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On the stability for implicit uncertain fractional integral equations with fuzzy concept

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Abstract

This paper aims to establish the existence and uniqueness results for two kinds of implicit uncertain fractional integral equations with the fuzzy concept via the fixed point theorem and successive approximation method. Besides, we also present various Ulam-Hyers stability and Ulam-Hyers-Rassias stability results of two kinds of implicit fuzzy fractional integral equations. Finally, some examples are given to illustrate our main results.

Keywords: Stability theory, successive approximation, fuzzy fractional integral equations, implicit fuzzy fractional integral equations.

1 Introduction

The subject of fractional calculus and fractional differential equations has become a rapidly growing area and has found applications in various areas of engineering, mathematics, physics and bioengineering, and other applied sciences. Besides, appreciable attentions have been paid to the investigation of fundamental theories of implicit fractional differential equations involving Caputo and Riemann-Liouville fractional derivatives (see [1, 12, 13, 14, 29, 41, 52] and references cited therein). In the last decades, the fuzzy calculus and fuzzy differential equations (FFEs) have received increasing attention in the scientific community and play an essential role in the areas of dynamical systems having uncertain factors. For the fundamental theories of fuzzy analysis and FFEs, we refer the readers to the papers [2, 6, 7, 11, 15, 17, 51, 50], and references cited therein. Based on the advantages and development of fractional calculus, the basic theory of fractional calculus in the field of FFEs has attracted many mathematicians and the investigation of this field has become an crucial subject in the mathematical analysis area with the fuzzy concept. There was a significant development in the theory of fuzzy fractional analysis, especially in the field of fuzzy fractional differential equations (FFDEs) involving Riemann-Liouville fractional derivative basing on the Hukuhara difference; see for instance the papers of Allahviranloo et al. [9], Khastan, Nieto and Rodríguez-López [26]. Recently, particular attention has been given to the fundamental theories of fuzzy fractional analysis according to the approaches of generalized Hukuhara derivative of fuzzy functions, and to the investigations of the existence and stability of solution of FDEs and partial FDEs via Caputo fractional derivative concept; see for instance the papers by Alikhani et al. [5], Allahviranloo et al. [8], Fard et al. [16], Hoa [23], Long et al. [31]-[34], Lupulescu et al. [20, 21, 35], Mazandarani [37, 39], Najariyan et al. [40], Salahshour et al. [44] and the references therein. Very recently, the methods for finding the exact and numerical solutions of FFDEs are presented in [36] with a modified fractional Euler method, in [19, 20] with the modified Adams-Bashforth-Moulton method, in [3, 4] with the methods based on operational matrix of shifted Chebyshev polynomials and the spectral tau, in [10] with the method of fuzzy Laplace transforms.

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Recently, the investigation of Ulam-Hyers' type stability became the object of research by several mathematicians and the study of this area has grown to be one of the most important subjects in the mathematical analysis area since it is quite useful in many applications such as numerical analysis, optimization, biology, and economics, where finding the exact solutions is quite difficult. For the advanced contribution to such problems, we refer the readers to Abbas et al. [1], Huang et al. [24], Hyers [25], Kucche et al. [28]-[29], Rassias [42], Rus [43], Shah and Zada [45, 57], Sousa et al. [48, 49] and Wang et al. [54, 55]. To the best of our knowledge, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities study of implicit fractional fuzzy integral equations (FFIEs) and differential equations have not yet been investigated, however, there are only a few works on Ulam types stability for fuzzy functional equations, fuzzy differential equations and impulsive fuzzy differential equations with integer-order (see Liu et al. [30], Long et al. [34], Shen et al. [46, 47], Wu et al. [56], and the references therein). With the expansion of the Ulam-Hyers's stability concepts and an increasing number of papers about the area of FFDEs, the results about the existence, uniqueness, and stability of the Ulam-Hyers-Rassias's type of solutions the above problems are interesting subjects. So, in this work, we shall investigate the existence of solution and stability theory of the following fractional implicit integral equations in the fuzzy setting:

$$u(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, u(s), {}^{RL}\mathcal{D}_{a+}^{\alpha} u(s)) ds, \quad t \in J, \tag{1}$$

and

$$u(t) \ominus_{gH} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, u(s), {}^{C}\mathcal{D}_{a+}^{\alpha} u(s)) ds, \quad t \in J, \tag{2}$$

where $J = (a, b]$ is a finite interval on the half-axis $\mathbb{R}^+$, $f \in C(J \times E \times E, E)$; ${}^{RL}\mathcal{D}_{a+}^{\alpha} u$ and ${}^{C}\mathcal{D}_{a+}^{\alpha} u$ denote the Riemann-Liouville fuzzy fractional derivative and the Caputo fuzzy fractional derivative of the fuzzy function u, respectively. The main object of the paper is to analyze the various types of stability, that is, Ulam-Hyers, Ulam-Hyers-Rassias in the interval $[a, b]$ for the fractional integral equation with the fuzzy concept (1) and (2) via fixed point theorem and successive approximation method.

2 Fundamental theorems

In this section, we recall some basic definitions of fuzzy analysis and fuzzy-type fractional theory which are used in this paper (see [11]). In the following, we consider the space of fuzzy numbers E, which consists of normal, fuzzy convex, upper semicontinuous and compactly supported fuzzy subsets of $\mathbb{R}$. Denote by $[u]^r = \{z \in \mathbb{R} \mid u(z) \geq r\}$ the $r-$level set of u, where $r \in (0, 1]$, and $[u]^0 = \overline{\{z \in \mathbb{R} \mid u(z) > 0\}}$. Then, the $r-$level set of u, $[u]^r = [\underline{u}(r), \overline{u}(r)]$, is a bounded closed interval, for any $r \in [0, 1]$. The Hausdorff metric on E is defined by

$$D_0[u_1, u_2] = \sup_{0 \leq r \leq 1} H([u_1]^r, [u_2]^r), \quad \forall u_1, u_2 \in E,$$

where $H([u_1]^r, [u_2]^r) = \max\{|\underline{u}_1(r) - \underline{u}_2(r)|, |\overline{u}_1(r) - \overline{u}_2(r)|\}$.

