

## Extended graphs based on $KM$ -fuzzy metric spaces

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### Abstract

This paper, applies the concept of  $KM$ -fuzzy metric spaces and introduces a novel concept of  $KM$ -fuzzy metric graphs based on  $KM$ -fuzzy metric spaces. This study, investigates the finite  $KM$ -fuzzy metric spaces with respect to metrics and  $KM$ -fuzzy metrics and constructs  $KM$ -fuzzy metric spaces on any given non-empty sets. It tries to extend the concept of  $KM$ -fuzzy metric spaces to a larger class of  $KM$ -fuzzy metric spaces such as union and product of  $KM$ -fuzzy metric spaces and in this regard investigates a class of products of  $KM$ -fuzzy metric graphs.

**Keywords:**  $KM$ -Fuzzy metric space, t-norm, (derivable)  $KM$ -fuzzy metric graph, C-graphable.

## 1 Introduction

As a generalization of the classical set theory, fuzzy set theory was introduced by Zadeh to deal with uncertainties[21]. Fuzzy set theory plays an important role in modeling and controlling unsure systems in nature, society and industry. Fuzzy set theory also plays a vital role in complex phenomena which is not easily characterized by classical set theory. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology. Fuzzy topology is a fundamental branch of fuzzy theory which has become an area of active research in the last years because of its wide range of applications. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of view. There have been several attempts to find an appropriate notion of metric spaces in fuzzy setting ([4, 6, 7, 14]), all of them being generalizations of the notion of metric spaces. They have introduced and studied the notion of  $KM$ -fuzzy metric spaces with respect to the concept of t-norms. Furthermore, the class of topological spaces that are fuzzy metrizable agrees with the class of metrizable-topological spaces. In 1975, Kramosil and Michalek introduced in [14], the notion of fuzzy metric spaces as  $KM$ -fuzzy metric spaces which is a generalization of the notion of crisp metrics and M. Grabiec added a lost condition to notion of  $KM$ -fuzzy metric spaces and completed this notation in [7]. In 2015, I. Mardones-Perez et al. studied the degree in which some topological and uniform properties of  $KM$ -fuzzy pseudometric spaces are satisfied and established some relations between  $KM$ -fuzzy pseudometric spaces and some particular fuzzy structures which appear naturally, the so called fuzzifying structures[17]. Fuzzy graphs introduced by Rosenfeld are finding an increasing number of applications in modelling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the difference between the traditional numerical models used in engineering and sciences[20]. The generalization of the fuzzy graphs is noticed by some researchers and more subject, such as fuzzy graph based on t-norm, intuitionistic fuzzy threshold graphs, m-polar fuzzy graph. Mordeson et al. generalized the definition of a fuzzy graph by replacing minimum in the basic definitions with an arbitrary t-norm. They developed a measure on the susceptibility of trafficking in persons for networks by using a t-norm other than minimum[19]. Further materials regarding graphs and hypergraphs are available in the literature too [1, 2, 3, 5, 8, 9, 10, 11, 13, 15, 18].

Regarding these points, we introduce the concept of C-graphable set and show that every non-empty set  $X$  is a C-graphable set. Also we proved that for every given set with respect to the concept of C-graphable sets one can

construct a metric space. The fuzzy metric spaces are not necessarily finite space, so one of our motivation of this work is a construction of finite  $KM$ -fuzzy metric space. This study presents a concept of  $KM$ -fuzzy metric graph. The main our motivation of this work is the concept of  $KM$ -fuzzy graph based on  $t$ -norm such as Domby  $t$ -norm, Godel  $t$ -norm and etc. We applied the notation of  $KM$ -fuzzy metric space to generate of finite  $KM$ -fuzzy metric graph. It is extended some production operations on  $KM$ -fuzzy metric spaces and so it is generalized on  $KM$ -fuzzy metric graphs.

## 2 Preliminaries

In this section, we recall some definitions and results, which we need use in what follows.

**Definition 2.1.** [12, 16] Let  $G_1 = (V_1, E_1)$ ,  $G_2 = (V_2, E_2)$  be simple graphs,  $(x_1, x_2)$ ,  $(y_1, y_2) \in V_1 \times V_2$ , where  $V_1 \times V_2$  is the vertex set of the following graphs:

(i) categorical(tensor, direct, cardinal, Kronecker) product graph  $G_1 \times G_2$ :

$$E(G_1 \times G_2) = \{(x_1, x_2)(y_1, y_2) \mid x_1y_1 \in E_1 \text{ and } x_2y_2 \in E_2\};$$

(ii) Cartesian product graph  $G_1 \otimes G_2$ :

$$E(G_1 \otimes G_2) = \{(x_1, x_2)(y_1, y_2) \mid (x_1 = y_1 \text{ and } x_2y_2 \in E_2) \text{ or } (x_1y_1 \in E_1 \text{ and } x_2 = y_2)\};$$

(iii) semi-strong product graph  $G_1 \bullet G_2$ :

$$E(G_1 \bullet G_2) = \{(x_1, x_2)(y_1, y_2) \mid (x_1 = y_1 \text{ and } x_2y_2 \in E_2) \text{ or } (x_1y_1 \in E_1 \text{ and } x_2y_2 \in E_2)\};$$

(iv) strong product (symmetric composition) graph  $G_1 \odot G_2$ :

$$E(G_1 \odot G_2) = E(G_1 \otimes G_2) \cup E(G_1 \times G_2);$$

(v) lexicographic product (composition) graph  $G_1 \circ G_2(G_1.G_2, G_1[G_2])$ :

$$E(G_1 \circ G_2) = \{(x_1, x_2)(y_1, y_2) \mid (x_1y_1 \in E_1) \text{ or } (x_1 = y_1 \text{ and } x_2y_2 \in E_2)\};$$

(vi) union graph  $G_1 \cup G_2$ :

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2); \text{ and } E(G_1 \cup G_2) = E(G_1) \cup E(G_2);$$

(vii) join product graph  $G_1 + G_2$ :

$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup E'$ , where  $E'$  is the set of all line joining  $V_1$  with  $V_2$ .

**Definition 2.2.** [18] A fuzzy graph  $G = (V, \sigma, \mu)$  is an algebraic structure of non-empty set  $V$  together with a pair of functions  $\sigma : V \rightarrow [0, 1]$  and  $\mu : V \times V \rightarrow [0, 1]$  such that for all  $x, y \in V$ ,  $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ . It is called  $\sigma$  as fuzzy vertex set and  $\mu$  as fuzzy edge set of  $G$ .

**Definition 2.3.** [17] A triplet  $(X, \rho, T)$  is called a  $KM$ -fuzzy metric space, if  $X$  is an arbitrary non-empty set,  $T$  is a left-continuous  $t$ -norm and  $\rho : X^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  is a fuzzy set, such that for each  $x, y, z, \in X$  and  $t, s \geq 0$ , we have:

- (i)  $\rho(x, y, 0) = 0$ ,
- (ii)  $\rho(x, x, t) = 1$  for all  $t > 0$ ,
- (iii)  $\rho(x, y, t) = \rho(y, x, t)$  (commutative property),
- (iv)  $T(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t + s)$  (triangular inequality),
- (vi)  $\rho(x, y, -) : \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  is a left-continuous map,
- (vii)  $\rho(x, y, t) \rightarrow 1$ , when  $t \rightarrow \infty$ .
- (viii)  $\rho(x, y, t) = 1, \forall t > 0$  implies that  $x = y$ .

If  $(X, \rho, T)$  satisfies in conditions (i)-(vii), then it is called  $KM$ -fuzzy pseudometric space and  $\rho$  is called a  $KM$ -fuzzy pseudometric.

**Theorem 2.4.** [17] Let  $(X, \rho, T)$  be a  $KM$ -fuzzy metric space. Then  $\rho(x, y, -) : \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  is a non-decreasing map.

### 3 Finite KM-fuzzy metric space

In this section, we apply the concept of KM-fuzzy metric spaces and construct a new class of KM-fuzzy metric spaces under operation product and union of KM-fuzzy metric spaces. In addition, for any given non-empty set we construct KM-fuzzy metric space with respect to  $\alpha$ -discrete metric, where  $\alpha \in \mathbb{R}^+$ . From now on, for all  $x, y \in [0, 1]$  we consider  $T_{min}(x, y) = \min\{x, y\}$ ,  $T_{pr}(x, y) = xy$ ,  $T_{lu}(x, y) = \max(0, x + y - 1)$ ,  $T_{do}(x, y) = \frac{xy}{x + y - xy}$  and  $\mathcal{C}_T = \{T : [0, 1] \times [0, 1] \rightarrow [0, 1] \mid T \text{ is a left-continuous t-norm}\}$ .

**Theorem 3.1.** *If  $(X, \rho, T_{min})$  is a KM-fuzzy metric space and  $T \in \mathcal{C}_T$ . Then  $(X, \rho, T)$  is a KM-fuzzy metric space.*

*Proof.* Let  $x, y, z \in X$ ,  $r, s \in \mathbb{R}^{\geq 0}$  and  $T \in \mathcal{C}_T$ . Since for all  $x, y \in [0, 1]$ ,  $T(x, y) \leq T_{min}(x, y)$ , we get that  $T(\rho(x, y, t), \rho(y, z, s)) \leq T_{min}(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t + s)$ . Hence  $(X, \rho, T)$  is a KM-fuzzy metric space.  $\square$

Let  $X$  be an arbitrary set and  $\alpha \in \mathbb{R}^+$ . For all  $x, y \in X$ , define  $d_\alpha : X \times X \rightarrow \mathbb{R}$  by  $d_\alpha(x, y) = 0$ , where  $x = y$  and  $d_\alpha(x, y) = \alpha$ , where  $x \neq y$  as an  $\alpha$ -discrete metric. So we have the following theorem.

