Iranian Journal of Fuzzy Systems

Volume 18, Number 1, (2021), pp. 75-85

Original paper

Fuzzy closure operators and their applications

S. W. Han[1] and R. R. Wang[2]

[1,2] *Department of Mathematics, Shaanxi Normal University, Xi'an 710119, PR China*

hansw@snnu.edu.cn, wrr2017@163.com

Abstract

Fuzzy closure operators play a significant role in fuzzy order theory. This paper aims to further enrich and improve the study of fuzzy closure operators. Based on the work of Bělohlávek and Yao, we shall continue to study the relative properties of fuzzy closure operators. First, we shall consider the extensions of L-subsets via fuzzy closure operators. Then we give an application of fuzzy closure operators, that is, by fuzzy closure operators we shall prove that the category **CFPos** of complete fuzzy posets and their fuzzy-join preserving maps is a reflective full subcategory of $\mathbf{FPos}_{\sqcup}$, where $\mathbf{FPos}_{\sqcup}$ denotes the category of fuzzy posets and their fuzzy-existing-join preserving maps.

Keywords: Fuzzy poset, complete fuzzy poset, fuzzy closure operator, extension of L-subset, reflective subcategory.

1 Introduction

Closure operators (closure systems) have been studied in many mathematical branches such as topology, poset and logic. In the classical order theory, they are particularly related to theoretical computer science and have a close relation to Galois connections [14]. In 1971, Zadeh introduced his fuzzy order [25]. From then on, different kinds of fuzzy orders have been introduced and studied by many authors [3, 4, 5, 6, 8, 9, 10, 11, 17, 26, 27]. Fuzzy partial order can be seen as a generalization of the crisp partial order. In [2], Bělohlávek defined and studied a fuzzy closure operator on a set X as a map $C\colon L^X \longrightarrow L^X$ satisfying the following conditions:
(C1) $\forall A \in L^X,\ A \leq C(A)$;
(C2) $\forall A_1, A_2 \in L^X,\ sub_X(A_1, A_2) \leq sub_X(C(A_1), C(A_2))$;
(C3) $\forall A \in L^X, C(A) = C(C(A))$,
where L is a complete residuated lattice (or a commutative unital quantale). Yao and Lu directly defined fuzzy closure operators on fuzzy posets. In fact, we can easily check that Bělohlávek's fuzzy closure operator on X is precisely a fuzzy closure operator on the fuzzy poset (L^X, sub_X). As we all know, closure operators are an important tool to study posets, and fuzzy closure operators are exactly the correspondence of closure operators in fuzzy posets. From this point of view, the need for development of fuzzy closure operators on fuzzy posets becomes apparent. In this paper, based on the work of Bělohlávek and Yao, we shall continue to study the relative properties of fuzzy closure operators on fuzzy posets. Then we give an application of fuzzy closure operators.

2 Preliminaries

In order to establish a foundation for this paper, we need to review some definitions and results.

An L-subset A on a universe set X is any map $A\colon X \to L$, where L is an appropriate set of truth values. In general, L has to be equipped with some structure. Throughout this paper, we always assume that L is a complete residuated lattice (or a commutative unital quantale) [20, 22], which is an algebra $L = (L, \wedge, \vee, *, \to, 0, 1)$ such that (1) $(L, \wedge, \vee, 0, 1)$ is a complete lattice with the least element 0 and the greatest element 1; (2) $(L, *, 1)$ is a commutative

Corresponding Author: S. W. Han
Received: January 2020; Revised: June 2020; Accepted: August 2020.

monoid with the identity 1; (3) $*$ and $\to$ form an adjoint pair, that is,

$$x * y \le z \text{ iff } x \le y \to z,$$

holds for all $x, y, z \in L$. The following proposition gives some basic properties of complete residuated lattices.

Proposition 2.1. [12, 16] *Suppose that L is a complete residuated lattice. Then for all $a, b, c \in L, \{a_i\}_{i\in I} \subseteq L$, the following statements hold*
(1) $1 \to a = a$;
(2) $0 \to a = 1$;
(3) $a \le b \Longleftrightarrow a \to b = 1$;
(4) $a \le b \Longrightarrow c \to a \le c \to b$;
(5) $a \le b \Longrightarrow b \to c \le a \to c$;
(6) $(\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I} (a_i \to b)$;
(7) $a \to (\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} (a \to a_i)$;
(8) $a \to (b \to c) = (a * b) \to c$;
(9) $a * (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} (a * b_i)$.

Let X be a set, L^X denote the set of all L-subsets on X, then L^X is a complete lattice under the pointwise order

$$f \le g \text{ in } L^X \Longleftrightarrow f(x) \le g(x) \text{ for all } x \in X.$$

Definition 2.2. [24] *Let X be a nonempty set. Then $e : X \times X \to L$ is called a fuzzy partial order provided that e satisfies the following conditions:*
(E1) $\forall x \in X,\ e(x, x) = 1$;
(E2) $\forall x, y, z \in X,\ e(x, y) * e(y, z) \le e(x, z)$;
(E3) $\forall x, y \in X,\ e(x, y) = e(y, x) = 1$ *implies* $x = y$.

The pair (X, e) is called a fuzzy partially ordered set (a fuzzy poset for short), usually written as X if there would be no confusion about the partial order e. As shown by [24], given a fuzzy poset (X, e), we let $\le_e = \{(x, y) \in X \times X : e(x, y) = 1\}$. Then $(X, \le_e)$ is a crisp poset.

Remark 2.3. *Let (X, e) be a fuzzy poset. Then*
(1) If $x \le_e y$, then $e(z, x) \le e(z, y)$ and $e(y, z) \le e(x, z)$ for all $z \in X$.
(2) If $e(x_1, y) = e(x_2, y)$ for all $y \in X$, then $x_1 = x_2$.

In the following, we shall give some standard examples of fuzzy posets.

Example 2.4. *(1) Let $(X, \le)$ be a crisp poset and $e_{\le} : X \times X \to L$ be a map defined by*

$$e_{\le}(x, y) = \begin{cases} 1 & \textit{if } x \le y, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $(X, e_{\le})$ is a fuzzy poset.

(2) Let (X, e) be a fuzzy poset and $Y \subseteq X$. Then $(Y, e|_Y)$ is a fuzzy poset, where $e|_Y$ is the restriction of e to $Y \times Y$.

