$$\begin{aligned} P_{J_2}(\pi^{-1}_{J_2,J_1}(A \times B)) &= P_{J_2}(\{(t_1, t_2, t_3) : (t_1, t_3) \in A \times B\}) \\ &= P_{J_2}(A \times R \times B) \\ &= \mathbf{m}(x_1(A) \cdot x_2(R) \cdot x_3(B)) \\ &= \mathbf{m}(x_1(A) \cdot x_3(B)) \\ &= P_{J_1}(A \times B). \end{aligned}$$

$$P_{J_2} \circ \pi^{-1}_{J_2,J_1} = P_{J_1}.$$

Proposition 4.6. *The family $\{P_J \mid \emptyset \neq J \subset N, J$ finite$\}$ satisfies the Kolmogorov consistency condition, i.e.*

$$P_{J_2}(\pi^{-1}_{J_2,J_1}(A)) = P_{J_1}(A)$$

whenever $J_1 \subset J_2$, $A \in \mathcal{B}(R^{|J_1|})$, where $\pi_{J_2,J_1} : R^{|J_2|} \to R^{|J_1|}$ is the projection.

Proof. P_{J_1} and $P_{J_2} \circ \pi^{-1}_{J_2,J_1}$ are two measures on $\mathcal{B}(R^{|J_1|})$ coinciding on the family of all rectangles. □

At this point we may use the Kolmogorov consistency theorem.

Proposition 4.7. *Let $\mathcal{C}$ be the family of all cylinders in R^N, i.e.*

$$\mathcal{C} = \{\pi^{-1}_J(A) \mid \emptyset \neq J \subset N, J \text{ finite}, A \in \mathcal{B}(R^{|J|})\}.$$

Then there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that $P(\pi^{-1}_J(A)) = P_J(A)$, for each cylinders $\pi^{-1}_J(A)$. Particularly

$$P(\{(t_n)_1^\infty : t_i \in A_i, i = 1,2,\ldots,n\}) = \mathbf{m}(h_n(A_1 \times \ldots \times A_n)) = \mathbf{m}(x_1(A_1) \cdot \ldots \cdot x_n(A_n)).$$

Proof. It follows by the Kolmogorov theorem and *Proposition 4.6*. □

Proposition 4.8. *Define the coordinate function $\xi_n : R^N \to R$ by the formula $\xi_n((t_i)_1^\infty) = t_n$. Then ξ_n is a random variable with respect to $\sigma(\mathcal{C})$ such that $P_{\xi_n} = \mathbf{m} \circ x_n = \mathbf{m}_{x_n}$.*

Proof. If $A \in \mathcal{B}(R)$, then $\xi_n^{-1}(A) = \{(t_i)_1^\infty; t_n \in A\} = \pi^{-1}_{\{n\}}(A) \in \mathcal{C}$. Moreover

$$P_{\xi_n}(A) = P(\xi_n^{-1}(A)) = P(\pi^{-1}_{\{n\}}(A)) = P_{\{n\}}(A) = \mathbf{m}(x_n(A)) = \mathbf{m}_{x_n}(A).$$

□

Remark 4.9. *By the preceding procedure to each sequence $(x_n)_n$ we can construct a sequence $(\xi_n)_n$ of a random variables.*

5 Lower and upper limits, m-almost everywhere convergence

In [4] we defined the notions of lower and upper limits for a sequence of IF-observables and showed the connection between two kinds of **m**-almost everywhere convergence.

Definition 5.1. *We shall say that a sequence $(x_n)_n$ of IF-observables has $\limsup\limits_{n\to\infty}$, if there exists an IF-observable $\overline{x} : \mathcal{B}(R) \to \mathcal{F}$ such that*

$$\overline{x}((-\infty,t)) = \bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for every $t \in R$. We write $\overline{x} = \limsup\limits_{n\to\infty} x_n$.

Note that if another IF-observable y satisfies the above condition, then $\mathbf{m} \circ y = \mathbf{m} \circ \overline{x}$.