Definition 2.1. Basing on Zadeh's extension principle, the addition and scalar multiplication in E are defined as follows:

$$[u_1 + u_2] = [u_1]^r + [u_2]^r = \{\omega_1 + \omega_2 \mid \omega_1 \in [u_1]^r, \omega_2 \in [u_2]^r\},$$
$$[k \cdot u_1]^r = k[u_1]^r = \{k\omega \mid \omega \in [u]^r\}, \forall r \in [0, 1].$$

Also, the following properties and for any $u_1, u_2, u_3, u_4, u \in E$, $k_1, k_2, k \in \mathbb{R}$ hold (see [19]):

(i) $u_1 + (u_2 + u_3) = (u_1 + u_2) + u_3$; $(k_1 + k_2)u_1 = k_1 u_2 + k_2 u_2$; $k(u_1 + u_2) = ku_1 + ku_2$.
(ii) $D_0[u_1 + u_3, u_2 + u_3] = D_0[u_1, u_2]$; $D_0[u_1 + u_2, u_3 + u_4] \leq D_0[u_1, u_3] + D_0[u_2, u_4]$.
(iii) $D_0[u_1 + u_2, \hat{0}] \leq D_0[u_2, \hat{0}] + D_0[u_2, \hat{0}]$; $D_0[ku_1, ku_2] = |k| D_0[u_1, u_2]$.
(iv) $D_0[k_1 u, k_2 u] = |k_1 - k_2| D_0[u, \hat{0}]$, where $k_1 k_2 \geq 0$.

Definition 2.2. [11] Let $u_1, u_2 \in E$. The Hukuhara difference between u_1 and u_2 has been introduced as a set $u_3 \in E$ for which $u_1 \ominus u_2 = u_3 \Leftrightarrow u_1 = u_2 + u_3$. It is well-known that the Hukuhara difference is unique, but it does not always exist. Then, a generalization of the Hukuhara difference (which is called generalized Hukuhara difference) aims to overcome this situation. The generalized Hukuhara difference of two fuzzy numbers $u_1, u_2 \in E$ (gH-difference for short) is defined as follows:

$$u_1 \ominus_{gH} u_2 = u_3 \Leftrightarrow \begin{cases} \text{(i)} \quad u_1 = u_2 + u_3, \\ \\ \text{or} \quad \text{(ii)} \quad u_2 = u_1 + (-1)u_3. \end{cases} \tag{3}$$

In terms of the $r-$levels, we notice (see [53]):

$$[u_1 \ominus_{gH} u_2]^r = [\min\{\underline{u}_1(r) - \underline{u}_2(r), \overline{u}_1(r) - \overline{u}_2(r)\}, \max\{\underline{u}_1(r) - \underline{u}_2(r), \overline{u}_1(r) - \overline{u}_2(r)\}], \forall r \in [0,1],$$

and the conditions for the existence of $u_1 \ominus_{gH} u_2$ in the case (i) are

$$\begin{cases} \underline{u}_3(r) = \underline{u}_1(r) - \underline{u}_2(r) \, \text{and} \, \overline{u}_3(r) = \overline{u}_1(r) - \overline{u}_2(r), \forall r \in [0,1], \\ \text{with} \, \underline{u}_3(r) \, \text{increasing}, \, \overline{u}_3(r) \, \text{decreasing}, \, \underline{u}_3(r) \leq \overline{u}_3(r). \end{cases}$$

and in the case (ii) are

$$\begin{cases} \underline{u}_3(r) = \overline{u}_1(r) - \overline{u}_2(r) \, \text{and} \, \overline{u}_3(r) = \underline{u}_1(r) - \underline{u}_2(r), \forall r \in [0,1], \\ \text{with} \, \underline{u}_3(r) \, \text{increasing}, \, \overline{u}_3(r) \, \text{decreasing}, \, \underline{u}_3(r) \leq \overline{u}_3(r). \end{cases}$$

Remark 2.3. Let $\lambda_1, \lambda_2 \in \mathbb{R}^+$ and $u \in E$. If $\lambda_1 > \lambda_2$, then $\lambda_1 u \ominus \lambda_2 u = (\lambda_1 - \lambda_2)u$.

Definition 2.4. [21] Let $u : [a,b] \to E$. We define the diameter of the r-level set of the fuzzy function u as $d([u(t)]^r) = \overline{u}(r,t) - \underline{u}(r,t)$. The fuzzy function u is said to be d-increasing or d-decreasing on $[a,b]$ if for every $r \in [0,1]$ the diameter of the fuzzy function $u(t)$ is nondecreasing or nonincreasing, respectively, on $[a,b]$. If u is d-increasing or d-decreasing on $[a,b]$, then we say that u is d-monotone on $[a,b]$.

Definition 2.5. [11] Let $u : (a,b) \to E$ and $t_0 \in (a,b)$. The fuzzy function u is said to be generalized Hukuhara differentiable at t_0, if there exists an element $u'(t_0) \in E$ such that

$$u'(t) = \lim_{h \to 0} \frac{u(t_0 + h) \ominus_{gH} u(t_0)}{h}. \tag{4}$$

Let $C([a,b], E)$ be the space of continuous fuzzy function u on $[a,b]$ with the supremum: $D_*[u, \hat{0}] = \sup_{t \in [a,b]} D_0[u(t), \hat{0}]$ and $L([a,b], E)$ be the set of all fuzzy functions $u : [a,b] \to E$ such that the functions $t \mapsto D_0[u(t), \hat{0}]$ belongs to $L[a,b]$.