(3) Let X be a set. For all $A, B \in L^X$, the subsethood degree [15] *of A in B is defined by $sub_X(A, B) = \bigwedge_{x\in X} (A(x) \to B(x))$. Then (L^X, sub_X) is a fuzzy poset, and it is easy to verify that $A \le B \Longleftrightarrow A \le_{sub_X} B$.*

Definition 2.5. [26] *Let (X, e) be a fuzzy poset, $x_0 \in X$ and $A \in L^X$. The element x_0 is called a join (resp., meet) of A, in symbols $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$), if*
(1) $\forall x \in X,\ A(x) \le e(x, x_0)$ (resp., $A(x) \le e(x_0, x)$);
(2) $\forall y \in X,\ \bigwedge_{x\in X} (A(x) \to e(x, y)) \le e(x_0, y)$ (resp., $\bigwedge_{x\in X} (A(x) \to e(y, x)) \le e(y, x_0)$).

Proposition 2.6. [23] *Let (X, e) be a fuzzy poset, $x_0 \in X$, $A \in L^X$. Then*
(1) $x_0 = \sqcup A$ iff $\forall y \in X,\ e(x_0, y) = \bigwedge_{x\in X} (A(x) \to e(x, y))$.
(2) $x_0 = \sqcap A$ iff $\forall y \in X,\ e(y, x_0) = \bigwedge_{x\in X} (A(x) \to e(y, x))$.

As mentioned in [23], if the join (or meet) of A exists, then it is unique.

Definition 2.7. [18, 26] *Let (X, e) be a fuzzy poset. Then $A \in L^X$ is called a fuzzy lower set if $\forall x, y \in X$, $A(x) * e(y, x) \leq A(y)$.*

Remark 2.8. *(1) For $x \in X$, $l_x \in L^X$ is defined by $\forall y \in X,\ l_x(y) = e(y, x)$. Then l_x is a fuzzy lower set and $\sqcup l_x = x$* (see [26, 27]).

(2)With respect to the order $\leq_e$, one can see that a fuzzy lower subset A of X is order-reversing, that is, $x \leq_e y \Longrightarrow A(y) \leq A(x)$.

Definition 2.9. [3, 4] *A fuzzy poset (X, e) is called complete if for all $A \in L^X$, $\sqcup A$ and $\sqcap A$ exist.*

Proposition 2.10. [23] *Let (X, e) be a fuzzy poset. Then the following statements are equivalent:*
(1) (X, e) is complete;
(2) for any $A \in L^X$, $\sqcup A$ exists;
(3) for any $A \in L^X$, $\sqcap A$ exists.

For each map $f \colon X \to Y$, we have a map $f_L^{\rightarrow} \colon L^X \to L^Y$ (called *L-forward powerset operator* [19]), defined by $\forall A \in L^X,\ y \in Y,\ f_L^{\rightarrow}(A)(y) = \bigvee\limits_{f(x)=y} A(x)$. The right adjoint to $f_L^{\rightarrow}$ is denoted by $f_L^{\leftarrow}$ (called *L-backward powerset operator* [19]) and given by $\forall B \in L^Y,\ f_L^{\leftarrow}(B) = B \circ f$.

Definition 2.11. [26] *Let (X, e_X) and (Y, e_Y) be two fuzzy posets. Then a map $f : X \to Y$ is said to be*
(1) order-preserving if $\forall x, y \in X$, $e_X(x, y) \leq e_Y(f(x), f(y))$;
(2) fuzzy-existing-join preserving if $f(\sqcup A) = \sqcup f_L^{\rightarrow}(A)$ holds for each $A \in L^X$ with $\sqcup A$ existing.

By Theorem 4.5 in [23], we know that if f is fuzzy-existing-join preserving, then f is order-preserving.

Definition 2.12. [24] *Let (X, e) be a fuzzy poset. An order-preserving map $j \colon X \to X$ is called a fuzzy closure operator (resp., fuzzy interior operator) on X if*

$$e(x, j(x)) = e(j(j(x)), j(x)) = 1 \text{ (resp., } e(j(x), x) = e(j(j(x)), j(x)) = 1),$$

holds for all $x \in X$.

The set of all fuzzy closure operators on X is denoted by $Cl_L(X)$. By Theorem 4.2 in [24], we see that j is a fuzzy closure operator on (X, e) if and only if j is order-preserving on (X, e) and j is a (classical) closure operator on $(X, \leq_e)$. Further, we can also show that $e(x, j(y)) = e(j(x), j(y))$ for all $x, y \in X$.

Definition 2.13. [24] *Let (X, e_X), (Y, e_Y) be two fuzzy posets and $f \colon X \to Y$, $g \colon Y \to X$ be two order-preserving maps. The pair (f, g) is called a fuzzy Galois connection between X and Y if $\forall x \in X,\ y \in Y$,*

$$e_Y(f(x), y) = e_X(x, g(y)),$$

where f is called the left adjoint of g and dually g is called the right adjoint of f.

Remark 2.14. *(1) Fuzzy closure operators have a close relationship with fuzzy Galois connections. To see this, let (f, g) be a fuzzy Galois connection between (X, e_X) and (Y, e_Y). Then by Theorem 4.3 in [24], it follows that $g \circ f \in Cl_L(X)$ and $f \circ g$ is a fuzzy interior operator on (Y, e_Y).*

(2) For a given map $f \colon X \to Y$, we have that $(f_L^{\rightarrow}, f_L^{\leftarrow})$ is a fuzzy Galois connection between (L^X, sub_X) and (L^Y, sub_Y). Further, we have that $f_L^{\leftarrow} \circ f_L^{\rightarrow}$ is a fuzzy closure operator on L^X, and $f_L^{\rightarrow} \circ f_L^{\leftarrow}$ is a fuzzy interior operator on L^Y.

Definition 2.15. [2] *$\mathcal{C} \subseteq L^X$ is called a fuzzy closure system if $\forall A \in L^X$,*

$$\bigcap_{B \in \mathcal{C}} (sub_X(A, B) \to B) \in \mathcal{C},$$

where $\forall x \in X$, $(\bigcap\limits_{B \in \mathcal{C}} (sub_X(A, B) \to B))(x) = \bigwedge\limits_{B \in \mathcal{C}} (sub_X(A, B) \to B(x))$.

Fuzzy closure operators and fuzzy closure systems are corresponding to each other. To see this, let j be a fuzzy closure operator on L^X. Then by Lemma 3.2 in [2], we see that $\mathcal{C}_j = \{A \in L^X |\ j(A) = A\}$ is a fuzzy closure system. Conversely, if $\mathcal{C}$ is a fuzzy closure system, we can define a fuzzy closure operator $j_{\mathcal{C}}$ on L^X as follows $\forall A \in L^X,\ x \in X,\ j_{\mathcal{C}}(A)(x) = \bigwedge\limits_{B \in \mathcal{C}} (sub_X(A, B) \to B(x))$. Bělohlávek in [2] had proved that $j_{\mathcal{C}_j} = j, \mathcal{C}_{j_{\mathcal{C}}} = \mathcal{C}$.