**Theorem 3.2.** *Let  $X$  be an arbitrary set and  $|X| \geq 2$ . Then there exists a fuzzy set  $\rho : X^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$ , such that  $(X, \rho, T_{min})$  is a KM-fuzzy metric space.*

*Proof.* Let  $|X| \geq 2$  and  $\alpha \in \mathbb{R}^+$  be a fixed element. Clearly  $(X, d_\alpha)$  is a metric space, now for all  $x, y \in X, 0 \neq m, s, t \in \mathbb{R}^{\geq 0}$ , define  $\rho : X^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  by  $\rho(x, y, 0) = 0$  and  $\rho(x, y, t > 0) = \frac{\varphi(t)}{\varphi(t) + md_\alpha(x, y)}$ , where  $\varphi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$  is an increasing continuous function and for all  $x, y \in X$ , we have  $\varphi(t) + md_\alpha(x, y) \neq 0$  and  $\varphi(t) \rightarrow 0$ , whence  $t \rightarrow 0$ . Now, we show that  $(X, \rho, T_{min})$  is a KM-fuzzy metric space. We prove only the triangular inequality and for all  $x, y, z \in X$ , consider the five cases  $x = y = z, x = y \neq z, x = z \neq y, x \neq y = z$  and  $x \neq y \neq z$ . In all cases for  $0 \in \{t, s\}$  it is clear, now for  $0 \notin \{t, s\}$  we investigate it. For  $x = y \neq z$ , since  $\varphi(t + s) \geq \varphi(s)$ , we have  $\varphi(t + s)(\varphi(s) + m\alpha) - \varphi(s)(\varphi(t + s) + m\alpha) \geq 0$  and so  $\frac{\varphi(s)}{\varphi(s) + m\alpha} \leq \frac{\varphi(t + s)}{\varphi(t + s) + m\alpha}$ . If  $x \neq y \neq z$ , then  $d_\alpha(x, y) = d_\alpha(z, y) = d_\alpha(x, z) = \alpha$ . Since  $\varphi$  is an increasing map, we get that  $m\alpha\varphi(t) \leq m\alpha\varphi(t + s)$  and it implies that  $\varphi(t)(\varphi(t + s) + m\alpha) \leq \varphi(t + s)(\varphi(t) + m\alpha)$  and so  $\frac{\varphi(t)}{\varphi(t) + m\alpha} \leq \frac{\varphi(t + s)}{\varphi(t + s) + m\alpha}$ , which means that  $\rho(x, y, t) \leq \rho(x, z, t + s)$ . By a similar way,  $\rho(z, y, s) \leq \rho(x, z, t + s)$  and so  $T_{min}(\rho(x, y, t), \rho(z, y, s)) \leq \rho(x, z, t + s)$ .

The other cases, are similar and so  $(X, \rho, T_{min})$  is a KM-fuzzy metric space.  $\square$

**Corollary 3.3.** *Let  $X$  be an arbitrary set and  $|X| \geq 2$ . Then there exists a fuzzy set  $\rho : X^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$ , such that for all  $T \in \mathcal{C}_T$ ,  $(X, \rho, T)$  is a KM-fuzzy metric space.*

**Example 3.4.** *Let  $X = \{a, b, c\}$ . For all  $x, y \in X$ , define  $\rho(x, y, 0) = 0$  and  $\rho(x, y, t > 0) = \frac{t^5}{t^5 + d_3(x, y)}$ . Then by Corollary 3.3,  $(X, \rho, T_{pr})$  is a KM-fuzzy metric space.*

#### 3.1 Finite KM-fuzzy metric space based on metric

In this subsection, we apply the concept of finite metric for constructing KM-fuzzy metric space on any given non-empty set.

**Definition 3.5.** *Let  $X$  be a finite set. We say that  $X$  is a C-graphable set, if  $G = (X, E)$  is a connected graph, where  $E \subseteq X \times X$  and  $G = (X, E)$  is called an  $X$ -derived graph. Let  $\mathcal{G}_X$  be the set of all connected graphs which are constructed on  $X$  as the set of vertices, so we have the following results.*

*Let  $G = (X, E)$  be a connected graph. For all  $x, y \in X$ , define  $d^g(x, y) = \min\{|P_{x,y}| \text{ where } P_{x,y} \text{ is a path between } x, y\}$ . Obviously,  $d^g$  is a metric on  $X$ .*

**Theorem 3.6.** *Let  $X$  be a finite set and  $|X| \geq 2$ . Then there exists a non-discrete metric  $d$  on  $X$  such that  $(X, d)$  is a metric space.*

*Proof.* Let  $|X| \geq 2$ . Clearly,  $X$  is a C-graphable set and so there exists a graph  $G = (X, E) \in \mathcal{G}_X$ . For all  $x, y \in X$ , define  $d(x, y) = d^g(x, y)$ . Clearly  $(X, d^g)$  is a metric space.  $\square$

**Corollary 3.7.** *Let  $n \in \mathbb{N}, X$  be a set and  $|X| = n$ .*

- (i) If  $G = (X, E) \cong K_n$  is the complete graph, then for metric spaces  $(X, d^g)$  and  $(X, d_1)$ , we have  $d^g = d_1$ .
- (ii) If  $G = (X, E) \cong C_n$  is the cycle graph, then for metric spaces  $(X, d^g)$  and  $(X, d_1)$ , we have  $d_1 \leq d^g \leq d \lfloor \frac{n}{2} \rfloor$ .

**Theorem 3.8.** Let  $X$  be a non-empty set. Then there exists a fuzzy subset  $\rho : X^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$ , such that  $(X, \rho, T_{pr})$  is a  $KM$ -fuzzy metric space.

*Proof.* Let  $|X| \geq 2$ . Then clearly,  $X$  is a  $C$ -graphable set and by Theorem 3.6,  $(X, d^g)$  is a metric space. For all  $x, y \in X$  and for all  $0 \neq m, t \in \mathbb{R}^{\geq 0}$ , define  $\rho(x, y, 0) = 0$  and  $\rho(x, y, t > 0) = \frac{\varphi(t)}{\varphi(t) + md^g(x, y)}$ , where  $\varphi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$  is an increasing continuous function,  $\varphi(t) + md^g(x, y) \neq 0$  and  $\varphi(t) \rightarrow 0$ , whence  $t \rightarrow 0$ . Now, we show that  $(X, \rho, T_{pr})$  is a  $KM$ -fuzzy metric space and in this regard, only prove triangular inequality property. Let  $x, y, z \in X$ . For  $0 \in \{t, s\}$  it is clear, now for  $0 \notin \{t, s\}$  we investigate it. Since for all  $s, t, m \in \mathbb{R}^+$ ,  $\varphi(t+s)\varphi(s)md^g(x, y) + \varphi(t+s)\varphi(t)md^g(y, z) \geq \varphi(t)\varphi(s)md^g(x, y) + \varphi(s)\varphi(t)md^g(y, z) \geq \varphi(s)\varphi(t)md^g(x, z)$ , and  $m^2d^g(y, z)d^g(y, z)\varphi(t+s) > 0$  we get that  $T_{pr}(\frac{\varphi(t)}{\varphi(t) + md^g(x, y)}, \frac{\varphi(s)}{\varphi(s) + md^g(y, z)}) \leq \frac{\varphi(t+s)}{\varphi(t+s) + md^g(x, z)}$ . It follows that  $T_{pr}(\rho(x, y, t), \rho(y, z, s)) \leq \rho(x, z, t+s)$  and so  $(X, \rho, T_{pr})$  is a  $KM$ -fuzzy metric space.  $\square$

**Corollary 3.9.** Let  $X$  be a non-empty set. Then there exists a fuzzy subset  $\rho : X^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$ , such that for all left-continuous  $t$ -norm  $T \leq T_{pr}$ ,  $(X, \rho, T)$  is a  $KM$ -fuzzy metric space.

### 3.2 Operations on $KM$ -fuzzy metric spaces

In this subsection, we extend  $KM$ -fuzzy metric spaces to union and product of  $KM$ -fuzzy metric spaces. Let  $(X_1, \rho_1, T)$  and  $(X_2, \rho_2, T)$  be  $KM$ -fuzzy metric spaces,  $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$  and  $t \in \mathbb{R}^{\geq 0}$ . For an arbitrary  $T \in \mathcal{C}_T$ , define  $T(\rho) : (X_1 \times X_2)^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  by  $T(\rho)((x_1, y_1), (x_2, y_2), t) = T(\rho_1(x_1, x_2, t), \rho_2(y_1, y_2, t))$ . So we have the following theorem.

**Theorem 3.10.** Let  $(X_1, \rho_1, T)$  and  $(X_2, \rho_2, T)$  be  $KM$ -fuzzy metric spaces. Then  $(X_1 \times X_2, T_{min}(\rho), T)$  is a  $KM$ -fuzzy metric space.

*Proof.* Let  $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X_1 \times X_2$  and  $t, s \in \mathbb{R}^{\geq 0}$ .

(i) Since for all  $x_1, x_2 \in X_1, y_1, y_2 \in X_2, \rho_1(x_1, x_2, 0) = 0$  and  $\rho_2(y_1, y_2, 0) = 0$ , we have  $T_{min}(\rho)((x_1, y_1), (x_2, y_2), 0) = 0$ .

(ii)  $T_{min}(\rho)((x_1, y_1), (x_2, y_2), t) = 1$  if and only if  $T_{min}(\rho_1(x_1, x_2, t), \rho_2(y_1, y_2, t)) = 1$  if and only if  $\rho_1(x_1, x_2, t) = \rho_2(y_1, y_2, t) = 1$  if and only if  $(x_1, y_1) = (x_2, y_2)$ .

(iii) It is clear that  $T_{min}(\rho)$  is a commutative map.

(iv)

$$\begin{aligned} & T(T_{min}(\rho)((x_1, y_1), (x_2, y_2), t), T_{min}(\rho)((x_2, y_2), (x_3, y_3), s)) \\ &= T(T_{min}(\rho_1(x_1, x_2, t), \rho_2(y_1, y_2, t)), T_{min}(\rho_1(x_2, x_3, s), \rho_2(y_2, y_3, s))) \\ &\leq T_{min}(T(\rho_1(x_1, x_2, t), \rho_1(x_2, x_3, s)), T(\rho_2(y_1, y_2, t), \rho_2(y_2, y_3, s))) \\ &\leq T_{min}(\rho_1(x_1, x_3, t+s), \rho_2(y_1, y_3, t+s)) \\ &= T_{min}(\rho)((x_1, y_1), (x_3, y_3), t+s). \end{aligned}$$

(v) Since  $\rho_1, \rho_2$  are left-continuous maps, we get that  $\rho$  is a left-continuous map.