Definition 2.6. [21] Let $u \in L([a,b], E)$. The Riemann-Liouville fractional integral of order $\alpha > 0$ of the fuzzy function u is defined as follows:

$$(\Im^{\alpha}_{a^+} u)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds, \quad t > a,$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

Definition 2.7. [21] Let $u \in L([a,b], E)$ and $\alpha \in (0,1)$. The Riemann-Liouville fractional derivative at $t \in (a,b)$ of order $\alpha \in (0,1)$ of the fuzzy function u is defined by

$$\left({}^{RL}\mathcal{D}^{\alpha}_{a^+} u\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} u(s) ds \right)'. \tag{5}$$

Theorem 2.8. (see Proposition 2 in [21]) Let $u \in L([a,b], E)$. If either $(\Im^{1-\alpha}_{a^+} u)(t)$ is $d-$increasing or $(\Im^{1-\alpha}_{a^+} u)(t)$ is $d-$decreasing on $(a,b]$, then

$$\Im^{\alpha}_{a^+} \left({}^{RL}\mathcal{D}^{\alpha}_{a^+} u\right)(t) = u(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} (\Im^{1-\alpha}_{a^+} u)(a).$$

Definition 2.9. (see [21]) Let $u \in L([a,b], E)$ be a fuzzy function such that ${}^{RL}\mathcal{D}^{\alpha}_{a^+} u$, $\alpha \in (0,1)$, exists on (a,b). The Caputo fractional derivative of the fuzzy function u is defined as follows:

$$\left({}^{C}\mathcal{D}^{\alpha}_{a^+} u\right)(t) = \left({}^{RL}\mathcal{D}^{\alpha}_{a^+} [u(\cdot) \ominus_{gH} u(a)]\right)(t).$$

Theorem 2.10. (see Corollary 1 in [21]) If u is a $d-$monotone fuzzy function and $\alpha \in (0,1)$, then

$$\Im^{\alpha}_{a^+} \left({}^{C}\mathcal{D}^{\alpha}_{a^+} u\right)(t) = u(t) \ominus_{gH} u(a).$$

3 Stability for fractional integral equations (1)

This section presents the existence and the Ulam-Hyers stability of solution for the following implicit fuzzy fractional integral equations:

$$u(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, u(s), {}^{RL}\mathcal{D}^{\alpha}_{a^+} u(s)) ds, \quad t \in J, \tag{6}$$

where $u_0 \in E, \alpha \in (0,1), f : J \times E \times E \to E$ is continuous. We say that $u \in C(J,E)$ is a solution of (6) if $\lim_{t\to a^+} \Im^{1-\alpha}_{a^+} u(t) = u_0$ and it satisfies integral equations (6) for all $t \in J$. Let $\gamma \in (0,1)$, Denote by $C_\gamma([a,b],E)$ the weighted spaces of continuous fuzzy functions given by $C_\gamma([a,b],E) = \{z(t) : (t-a)^{1-\gamma} z(t) \in C([a,b],E)\}$, with the distance $D_\gamma[z, \hat{0}] = \sup_{t\in[a,b]} D_0[(t-a)^{1-\gamma} z(t), \hat{0}]$. Denote by $BC([a,b],E)$ the set of all continuous and bounded fuzzy functions on $[a,b]$. We denote $BC_\gamma([a,b],E)$ the weighted space of the fuzzy function in $BC([a,b],E)$, which is defined by $BC_\gamma([a,b],E) = \{v : [a,b] \to E : (t-a)^{1-\gamma} v(t) \in BC([a,b],E)\}$.

Theorem 3.1. (see Theorem 4.1 in [22]) Suppose that $f \in C([a,b],E)$ and u is a fuzzy function such that $u_{1-\alpha}(t) := (\Im^{1-\alpha}_{a^+} u)(t)$ is continuous and it has a monotone diameter on $[a,b]$. If u satisfies (6) provided that $d([u(t)]^r) - d([\varphi(t)]^r)$, where $\varphi(t) := ((t-a)^{\alpha-a}/\Gamma(\alpha)) u(a)$, has a constant sign on $[a,b]$, then $u(t)$ is a solution of the following problem

$${}^{RL}\mathcal{D}^{\alpha}_{a^+} u(t) = f(t, u(t), {}^{RL}\mathcal{D}^{\alpha}_{a^+} u(t)), \quad \Im^{1-\alpha}_{a^+} u(a) := \lim_{t\to a^+} \Im^{1-\alpha}_{a^+} u(t) = u_0. \tag{7}$$

We set

$$(\mathcal{F}v)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, v(s), {}^{RL}\mathcal{D}^{\alpha}_{a^+} v(s)) ds.$$

Similar to Remark 4.1 in [22], we also have the remark as follows.

Remark 3.2. Denote $J := [a,b]$. According to Eq. (6) and the definition of the generalized Hukuhara difference, we see that:

+ If $\Im^{1-\alpha}_{a^+} u(t)$ has an increasing diameter on J, then (6) becomes

$$u(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} u_0 + (\mathcal{F}v)(t). \tag{8}$$

+ If $\Im^{1-\alpha}_{a^+} u(t)$ has a decreasing diameter on J, then (6) becomes

$$u(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} u_0 \ominus (-1)(\mathcal{F}v)(t). \tag{9}$$

  provided that the Hukuhara difference in Eq. (9) exists on J.

To become simpler, in this section, we will examine only the general form (6) instead of the two separate problems (8) and (9). Let $\varepsilon > 0$ and $\varphi : J \to \mathbb{R}^+$ be continuous. We consider the inequalities as follows:

$$D_0[v(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} v_0, (\mathcal{F}v)(t)] \le \varepsilon, \ t \in J. \tag{10}$$

$$D_0[v(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} v_0, (\mathcal{F}v)(t)] \le \varepsilon\varphi(t), \ t \in J. \tag{11}$$

Now, we present the definitions of the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability of (6), which are considered as an extension of the Ulam stability of FFDEs (see Liu et al. [30] and Shen [46]).