Definition 2.16. Let (X, e) be a fuzzy poset. Then a fuzzy lower set $B \in L^X$ is called a *fuzzy D-ideal* (called a $\mathcal{K}$-*set* in [21]) if $sub_X(A, B) \leq B(\sqcup A)$ holds for each $A \in L^X$ with $\sqcup A$ existing.

Let (X, e) be a fuzzy poset, A be a subset of X, and χ_A denote the *characteristic function* of A, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.17. (1) If A is a subset of X with $\sqcup\chi_A$ existing in (X, e), then $\bigvee A$ exists in $(X, \leq_e)$ and $\bigvee A = \sqcup\chi_A$.

(2) If A is a subset of X with $\sqcap\chi_A$ existing in (X, e), then $\bigwedge A$ exists in $(X, \leq_e)$ and $\bigwedge A = \sqcap\chi_A$.

(3) When L is a two-element lattice, a fuzzy poset (X, e) is precisely a crisp poset $(X, \leq)$, that is, $\leq = \leq_e$ and $e = e_{\leq}$. For each L-subset $A \in L^X$, there exists a unique subset $\hat{A}$ of X such that $A = \chi_{\hat{A}}$ and A is a fuzzy D-ideal if and only if $\hat{A}$ is a D-ideal [7].

(4) For any $x \in X, A \in L^X$, l_x and A^l are fuzzy fuzzy D-ideals in (X, e), where $A^l(x) = \bigwedge_{y \in X} (A(y) \to e(x, y))$.

Let $\mathcal{S}_X$ denote the set of all fuzzy D-ideals of X. Then $\mathcal{S}_X$ is a fuzzy closure system [21]. We let $j_{\mathcal{S}}$ denote the fuzzy closure operator on L^X induced by $\mathcal{S}_X$. For the notions and concepts, which are not explained here, please refer to [1, 2, 16, 24].

3 The extensions of L-subsets via fuzzy closure operators

Let X_j denote the set of all fixed elements of a fuzzy closure operator j on a fuzzy poset (X, e). Then by Example 2.4(2), $(X_j, e|_{X_j})$ is a fuzzy poset. Later, for the sake of convenience in writing, $e|_{X_j}$ is simply written as e. In the following, we shall consider the extensions of L-subsets of X_j into X.

3.1 The first type of extensions

For an L-subset A on X_j, we shall consider its first type of extensions.

Definition 3.1. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. Then $A_j \in L^X$ is called the first type of extensions of A, where $A_j = A \circ j$.*

Remark 3.2. *Obviously, $A_j(x) = A(x)$ whenever $x \in X_j$.*

Lemma 3.3. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. Then $\bigwedge_{x \in X_j} (A(x) \to e(x, y)) = \bigwedge_{x \in X} (A_j(x) \to e(x, y))$ holds for any $y \in X$.*

Proof. By Remark 3.2, we see that $\bigwedge_{x \in X_j} (A(x) \to e(x, y)) \geq (\bigwedge_{x \in X_j} (A(x) \to e(x, y))) \wedge (\bigwedge_{x \notin X_j} (A_j(x) \to e(x, y))) = \bigwedge_{x \in X} (A_j(x) \to e(x, y))$. On the other hand, by Proposition 2.1 and Remark 2.3 we have that $A_j(x) \to e(j(x), y) \leq A_j(x) \to e(x, y) \Longrightarrow \bigwedge_{x \in X_j} (A(x) \to e(x, y)) = \bigwedge_{x \in X} (A(j(x)) \to e(j(x), y)) \leq \bigwedge_{x \in X} (A_j(x) \to e(x, y))$. Thus, $\bigwedge_{x \in X_j} (A(x) \to e(x, y)) = \bigwedge_{x \in X} (A_j(x) \to e(x, y))$. □

Proposition 3.4. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. If $\sqcup A_j$ exists in (X, e), then $\sqcup A$ exists in (X_j, e) and $\sqcup A = j(\sqcup A_j)$.*

Proof. Since $\sqcup A_j$ exists in (X, e), we let $x_0 = \sqcup A_j$. By Definition 2.5, we have that

(1) $\forall x \in X,\ A_j(x) \leq e(x, x_0)$, which implies that $\forall x \in X_j,\ A(x) = A_j(x) \leq e(x, x_0) \leq e(j(x), j(x_0)) = e(x, j(x_0))$.

(2) $\forall y \in X,\ \bigwedge_{x \in X} (A_j(x) \to e(x, y)) \leq e(x_0, y)$, which implies that

$$\forall y \in X_j,\ \bigwedge_{x \in X_j} (A(x) \to e(x, y)) = \bigwedge_{x \in X} (A_j(x) \to e(x, y)) \leq e(x_0, y) \leq e(j(x_0), j(y)) = e(j(x_0), y).$$

By the above description, we have that $\sqcup A$ exists in (X_j, e) and $\sqcup A = j(x_0)$, that is, $\sqcup A = j(\sqcup A_j)$. □

Lemma 3.5. *Let $(X, \leq)$ be a poset, $j: X \to X$ be a map. Then j is a closure operator on $(X, \leq)$ if and only if j is a fuzzy closure operator on $(X, e_{\leq})$.*

Proof. The proof follows from Theorem 4.2 in [24]. □

In what follows we shall consider the relationship between the meets of A and its extension A_j for L-subsets A on X_j.

Example 3.6. Let $X = \{a, b, c, d, e, 1\}$ be a poset with the partial order $\leq$ determined by Figure 1.

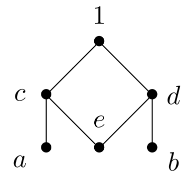

Figure 1

(1) Define a map $j_1 : X \to X$ such as $j_1(a) = j_1(c) = c$, $j_1(b) = j_1(d) = d$, $j_1(e) = e, j_1(1) = 1$. It is easy to check that j_1 is a closure operator and $X_{j_1} = \{e, c, d, 1\}$. Let $A : X_{j_1} \to L$ be a map with $A = \chi_{\{c,d,1\}}$. Then by Remark 2.17(3) we have $A_{j_1} = \chi_{\{a,b,c,d,1\}}$. By Remark 2.17(2),(3), we have that $\sqcap A$ exists in $(X_{j_1}, e_{\leq})$ and $\sqcap A = e$, but $\sqcap A_{j_1}$ does not exist in $(X, e_{\leq})$.