Definition 5.2. *A sequence $(x_n)_n$ of IF-observables has $\liminf\limits_{n\to\infty}$, if there exists an IF-observable $\underline{x}$ such that*

$$\underline{x}((-\infty,t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)$$

for all $t \in R$. Notation: $\underline{x} = \liminf\limits_{n\to\infty} x_n$.

Proposition 5.3. *A sequence $(x_n)_n$ of an IF-observables converges **m**-almost everywhere to 0 if and only if*

$$\mathbf{m}\Big(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\Big(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t - \frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\big(0_{\mathcal{F}}((-\infty,t))\big),$$

for every $t \in R$.

Proof. In [4] *Proposition 4.1.* □

In accordance with *Proposition 5.3* we can extend the notion of **m**-almost everywhere convergence by the following way.

Definition 5.4. *A sequence $(x_n)_n$ of an IF-observables converges* **m**-*almost everywhere to an IF-observable x, if*

$$\mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\Big(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)\Big) = \mathbf{m}\big(x((-\infty,t))\big),$$

for every $t \in R$.

The next theorem is important for the proof of the Individual ergodic theorem in intuitionistic fuzzy case, where we work with the sequence of several IF-observables induced by the Borel function.

Theorem 5.5. *Let $(x_n)_n$ be a sequence of IF-observables, $(\xi_n)_n$ be the sequence of corresponding projections, $(g_n)_n$ be a sequence of Borel measurable functions $g_n : R^n \to R$. If the sequence $\big(g_n(\xi_1,\ldots,\xi_n)\big)_n$ converges P-almost everywhere, then the sequence $\big(g_n(x_1,\ldots,x_n)\big)_n$ converges* **m**-*almost everywhere and*

$$\mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big) = \mathbf{m}\Big(\liminf_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big)$$

for each $t \in R$. Moreover

$$P\Big(\{u \in R^N : \limsup_{n\to\infty} g_n\big(\xi_1(u),\ldots,\xi_n(u)\big) < t\}\Big) = \mathbf{m}\Big(\limsup_{n\to\infty} g_n(x_1,\ldots,x_n)\big((-\infty,t)\big)\Big)$$

for each $t \in R$.

Proof. In [5] *Theorem 5.1.* □

6 Individual ergodic theorem

First we recall the classical Individual ergodic theorem. Let (X,σ,P) be a probability space, $T : X \to X$ be a measure preserving transformation (i.e. $A \in \sigma$ implies $T^{-1}(A) \in \sigma$ and $P\big(T^{-1}(A)\big) = P(A)$), $\xi : X \to R$ be an integrable random variable. Then there exists an integrable random variable ξ^* such that the following conditions are satisfied:

(i) $E(\xi) = E(\xi^*)$,

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}(\xi \circ T^i) = \xi^*$ P-almost everywhere,

(iii) $\xi^* = \xi \circ T$ P-almost everywhere.

We defined the IF-mean value of an IF-observable and **m**-almost everywhere convergence in the previous sections. Now we must define a transformation preserving an IF-state **m**.

Definition 6.1. *Let $(\mathcal{F},\cdot)$ be a family of IF-events with product,* **m** *be an IF-state. Then a mapping $\tau : \mathcal{F} \to \mathcal{F}$ is said to be a* **m**-*preserving transformation, if the following conditions are satisfied:*

(i) $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$;

(ii) *if* $\mathbf{A},\mathbf{B} \in \mathcal{F}$ *and* $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$, *then* $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$ *and* $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$;

(iii) *if* $\mathbf{A}_n \nearrow \mathbf{A}$, $\mathbf{A}_n, \mathbf{A} \in \mathcal{F}$, $n \in \mathbf{N}$, *then* $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$;

(iv) $\mathbf{m}\big(\tau(\mathbf{A}) \cdot \tau(\mathbf{B})\big) = \mathbf{m}(\mathbf{A} \cdot \mathbf{B})$ *for each* $\mathbf{A},\mathbf{B} \in \mathcal{F}$.

The next theorem says about a representation of **m**-preserving transformation.