Definition 3.3. The problem (6) is called

(i) Ulam-Hyers stability if there exists a real number $K_f > 0$ such that for each $\varepsilon > 0$ and for every solution $v \in C(J,E)$ of (10) there is a solution $u \in C(J,E)$ of (6) satisfying

$$D_0[u(t), v(t)] \le \varepsilon K_f, \ t \in J.$$

(ii) Ulam-Hyers-Rassias stability concerning φ if there exists $K_{f,\varphi} > 0$ such that for every ε and for every solution $v \in C(J,E)$ of (11) there is a solution $u \in C(J,E)$ of (6) satisfying

$$D_0[u(t), v(t)] \leq \varepsilon K_{f,\varphi}\varphi(t),\ t \in J.$$

Remark 3.4. From the inequalities (10)-(11), we notice:

(i) If $v \in C(J,E)$ is a solution of (10), then there is $\delta \in C(J,E)$ satisfying $D_0[\delta(t), \hat{0}] \leq \varepsilon$, $\forall t \in J$, and $v(t) \ominus_{gH} \dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} v_0 = (\mathcal{F}v)(t) + \delta(t)$.

(ii) If $v \in C(J,E)$ is a solution of (11), then there is $\delta \in C(J,E)$ satisfying $D_0[\delta(t), \hat{0}] \leq \varepsilon\varphi(t)$, $\forall t \in J$, and $v(t) \ominus_{gH} \dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} v_0 = (\mathcal{F}v)(t) + \delta(t)$.

Theorem 3.5. Let $f : J \times E \times E \to E$ be continuous on $BC_\gamma(J,E)$ for $t \in J$. Suppose that the following hypotheses hold:
(i) There exists a real-valued continuous function $P : J \to \mathbb{R}^+$ such that

$$D_0[f(t,w,z), \hat{0}] \leq P(t) \frac{D_0[w,\hat{0}] + D_0[z,\hat{0}]}{1 + D_0[w,\hat{0}] + D_0[z,\hat{0}]},$$

where $t \in J$ and $w, z \in E$.
(ii) There exists a positive constant C_φ and a function $q : J \to \mathbb{R}^+$ is continuous such that

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\varphi(s)ds \leq C_\varphi \varphi(t), \text{ and } P(t) \leq \varepsilon q(t)\varphi(t).$$

(iii) There exists a real-valued function $\psi \in C(J, \mathbb{R}^+)$ and a constant $L \in (0,1)$ such that

$$D_0[f(t,w,z), f(t,\hat{w},\hat{z})] \leq (t-a)^{1-\alpha}(\psi(t)D_0[w,\hat{w}] + LD_0[z,\hat{z}]),$$

where $t \in J$, and $w, z, \hat{w}, \hat{z} \in E$.
If $\lambda := \dfrac{1}{1-L} \dfrac{(b-a)^\alpha}{\Gamma(\alpha+1)} \sup_{t\in J} \psi(t) < 1$, then the problem (6) has a unique solution and Ulam-Hyers-Rassias stability.

Proof. By employing Schauder's Theorem, from hypotheses (i) and (ii), we will verify (6) has at least one solution on J and Ulam-Hyers-Rassias stability. Denote $\gamma = 1 - \alpha$ and we consider $\mathbb{T} : BC_\gamma(J,E) \to BC_\gamma(J,E)$ given by

$$(\mathbb{T}u)(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, u(s), {}^{RL}\mathcal{D}^\alpha_{a+}u(s))ds. \tag{12}$$

From Theorem 3.1, we observe that (12) can be expressed as follows:

$$(\mathbb{T}u)(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds, \tag{13}$$

where $g(\cdot) \in BC_\gamma(J,E)$ with $g(t) = f(t, u(t), g(t))$, $\forall t \in J$. We set $r = \dfrac{D_0[u_0,\hat{0}]}{\Gamma(\alpha)} + P^* \dfrac{(b-a)}{\Gamma(\alpha+1)}$, where $P^* = \sup_{t\in J} P(t)$. The operator $\mathbb{T}$ maps $BC_\gamma(J,E)$ to $BC_\gamma(J,E)$. Indeed, the operator $\mathbb{T}u$ is continuous on J for any $u \in BC_\gamma(J,E)$, and for each $t \in J$ one has

$$D_0[(t-a)^\gamma((\mathbb{T}u)(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} u_0), \hat{0}] \leq \frac{(t-a)^\gamma}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0[f(s,u(s), {}^{RL}\mathcal{D}^\alpha_{a+}u(s)), \hat{0}] \leq P^* \frac{(t-a)}{\Gamma(\alpha+1)}.$$

This yields that $D_\gamma[\mathbb{T}u,\hat{0}] \le \dfrac{D_0[u_0,\hat{0}]}{\Gamma(\alpha)} + P^*\dfrac{(b-a)}{\Gamma(\alpha+1)}$. Thus $D_\gamma[\mathbb{T}u,\hat{0}] \le r$. Hence, $\mathbb{T}u \in BC_\gamma(J,E)$. This implies that the operator $\mathbb{T}$ transforms the ball $B_r := B(\hat{0},r) = \{v \in BC_\gamma(J,E) \,|\, D_\gamma[v,\hat{0}] \le r\}$ into itself. Several the following steps will be given to show that the operator $\mathbb{T}$ satisfies all conditions of Schauder's Theorem.

+ The operator $\mathbb{T}$ is continuous. Indeed, let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_r. Then, for each $t \in J$ and from (13), we have

$$D_0[(t-a)^\gamma(\mathbb{T}u_n)(t),(t-a)^\gamma(\mathbb{T}u)(t)] = D_0\left[(t-a)^\gamma((\mathbb{T}u_n)(t)\ominus_{gH}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}u_0),(t-a)^\gamma((\mathbb{T}u)(t)\ominus_{gH}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}u_0)\right]$$
$$\le \frac{(t-a)^\gamma}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}D_0[g_n(s),g(s)]ds,$$

where $g_n, g \in BC_\gamma(J,E)$ such that

$$g_n(t) = f(t,u_n(t),g_n(t)) \quad \text{and} \quad g(t) = f(t,u(t),g(t)).$$

Since $u_n \to u$ as $n \to \infty$ and f is a continuous fuzzy function, by the Lebesgue dominated convergence theorem we have $D_0[g_n(t),g(t)] \to 0$ as $n \to \infty$. This yields that $D_\gamma[\mathbb{T}u_n,\mathbb{T}u] \to 0$ as $n \to \infty$.