(2) Define a map $j_2 : X \to X$ such as $j_2(a) = a$, $j_2(c) = c, j_2(e) = e$, $j_2(b) = j_2(d) = j_2(1) = 1$. It is easy to check that j_2 is a closure operator and $X_{j_2} = \{a, c, e, 1\}$. Let $A: X_{j_2} \to L$ be a map with $A = \chi_{\{a,e\}}$. Then by Remark 2.17(3) we have $A_{j_2} = \chi_{\{a,e\}}$. By Remark 2.17(2),(3), we have that $\sqcap A$ and $\sqcap A_{j_1}$ do not exist.

(3) For the closure operator j_2 defined as (2), we let $A: X_{j_2} \to L$ be a map with $A = \chi_{\{1\}}$. Then by Remark 2.17(3) we have $A_{j_2} = \chi_{\{b,d,1\}}$. By Remark 2.17(2),(3), we have that $\sqcap A$ exists in $(X_{j_2}, e_{\leq})$ and $\sqcap A = 1$, $\sqcap A_{j_2}$ exists in $(X, e_{\leq})$ and $\sqcap A_{j_2} = b$. Further, we have $j_2(\sqcap A_{j_2}) = \sqcap A$.

Example 3.7. *Let $X = \{a, b, c, d, e, k, 1\}$ be a poset with the partial order $\leq$ determined by Figure 2.*

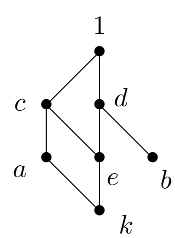

Figure 2

Define a map $j : X \to X$ such as $j(a) = j(c) = j(1) = 1$, $j(k) = j(e) = e$, $j(d) = d, j(b) = b$. It is easy to check that j is a closure operator and $X_j = \{b, d, e, 1\}$. Let $A : X_j \to L$ be a map with $A = \chi_{\{d,1\}}$. Then by Remark 2.17(3) we have $A_j = \chi_{\{a,c,d,1\}}$. By Remark 2.17(2),(3), we have that $\sqcap A$ exists in $(X_j, e_{\leq})$ and $\sqcap A = d$, $\sqcap A_j$ exists in $(X, e_{\leq})$ and $\sqcap A_j = k$, but $j(k) = e \neq d$, that is, $j(\sqcap A_j) \neq \sqcap A$.

Example 3.8. *Let $X = \{0, a, b, c, d, e, k, 1\}$ be a poset with the partial order $\leq$ determined by Figure 3.*

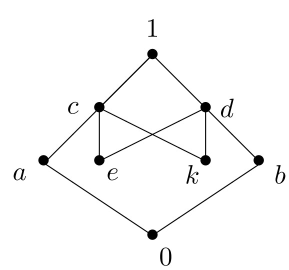

Figure 3

Define a map $j : X \to X$ such as $j(a) = c$, $j(b) = d$ and $j(x) = x$ whenever $x \in \{0, c, d, e, k, 1\}$. It is easy to check that j is a closure operator and $X_j = \{0, c, d, e, k, 1\}$. Let $A : X_j \to L$ be a map with $A = \chi_{\{c,d\}}$. Then by Remark 2.17(3) we have $A_j = \chi_{\{a,b,c,d\}}$. By Remark 2.17(2),(3), we have that $\sqcap A_j$ exists in $(X, e_{\leq})$ and $\sqcap A_j = 0$, but $\sqcap A$ does not exist in $(X_j, e_{\leq})$.

3.2 The second type of extensions

In the following, we shall investigate the second type of extensions for L-subsets on X_j.

Definition 3.9. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. Then $A_{j_0} \in L^X$ is called the second type of extensions of A, where*

$$\forall x \in X, A_{j_0}(x) = \begin{cases} A(x) & \text{if } x \in X_j, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.10. *(1) From Examples 3.7, 3.8 and Proposition 3.14, it follows that there is no direct relation between the first type of extensions A_j and the second type of extensions A_{j_0}.*

(2) $A_{j_0} \leq A_j$ and $A_{j_0}|_{X_j} = A_j|_{X_j} = A$.

Proposition 3.11. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. Then $\sqcup A_{j_0}$ exists in (X, e) if and only if $\sqcup A_j$ exists in (X, e). Further, $\sqcup A_j = \sqcup A_{j_0}$ whenever $\sqcup A_j$ or $\sqcup A_{j_0}$ exists in (X, e).*

Proof. The proof directly follows from Lemma 3.3 and the following fact that

$$\bigwedge_{x \in X_j} (A(x) \to e(x, y)) = \bigwedge_{x \in X} (A_{j_0}(x) \to e(x, y)),$$

holds for any $y \in X$. □

Corollary 3.12. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. If $\sqcup A_{j_0}$ exists in (X, e), then $\sqcup A$ exists in (X_j, e) and $\sqcup A = j(\sqcup A_{j_0})$.*

In what follows we shall consider the relationship between $\sqcap A$ and $\sqcap A_{j_0}$.

Lemma 3.13. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. If $\sqcap A_{j_0}$ exists in (X, e), then $\sqcap A$ exists in (X_j, e) and $\sqcap A = j(\sqcap A_{j_0})$.*

Proof. We will first prove the fact that $\bigwedge_{x \in X_j} (A(x) \to e(y, x)) = \bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x))$ holds for any $y \in X$.

Indeed:

$$\begin{aligned} \bigwedge_{x \in X_j} (A(x) \to e(y, x)) &= \Big(\bigwedge_{x \in X_j} (A(x) \to e(y, x))\Big) \wedge 1 \\ &= \Big(\bigwedge_{x \in X_j} (A(x) \to e(y, x))\Big) \wedge \Big(\bigwedge_{x \notin X_j} (A_{j_0}(x) \to e(y, x))\Big) \\ &= \bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x)). \end{aligned}$$

Since $\sqcap A_{j_0}$ exists in (X, e), we let $x_0 = \sqcap A_{j_0}$. Then by Definition 2.5, we have that

(1) $\forall x \in X, A_{j_0}(x) \leq e(x_0, x)$, which implies that $\forall x \in X_j,\ A(x) = A_{j_0}(x) \leq e(x_0, x) \leq e(j(x_0), j(x)) = e(j(x_0), x)$.