(vi) Let  $t \rightarrow \infty$ . Then  $\text{Lim } T_{min}(\rho_1(x_1, x_2, t), \rho_2(y_1, y_2, t)) = T_{min}(\text{Lim } \rho_1(x_1, x_2, t), \text{Lim } \rho_2(y_1, y_2, t)) = T_{min}(1, 1) = 1$ . Thus  $(X_1 \times X_2, T_{min}(\rho), T)$  is a  $KM$ -fuzzy metric space.  $\square$

Let  $X_1 \cap X_2 = \emptyset$ ,  $(X_1, \rho_1, T)$  and  $(X_2, \rho_2, T)$  be  $KM$ -fuzzy metric spaces,  $x, y \in X_1 \cup X_2$  and  $t \in \mathbb{R}^{\geq 0}$ . Consider  $\epsilon(x, y, t) = \bigwedge_{\substack{x, u \in X_1 \\ y, v \in X_2}} (\rho_1(x, u, t) \wedge \rho_2(y, v, t))$ , define  $\rho_1 \cup \rho_2 : (X_1 \cup X_2)^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  by

$$(\rho_1 \cup \rho_2)(x, y, t) = \begin{cases} \rho_1(x, y, t) & \text{if } x, y \in X_1, \\ \rho_2(x, y, t) & \text{if } x, y \in X_2, \\ \epsilon(x, y, t) & \text{if } x \in X_1, y \in X_2, \end{cases}.$$

So we have the following theorem.

**Theorem 3.11.** *Let  $(X_1, \rho_1, T)$  and  $(X_2, \rho_2, T)$  be KM-fuzzy metric spaces. Then  $(X_1 \cup X_2, \rho_1 \cup \rho_2, T)$  is a KM-fuzzy metric space, where  $X_1 \cap X_2 = \emptyset$ .*

*Proof.* Let  $x, y, z \in X_1 \cup X_2$  and  $t, s \in \mathbb{R}^{\geq 0}$ . We only prove the triangular inequality property and other cases are immediate. Let  $x, y \in X_1$  (for  $x, y \in X_2$ , one can prove in a similar way), then  $T((\rho_1 \cup \rho_2)(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) = T(\rho_1(x, y, t), (\rho_1 \cup \rho_2)(y, z, s))$ . If  $z \in X_1$ , then  $T((\rho_1 \cup \rho_2)(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) = T(\rho_1(x, y, t), \rho_1(y, z, s)) \leq \rho_1(x, z, t+s) = (\rho_1 \cup \rho_2)(x, z, t+s)$ . If  $z \in X_2$ , then  $T((\rho_1 \cup \rho_2)(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) = T(\rho_1(x, y, t), \epsilon) \leq \epsilon = (\rho_1 \cup \rho_2)(x, z, t+s)$ . Let  $x \in X_1, y \in X_2$ . Then  $T((\rho_1 \cup \rho_2)(x, y, t), (\rho_1 \cup \rho_2)(y, z, s)) = T(\epsilon, (\rho_1 \cup \rho_2)(y, z, s))$ . If  $z \in X_2$ , since  $x \in X_1$  and  $y \in X_2$ , we get that  $(\rho_1 \cup \rho_2)(x, z, t+s) = \epsilon$  and so  $T(\epsilon, (\rho_1 \cup \rho_2)(y, z, s)) = T(\epsilon, \rho_2(y, z, s)) \leq \epsilon = (\rho_1 \cup \rho_2)(x, z, t+s)$ . If  $z \in X_1$ , since  $x \in X_1$  and  $y \in X_2$ , we get that  $(\rho_1 \cup \rho_2)(x, z, t+s) \neq \epsilon$  and so  $T(\epsilon, (\rho_1 \cup \rho_2)(y, z, s)) = T(\epsilon, \epsilon) \leq \epsilon \leq \rho_1(x, z, t+s) = (\rho_1 \cup \rho_2)(x, z, t+s)$ . It follows that  $(X_1 \cup X_2, \rho_1 \cup \rho_2, T)$  is a KM-fuzzy metric space.  $\square$

**Corollary 3.12.** *Let  $(X_1, \rho, T)$  and  $(X_2, \rho, T)$  be KM-fuzzy metric spaces, where  $X_1 \cap X_2 = \emptyset$ . Then  $(X_1 \cup X_2, \rho, T)$  is a KM-fuzzy metric space.*

**Theorem 3.13.** *Let  $(X, \rho, T)$  be a KM-fuzzy metric space and  $\varphi$  be a bijection on  $X$ . Then there exists a fuzzy subset  $\rho' : \varphi(X)^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  such that  $(\varphi(X), \rho', T)$  is a KM-fuzzy metric space.*

*Proof.* Let  $x, y, X$  and  $t \in \mathbb{R}^{\geq 0}$ . Define  $\rho' : \varphi(X)^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  by  $\rho'(\varphi(x), \varphi(y), t) = \rho(x, y, t)$ . It is clear that  $(\varphi(X), \rho', T)$  is a KM-fuzzy metric space.  $\square$

## 4 KM-Fuzzy metric graph

In this section, we introduce a novel concept as KM-fuzzy metric graphs and analyse some their properties. So it is shown that on any simple connected graph there exists a KM-fuzzy metric graph.

**Definition 4.1.** *Let  $(V, \rho, T)$  be a KM-fuzzy metric space and  $G^* = (V, E)$  be a simple graph. Then  $G = (\sigma, \mu, \rho, T)$  is called a KM-fuzzy metric graph (a strong KM-fuzzy metric graph) on  $G^*$ , if  $\sigma : V \rightarrow [0, 1]$ ,  $\mu : E \rightarrow [0, 1]$ , there exists some time  $t \in \mathbb{R}^{\geq 0}$  (for  $t = 0$ , we call starting time) such that for all  $xy \in E$ , we have  $T(\mu(xy), T(\sigma(x), \sigma(y))) \leq \rho(x, y, t) ( T(\mu(xy), T(\sigma(x), \sigma(y))) = \rho(x, y, t) )$ . We call  $\sigma$  as a KM-fuzzy metric vertex set of  $G$  and  $\mu$  as a KM-fuzzy metric edge set of  $G$ .*

**Proposition 4.2.** *Let  $G = (\sigma, \mu, \rho, T)$  be a KM-fuzzy metric graph on  $G^* = (V, E)$ . Then for all  $xy \in E$ , for starting time we have  $\mu(xy) = 0$  or  $\sigma(x) = 0$  or  $\sigma(y) = 0$ .*

*Proof.* Let  $xy \in E$ . Since  $G = (\sigma, \mu, \rho, T)$  is a KM-fuzzy metric graph on  $G^* = (V, E)$ , we get that  $T(\mu(xy), T(\sigma(x), \sigma(y))) \leq \rho(x, y, 0)$ . Hence  $T(\mu(xy), T(\sigma(x), \sigma(y))) = 0$  and so  $\mu(xy) = 0$  or  $\sigma(x) = 0$  or  $\sigma(y) = 0$ .  $\square$

Let  $G = (\sigma, \mu, \rho, T)$  be a KM-fuzzy metric graph on  $G^* = (V, E)$ . Consider  $\alpha_{min} = \bigwedge_{xy \in E} T(\mu(xy), T(\sigma(x), \sigma(y)))$ . Then we have the following result.

**Theorem 4.3.** *Let  $(V, \rho, T)$  be a KM-fuzzy metric space,  $G^* = (V, E)$  be a simple graph and  $G = (\sigma, \mu, \rho, T)$  be a KM-fuzzy metric graph on  $G^* = (V, E)$ . For any  $\alpha \leq \alpha_{min}$ ,  $G^\alpha = (\sigma^\alpha, \mu^\alpha)$  is a subgraph of  $G^* = (V, E)$ , where  $\sigma^\alpha = \{x \in V \mid \sigma(x) \geq \alpha\}$  and  $\mu^\alpha = \{xy \in E \mid \mu(xy) \geq \alpha\}$ .*

*Proof.* Let  $xy \in E$ . Since  $T(\mu(xy), T(\sigma(x), \sigma(y))) \leq T_{min}(\mu(xy), \sigma(x), \sigma(y))$ , we get that  $\mu(xy) \geq \alpha_{min} \geq \alpha$ ,  $\sigma(x) \geq \alpha_{min} \geq \alpha$  and  $\sigma(y) \geq \alpha_{min} \geq \alpha$ . Hence for any  $\alpha \leq \alpha_{min}$ , we have  $\mu^\alpha \subseteq \sigma^\alpha \times \sigma^\alpha$  and so  $G^\alpha = (\sigma^\alpha, \mu^\alpha)$  is a subgraph of  $G^* = (V, E)$ .  $\square$

**Theorem 4.4.** *Let  $(V, \rho, T)$  be a KM-fuzzy metric space and  $G^* = (V, E)$  be a simple graph.*

(i) *If  $\mu \leq \rho$ , then  $G = (\sigma, \mu, \rho, T)$  is a KM-fuzzy metric graph on  $G^*$ .*

(ii) *If  $T = T_{min}$ ,  $G = (\sigma, \mu, \rho, T_{min})$  is a KM-fuzzy metric graph on  $G^*$  and  $\mu > \rho$ , then  $G = (\sigma, \mu)$  is not a fuzzy graph on  $G^*$ .*

(iii) If  $T = T_{min}$  and  $G = (\sigma, \mu, \rho, T_{min})$  is a strong KM-fuzzy metric graph on  $G^*$ , then  $G = (\sigma, \mu)$  is a KM-fuzzy graph on  $G^*$  if and only if  $\rho \equiv \mu$ .

*Proof.* Let  $x, y \in V$ . Then for some  $t \in \mathbb{R}^{\geq 0}$ :

(i) Since  $T(\mu(xy), T(\sigma(x), \sigma(y))) \leq \mu(xy)$ , then  $\mu \leq \rho$  implies that  $T(\mu(xy), T(\sigma(x), \sigma(y))) \leq \rho(x, y, t)$ . So  $G = (\sigma, \mu, \rho, T)$  is a KM-fuzzy metric graph on  $G^*$ .

(ii) Let  $G = (\sigma, \mu)$  be a fuzzy graph on  $G^*$ . For all  $xy \in E$ , since  $G = (\sigma, \mu, \rho, T_{min})$  is a KM-fuzzy metric graph on  $G^*$ , using  $\mu(xy) \leq T_{min}(\sigma(x), \sigma(y))$ , we get that  $\mu(xy) = T_{min}(\mu(xy), T_{min}(\sigma(x), \sigma(y))) \leq \rho(x, y, t)$  which it is a contradiction.