+ $\mathbb{T}(B_r)$ is uniformly bounded and equicontinuous on J. Indeed, since $\mathbb{T}(B_r) \subset B_r$ and B_r is bounded, $\mathbb{T}(B_r)$ is uniformly bounded. Now, let $a < t_1 < t_2 \le b$ and let $u \in B_r$. From (13) one has

$$\begin{aligned}
&D_0[(t_2-a)^\gamma(\mathbb{T}u)(t_2),(t_1-a)^\gamma(\mathbb{T}u)(t_1)]\\
&= D_0[(t_2-a)^\gamma((\mathbb{T}u)(t_2)\ominus_{gH}\frac{(t_2-a)^{\alpha-1}}{\Gamma(\alpha)}u_0),(t_1-a)^\gamma((\mathbb{T}u)(t_1)\ominus_{gH}\frac{(t_1-a)^{\alpha-1}}{\Gamma(\alpha)}u_0)]\\
&= \frac{1}{\Gamma(\alpha)}D_0\left[(t_2-a)^\gamma\int_a^{t_2}(t_2-s)^{\alpha-1}g(s)ds,(t_1-a)^\gamma\int_a^{t_1}(t_1-s)^{\alpha-1}g(s)ds\right]\\
&\le \frac{1}{\Gamma(\alpha)}(t_2-a)^\gamma\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}D_0[g(s),\hat{0}]ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left|(t_2-a)^\gamma(t_2-s)^{\alpha-1}-(t_1-a)^\gamma(t_1-s)^{\alpha-1}\right|D_0[g(s),\hat{0}]ds\\
&\le \frac{P^*(t_2-a)^\gamma}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha + \frac{P^*}{\Gamma(\alpha+1)}|(t_2-t_1)-(t_2-a)^\alpha(t_2-t_1)^\alpha|. \qquad (14)
\end{aligned}$$

We notice the right hand side of (14) tends to 0 as $t_1 \to t_2$. According to Arzelá-Ascoli Theorem, it follows that the operator $\mathbb{T}: B_r \to B_r$ is continuous and compact. Based on Schauder's Theorem, we deduce $\mathbb{T}$ has a fixed point u which is solution of (6). Next, to show the Ulam-Hyers-Rassias stability of (6) we will use hypothesis (ii). Let u^* is a solution of (6) on J, then one has

$$u^*(t)\ominus_{gH}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}u_0 = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,u^*(s),{}^{RL}\mathcal{D}^\alpha_{a^+}u^*(s))ds. \qquad (15)$$

Let v be a solution of (11), then basing on Remark 3.4-(ii) and from (15) and hypothesis (ii) one has

$$\begin{aligned}
D_0[u^*(t),v(t)] &= D_0[u^*(t)\ominus_{gH}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}u_0, v(t)\ominus_{gH}\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}u_0]\\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}D_0\left[f(s,u^*(s),{}^{RL}\mathcal{D}^\alpha_{a^+}u^*(s)),f(s,v(s),{}^{RL}\mathcal{D}^\alpha_{a^+}v(s))\right]ds + \varepsilon\varphi(t)\\
&\le \frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}P(s)ds + \varepsilon\varphi(t)
\end{aligned}$$

$$\leq \varepsilon\varphi(t) + \frac{2\varepsilon q^*}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\varphi(s)ds$$
$$\leq \varepsilon(1 + 2q^* C_\varphi)\varphi(t),$$

where $q^* := \sup_{t\in J} q(t)$. Then, if we set $K_{f,\varphi} = (1 + 2q^* C_\varphi)$, then the problem (6) is Ulam-Hyers Rassias stable. Next, the unique property of solution to the problem (6) will be shown by using hypothesis (iii). To show this assertion, we reconsider the operator defined as in (12) and from hypothesis (iii) we obtain

$$D_0[(\mathbb{T}u^*)(t), (\mathbb{T}v)(t)] = D_0[(\mathbb{T}u^*)(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}, (\mathbb{T}\hat{u})(t) \ominus_{gH} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}]$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0\left[f(s, u^*(s), {}^{RL}\mathcal{D}^\alpha_{a^+} u^*(s)), f(s, \hat{u}(s), {}^{RL}\mathcal{D}^\alpha_{a^+}\hat{u}(s))\right] ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left[(s-a)^{1-\alpha}(\psi(s)D_0[u^*(s), \hat{u}(s)] + LD_0[{}^{RL}\mathcal{D}^\alpha_{a^+}u^*(s), {}^{RL}\mathcal{D}^\alpha_{a^+}\hat{u}(s)])\right] ds. \tag{16}$$

On the other hand, by Theorem 3.1 one has

$$D_\gamma[{}^{RL}\mathcal{D}^\alpha_{a^+}u^*, {}^{RL}\mathcal{D}^\alpha_{a^+}\hat{u}] \leq \psi(t)D_\gamma[u^*, \hat{u}] + LD_\gamma[{}^{RL}\mathcal{D}^\alpha_{a^+}u^*, {}^{RL}\mathcal{D}^\alpha_{a^+}\hat{u}] \tag{17}$$

and so we obtain

$$D_\gamma[{}^{RL}\mathcal{D}^\alpha_{a^+}u^*, {}^{RL}\mathcal{D}^\alpha_{a^+}\hat{u}] \leq \frac{\psi(t)}{1-L} D_\gamma[u^*, \hat{u}].$$

Set $\psi^* = \sup_{t\in J}\psi(t)$. Then, from (16) and (17) we get

$$D_\gamma[\mathbb{T}u^*, \mathbb{T}v] \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \frac{\psi^*}{1-L} D_\gamma[u^*, \hat{u}].$$

Therefore, the operator $\mathbb{T} : BC_\gamma(J, E) \to BC_\gamma(J, E)$ is contractive. □

Remark 3.6. In the Theorem 3.5, if we set $\varphi(\cdot) = 1$ on J and $K_f := (1 + 2q^* C_\varphi)$, then by Definition 3.3 we deduce that the problem (6) is Ulam-Hyers stable.