(2) $\forall y \in X, \bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x)) \leq e(y, x_0)$, which implies that

$$\forall y \in X_j,\ \bigwedge_{x \in X_j} (A(x) \to e(y, x)) = \bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x)) \leq e(y, x_0) \leq e(y, j(x_0)).$$

Therefore, by Definition 2.5, we have that $\sqcap A$ exists in (X_j, e) and $\sqcap A = j(x_0)$, that is $\sqcap A = j(\sqcap A_{j_0})$. □

Proposition 3.14. *Let j be a fuzzy closure operator on a fuzzy poset (X, e), $A \in L^{X_j}$. If $\sqcap A_{j_0}$ exists in (X, e), then $\sqcap A$ exists in (X_j, e) and $\sqcap A = \sqcap A_{j_0}$.*

Proof. Since $\sqcap A_{j_0}$ exists in (X, e), we let $x_0 = \sqcap A_{j_0}$. By Lemma 3.13, we know that $\sqcap A = j(x_0)$. Thus it suffices to prove that $x_0 = j(x_0)$, that is, $\bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x)) = e(y, j(x_0))$ holds for any $y \in X$.

On one hand, we have that $\bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x)) = e(y, x_0) \leq e(y, j(x_0))$. On the other hand, we have that $\forall x \in X$, (1) if $x \notin X_j$, then $A_{j_0}(x) = 0 \leq e(j(x_0), x)$; (2) if $x \in X_j$, then $A_{j_0}(x) = A(x) \leq e(j(x_0), x)$, which implies that $A_{j_0}(x) \to e(y, x) \geq e(j(x_0), x) \to e(y, x) \geq e(y, j(x_0))$, that is, $\bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x)) \geq e(y, j(x_0))$. Thus, we have $\bigwedge_{x \in X} (A_{j_0}(x) \to e(y, x)) = e(y, j(x_0))$. □

Corollary 3.15. *If (X, e) is a complete fuzzy poset, then so is (X_j, e).*

Remark 3.16. *In general, the converse of Proposition 3.14 does not hold. For instance, we let (X, e) be the fuzzy poset in Example 3.6, $j = j_1$. We further assume that A is the L-subset on X_j in Example 3.6(1). Then $\sqcap A$ exists in (X_j, e) and $\sqcap A = e$, but $\sqcap A_{j_0}$ does not exist in (X, e).*

4 CFPos as a reflective full subcategory of $\mathbf{FPos}_{\sqcup}$.

In [13], Gao et al. discussed the relationship between fuzzy closure operators and fuzzy closure L-systems based on complete residuated lattice. In this section, we shall consider an application of fuzzy closure operators. Let (X, e) be a fuzzy poset. Then (L^X, sub_X) is a complete fuzzy poset, where for $\mathcal{B} \in L^{L^X}$, $x \in X$, $(\sqcup\mathcal{B})(x) = \bigvee_{A \in L^X} (\mathcal{B}(A) * A(x))$ and $(\sqcap\mathcal{B})(x) = \bigwedge_{A \in L^X} (\mathcal{B}(A) \to A(x))$. By Corollary 3.15, we have that $(\mathcal{S}_X, sub_X)$ is also a complete fuzzy poset. Further, we have that $\sqcup\mathcal{A} = j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})$ for each $\mathcal{A} \in L^{\mathcal{S}_X}$.

Lemma 4.1. *Let (X, e) be a fuzzy poset, $A, B \in L^X$. If $\sqcup A$ and $\sqcup B$ exist in (X, e), then $sub_X(A, B) \le e(\sqcup A, \sqcup B)$.*

Proof. Since $B(x) \le e(x, \sqcup B)$ for all $x \in X$, it follows from Proposition 2.1 and Proposition 2.6 that $sub_X(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)) \le \bigwedge_{x \in X} (A(x) \to e(x, \sqcup B)) = e(\sqcup A, \sqcup B)$. □

Lemma 4.2. *Let $(X, e_X), (Y, e_Y)$ be two fuzzy posets, $f : X \to Y$ be a fuzzy-existing-join preserving map, and $B \in \mathcal{S}_Y$. Then $f_L^{\leftarrow}(B) \in \mathcal{S}_X$, that is, $f_L^{\leftarrow}(B)$ is a fuzzy D-ideal in (X, e_X).*

Proof. For all $x, y \in X$, we have that $f_L^{\leftarrow}(B)(x) * e_X(y, x) = B(f(x)) * e_X(y, x) \le B(f(x)) * e_X(f(y), f(x)) \le B(f(y)) = f_L^{\leftarrow}(B)(y)$, that is, $f_L^{\leftarrow}(B)$ is a fuzzy lower set.

Next, we let $A \in L^X$ with $\sqcup A$ existing. Then since $f(\sqcup A) = \sqcup f_L^{\rightarrow}(A)$ and B is a fuzzy D-ideal, by Remark 2.3 and Remark 2.14 we have that

$$sub_X(A, f_L^{\leftarrow}(B)) \le sub_Y(f_L^{\rightarrow}(A), f_L^{\rightarrow}(f_L^{\leftarrow}(B))) \le sub_Y(f_L^{\rightarrow}(A), B) \le B(\sqcup f_L^{\rightarrow}(A)) = B(f(\sqcup A)) = f_L^{\leftarrow}(B)(\sqcup A).$$

Thus, $f_L^{\leftarrow}(B)$ is a fuzzy D-ideal. □

Lemma 4.3. *Let $(X, e_X), (Y, e_Y)$ be two fuzzy posets and $f\colon X \to Y$ be an order-preserving map. If (Y, e_Y) is a complete fuzzy poset, then $e_Y(\sqcup f_L^{\rightarrow}(A), a) = sub_Y(f_L^{\rightarrow}(A), l_a)$ for all $a \in Y$, $A \in L^X$.*

Proof. The proof is straightforward. □

Proposition 4.4. *Let $(X, e_X), (Y, e_Y)$ be two fuzzy posets and $f\colon X \to Y$ be a fuzzy-existing-join preserving map. If (Y, e_Y) is a complete fuzzy poset, then $\sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})) = \sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})$ for all $\mathcal{A} \in L^{\mathcal{S}_X}$.*

Proof. Let $\mathcal{A} \in L^{\mathcal{S}_X}$. Then we have

(1) $e_Y(\sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), \sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}))) = 1$.

Indeed: Since $\sqcup\mathcal{A}_{j_{\mathcal{S}}} \le j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})$, by Lemma 4.1 we have

$$1 = sub_X(\sqcup\mathcal{A}_{j_{\mathcal{S}}}, j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})) \le sub_Y(f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}))) \le e_Y(\sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), \sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}))).$$

(2) $e_Y(\sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})), \sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})) = 1$.