(iii)  $G = (\sigma, \mu)$  is a fuzzy graph on  $G^*$  if and only if for all  $xy \in E$ ,  $\mu(xy) \leq T_{min}(\sigma(x), \sigma(y))$  if and only if  $T_{min}(\mu(xy), T_{min}(\sigma(x), \sigma(y))) = \mu(xy)$  if and only if  $\rho(x, y, t) = \mu(xy)$ .  $\square$

**Corollary 4.5.** Let  $G = (\sigma, \mu, \rho, T)$  be a KM-fuzzy metric connected graph on  $G^* = (V, E)$ . Then for starting time  $G = (\sigma, \mu)$  is not a fuzzy graph on  $G^*$ .

**Theorem 4.6.** Let  $(V, \rho, T)$  be a KM-fuzzy metric space,  $G^* = (V, E)$  be a simple graph and  $xy \in E$ . Then for  $\sigma : V \rightarrow [0, 1]$  and  $\mu : E \rightarrow [0, 1]$ ,

(i) If  $\sigma(x) + \sigma(y) \leq 1$ , then  $G = (\sigma, \mu, \rho, T_{lu})$  is a KM-fuzzy metric graph on  $G^*$ .

(ii) If  $\mu(xy) + 1 \leq \mu(xy) + \sigma(x) + \sigma(y) \leq 2$ , then  $G = (\sigma, \mu, \rho, T_{lu})$  is a KM-fuzzy metric graph on  $G^*$ .

*Proof.* Let  $x, y \in V$ . Then for some  $t \in \mathbb{R}^{\geq 0}$ :

(i)  $T_{lu}(\mu(xy), T_{lu}(\sigma(x), \sigma(y))) = \max(0, \mu(xy) + T_{lu}(\sigma(x), \sigma(y)) - 1) = \max(0, \mu(xy) + \max(0, \sigma(x) + \sigma(y) - 1) - 1)$ . If  $\sigma(x) + \sigma(y) \leq 1$ , then  $T_{lu}(\mu(xy), T_{lu}(\sigma(x), \sigma(y))) = \max(0, \mu(xy) - 1) = 0$ , since for all  $x, y \in V$  we have  $\mu(xy) \leq 1$ . It concludes that for any time  $t \in \mathbb{R}^{\geq 0}$  get that  $T_{lu}(\mu(xy), T_{lu}(\sigma(x), \sigma(y))) \leq \rho(x, y, t)$ .

(ii) Because  $\mu(xy) + 1 \leq \mu(xy) + \sigma(x) + \sigma(y) \leq 2$ , we get that  $\sigma(x) + \sigma(y) \geq 1$  and by item (i), have  $T_{lu}(\mu(xy), T_{lu}(\sigma(x), \sigma(y))) = T_{lu}(0, \mu(xy) + \sigma(x) + \sigma(y) - 2) = 0$ . It follows that  $G = (\sigma, \mu, \rho, T_{lu})$  is a KM-fuzzy metric graph on  $G^*$ .  $\square$

**Theorem 4.7.** Let  $G^* = (V, E)$  be a simple connected graph. Then there exists a KM-fuzzy metric graph on  $G^*$ .

*Proof.* Let  $\varphi : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$  be an increasing continuous function. By Corollary 3.9, for a fuzzy subset  $\rho : V^2 \times \mathbb{R}^{\geq 0} \rightarrow [0, 1]$  by  $\rho(x, y, t) = \begin{cases} \frac{\varphi(t)}{\varphi(t) + d^g(x, y)} & \text{if } t > 0 (\varphi(t) \geq 2) \\ 0 & \text{if } t = 0 \end{cases}$ , there exists a continuous t-norm  $T = T_{pr}$ , such that  $(V, \rho, T)$

is a KM-fuzzy metric space. Consider  $n = \max\{d^g(x, y) \mid xy \in E\}$  and for any fuzzy subset  $\sigma : V \rightarrow [0, 1]$ , define  $\sigma' : V \rightarrow [0, 1]$  by  $\sigma'(x) = 10^{-n-2}\sigma(x)$  and  $\mu : E \rightarrow [0, 1]$  by  $\mu(xy) = \sigma'(x) + \sigma'(y)$ . By Corollary 3.9, since for all  $x, y \in V$ ,  $\sigma(x) + \sigma(y) \leq 2$ , we get that  $T(\mu(xy), T(\sigma'(x), \sigma'(y))) \leq T_{pr}(\mu(xy), T_{pr}(\sigma'(x), \sigma'(y))) = (\sigma'(x) + \sigma'(y))(\sigma'(x)\sigma'(y)) = (\sigma(x) + \sigma(y))(\sigma(x)\sigma(y))10^{-3n-6} \leq \frac{\varphi(t)}{\varphi(t) + d^g(x, y)}$ . Therefore, for after starting time  $G = (\sigma', \mu, \rho, T)$  is a KM-fuzzy metric graph on  $G^*$ .  $\square$

**Example 4.8.** Let  $X = \{a, b, c, d, e\}$ . Consider an  $X$ -derived graph  $G^* = (X, E)$  in Figure 1. For all  $x, y \in$

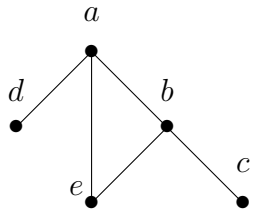


Figure 1:  $\{a, b, c, d, e\}$ -derived graph  $G^* = (X, E)$ .

$X$ , take  $\varphi(t) = t$ , in Theorem 3.8. Thus  $(X, \rho, T_{pr})$  is a KM-fuzzy metric space. Define a fuzzy subset  $\sigma = \{(a, 0.1), (b, 0.2), (c, 0.3), (d, 0.4), (e, 0.5)\}$  and so for half time  $t = \frac{1}{2}$ , obtain a KM-fuzzy metric graph on  $\{a, b, c, d, e\}$ -derived graph  $G = (X, E)$  in Figure 2.

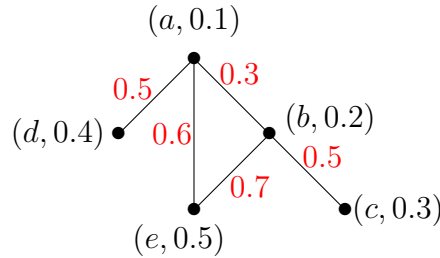


Figure 2:  $KM$ -fuzzy metric graph  $G = (\sigma, \mu, \rho, T_{pr})$  on half time  $\frac{1}{2}$ .

### 4.1 Operations on $KM$ -fuzzy metric graphs

In this section, for any given two  $KM$ -fuzzy metric graphs, define some product operations and show that the product of  $KM$ -fuzzy metric graphs is a  $KM$ -fuzzy metric graph. From now on, we consider  $G_1 = (\sigma_1, \mu_1, \rho_1, T)$ ,  $G_2 = (\sigma_2, \mu_2, \rho_2, T)$  as  $KM$ -fuzzy metric graphs on simple graphs  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively.

**Definition 4.9.** Let  $G_1, G_2$  be  $KM$ -fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Define the categorical product (tensor product) of fuzzy subsets  $\sigma_1 \times \sigma_2 : V(G_1^* \times G_2^*) \rightarrow [0, 1]$  by  $(\sigma_1 \times \sigma_2)(x_1, x_2) = T_{min}(\sigma_1(x_1), \sigma_2(x_2))$  and  $\mu_1 \times \mu_2 : E(G_1^* \times G_2^*) \rightarrow [0, 1]$  by  $(\mu_1 \times \mu_2)((x_1, x_2)(y_1, y_2)) = T_{min}(\mu_1(x_1y_1), \mu_2(x_2y_2))$ .

**Example 4.10.** Consider the  $KM$ -fuzzy metric spaces  $(V_1 = \{-1, -2\}, \rho_1, T_{pr})$ ,  $(V_2 = \{3, 4, 5\}, \rho_2, T_{pr})$ , where  $\rho_1(-1, -1, t > 0) = 1, \rho_1(-2, -2, t > 0) = 1, \rho_1(-1, -2, t > 0) = \frac{1+t}{2+t}, \rho_1(x, y, 0) = 0, x, y \in V_1$  and for all  $x, y \in \{3, 4, 5\}, \rho_2(x, y, t) = \begin{cases} \frac{\min\{x, y\} + t}{\max\{x, y\} + t} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$ . For the  $KM$ -fuzzy metric graphs  $G_1 = (\sigma_1, \mu_1, \rho_1, T_{pr})$  in unit time  $t_1 = 1$  and  $G_2 = (\sigma_2, \mu_2, \rho_2, T_{pr})$  in unit time  $t_2 = 1$  on  $G_1^*$  and  $G_2^*$  in Figure 3, we obtain the  $KM$ -fuzzy

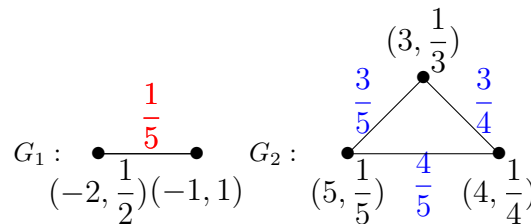


Figure 3:  $KM$ -fuzzy metric graphs  $G_1 = (\sigma_1, \mu_1, \rho_1, T_{pr}), G_2 = (\sigma_2, \mu_2, \rho_2, T_{pr})$  for  $t = 1$ .

metric graph  $G_1 \times G_2$  in Figure 4.

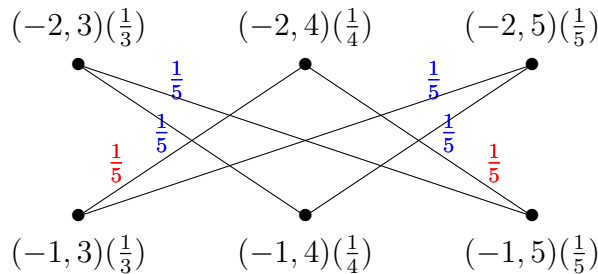


Figure 4:  $KM$ -fuzzy metric graph  $G_1 \times G_1 = (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2, T_{min}(\rho), T_{pr})$  for  $t = 1$ .