Example 3.7. Let $\alpha \in (0, 1), \lambda \in (-1, 1)\backslash\{0\}$. Let $v \in E$ be such that $\Im^\alpha_{0^+}v(t)$ is $d-$monotone on $(0, 1]$, and satisfy the implicit fuzzy fractional integral inequality as follows:

$$D_0\Big[v(t) \ominus_{gH} \frac{t^{\alpha-1}}{\Gamma(\alpha)}v_0, \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(\frac{\lambda}{2}v(s) + \frac{1}{2}{}^{RL}\mathcal{D}^\alpha_{0^+}v(s)\big)ds\Big] \leq \varepsilon E_{\alpha,1}(5t^\alpha), \tag{18}$$

where $t \in (0, 1]$, $v_0 \in E$ is given and $\varepsilon > 0$. Setting $f(t, v, {}^{RL}\mathcal{D}^\alpha_{0^+}v(t)) = (\lambda/2)v(t) + \frac{1}{2}{}^{RL}\mathcal{D}^\alpha_{0^+}v(t)$ and $\varphi(t) = E_{\alpha,1}(5t^\alpha)$. By taking $P^* = \sup_{t\in(0,1]}[(\lambda/2)v(t) + \frac{1}{2}{}^{RL}\mathcal{D}^\alpha_{0^+}v(t)]$, then one can define the value of $q^* = P^*/\varepsilon$. In addition, one notices that $\psi(t) = |\lambda|/2$, $L = 1/2$ and

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}E_{\alpha,1}(5s^\alpha)ds = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^\infty \int_0^t (t-s)^{\alpha-1} \frac{5^k s^{\alpha k}}{\Gamma(\alpha k+1)} ds = \sum_{k=0}^\infty \frac{5^k t^{\alpha(k+1)}}{\Gamma(\alpha(k+1)+1)}$$
$$\overset{j:=k+1}{\leqslant} \frac{1}{5}E_{\alpha,1}(5t^\alpha) =: C_\varphi E_{\alpha,1}(5t^\alpha). \tag{19}$$

Then, one has that $C := \frac{(1-0)^\alpha}{(1-L)} \sup_{t\in(0,1]}\psi(t) = \lambda \in (0, 1)$. Next, in this example, basing on Remark 3.2 we will examine two cases as follows:

Case 1. Assume that $\lambda \in (0, 1)$ and the fuzzy function v has an increasing diameter, which satisfies the implicit fuzzy fractional integral inequality as follows:

$$D_0\Big[v(t) \ominus \frac{t^{\alpha-1}}{\Gamma(\alpha)}v_0, \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\big(\frac{\lambda}{2}v(s) + \frac{1}{2}{}^{RL}\mathcal{D}^\alpha_{0^+}v(s)\big)ds\Big] \leq \varepsilon E_{\alpha,1}(8(t-a)^\alpha). \tag{20}$$

We notice the hypotheses of Theorem 3.5 hold, hence there is a unique solution u satisfying $u(0) = v_0$ and

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(\frac{\lambda}{2} u(s) + \frac{1}{2} {}^{RL}\mathcal{D}^{\alpha}_{0+} u(s) \big) ds, \tag{21}$$

and

$$D_0[v(t), u(t)] \le \varepsilon (1 + \frac{2q^*}{5}) E_{\alpha,1}(5t^{\alpha}), \quad \forall t \in (0, 1]. \tag{22}$$

In addition, it follows from Theorem 2.8 and Remark 2.3 that Eq. (21) becomes

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} v_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds. \tag{23}$$

Then, based on the successive approximation method defined as in Subsection 3.2 in [9], one can get the exact solution of (23) as follows:

$$u(t) = v_0 t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}), \quad t \in (0, 1].$$

The graph of the solution is shown in Fig. 1 . In our numerical simulation, the parameters are given by: $\lambda = 1, u_0 = (1, 2, 3), \alpha = 0.75$.

Case 2. Assume that $\lambda \in (-1, 0)$ and the fuzzy function v has a decreasing diameter, which satisfies the implicit fuzzy fractional integral inequality as follows:

$$D_0\left[(-1)\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} v_0 \ominus v(t) \right), \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(\frac{\lambda}{2} v(s) + \frac{1}{2} {}^{RL}\mathcal{D}^{\alpha}_{0+} v(s) \big) ds \right] \le \varepsilon E_{\alpha,1}(8(t-a)^{\alpha}). \tag{24}$$

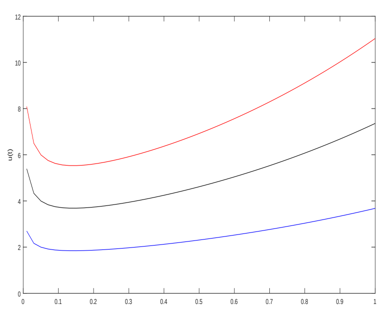

Figure 1: The graph of $u(t)$ in Case 1.

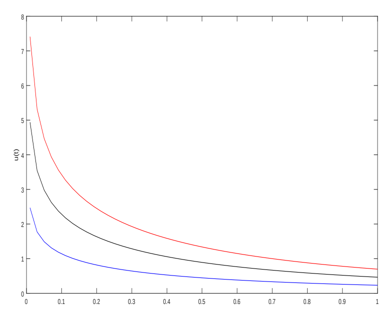

Figure 2: The graph of $u(t)$ in Case 2.