Theorem 6.2. *Let $\tau : \mathcal{F} \to \mathcal{F}$ be a $\mathbf{m}$-preserving transformation, where $\mathbf{m}$ is an IF-state. For each $\mathbf{A} = (\mu_A, \nu_A) \in \mathcal{F}$ denote*

$$\tau(\mathbf{A}) = \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big).$$

Then the mappings $\tau^\flat, \tau^\sharp : \mathcal{T} \to \mathcal{T}$ are the measure preserving transformations in a tribe $\mathcal{T} = \{f : \Omega \to [0,1]; f$ is $\mathcal{S}-$ measurable$\}$.

Proof. Let $\tau : \mathcal{F} \to \mathcal{F}$ be a $\mathbf{m}$-preserving transformation. Then from *Definition 6.1* we obtain that the mapping τ is satisfying the four conditions:

(i) Let $\tau\big((1_\Omega, 0_\Omega)\big) = (1_\Omega, 0_\Omega)$. Then $(1_\Omega, 0_\Omega) = \tau\big((1_\Omega, 0_\Omega)\big) = \big(\tau^\flat(1_\Omega), 1_\Omega - \tau^\sharp(1_\Omega)\big)$ and therefore we have $1_\Omega = \tau^\flat(1_\Omega), 1_\Omega = \tau^\sharp(1_\Omega)$.

(ii) Let $\mathbf{A}, \mathbf{B} \in \mathcal{F}$ and $\mathbf{A} \odot \mathbf{B} = (0_\Omega, 1_\Omega)$. Then $(0_\Omega, 1_\Omega) = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B) = \big((\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega, (\nu_A + \nu_B) \wedge 1_\Omega\big)$.

$$0_\Omega = (\mu_A + \mu_B - 1_\Omega) \vee 0_\Omega \quad \text{and} \quad 1_\Omega = (\nu_A + \nu_B) \wedge 1_\Omega.$$

Therefore,

$$\begin{aligned} \mu_A + \mu_B &\leq 1_\Omega &(1)\\ \nu_A + \nu_B &\geq 1_\Omega. &(2) \end{aligned}$$

By (2) we obtain

$$(1_\Omega - \nu_A) \odot (1_\Omega - \nu_B) = \big(1_\Omega - (\nu_A + \nu_B)\big) \vee 0_\Omega = 0_\Omega.$$

Since $\tau(\mathbf{A}) \odot \tau(\mathbf{B}) = (0_\Omega, 1_\Omega)$, we get

$$\begin{aligned} (0_\Omega, 1_\Omega) &= \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big) \odot \big(\tau^\flat(\mu_B), 1_\Omega - \tau^\sharp(1_\Omega - \nu_B)\big)\\ &= \big(\tau^\flat(\mu_A) \odot \tau^\flat(\mu_B), (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \oplus (1_\Omega - \tau^\sharp(1_\Omega - \nu_B))\big) \end{aligned}$$

$$\begin{aligned} 0_\Omega &= \tau^\flat(\mu_A) \odot \tau^\flat(\mu_B)\\ 1_\Omega &= (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \oplus (1_\Omega - \tau^\sharp(1_\Omega - \nu_B))\\ &= \big(1_\Omega - \tau^\sharp(1_\Omega - \nu_A) + 1_\Omega - \tau^\sharp(1_\Omega - \nu_B)\big) \wedge 1_\Omega. \end{aligned}$$

Therefore $\tau^\sharp(1_\Omega - \nu_A) + \tau^\sharp(1_\Omega - \nu_B) \leq 1_\Omega$ and

$$\tau^\sharp(1_\Omega - \nu_A) \odot \tau^\sharp(1_\Omega - \nu_B) = \big(\tau^\sharp(1_\Omega - \nu_A) + \tau^\sharp(1_\Omega - \nu_B) - 1_\Omega\big) \vee 0_\Omega = 0_\Omega.$$

Finally since $\tau(\mathbf{A} \oplus \mathbf{B}) = \tau(\mathbf{A}) \oplus \tau(\mathbf{B})$, then

$$\begin{aligned} \tau\big((\mu_A \oplus \mu_B, \nu_A \odot \nu_B)\big) &= \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big) \oplus \big(\tau^\flat(\mu_B), 1_\Omega - \tau^\sharp(1_\Omega - \nu_B)\big),\\ \big(\tau^\flat(\mu_A \oplus \mu_B), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A \odot \nu_B)\big) &= \big(\tau^\flat(\mu_A) \oplus \tau^\flat(\mu_B), (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \odot (1_\Omega - \tau^\sharp(1_\Omega - \nu_B))\big). \end{aligned}$$