For each $a \in Y$, it is easy to verify that l_a is a fuzzy D-ideal in (Y, e_Y). From Lemma 4.2, we have that $f_L^{\leftarrow}(l_a)$ is a fuzzy D-ideal in (X, e_X). By Remark 2.3 and Remark 2.14, we have that

$$\begin{aligned}
e_Y(\sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), a) &= sub_Y(f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), l_a) && \text{Lemma 4.3}\\
&\le sub_X(f_L^{\leftarrow}(f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})), f_L^{\leftarrow}(l_a))\\
&\le sub_X(\sqcup\mathcal{A}_{j_{\mathcal{S}}}, f_L^{\leftarrow}(l_a))\\
&= \bigwedge_{A \in L^X} (\mathcal{A}_{j_{\mathcal{S}}}(A) \to sub_X(A, f_L^{\leftarrow}(l_a)))\\
&= \bigwedge_{A \in \mathcal{S}_X} (\mathcal{A}(A) \to sub_X(A, f_L^{\leftarrow}(l_a))) && \text{Lemma 3.3}\\
&= sub_X(\sqcup\mathcal{A}, f_L^{\leftarrow}(l_a))\\
&= sub_X(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), f_L^{\leftarrow}(l_a))\\
&\le sub_Y(f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})), f_L^{\rightarrow}(f_L^{\leftarrow}(l_a)))\\
&\le sub_Y(f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})), l_a)\\
&= e_Y(\sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})), a). && \text{Lemma 4.3}
\end{aligned}$$

Taking $a = \sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})$, by the above description we have that $1 = e_Y(\sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})), \sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}))$.

Therefore, by (E3) we have that $\sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})) = \sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})$. □

Lemma 4.5. [23] *Let (X, e) be a fuzzy poset. Then $e(x, y) = \bigwedge_{z \in X} (e(z, x) \to e(z, y))$ for all $x, y \in X$.*

Lemma 4.6. *Let (X,e) be a fuzzy poset and $C\in\mathcal{S}_X$. Then $sub_X(A,C)=C(\sqcup A)=sub_X(l_{\sqcup A},C)$ holds for each $A\in L^X$ with $\sqcup A$ existing.*

Proof. We assume that C is a fuzzy D-ideal in (X,e). On one hand, we can easily show that $C(\sqcup A)=\bigwedge_{x\in X}(e(x,\sqcup A)\to C(x))=\bigwedge_{x\in X}(l_{\sqcup A}(x)\to C(x))=sub_X(l_{\sqcup A},C)$. On the other hand, since $A(x)\le e(x,\sqcup A)$ we have that $sub_X(A,C)\le C(\sqcup A)=\bigwedge_{x\in X}(e(x,\sqcup A)\to C(x))\le\bigwedge_{x\in X}(A(x)\to C(x))=sub_X(A,C)$, which implies that $sub_X(A,C)=C(\sqcup A)=sub_X(l_{\sqcup A},C)$. □

Proposition 4.7. *Let (X,e) be a fuzzy poset, $u:X\to\mathcal{S}_X$ be a map defined by $u(x)=l_x$. Then u is fuzzy-existing-join preserving.*

Proof. Let $A\in L^X$ with $\sqcup A$ existing, $C\in\mathcal{S}_X$. Then we have that

$$\begin{aligned}
sub_X(\sqcup u_L^{\rightarrow}(A),C) &= \bigwedge_{B\in\mathcal{S}_X}((u_L^{\rightarrow}(A))(B)\to sub_X(B,C))\\
&= \bigwedge_{B\in\mathcal{S}_X}((\bigvee_{l_x=B}A(x))\to sub_X(B,C))\\
&= \bigwedge_{B\in\mathcal{S}_X}\bigwedge_{l_x=B}(A(x)\to sub_X(B,C))\\
&= \bigwedge_{x\in X}(A(x)\to sub_X(l_x,C))\\
&= \bigwedge_{x\in X}(A(x)\to C(\sqcup l_x)) \qquad \text{Lemma 4.6}\\
&= \bigwedge_{x\in X}(A(x)\to C(x))\\
&= sub_X(A,C)=sub_X(l_{\sqcup A},C)\\
&= sub_X(u(\sqcup A),C),
\end{aligned}$$

which implies that $u(\sqcup A)=\sqcup u_L^{\rightarrow}(A)$. □

Lemma 4.8. *Let (X,e) be a fuzzy poset, $A\in\mathcal{S}_X$. Then $\sqcup(l_A\wedge\chi_{u(X)})=A$ where $u(X)=\{l_x:x\in X\}$.*

Proof. Since $(\mathcal{S}_X,sub_X)$ is a complete fuzzy poset, we have that for $A\in\mathcal{S}_X$, the fuzzy join of the map $l_A\wedge\chi_{u(X)}:\mathcal{S}_X\to L$ defined pointwisely exists, that is, $\sqcup(l_A\wedge\chi_{u(X)})$ exists in $(\mathcal{S}_X,sub_X)$.

For any $B\in\mathcal{S}_X$, we have

$$\begin{aligned}
sub_X(\sqcup(l_A\wedge\chi_{u(X)}),B) &= \bigwedge_{T\in\mathcal{S}_X}((l_A\wedge\chi_{u(X)})(T)\to sub_X(T,B))\\
&= \bigwedge_{T\in\mathcal{S}_X}((l_A(T)\wedge\chi_{u(X)}(T))\to sub_X(T,B))\\
&= \bigwedge_{T\in u(X)}(l_A(T)\to sub_X(T,B))\\
&= \bigwedge_{T\in u(X)}(sub_X(T,A)\to sub_X(T,B))\\
&= \bigwedge_{x\in X}(sub_X(l_x,A)\to sub_X(l_x,B))\\
&= \bigwedge_{x\in X}(A(x)\to B(x)) \qquad \text{Lemma 4.6}\\
&= sub_X(A,B),
\end{aligned}$$

which implies that $\sqcup(l_A\wedge\chi_{u(X)})=A$. □

Let **FPos** ($\mathbf{FPos}_{\sqcup}$) denote the category of fuzzy posets and order-preserving (fuzzy-existing-join preserving) maps, and **CFPos** denote the category of complete fuzzy posets and fuzzy-join preserving maps. Obviously, **CFPos** is a full subcategory of $\mathbf{FPos}_{\sqcup}$. **CFPos** is a subcategory of **FPos**, but not full.

Theorem 4.9. ***CFPos*** *is a reflective full subcategory of* $\boldsymbol{FPos}_{\sqcup}$.