**Theorem 4.11.** Let  $G_1$  and  $G_2$  be  $KM$ -fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Then  $G_1 \times G_2 = (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2, T_{min}(\rho), T)$  is a  $KM$ -fuzzy metric graph on  $G_1^* \times G_2^*$ .

*Proof.* Firstly, by Theorem 3.10,  $(V_1 \times V_2, T_{\min}(\rho), T)$  is a *KM*-fuzzy metric space. Let  $(x_1, x_2), (y_1, y_2) \in E(G_1^* \times G_2^*)$ . Since  $G_1$  is a *KM*-fuzzy metric graph on  $G_1^*$  and  $G_2$  is a *KM*-fuzzy metric graph on  $G_2^*$ , for some  $t_1, t_2 \in \mathbb{R}^{\geq 0}$ , we get that  $T((\mu_1 \times \mu_2)((x_1, x_2)(y_1, y_2)), T((\sigma_1 \times \sigma_2)(x_1, x_2), (\sigma_1 \times \sigma_2)(y_1, y_2))) = T(T_{\min}(\mu_1(x_1y_1), \mu_2(x_2y_2)), T((T_{\min}(\sigma_1(x_1), \sigma_2(x_2)), (T_{\min}(\sigma_1(y_1), \sigma_2(y_2)))))) \leq T(\mu_1(x_1y_1), T(\sigma_1(x_1), \sigma_1(y_1))) \leq \rho_1(x_1, y_1, t_1)$  and  $T((\mu_1 \times \mu_2)((x_1, x_2)(y_1, y_2)), T((\sigma_1 \times \sigma_2)(x_1, x_2), (\sigma_1 \times \sigma_2)(y_1, y_2))) = T(T_{\min}(\mu_1(x_1y_1), \mu_2(x_2y_2)), T((T_{\min}(\sigma_1(x_1), \sigma_2(x_2)), (T_{\min}(\sigma_1(y_1), \sigma_2(y_2)))))) \leq T(\mu_2(x_2y_2), T(\sigma_2(x_2), \sigma_2(y_2))) \leq \rho_2(x_2, y_2, t_2)$ . Consider  $t = \max\{t_1, t_2\}$ , so by Theorem 2.4, we obtain  $T((\mu_1 \times \mu_2)((x_1, x_2), (y_1, y_2)), T((\sigma_1 \times \sigma_2)(x_1, x_2), (\sigma_1 \times \sigma_2)(y_1, y_2))) \leq T_{\min}(\rho_1(x_1, y_1, t_1), \rho_2(x_2, y_2, t_2)) \leq T_{\min}(\rho)((x_1, x_2), (y_1, y_2), t)$ . Thus  $G_1 \times G_2 = (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2, T_{\min}(\rho), T)$  is a *KM*-fuzzy metric graph on  $G_1^* \times G_2^*$ .  $\square$

**Definition 4.12.** Let  $G_1, G_2$  be *KM*-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Define the Cartesian product (or product) fuzzy subsets  $\sigma_1 \otimes \sigma_2 : V_1 \times V_2 \rightarrow [0, 1]$  by  $(\sigma_1 \otimes \sigma_2)(x_1, x_2) = T_{\min}(\sigma_1(x_1), \sigma_2(x_2))$  and  $\mu_1 \otimes \mu_2 : E_1 \times E_2 \rightarrow [0, 1]$  by

$$(\mu_1 \otimes \mu_2)((x, x_2)(x, y_2)) = T_{\min}(\sigma_1(x), \mu_2(x_2y_2)), (\mu_1 \otimes \mu_2)((x_1, y_1)(y_1, y)) = T_{\min}(\sigma_2(y), \mu_1(x_1y_1)).$$

**Example 4.13.** Consider the *KM*-fuzzy metric graphs  $G_1$  and  $G_2$  in Example 4.10. So we obtain the *KM*-fuzzy metric graph  $G_1 \otimes G_2$  in Figure 5.

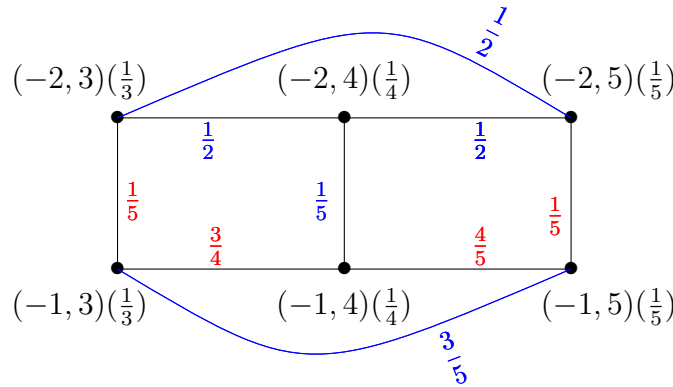


Figure 5: *KM*-fuzzy metric graph  $G_1 \otimes G_1 = (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2, T_{\min}(\rho), T_{pr})$  for  $t = 1$ .

**Theorem 4.14.** Let  $G_1$  and  $G_2$  be *KM*-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively.

(i) If  $(G_1^* \otimes G_2^*, T(\rho), T)$  is a *KM*-fuzzy metric space, then  $T(\rho) = \rho_1$  or  $T(\rho) = \rho_2$ , where  $T \in \mathcal{C}_T$ .

(ii)  $G_1 \otimes G_2 = (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2, T_{\min}(\rho), T)$  is a *KM*-fuzzy metric graph on  $G_1^* \otimes G_2^*$ .

*Proof.* (i) Let  $(x_1, x_2), (y_1, y_2) \in E(G_1^* \otimes G_2^*)$ . Then  $x_1 = y_1$  and  $x_2y_2 \in E(G_2^*)$  or  $x_2 = y_2$  and  $x_1y_1 \in E(G_1^*)$ . If  $x_1 = y_1$  and  $x_2y_2 \in E(G_2^*)$ , then  $T(\rho)((x_1, x_2), (x_1, y_2), t) = T(\rho_1(x_1, x_1, t), \rho_2(x_2, y_2, t)) = T(1, \rho_2(x_2, y_2, t)) = \rho_2(x_2, y_2, t)$ . If  $x_2 = y_2$  and  $x_1y_1 \in E(G_1^*)$ , then  $T(\rho)((x_1, x_2), (y_1, x_2), t) = T(\rho_1(x_1, y_1, t), \rho_2(x_2, x_2, t)) = T(\rho_1(x_1, y_1, t), 1) = \rho_1(x_1, y_1, t)$ .

(ii) Firstly, by Theorem 3.10,  $(V_1 \times V_2, T_{\min}(\rho), T)$  is a *KM*-fuzzy metric space. Let  $(x_1, x_2), (y_1, y_2) \in E(G_1^* \otimes G_2^*)$ . Since  $G_1$  is a *KM*-fuzzy metric graph on  $G_1^*$  and  $G_2$  is a *KM*-fuzzy metric graph on  $G_2^*$ , for some  $t_1, t_2 \in \mathbb{R}^{\geq 0}$ , give  $t = \max\{t_1, t_2\}$ , so by item (i) and Theorem 2.4, we get that

$$\begin{aligned} & T((\mu_1 \otimes \mu_2)((x, x_2)(x, y_2)), T((\sigma_1 \otimes \sigma_2)(x, x_2), (\sigma_1 \otimes \sigma_2)(x, y_2))) \\ &= T(T_{\min}(\sigma_1(x), \mu_2(x_2y_2)), T(T_{\min}(\sigma_1(x), \sigma_2(x_2)), T_{\min}(\sigma_1(x), \sigma_2(y_2)))) \\ &\leq T(\mu_2(x_2y_2), T(\sigma_2(x), \sigma_2(y_2))) \\ &\leq \rho_2(x_2, y_2, t_2) \\ &\leq T_{\min}(\rho)((x, x_2)(x, y_2), t) \end{aligned}$$



and

$$\begin{aligned} & T((\mu_1 \otimes \mu_2)((x_1, y)(y_1, y)), T((\sigma_1 \otimes \sigma_2)(x_1, y), (\sigma_1 \otimes \sigma_2)(y_1, y)) \\ &= T(T_{min}(\sigma_2(y), \mu_1(x_1y_1)), T(T_{min}(\sigma_1(x_1), \sigma_2(y)), T_{min}(\sigma_1(y_1), \sigma_2(y)))) \\ &\leq T(\mu_1(x_1y_1), T(\sigma_1(x_1), \sigma_1(y_1))) \\ &\leq \rho_1(x_1, y_1, t_1) \\ &\leq T_{min}(\rho)((x_1, y)(y_1, y), t). \end{aligned}$$

Thus  $G_1 \otimes G_2 = (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2, T_{min}(\rho), T)$  is a KM-fuzzy metric graph on  $G_1^* \otimes G_2^*$ . □

**Definition 4.15.** Let  $G_1, G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Define the semi-strong product of fuzzy subsets  $\sigma_1 \bullet \sigma_2 : V_1 \times V_2 \rightarrow [0, 1]$  by  $(\sigma_1 \bullet \sigma_2)(x_1, x_2) = T_{min}(\sigma_1(x_1), \sigma_2(x_2))$  and  $\mu_1 \bullet \mu_2 : E_1 \times E_2 \rightarrow [0, 1]$  by

$$(\mu_1 \bullet \mu_2)((x, x_2)(x, y_2)) = T_{min}(\sigma_1(x), \mu_2(x_2y_2)), (\mu_1 \bullet \mu_2)((x_1, x_2)(y_1, y_2)) = T_{min}(\mu_1(x_1y_1), \mu_2(x_2y_2)).$$

**Example 4.16.** Consider the KM-fuzzy metric graphs  $G_1$  and  $G_2$  in Example 4.10. So we obtain the KM-fuzzy metric graph  $G_1 \bullet G_2$  in Figure 6.