Hence $\tau^\flat(\mu_A \oplus \mu_B) = \tau^\flat(\mu_A) \oplus \tau^\flat(\mu_B)$, and

$$1_\Omega - \tau^\sharp(1_\Omega - \nu_A \odot \nu_B) = (1_\Omega - \tau^\sharp(1_\Omega - \nu_A)) \odot (1_\Omega - \tau^\sharp(1_\Omega - \nu_B))$$

and using de Morgan rule: $f \odot g = 1_\Omega - (1_\Omega - f) \oplus (1_\Omega - g)$ on the second equality we obtain

$$1_\Omega - \tau^\sharp\big((1_\Omega - \nu_A) \oplus (1_\Omega - \nu_B)\big) = 1_\Omega - \tau^\sharp(1_\Omega - \nu_A) \oplus \tau^\sharp(1_\Omega - \nu_B).$$

Therefore

$$\tau^\sharp\big((1_\Omega - \nu_A) \oplus (1_\Omega - \nu_B)\big) = \tau^\sharp(1_\Omega - \nu_A) \oplus \tau^\sharp(1_\Omega - \nu_B).$$

(iii) If $\mathbf{A}_n = (\mu_{A_n}, \nu_{A_n}) \nearrow \mathbf{A} = (\mu_A, \nu_A)$, i.e. $\mu_{A_n} \nearrow \mu_A$ and $\nu_{A_n} \searrow \nu_A$, then $\tau(\mathbf{A}_n) \nearrow \tau(\mathbf{A})$.
Hence

$$\begin{aligned} \big(\tau^\flat(\mu_{A_n}), 1_\Omega - \tau^\sharp(1_\Omega - \nu_{A_n})\big) &\nearrow \big(\tau^\flat(\mu_A), 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\big)\\ \tau^\flat(\mu_{A_n}) &\nearrow \tau^\flat(\mu_A)\\ 1_\Omega - \tau^\sharp(1_\Omega - \nu_{A_n}) &\searrow 1_\Omega - \tau^\sharp(1_\Omega - \nu_A)\\ \tau^\sharp(1_\Omega - \nu_{A_n}) &\nearrow \tau^\sharp(1_\Omega - \nu_A). \end{aligned}$$

(iv) Let $\mathbf{m}\big(\tau(\mathbf{A})\cdot\tau(\mathbf{B})\big)=\mathbf{m}(\mathbf{A}\cdot\mathbf{B})$ for each $\mathbf{A},\mathbf{B}\in\mathcal{F}$. Then

$$\begin{aligned}\tau(\mathbf{A})\cdot\tau(\mathbf{B}) &= \mathbf{A}\cdot\mathbf{B},\\ \big(\tau^\flat(\mu_A),1_\Omega-\tau^\sharp(1_\Omega-\nu_A)\big)\cdot\big(\tau^\flat(\mu_B),1_\Omega-\tau^\sharp(1_\Omega-\nu_B)\big) &= (\mu_A,\nu_A)\cdot(\mu_B,\nu_B),\\ \big(\tau^\flat(\mu_A)\cdot\tau^\flat(\mu_B),1_\Omega-\tau^\sharp(1_\Omega-\nu_A)\cdot\tau^\sharp(1_\Omega-\nu_B)\big) &= (\mu_A\cdot\mu_B,1_\Omega-(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B)).\end{aligned}$$

Hence

$$\begin{aligned}\tau^\flat(\mu_A)\cdot\tau^\flat(\mu_B) &= \mu_A\cdot\mu_B,\\ 1_\Omega-\tau^\sharp(1_\Omega-\nu_A)\cdot\tau^\sharp(1_\Omega-\nu_B) &= 1_\Omega-(1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B),\\ \tau^\sharp(1_\Omega-\nu_A)\cdot\tau^\sharp(1_\Omega-\nu_B) &= (1_\Omega-\nu_A)\cdot(1_\Omega-\nu_B).\end{aligned}$$