Proof. Let (X,e_X) be a fuzzy poset. Then $(\mathcal{S}_X,sub_X)$ is a complete fuzzy poset. By Proposition 4.7, we have that $u:X\to\mathcal{S}_X$ is a $\mathbf{FPos}_{\sqcup}$-morphism. In the following, we need to show that $u:X\to\mathcal{S}_X$ has the universal property.

Let (Y,e_Y) be a complete fuzzy poset and $f:X\to Y$ be a fuzzy-existing-join preserving map. We define a map $\hat{f}:\mathcal{S}_X\to Y$ as follows

$$\hat{f}(A)=\sqcup f_L^{\rightarrow}(A)$$

for all $A \in \mathcal{S}_X$. Since (Y, e_Y) is a complete fuzzy poset, the map $\hat{f}$ is well defined.

Firstly, we shall prove that $\hat{f}$ is fuzzy-join preserving, that is, $\hat{f}(\sqcup\mathcal{A}) = \sqcup\hat{f}_L^{\rightarrow}(\mathcal{A})$ for all $\mathcal{A} \in L^{\mathcal{S}_X}$. By Proposition 4.4, we have that $\hat{f}(\sqcup\mathcal{A}) = \sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}) = \sqcup f_L^{\rightarrow}(j_{\mathcal{S}}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})) = \sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})$. In what follows we only need to prove that $\sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}) = \sqcup\hat{f}_L^{\rightarrow}(\mathcal{A})$.

Let $y \in Y$. Then we have

$$\begin{aligned}
e_Y(\sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), y) &= \bigwedge_{t\in Y} (f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}})(t) \rightarrow e_Y(t,y)) \\
&= \bigwedge_{t\in Y} ((\bigvee_{f(z)=t} (\sqcup\mathcal{A}_{j_{\mathcal{S}}})(z)) \rightarrow e_Y(t,y)) \\
&= \bigwedge_{t\in Y} \bigwedge_{f(z)=t} ((\sqcup\mathcal{A}_{j_{\mathcal{S}}})(z) \rightarrow e_Y(t,y)) \\
&= \bigwedge_{z\in X} ((\sqcup\mathcal{A}_{j_{\mathcal{S}}})(z) \rightarrow e_Y(f(z),y)) \\
&= \bigwedge_{z\in X} ((\bigvee_{A\in L^X} \mathcal{A}_{j_{\mathcal{S}}}(A) * A(z)) \rightarrow e_Y(f(z),y)) \\
&= \bigwedge_{z\in X} \bigwedge_{A\in L^X} ((\mathcal{A}_{j_{\mathcal{S}}}(A) * A(z)) \rightarrow e_Y(f(z),y)) \\
&= \bigwedge_{z\in X} \bigwedge_{A\in L^X} (\mathcal{A}_{j_{\mathcal{S}}}(A) \rightarrow (A(z) \rightarrow e_Y(f(z),y))) \\
&= \bigwedge_{A\in L^X} (\mathcal{A}_{j_{\mathcal{S}}}(A) \rightarrow (\bigwedge_{z\in X} (A(z) \rightarrow e_Y(f(z),y)))) \\
&= \bigwedge_{A\in L^X} (\mathcal{A}_{j_{\mathcal{S}}}(A) \rightarrow (\bigwedge_{z\in X} (A(z) \rightarrow f_L^{\leftarrow}(l_y)(z)))) \\
&= \bigwedge_{A\in L^X} (\mathcal{A}_{j_{\mathcal{S}}}(A) \rightarrow sub_X(A, f_L^{\leftarrow}(l_y))) \\
&= \bigwedge_{A\in \mathcal{S}_X} (\mathcal{A}(A) \rightarrow sub_X(A, f_L^{\leftarrow}(l_y))),
\end{aligned}$$

$$\begin{aligned}
e_Y(\sqcup\hat{f}_L^{\rightarrow}(\mathcal{A}), y) &= \bigwedge_{t\in Y} (\hat{f}_L^{\rightarrow}(\mathcal{A})(t) \rightarrow e_Y(t,y)) = \bigwedge_{t\in Y} ((\bigvee_{\hat{f}(A)=t} \mathcal{A}(A)) \rightarrow e_Y(t,y)) \\
&= \bigwedge_{t\in Y} \bigwedge_{\hat{f}(A)=t} (\mathcal{A}(A) \rightarrow e_Y(t,y)) = \bigwedge_{A\in\mathcal{S}_X} (\mathcal{A}(A) \rightarrow e_Y(\hat{f}(A),y)) \\
&= \bigwedge_{A\in\mathcal{S}_X} (\mathcal{A}(A) \rightarrow e_Y(\sqcup f_L^{\rightarrow}(A),y)) \\
&= \bigwedge_{A\in\mathcal{S}_X} (\mathcal{A}(A) \rightarrow (\bigwedge_{z\in Y} (f_L^{\rightarrow}(A)(z) \rightarrow e_Y(z,y)))) \\
&= \bigwedge_{A\in\mathcal{S}_X} \bigwedge_{z\in Y} (\mathcal{A}(A) \rightarrow (f_L^{\rightarrow}(A)(z) \rightarrow e_Y(z,y))) \\
&= \bigwedge_{A\in\mathcal{S}_X} \bigwedge_{z\in Y} (\mathcal{A}(A) \rightarrow ((\bigvee_{f(x)=z} A(x)) \rightarrow e_Y(z,y))) \\
&= \bigwedge_{A\in\mathcal{S}_X} \bigwedge_{z\in Y} \bigwedge_{f(x)=z} (\mathcal{A}(A) \rightarrow (A(x) \rightarrow e_Y(z,y))) \\
&= \bigwedge_{A\in\mathcal{S}_X} \bigwedge_{x\in X} (\mathcal{A}(A) \rightarrow (A(x) \rightarrow e_Y(f(x),y))) \\
&= \bigwedge_{A\in\mathcal{S}_X} (\mathcal{A}(A) \rightarrow \bigwedge_{x\in X} (A(x) \rightarrow e_Y(f(x),y))) \\
&= \bigwedge_{A\in\mathcal{S}_X} (\mathcal{A}(A) \rightarrow sub_X(A, f_L^{\leftarrow}(l_y))).
\end{aligned}$$

Thus, we have that $e_Y(\sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}), y) = e_Y(\sqcup\hat{f}_L^{\rightarrow}(\mathcal{A}), y)$, which implies $\sqcup f_L^{\rightarrow}(\sqcup\mathcal{A}_{j_{\mathcal{S}}}) = \sqcup\hat{f}_L^{\rightarrow}(\mathcal{A})$.