We notice the hypotheses of Theorem 3.5 hold, hence there is a unique solution u such that $u(0) = v_0$, and

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} v_0 \ominus \frac{(-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(\lambda u(s) + \frac{1}{2} {}^{RL}\mathcal{D}^{\alpha}_{0+} u(s) \big) ds \tag{25}$$

and the estimate (22). Similarly, it follows from Theorem 2.8 and Remark 2.3 that Eq. (53) becomes

$$u(t) = u_0 \ominus \frac{(-1)\lambda}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds. \tag{26}$$

Then, one also gets the following exact solution:

$$u(t) = v_0 t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}), \quad t \in (0, 1].$$

The graph of solution is illustrated by Fig. 2 . In our numerical simulation, the parameters are given by: $\lambda = -1, u_0 = (1, 2, 3), \alpha = 0.75$.

4 Stability for fractional integral equations (2)

In this part, based on the technique of successive approximation, we will show the Ulam-Heyers-Rassias stability of the following FFIEs:

$$u(t) \ominus_{gH} u_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f\left(s, u(s), {}^C\mathcal{D}_{a^+}^{\alpha} u(s)\right) ds, \quad t \in J, \tag{27}$$

where $u_0 \in E, \alpha \in (0,1), f : J \times E \times E \to E$ is continuous. Let $u \in C(J,E)$, then u is called a $d-$monotone solution of (27) if it satisfies integral equations (27) for all $t \in J$, $u(a) = u_0$, and u has a increasing or decreasing diameter of the $r-$level set on J. We set

$$(\mathcal{F}v)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, v(s), {}^C\mathcal{D}_{a^+}^{\alpha} v(s)) ds.$$

Remark 4.1. According to Eq. (27) and the definition of the generalized Hukuhara difference, we notice:

+ If $u(t)$ has a increasing diameter of the $r-$level set on J, Eq. (27) becomes

$$u(t) = u_0 + (\mathcal{F}v)(t). \tag{28}$$

+ If $u(t)$ has a decreasing diameter of the $r-$level set on J, Eq. (27) becomes

$$u(t) = u_0 \ominus (-1)(\mathcal{F}v)(t). \tag{29}$$

provided that the Hukuhara difference in this equation exists on J.

Similar to Section 3, we will investigate the general form (27) instead of the two separate problems (8) and (9). Let $\varepsilon > 0$ and $\varphi : J \to \mathbb{R}^+$ be continuous. We consider the inequalities as follows:

$$D_0[v(t) \ominus_{gH} v_0, (\mathcal{F}v)(t)] \le \varepsilon, \ t \in J. \tag{30}$$

$$D_0[v(t) \ominus_{gH} v_0, (\mathcal{F}v)(t)] \le \varphi(t), \ t \in J. \tag{31}$$

$$D_0[v(t) \ominus_{gH} v_0, (\mathcal{F}v)(t)] \le \varepsilon\varphi(t), \ t \in J. \tag{32}$$

Similar to Section 3, we also introduce the definitions of the Ulam type stability of problem (27).

Definition 4.2. The problem (27) is called

(i) Ulam-Hyers stability if there exists a real number $K_f > 0$ such that for every ε and for every solution $v \in C(J,E)$ of (30) there is a solution $u \in C(J,E)$ of (27) satisfying

$$D_0[u(t), v(t)] \le \varepsilon K_f, \ t \in J.$$

(ii) generalized Ulam-Hyers stability if there exists a continuous and positively-defined function K_f provided that $K_f(0) = 0$ such that for every ε and for every solution $v \in C(J,E)$ of (30) there is a solution $u \in C(J,E)$ of (27) satisfying

$$D_0[u(t), v(t)] \le K_f(\varepsilon), \ t \in J.$$

(iii) Ulam-Hyers-Rassias stability concerning φ if there exists a real number $K_{f,\varphi} > 0$ such that for every ε and for every solution $v \in C(J,E)$ of (32) there is a solution $u \in C(J,E)$ of (27) satisfying

$$D_0[u(t), v(t)] \le \varepsilon K_{f,\varphi}\varphi(t), \ t \in J.$$

(iv) generalized Ulam-Hyers-Rassias stability concerning φ if there exists $K_{f,\varphi} > 0$ such that for every ε and for every solution $v \in C(J,E)$ of (31) there is a solution $u \in C(J,E)$ of (27) satisfying

$$D_0[u(t), v(t)] \le K_{f,\varphi}\varphi(t), \ t \in J.$$

In Definition 4.2, we observe that: if (i) holds, then (ii) is also satisfied; if (iii) holds, then (iv) is also satisfied; if (iii) holds, then (i) is also satisfied provided that $\varphi(\cdot) = 1$.

Theorem 4.3. (see Theorem 3 in [21]) Let $f : J \times E \times E \to E$ belongs to $C(J, E)$. If a $d-$monotone fuzzy function u satisfies the integral fractional equation (27) and a function $t \mapsto (\mathcal{F}u)(t)$ has an increasing diameter on J, u is a solution of the following problem:

$$^{C}\mathcal{D}^{\alpha}_{a^+}u(t) = f(t, u(t), {}^{C}\mathcal{D}^{\alpha}_{a^+}u(t)), \quad u(a) = u_0.$$

Theorem 4.4. Let f be a continuous fuzzy function which satisfies the following condition: there exists a constant $L \in (0, 1)$ and a real-valued function $\psi \in C(J, \mathbb{R}^+)$ such that

$$D_0[f(t, w, z), f(t, \hat{w}, \hat{z})] \leq \psi(t)D_0[w, \hat{w}] + LD_0[z, \hat{z}], \tag{33}$$

where $t \in J$, $w, z, \hat{w}, \hat{z} \in E$. Then, the following assertions hold:

(A) For each ε, if $v : J \to E$ satisfies (30), then there is a unique solution $u : J \to E$ of (27) provided that $u_0 = v_0$, which satisfies

$$D_0[u(t), v(t)] \leq \varepsilon E_{\alpha,1}\left((\psi^*/(1 - L))(b - a)^{\alpha}\right), \quad \forall t \in J, \tag{34}$$

where $\psi^* = \sup_{t\in J} \psi(t)$. In addition, the problem (27) is also Ulam-Hyers and generalized Ulam-Hyers stable.