□

Theorem 6.3. (Individual ergodic theorem) *Let $(\mathcal{F},\cdot)$ be a family of IF-events with product,* $\mathbf{m}$ *be an IF-state. Let x be an integrable IF-observable and τ be an $\mathbf{m}$-preserving transformation. Then there exists an integrable IF-observable x^* such that*

(i) $\mathbf{E}(x)=\mathbf{E}(x^*)$,

(ii) $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=0}^{n-1}(\tau^i\circ x)=x^*$ $\mathbf{m}$*-almost everywhere.*

Proof. Let $x_n=\tau^{n-1}\circ x$ $(n=1,2,\ldots)$, i.e

$$x_1=x, x_2=\tau\circ x, x_3=\tau^2\circ x,\ldots. \tag{3}$$

Let us return to the Kolmogorov extension process (see *Proposition 4.7*). Let us consider the probability space $(R^N,\sigma(\mathcal{C}),P)$ such that

$$P\big(\{(t_i)_{i=1}^\infty : t_1\in A_1,\ldots,t_n\in A_n\}\big)=\mathbf{m}\big(x_1(A_1)\cdot\ldots\cdot x_n(A_n)\big) \tag{4}$$

for each $n\in N$ and $A_i\in\mathcal{B}(R)$.

Let $T:R^N\to R^N$ be the shift defined by the formula $T\big((t_n)_n\big)=(s_n)_n$, where $s_n=t_{n+1}$ $(n=1,2,\ldots)$.

Let $A=\{(t_i)_{i=1}^\infty : t_1\in A_1,\ldots,t_n\in A_n\}$ is the cylinder. In this case

$$T^{-1}(A)=\{(t_i)_{i=1}^\infty : T\big((t_i)_{i=1}^\infty\big)\in A\}=\{(t_i)_{i=1}^\infty : t_{1+1}\in A_1,\ldots,t_{n+1}\in A_n\}.$$

Therefore using (4), (3) and (iv) from *Definition 6.1* we have

$$\begin{aligned}P\big(T^{-1}(A)\big) &= \mathbf{m}\big(x_{1+1}(A_1)\cdot\ldots\cdot x_{n+1}(A_n)\big)\\ &= \mathbf{m}\big(\tau\circ x(A_1)\cdot\ldots\cdot\tau^n\circ x(A_n)\big)\\ &= \mathbf{m}\Big(\tau\big(x(A_1)\big)\cdot\ldots\cdot\tau\big(\tau^{n-1}\circ x(A_n)\big)\Big)\\ &= \mathbf{m}\Big(\tau\big(x_1(A_1)\big)\cdot\ldots\cdot\tau\big(x_n(A_n)\big)\Big)\\ &= \mathbf{m}\big(x_1(A_1)\cdot\ldots\cdot x_n(A_n)\big)=P(A).\end{aligned}$$

Hence the mapping T preserves the probability measure P, i.e. $P\big(T^{-1}(A)\big)=P(A)$.

Since the IF-observable $x=x_1$ is integrable, the first coordinate function ξ_1 defined by $\xi_1\big((t_i)_{i=1}^\infty\big)=t_1$ is integrable, too (see *Proposition 4.8* and *Definition 2.9*). Therefore by the Individual ergodic theorem there exists an integrable random variable ξ^* such that $E(\xi^*)=E(\xi)$ and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}(\xi\circ T^i)=\xi^* \qquad P-\text{almost everywhere.}$$

Of course $\xi\circ T^i=\xi_{i+1}$, hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\xi_j=\xi^* \qquad P-\text{almost everywhere.}$$