Next, we shall prove that $\hat{f} \circ u = f$. Let $x \in X$ and $y \in Y$. Then we have

$$\begin{aligned}
e_Y(\hat{f} \circ u(x), y) &= e_Y(\hat{f}(l_x), y) = e_Y(\sqcup f_L^{\rightarrow}(l_x), y) \\
&= \bigwedge_{z\in Y} ((f_L^{\rightarrow}(l_x))(z) \rightarrow e_Y(z,y)) \\
&= \bigwedge_{z\in Y} ((\bigvee_{f(t)=z} (l_x)(t)) \rightarrow e_Y(z,y)) \\
&= \bigwedge_{z\in Y} \bigwedge_{f(t)=z} (e_X(t,x) \rightarrow e_Y(z,y)) \\
&= \bigwedge_{t\in X} (e_X(t,x) \rightarrow e_Y(f(t),y)) \\
&= e_Y(f(x),y),
\end{aligned}$$

which implies $\hat{f} \circ u(x) = f(x)$, that is, $\hat{f} \circ u = f$.

Finally, we shall prove the uniqueness of $\hat{f}$. Assume that there exists another fuzzy-join preserving map $g : \mathcal{S}_X \to Y$ such that $g \circ u = f$. Then $\forall x \in X,\ \hat{f}(l_x) = g(l_x)$.

Fact 1: Let $A \in \mathcal{S}_X$. Then $\hat{f}_L^{\rightarrow}(l_A \wedge \chi_{u(X)}) = g_L^{\rightarrow}(l_A \wedge \chi_{u(X)})$.

Indeed: For any $y \in Y$, we have that

$$\begin{aligned}(g_L^{\rightarrow}(l_A \wedge \chi_{u(X)}))(y) &= \bigvee_{g(B)=y} (l_A \wedge \chi_{u(X)})(B) = \bigvee_{g(B)=y} l_A(B) \wedge \chi_{u(X)}(B) \\ &= \bigvee_{g(l_x)=y} sub_X(l_x, A) = \bigvee_{\hat{f}(l_x)=y} sub_X(l_x, A) \\ &= \bigvee_{\hat{f}(B)=y} l_A(B) \wedge \chi_{u(X)}(B) \\ &= \bigvee_{\hat{f}(B)=y} (l_A \wedge \chi_{u(X)})(B) \\ &= (\hat{f}_L^{\rightarrow}(l_A \wedge \chi_{u(X)}))(y),\end{aligned}$$

which implies that $\hat{f}_L^{\rightarrow}(l_A \wedge \chi_{u(X)}) = g_L^{\rightarrow}(l_A \wedge \chi_{u(X)})$.

It follows from Lemma 4.8 that $\sqcup(l_A \wedge \chi_{u(X)}) = A$. By **Fact 1**, we have that

$$g(A) = g(\sqcup(l_A \wedge \chi_{u(X)})) = \sqcup g_L^{\rightarrow}(l_A \wedge \chi_{u(X)}) = \sqcup \hat{f}_L^{\rightarrow}(l_A \wedge \chi_{u(X)}) = \hat{f}(\sqcup(l_A \wedge \chi_{u(X)})) = \hat{f}(A),$$

which implies that $g = \hat{f}$. □

Let (X, e) be a fuzzy poset, $\mathcal{L}_X$ denote the set of all fuzzy lower sets on X. By Definition 2.15, we can easily verify that $\mathcal{L}_X$ is a fuzzy closure system. Then there exists a fuzzy closure operator $j_{\mathcal{L}}$ on L^X such that $\mathcal{L}_X = \{A \in L^X : j_{\mathcal{L}}(A) = A\}$. From Corollary 3.15, it follows that $(\mathcal{L}_X, sub_X)$ is a complete fuzzy poset.

Theorem 4.10. ***CFPos*** *is a reflective subcategory of* ***FPos****.*

Proof. Let (X, e_X) be a fuzzy poset. By the above description, we have that $(\mathcal{S}_X, sub_X)$ is a complete fuzzy poset. We now define a map $v\colon X \to \mathcal{L}_X$. It is easy to check that v is order-preserving (v is not fuzzy-existing-join preserving), that is, v is a **FPos**-morphism. Similar to the proof of Theorem 4.9, we can show that $v\colon X \to \mathcal{L}_X$ has the universal property. Thus, **CFPos** is a reflective subcategory of **FPos**. □

5 Conclusion

In this paper, we mainly discuss two types of extensions of L-subsets via fuzzy closure operators, and prove that the set of all fixed elements of a fuzzy closure operator j in a complete fuzzy poset is again a complete fuzzy poset. As an application of fuzzy closure operators, we also investigate the relations between the category **CFPos** of complete fuzzy posets and fuzzy-join-preserving maps and the category **FPos** (**FPos**$_\sqcup$) of fuzzy posets and order-preserving (fuzzy-existing-join preserving) maps. In further work, we shall use fuzzy closure operators to investigate the tensor product of complete fuzzy posets.

Acknowledgement

The authors gratefully acknowledge the financial support of the National Natural Science Foundation of China (Grant nos. 11971286,11531009,11871320), and also wish to express their sincere thanks to the anonymous referees for their valuable comments and suggestions which improved the quality of the paper.

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Iranian Journal of Fuzzy Systems
Volume 18, Number 1 (2021), pp. 209

University of Sistan and Baluchestan

Fuzzy closure operators and their applications

S. W. Han and R. R. Wang

عملگرهای بستار فازی و کاربرد آن‌ها

چکیده. عملگرهای بستار فازی نقش عمده‌ای در نظریه ترتیب فازی ایفا می‌کنند. هدف این مقاله غنی‌سازی بیشتر و بهبود مطالعه عملگرهای فازی است. براساس Yao و Belohlavek مطالعه‌ی خواص نسبی عملگرهای بستار جبری فازی ادامه خواهیم داد. ابتدا، توسیع‌های L- زیرمجموعه‌ها را از طریق عملگرهای فازی در نظر خواهیم گرفت. سپس، کاربردی از عملگرهای فازی ارائه می‌دهیم، یعنی، به وسیله عملگرهای بستار فازی، ثابت می‌کنیم که رسته CFPos از مجموعه‌های مرتب جزئی تمام و نگاشت‌های حافظ الحاق فازی آن‌ها یک زیر رسته کامل انعکاسی از $FPos_{\sqcup}$ می‌باشد، که $FPos_{\sqcup}$ رسته مجموعه‌های مرتب جزئی فازی و نگاشت‌های حافظ الحاق موجود فازی آن‌ها را نشان می‌دهد.