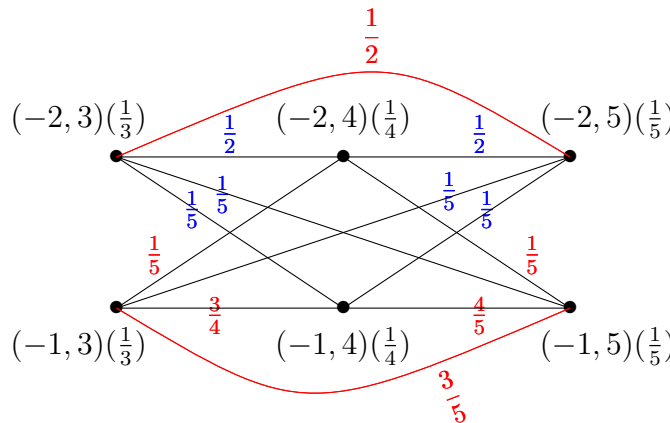


Figure 6: KM-fuzzy metric graph  $G_1 \bullet G_1 = (\sigma_1 \bullet \sigma_2, \mu_1 \bullet \mu_2, T_{min}(\rho), T_{pr})$  for  $t = 1$ .

**Theorem 4.17.** Let  $G_1$  and  $G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Then  $G_1 \bullet G_2 = (\sigma_1 \bullet \sigma_2, \mu_1 \bullet \mu_2, T_{min}(\rho), T)$  is a KM-fuzzy metric graph on  $G_1^* \bullet G_2^*$ .

*Proof.* It is similar to Theorems 4.11 and 4.14. □

**Definition 4.18.** Let  $G_1, G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Define the strong product of fuzzy subsets  $\sigma_1 \odot \sigma_2 : V_1 \times V_2 \rightarrow [0, 1]$  by  $(\sigma_1 \odot \sigma_2)(x_1, x_2) = T_{min}(\sigma_1(x_1), \sigma_2(x_2))$  and  $\mu_1 \odot \mu_2 : E_1 \times E_2 \rightarrow [0, 1]$  by  $(\mu_1 \odot \mu_2)((x, x_2)(x, y_2)) = T_{min}(\sigma_1(x), \mu_2(x_2y_2))$ ,  $(\mu_1 \odot \mu_2)((x_1, y)(y_1, y)) = T_{min}(\sigma_2(y), \mu_1(x_1y_1))$ ,  $(\mu_1 \odot \mu_2)((x_1, x_2)(y_1, y_2)) = T_{min}(\mu_1(x_1y_1), \mu_2(x_1x_2))$ .

**Example 4.19.** Consider the KM-fuzzy metric spaces  $(V_1 = \{1, 2\}, \rho_1, T_{min})$ ,  $(V_2 = \{3, 4, 5\}, \rho_2, T_{min})$ , where for all  $x, y \in \{1, 2\}$ ,  $\rho_1(x, y, 0) = 0, \rho_1(x, y, t > 0) = \frac{\min\{x, y\} + t}{\max\{x, y\} + t}$  and for all  $x, y \in \{3, 4, 5\}$ ,  $\rho_2(x, y, 0) = 0, \rho_2(x, y, t > 0) =$

$\begin{cases} 1 & \text{if } x = y \\ \frac{9+t}{10+t} & \text{if } x \neq y \end{cases}$ . For the KM-fuzzy metric graphs  $G_1$  with  $t_1 = 2$  and  $G_2$  with  $t_2 = 1$  on  $G_1^*$  and  $G_2^*$  in Figure 7, we obtain the KM-fuzzy metric graph  $G_1 \odot G_2$  in Figure 8.

**Theorem 4.20.** Let  $G_1$  and  $G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Then  $G_1 \odot G_2 = (\sigma_1 \odot \sigma_2, \mu_1 \odot \mu_2, T_{min}(\rho), T)$  is a KM-fuzzy metric graph on  $G_1^* \odot G_2^*$ .

*Proof.* It is similar to Theorems 4.11 and 4.14.



fuzzy subsets  $\sigma_1 \cup \sigma_2 : V_1 \cup V_2 \rightarrow [0, 1]$  and  $\mu_1 \cup \mu_2 : E_1 \cup E_2 \rightarrow [0, 1]$  by  $(\sigma_1 \cup \sigma_2)(x) = \begin{cases} \sigma_1(x) & \text{if } x \in V_1 \setminus V_2 \\ \sigma_2(x) & \text{if } x \in V_2 \setminus V_1 \\ T_{\min}(\sigma_1(x), \sigma_2(x)) & \text{if } x \in V_2 \cap V_1 \end{cases}$

and  $(\mu_1 \cup \mu_2)(xy) = \begin{cases} \mu_1(xy) & \text{if } xy \in E_1 \setminus E_2 \\ \mu_2(xy) & \text{if } xy \in E_2 \setminus E_1 \\ T_{\min}(\mu_1(xy), \mu_2(xy)) & \text{if } xy \in E_2 \cap E_1 \end{cases}$ .

**Example 4.25.** Consider the KM-fuzzy metric spaces  $(V_1 = \{-1, -2, 3, 4\}, \rho_1, T_{lu})$ ,  $(V_2 = \{5, 6, 7\}, \rho_2, T_{lu})$ , where for all  $x, y \in \{-1, -2, 3, 4\}$ ,  $\rho_1(x, y, 0) = 0, \rho_1(x, y, t > 0) = \frac{t}{t + d_1(x, y)}$  and for all  $x, y \in \{6, 7, 5\}$ ,  $\rho_2(x, y, 0) = 0, \rho_2(x, y, t > 0) = \frac{\min\{x, y\} + 2t}{\max\{x, y\} + 2t}$ . Clearly we have the KM-fuzzy metric graphs  $G_1$  with  $t_1 = 1$  and  $G_2$  with  $t_2 = 3$  on  $G_1^*$  and  $G_2^*$  in Figure 10. So we obtain a KM-fuzzy metric graphs  $G_1 \cup G_2$  with  $t = 3$  as in Figure 10.

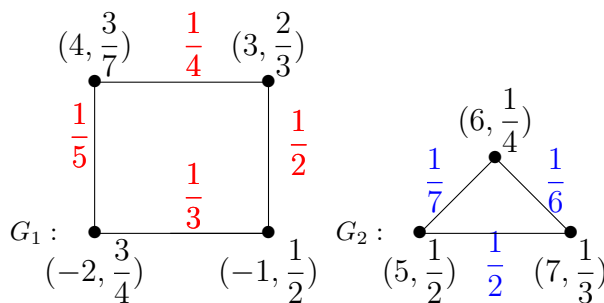


Figure 10: KM-fuzzy metric graphs  $G_1 = (\sigma_1, \mu_1, \rho_1, T_{lu})$ ,  $G_2 = (\sigma_2, \mu_2, \rho_2, T_{lu})$  with  $t_1 = 1, t_2 = 3$ .

**Theorem 4.26.** Let  $G_1 = (\sigma_1, \mu_1, \rho_1, T)$ ,  $G_2 = (\sigma_2, \mu_2, \rho_2, T)$  be KM-fuzzy metric graphs on simple graphs  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively. If  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  are two simple graphs, where  $V_1 \cap V_2 = \emptyset$ , then  $G_1 \cup G_2 = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2, \rho_1 \cup \rho_2, T)$  is a KM-fuzzy metric graph on  $G_1^* \cup G_2^*$ .

*Proof.* Firstly, by Theorem 3.11,  $(V_1 \cup V_2, T_{\min}(\rho), T)$  is a KM-fuzzy metric space. Let  $xy \in E(G_1^* \cup G_2^*)$ . Since  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$  are two simple graphs,  $xy \in E_1 \setminus E_2$  implies that  $(x, y \in V_1 \setminus V_2)$  and  $xy \in E_2 \setminus E_1$  implies that  $(x, y \in V_2 \setminus V_1)$ . Since  $G_1$  is a KM-fuzzy metric graph on  $G_1^*$  and  $G_2$  is a KM-fuzzy metric graph on  $G_2^*$ , for some  $t_1, t_2 \in \mathbb{R}^{\geq 0}$ , take  $t = \max\{t_1, t_2\}$  so by Theorems 2.4 and 3.11, if  $xy \in E_1 \setminus E_2$ , we have

$$T((\mu_1 \cup \mu_2)(xy), T((\sigma_1 \cup \sigma_2)(x), (\sigma_1 \cup \sigma_2)(y))) = T(\mu_1(xy), T(\sigma_1(x), \sigma_1(y))) \leq \rho_1(x, y, t) \leq (\rho_1 \cup \rho_2)(x, y, t).$$

In a similar way, if  $xy \in E_2 \setminus E_1$ , one can see that  $T_{\min}((\mu_1 \cup \mu_2)(xy), T_{\min}((\sigma_1 \cup \sigma_2)(x), (\sigma_1 \cup \sigma_2)(y))) \leq \rho_2(x, y, t) = (\rho_1 \cup \rho_2)(x, y, t)$ . Other cases are similar. Thus  $G_1 \cup G_2 = (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2, \rho_1 \cup \rho_2, T)$  is a KM-fuzzy metric graph on  $G_1^* \cup G_2^*$ .  $\square$

**Definition 4.27.** Let  $G_1, G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Define the semi-ring sum of fuzzy subsets  $\sigma_1 \otimes \sigma_2 : V_1 \cup V_2 \rightarrow [0, 1]$  and  $\mu_1 \otimes \mu_2 : E_1 \cup E_2 \rightarrow [0, 1]$  by  $(\sigma_1 \otimes \sigma_2)(x) = (\sigma_1 \cup \sigma_2)(x)$  and  $(\mu_1 \otimes \mu_2)(xy) = \begin{cases} \mu_1(xy) & \text{if } xy \in E_1 \setminus E_2 \\ \mu_2(xy) & \text{if } xy \in E_2 \setminus E_1 \\ 0 & \text{if } xy \in E_2 \cap E_1 \end{cases}$ .

**Theorem 4.28.** Let  $G_1$  and  $G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. If  $G_1^*$  and  $G_2^*$  are two simple graphs, where  $V_1 \cap V_2 = \emptyset$ , then  $G_1 \otimes G_2 = (\sigma_1 \otimes \sigma_2, \mu_1 \otimes \mu_2, \rho_1 \cup \rho_2, T)$  is a KM-fuzzy metric graph on  $G_1^* \cup G_2^*$ .