Put $g_n(u_1, \dots, u_n) = \frac{1}{n}\sum_{i=1}^{n} u_i$. By *Theorem 5.5* the sequence of IF-observables

$$\Big(g_n(x_1, \dots, x_n)\Big)_n = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)_n = \left(\frac{1}{n}\sum_{i=0}^{n-1} \tau^i \circ x\right)_n$$

is convergent $\mathbf{m}$-almost everywhere to the IF-observable $x^* = \limsup_{n\to\infty} g_n(x_1, \dots, x_n)$ and

$$P\big(\xi^*(-\infty, t)\big) = \mathbf{m}\big(x^*\big((-\infty, t)\big)\big)$$

for each $t \in R$. Since $P_{\xi^*} = \mathbf{m}_{x^*}$ and $P_{\xi_1} = \mathbf{m}_{x_1} = \mathbf{m}_x$, we obtain $\mathbf{E}(x) = E(\xi_1) = E(\xi^*) = \mathbf{E}(x^*)$. □

7 Conclusion

The paper is concerned with ergodic theory for family of intuitionistic fuzzy events. We proved the Individual ergodic theorem for intuitionistic fuzzy observables using $\mathbf{m}$-almost everywhere convergence, where $\mathbf{m}$ is an intuitinistic fuzzy state. Since the intuitionistic fuzzy probability $\mathcal{P}$ can be decomposited to two intuitionistic fuzzy states, then this results can be applied to $\mathcal{P}$-almost everywhere convergence, too.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets: Theory and applications*, Physica Verlag, New York, 1999.

[2] K. T. Atanassov, *On intuitionistic fuzzy sets*, Springer-Verlag Berlin Heidelberg, 2012.

[3] K. Čunderlíková, *The individual ergodic theorem on the IF-events with product*, Soft Computing - A Fusion of Foundations, Methodologies and Applications, **14**(3) (2010), 229-234.

[4] K. Čunderlíková, *Upper and lower limits and* $\mathbf{m}$*-almost everywhere convergence of intuitionistic fuzzy observables*, Notes on Intuitionistic Fuzzy Sets, **24**(4) (2018), 40-49.

[5] K. Čunderlíková, $\mathbf{m}$*-almost everywhere convergence of intuitionistic fuzzy observables induced by Borel measurable function*, Notes on Intuitionistic Fuzzy Sets, **25**(2) (2019), 29-40.

[6] K. Lendelová, *Conditional IF-probability*, Lawry J. et al. (eds), Advances in Soft Computing: Soft Methods for Integrated Uncertainty Modelling, **37** (2006), 275-283.

[7] K. Lendelová, B. Riečan, *Weak law of large numbers for IF-events*, B. De Baets, et al. (eds), Current Issues in Data and Knowledge Engineering, Warszawa, EXIT, (2004), 309-314.

[8] B. Riečan, *On a problem of Radko Mesiar: General form of IF-probabilities*, Fuzzy Sets and Systems, **152** (2006), 1485-1490.

[9] B. Riečan, *On the probability and random variables on IF events*, D. Ruan, et al. (eds), Applied Artifical Intelligence, Proc. 7th FLINS Conf. Genova, World Scientific, (2006), 138-145.

[10] B. Riečan, *Analysis of fuzzy logic models*, V. M. Koleshko, (ed), Intelligent Systems, INTECH, (2012), 219-244.

Iranian Journal of Fuzzy Systems
Volume 17, Number 5 (2020), pp. 200
University of Sistan and Baluchestan

Individual ergodic theorem for intuitionistic fuzzy observables using intuitionistic fuzzy state

K. Cunderlikova

قضیه ارگودیک منحصر بفرد برای مشاهده‌پذیرهای فازی شهودی با استفاده از حالت فازی شهودی

چکیده. نظریه ارگودیک منحصر بفرد روی σ-جبرها ساخته شده‌است. بعد از آن قضیه ارگودیک منحصر بفرد روی ساختارهای عمومی‌تر مانند MV-جبرها و ساختارهای کوانتم مورد مطالعه قرار گرفت. هدف از این مقاله، فرمول‌بندی قضیه ارگودیک منحصر بفرد برای مشاهده‌پذیرهای فازی شهودی با استفاده از m-تقریب همه‌جا همگرا، که m یک حالت فازی شهودی است. ساختار Kolmogorov برای مشاهده‌پذیرهای فازی را نیز نشان می‌دهیم.