*Proof.* It is similar to Theorem 4.26.  $\square$

**Definition 4.29.** Let  $G_1, G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. Define the join(or suspension) of fuzzy subsets  $\sigma_1 + \sigma_2 : V_1 \cup V_2 \rightarrow [0, 1]$  and  $\mu_1 + \mu_2 : E_1 \cup E_2 \rightarrow [0, 1]$  by  $(\sigma_1 + \sigma_2)(x) =$

$(\sigma_1 \cup \sigma_2)(x)$  and  $(\mu_1 + \mu_2)(xy) = \begin{cases} (\mu_1 \cup \mu_2)(xy) & \text{if } xy \in E_1 \cup E_2 \\ (\rho_1 \cup \rho_2)(x, y, t) & \text{if } xy \in E'(x \in V_1, y \in V_2) \end{cases}$ , where  $E'$  is the set of all edges joining the vertices of  $V_1$  and  $V_2$  and  $t \in \mathbb{R}^{\geq 0}$ .

**Example 4.30.** Consider the KM-fuzzy metric graphs  $G_1 = (\sigma_1, \mu_1)$  and  $G_2 = (\sigma_2, \mu_2)$  in Example 4.10. So we obtain the KM-fuzzy metric graph  $G_1 + G_2$  with  $t = 1$  in Figure 11.

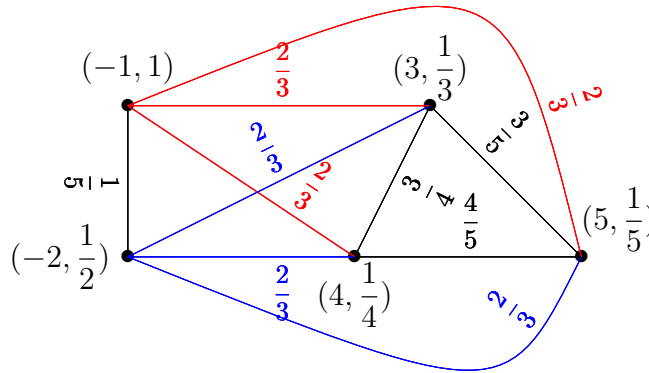


Figure 11: KM-fuzzy metric graph  $G_1 + G_2 = (\sigma_1 + \sigma_2, \mu_1 + \mu_2, \rho_1 \cup \rho_2, T_{pr})$  for  $t = 1$ .

**Theorem 4.31.** Let  $G_1$  and  $G_2$  be KM-fuzzy metric graphs on simple graphs  $G_1^*$  and  $G_2^*$ , respectively. If  $G_1^*$  and  $G_2^*$  are two simple graphs, where  $V_1 \cap V_2 = \emptyset$ , then  $G_1 + G_2 = (\sigma_1 + \sigma_2, \mu_1 + \mu_2, \rho_1 \cup \rho_2, T)$  is a KM-fuzzy metric graph on  $G_1^* + G_2^*$ .

*Proof.* Let  $xy \in E(G_1^* + G_2^*)$ . Then  $xy \in E_1 \setminus E_2, xy \in E_2 \setminus E_1$  or  $xy \in E'$ . We only consider  $xy \in E'$  and other cases are similar to Theorem 4.26. Since  $xy \in E'$ , we get that  $(x \in V_1 \setminus V_2, y \in V_2 \setminus V_1)$  or  $(y \in V_1 \setminus V_2, x \in V_2 \setminus V_1)$ . If  $x \in V_1 \setminus V_2, y \in V_2 \setminus V_1 (y \in V_1 \setminus V_2, x \in V_2 \setminus V_1$  is proved in a similar way), for some  $t_1, t_2 \in \mathbb{R}^{\geq 0}$ , take  $t = \max\{t_1, t_2\}$  so by Theorem 2.4, we have  $T((\mu_1 \cup \mu_2)(xy), T((\sigma_1 \cup \sigma_2)(x), (\sigma_1 \cup \sigma_2)(y))) \leq T((\rho_1 \cup \rho_2)(x, y, t), T(\sigma_1(x), \sigma_2(y))) \leq (\rho_1 \cup \rho_2)(x, y, t)$ .  $\square$

**Definition 4.32.** Let  $(V, \rho, T)$  be a KM-fuzzy metric space and  $G^* = (V, E)$  be a simple graph. If  $G = (\sigma, \mu, \rho, T)$  is a KM-fuzzy metric graph on  $G^*$ , then define the complement of fuzzy subsets  $\bar{\sigma} : V \rightarrow [0, 1]$  and  $\bar{\mu} : E \rightarrow [0, 1]$  by  $\bar{\sigma}(x) = \sigma(x)$  and  $\bar{\mu}(xy) = \rho(x, y, t) - T(\mu(xy), T(\sigma(x), \sigma(y)))$ , where  $x, y \in V$  and some  $t \in \mathbb{R}^{\geq 0}$  that  $T(\mu(xy), T(\sigma(x), \sigma(y))) \leq \rho(x, y, t)$ . We will denote the complement of a KM-fuzzy metric graph  $G = (\sigma, \mu, \rho, T)$ , by  $\bar{G} = (\bar{\sigma}, \bar{\mu}, \rho, T)$ .

**Theorem 4.33.** Let  $(V, \rho, T)$  be a KM-fuzzy metric space and  $G^* = (V, E)$  be a simple graph. If  $G = (\sigma, \mu, \rho, T)$  is a KM-fuzzy metric graph on  $G^*$ , then  $\bar{G} = (\bar{\sigma}, \bar{\mu}, \rho, T)$  is a KM-fuzzy metric graph.

*Proof.* Let  $x, y \in V$ . Since  $G$  is a KM-fuzzy metric graph on  $G^*$ , for some  $t \in \mathbb{R}^{\geq 0}$ ,  $T(\bar{\mu}(xy), T(\bar{\sigma}(x), \bar{\sigma}(y))) = T(\rho(x, y, t) - T(\mu(xy), T(\sigma(x), \sigma(y))), T(\sigma(x), \sigma(y))) \leq \rho(x, y, t) - T(\mu(xy), T(\sigma(x), \sigma(y))) \leq \rho(x, y, t)$ . It follows that  $\bar{G} = (\bar{\sigma}, \bar{\mu}, \rho, T)$  is a KM-fuzzy metric graph.  $\square$

**Theorem 4.34.** Let  $(V, \rho, T_{pr})$  be a KM-fuzzy metric space,  $G^* = (V, E)$  be a simple graph and  $G = (\sigma, \mu, \rho, T_{pr})$  be a KM-fuzzy metric graph on  $G^*$ . Then

(i)  $\bar{\mu} = \mu$  implies that  $\sigma \equiv 1$  or  $\mu(xy) = \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)}$ , where  $xy \in E$ .

(ii) If  $G$  is a strong KM-fuzzy metric graph, then  $\bar{\mu} = \mu$  implies that  $\rho \equiv \mu$ .

*Proof.* (i), (ii) Let  $x, y \in V$ . Then for some  $t \in \mathbb{R}^{\geq 0}$ ,  $\bar{\mu}(xy) = \mu(xy) \Leftrightarrow \rho(x, y, t) - (\rho(x, y, t) - \mu(xy)\sigma(x)\sigma(y))\sigma(x)\sigma(y) = \mu(xy) \Leftrightarrow \rho(x, y, t) - \rho(x, y, t)\sigma(x)\sigma(y) + \mu(xy)\sigma^2(x)\sigma^2(y) = \mu(xy) \Leftrightarrow \rho(x, y, t)(1 - \sigma(x)\sigma(y)) - \mu(xy)(1 - \sigma^2(x)\sigma^2(y)) = 0 \Leftrightarrow (1 - \sigma(x)\sigma(y))(\rho(x, y, t) - \mu(xy)(1 + \sigma(x)\sigma(y))) = 0 \Leftrightarrow (1 - \sigma(x)\sigma(y)) = 0$  or  $(\rho(x, y, t) - \mu(xy)(1 + \sigma(x)\sigma(y))) = 0$ . It follows that  $1 - \sigma(x)\sigma(y) = 0$  or  $\rho(x, y, t) - \mu(xy)(1 + \sigma(x)\sigma(y)) = 0$ . If  $1 - \sigma(x)\sigma(y) = 0$ , we obtain that  $\sigma(x) = \frac{1}{\sigma(y)}$  and

$\rho(x, y, t) - \mu(xy)(1 + \sigma(x)\sigma(y)) = 0$  implies that  $\mu(xy) = \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)}$ . In addition, if  $G$  is a strong KM-fuzzy metric graph, then  $\sigma \equiv 1$  concludes that  $\rho \equiv \mu$  and  $\mu(xy) = \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)}$  implies that  $\mu \equiv 0$  and so  $\rho \equiv \mu$ .  $\square$

Let  $(V, \rho, T)$  be a KM-fuzzy metric spaces and  $G^* = (V, E)$  be a simple graph. For simplify, we denote  $\bar{\mu} = \mu^1, \bar{\bar{\mu}} = \mu^2$  and for all  $n \in \mathbb{N}, \mu^{n-1} = \mu^n$ .

**Theorem 4.35.** *Let  $(V, \rho, T_{pr})$  be a KM-fuzzy metric space,  $G^* = (V, E)$  be a simple graph and  $G = (\sigma, \mu, \rho, T_{pr})$  be a strong KM-fuzzy metric graph on  $G^*$ . Then*

$$\mu^n(xy) = \begin{cases} 0 & \text{if } n = 1, \\ \rho(x, y, t) & \text{if } n = 2, \\ \mu^{n-1}(xy) + \rho(x, y, t)(-\sigma(x)\sigma(y))^{n-2} & \text{if } n \geq 3. \end{cases}$$

*Proof.* Let  $x, y \in V$ . For some  $t \in \mathbb{R}^{\geq 0}$ , since  $G = (\sigma, \mu, \rho, T_{pr})$  is a strong KM-fuzzy metric graph on  $G^*$ , we get that  $\mu^1(xy) = \rho(x, y, t) - \mu(xy)\sigma(x)\sigma(y) = 0, \mu^2(xy) = \rho(x, y, t) - 0\sigma(x)\sigma(y) = \rho(x, y, t), \mu^3(xy) = \rho(x, y, t) - \rho(x, y, t)\sigma(x)\sigma(y), \mu^4(xy) = \rho(x, y, t) - (\rho(x, y, t)\sigma(x)\sigma(y)) + \rho(x, y, t)\sigma^2(x)\sigma^2(y)$ , and by induction, for all  $n \geq 3$  we get that  $\mu^n(xy) = \mu^{n-1}(xy) + \rho(x, y, t)(-\sigma(x)\sigma(y))^{n-2}$ .  $\square$

**Corollary 4.36.** *Let  $(V, \rho, T_{pr})$  be a KM-fuzzy metric space,  $G^* = (V, E)$  be a simple graph and  $G = (\sigma, \mu, \rho, T_{pr})$  be a strong KM-fuzzy metric graph on  $G^*$ . Then*

(i)  $\lim(\mu^n(xy)) = \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)}$ , when  $n \rightarrow \infty$ .

(ii)  $\lim(\mu^n(x^2)) = \frac{1}{1 + \sigma^2(x)}$ , when  $n \rightarrow \infty$ .

*Proof.* Applying Theorem 4.35, for all  $k \geq 3$  we have  $\mu^k(xy) - \mu^{k-1}(xy) = \rho(x, y, t)(-\sigma(x)\sigma(y))^{k-2}$ . Thus  $\sum_{k=3}^n (\mu^k(xy) - \mu^{k-1}(xy)) = \sum_{k=3}^n (\rho(x, y, t)(-\sigma(x)\sigma(y))^{k-2}) \Rightarrow \mu^k(xy) - \rho(x, y, t) =$

$$-\rho(x, y, t)\sigma(x)\sigma(y)\left(\frac{1 - (\sigma(x)\sigma(y))^{n-2}}{1 - \sigma(x)\sigma(y)}\right) \text{ and so } n \rightarrow \infty, \text{ implies that } \lim(\mu^n(xy)) = \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)}. \quad \square$$

**Definition 4.37.** *Let  $G = (\sigma, \mu, \rho, T), G_1 = (\sigma_1, \mu_1, \rho_1, T')$  and  $G_2 = (\sigma_2, \mu_2, \rho_2, T')$  be KM-fuzzy metric graphs on simple graphs  $G^* = (V, E), G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively.*

(i) *A bijective mapping  $\varphi : V_1 \rightarrow V_2$  which for all  $xy \in E_1$ , for all  $t \in \mathbb{R}^{\geq 0}, \sigma_1(x) = \sigma_2(\varphi(x)), \mu_1(xy) = \mu_2(\varphi(x)\varphi(y))$  and  $\rho_1(x, y, t) = \rho_2(\varphi(x), \varphi(y), t)$  is called an isomorphism  $\varphi : G_1 \rightarrow G_2$  of KM-fuzzy metric graphs and we will denote it by  $G_1 \cong G_2$ .*

(ii)  *$G$  is said to be a self-complementary KM-fuzzy metric graph, if  $\bar{G} = (\bar{\sigma}, \bar{\mu}) \cong G = (\sigma, \mu)$ .*

**Theorem 4.38.** *Let  $(V, \rho, T)$  be a KM-fuzzy metric space,  $G^* = (V, E)$  be a simple graph and  $G = (\sigma, \mu, \rho, T)$  be a self-complementary KM-fuzzy metric graph on  $G^*$ . Then*

(i)  $\sum_{x \neq y} \mu(xy) + \sum_{x \neq y} T(\mu(xy), T(\sigma(x), \sigma(y))) = \sum_{x \neq y} \rho(x, y, t)$ .

(ii) *If  $T = T_{min}$  and  $G = (\sigma, \mu)$  is a fuzzy graph on  $G^*$ , then  $\sum_{x \neq y} \mu(xy) = \frac{1}{2} \sum_{x \neq y} \rho(x, y, t)$ .*

*Proof.* Let  $x, y \in V$  and  $t \in \mathbb{R}^{\geq 0}$ . (i) Since  $G$  is a self-complementary KM-fuzzy metric graph, then there exists an isomorphism  $\varphi : V \rightarrow V$  such that for all  $x \in V, \sigma(\varphi(x)) = \sigma(x)$  and for all  $xy \in E, \mu(\varphi(x)\varphi(y)) = \varphi(xy)$ . Hence  $\bar{\mu}(\varphi(x)\varphi(y)) = \rho(\varphi(x), \varphi(y), t) - T(\mu(\varphi(x)\varphi(y)), T(\sigma(x), \sigma(y))) \Rightarrow \mu(xy) = \rho(\varphi(x), \varphi(y), t) - T(\mu(\varphi(x)\varphi(y)), T(\sigma(x), \sigma(y))) \Rightarrow \sum_{x \neq y} \mu(xy) + \sum_{x \neq y} T(\mu(\varphi(x)\varphi(y)), T(\sigma(x), \sigma(y))) = \sum_{x \neq y} \rho(x, y, t)$ .

(ii) Since  $T = T_{min}$  and  $G = (\sigma, \mu)$  is a fuzzy graph on  $G^*$ , we get that  $T_{min}(\mu(xy), T_{min}(\sigma(x), \sigma(y))) = \mu(xy)$  and by item (i), we have

$$\begin{aligned} & \sum_{x \neq y} \mu(xy) + \sum_{x \neq y} T_{min}(\mu(\varphi(x)\varphi(y)), T_{min}(\sigma(x), \sigma(y))) = \sum_{x \neq y} \rho(x, y, t) \Rightarrow \sum_{x \neq y} \mu(xy) + \sum_{x \neq y} \mu(\varphi(x)\varphi(y)) \\ = & \sum_{x \neq y} \rho(x, y, t) \Rightarrow \sum_{x \neq y} \mu(xy) + \sum_{x \neq y} \mu(xy) = \sum_{x \neq y} \rho(x, y, t) \Rightarrow 2 \sum_{x \neq y} \mu(xy) = \sum_{x \neq y} \rho(x, y, t). \end{aligned}$$

□

**Theorem 4.39.** Let  $(V, \rho, T_{pr})$  be a  $KM$ -fuzzy metric space,  $G^* = (V, E)$  be a simple graph and  $G = (\sigma, \mu, \rho, T_{pr})$  be a  $KM$ -fuzzy metric graph on  $G^*$ . If  $\mu(xy) = \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)}$ , then  $G$  is a self-complementary  $KM$ -fuzzy metric graph.

*Proof.* Let  $x, y \in V$  and  $t \in \mathbb{R}^{\geq 0}$ . Then the identity mapping  $I : V \rightarrow V$  is an isomorphism from  $G$  to  $\bar{G}$ . So  $\bar{\mu}(I(x)I(y)) = \bar{\mu}(xy) = \rho(x, y, t) - \mu(xy)\sigma(x)\sigma(y) = \rho(x, y, t) - \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)}\sigma(x)\sigma(y) = \frac{\rho(x, y, t)}{1 + \sigma(x)\sigma(y)} = \mu(xy)$ . It concludes that  $G$  is a self-complementary  $KM$ -fuzzy metric graph. □

**Theorem 4.40.** The complements of two isomorphic  $KM$ -fuzzy metric graphs are isomorphic  $KM$ -fuzzy metric graphs and conversely.

*Proof.* Let  $(V_1, \rho_1, T), (V_2, \rho_2, T)$  be  $KM$ -fuzzy metric spaces,  $G_1 = (\sigma_1, \mu_1, \rho_1, T)$  and  $G_2 = (\sigma_2, \mu_2, \rho_2, T)$  be isomorphic  $KM$ -fuzzy metric graphs on simple graphs  $G_1^* = (V_1, E_1)$  and  $G_2^* = (V_2, E_2)$ , respectively. Then there exists a bijective mapping  $\varphi : V_1 \rightarrow V_2$  which for all  $xy \in E_1, \sigma_1(x) = \sigma_2(\varphi(x)), \mu_1(xy) = \mu_2(\varphi(x)\varphi(y))$  and  $\rho_1(x, y, t) = \rho_2(\varphi(x), \varphi(y), t)$ . Thus,  $\bar{\mu}_1(xy) = \rho_1(x, y, t) - T(\mu_1(xy), T(\sigma_1(x), \sigma_1(y))) = \rho_2(\varphi(x), \varphi(y), t) - T(\mu_2(\varphi(x)\varphi(y)), T(\sigma_2(\varphi(x)), \sigma_2(\varphi(y)))) = \bar{\mu}_2(\varphi(x)\varphi(y))$ . Hence  $\bar{G}_1 \cong \bar{G}_2$ . Similarly, we can prove the converse part. □

## 5 Conclusions

The current paper has introduced a novel concept fuzzy algebra as  $KM$ -fuzzy metric graph and a new generalization of graphs based on  $KM$ -fuzzy metric spaces. This work extended and obtained some properties in  $KM$ -fuzzy metric spaces. Also it showed that every non empty set converted to a  $KM$ -fuzzy metric space, the product and union of  $KM$ -fuzzy metric spaces is a  $KM$ -fuzzy metric space, the extended  $KM$ -fuzzy metric spaces are constructed using the some algebraic operations on  $KM$ -fuzzy metric spaces, the concept of complement of  $KM$ -fuzzy metric graph is defined and is investigated some its properties and by using the notation of bijections, the notation of isomorphisms on  $KM$ -fuzzy metric graphs are defined and extracted the self-complementary of  $KM$ -fuzzy metric graphs. We hope that these results are helpful for further studies in theory of graphs. In our future studies, we hope to obtain more results regarding intuitionistic metric graphs, neutrosophic metric graphs,  $KM$ -fuzzy metric hypergraphs, bipolar  $KM$ -fuzzy metric graphs, automorphism  $KM$ -fuzzy metric graphs and their applications.

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## Extended graphs based on KM-fuzzy metric spaces

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### گراف‌های توسیع یافته روی فضاهای متریک KM-فازی

**چکیده.** در این مقاله، مفهوم فضاهای متریک KM-فازی را به کار می‌گیریم و مفهوم نوین گراف‌های متریک KM-فازی را روی فضاهای متریک KM-فازی معرفی می‌کنیم. این مطالعه، با توجه به مترهای عادی و مترهای KM-فازی روی فضاهای متریک KM-فازی متناهی تحقیق می‌کند و فضاهای متریک KM-فازی را روی مجموعه‌های ناتهی دلخواه می‌سازد. در این مقاله تلاش می‌شود تا مفهوم فضاهای متریک KM-فازی را به رده بزرگتری از فضاهای متریک فازی مانند اجتماع و ضرب فضاهای متریک KM-فازی توسیع دهد و در این راستا روی رده ضرب‌هایی از گراف‌های متریک KM-فازی تحقیق می‌کند.