(B) In addition, suppose that there exists a constant $C_\varphi > 0$ provided that $0 < \lambda := \frac{C_\varphi \psi^*}{(1-L)} < 1$ and for $t \in J$

$$\frac{1}{\Gamma(\alpha)}\int_a^t \frac{\varphi(s)}{(t - s)^{1-\alpha}}ds \leq C_\varphi \varphi(t).$$

Then, for each ε, if v satisfies (32), there is a unique solution u of (27) provided that $u_0 = v_0$, and the problem (27) has Ulam-Hyers-Rassias stability. This also yields the generalized Ulam-Hyers-Rassias stability of (27).

Proof. By assumptions of assertions (A) and (B) of theorem, similar to Remark 3.4 we notice that if a fuzzy function v satisfies the inequality (30) and (32), respectively, then there exists fuzzy functions $\delta_1(t), \delta_2(t)$, respectively, such that $D_0[\delta_1(t), \hat{0}] \leq \varepsilon, D_0[\delta_2(t), \hat{0}] \leq \varepsilon\varphi(t)$ for all $t \in J$, and

$$v(t) \ominus_{gH} v_0 = \frac{1}{\Gamma(\alpha)}\int_a^t (t - s)^{\alpha-1}f(s, v(s), {}^{C}\mathcal{D}^{\alpha}_{a^+}v(s))ds + \delta_1(t), \quad t \in J, \tag{35}$$

$$v(t) \ominus_{gH} v_0 = \frac{1}{\Gamma(\alpha)}\int_a^t (t - s)^{\alpha-1}f(s, v(s), {}^{C}\mathcal{D}^{\alpha}_{a^+}v(s))ds + \delta_2(t), \quad t \in J, \tag{36}$$

respectively. Based on Theorem 4.3, the fuzzy fractional integral equations (35) and (36) can be expressed by:

$$v(t) \ominus_{gH} v_0 = \frac{1}{\Gamma(\alpha)}\int_a^t (t - s)^{\alpha-1}g(s)ds + \delta_i(t), \; i \in \{1, 2\}, \tag{37}$$

where $g \in C(J, E)$ with $g(t) = f\,(t, v(t), g(t))$. Next, we set $u_0(t) = v(t)$ and we define the sequence $\{u_n\}_{n\in\mathbb{N}}$ as follows:

$$u_n(t) \ominus_{gH} v_0 = \frac{1}{\Gamma(\alpha)}\int_a^t (t - s)^{\alpha-1}g_{n-1}(s)ds, \quad t \in J, \tag{38}$$

where $g_{n-1}(\cdot) \in C(J, E)$ is expressed by:

$$g_{n-1}(t) := f(t, u_{n-1}(t), g_{n-1}(t)).$$

- Prove the assertion (A): For $n=1$ and from (37), since $u_0(t)=v(t)$, one has

$$D_0[u_1(t),u_0(t)] = D_0[u_1(t)\ominus_{gH} v_0, u_0(t)\ominus_{gH} v_0] = D_0\left[\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} g_0(s)ds, \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} g(s)ds + \delta_1(t)\right] \leq \varepsilon.$$

From (38) and by 33, for any $t\in J$ and $n=2,3,4,\ldots$, we obtain

$$\begin{aligned} D_0[g_{n-1}(t),g_{n-2}(t)] &= D_0[f(t,u_{n-1}(t),g_{n-1}(t)), f(t,u_{n-2}(t),g_{n-2}(t))] \\ &\leq \psi(t)D_0[u_{n-1}(t),u_{n-2}(t)] + LD_0[g_{n-1}(t),g_{n-2}(t)] \\ &\leq \frac{\psi(t)}{(1-L)}D_0[u_{n-1}(t),u_{n-2}(t)], \end{aligned}$$

and this yields that

$$D_0[u_n(t),u_{n-1}(t)] = D_0[u_n(t)\ominus v_0, u_{n-1}(t)\ominus_{gH} v_0] \leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}D_0[g_{n-1}(s),g_{n-2}(s)]ds$$

$$\leq \frac{\psi^*}{(1-L)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}D_0[u_{n-1}(s),u_{n-2}(s)]ds,$$

where $\psi^*=\sup_{t\in J}\psi(t)$. So, from (48) and for $n=2$, one has $D_0[u_2(t),u_1(t)]\leq \varepsilon\frac{\psi^*}{(1-L)}\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}$, and for $n=3$ one also gets $D_0[u_3(t),u_2(t)]\leq \varepsilon\left(\frac{\psi^*}{1-L}\right)^2\frac{(t-a)^{2\alpha}}{\Gamma(2\alpha+1)}$. By employing the principle of mathematical induction, for $n\geq 4$ one gets

$$D_0[u_n(t),u_{n-1}(t)] \leq \varepsilon\left(\frac{\psi^*}{1-L}\right)^{n-1}\frac{(t-a)^{\alpha(n-1)}}{\Gamma((n-1)\alpha+1)}. \tag{39}$$

Furthermore, if we assume that (39) is hold, then by the principle of mathematical induction one also obtains

$$D_0[u_{n+1}(t),u_n(t)] \leq \varepsilon\left(\frac{\psi^*}{1-L}\right)^{n}\frac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{40}$$

Then, one obtains

$$\sum_{n=0}^{\infty} D_0[u_{n+1}(t)\ominus_{gH} u_n(t),\hat{0}] = \sum_{n=0}^{\infty} D_0[u_{n+1}(t),u_n(t)] \leq \varepsilon\sum_{n=0}^{\infty}\frac{[(\psi^*/(1-L))(t-a)^\alpha]^n}{\Gamma(n\alpha+1)}. \tag{41}$$

We notice that the right-hand side of (41) converges to Mittag-Leffler function $E_{\alpha,1}\left((\psi^*/(1-L))(t-a)^\alpha\right)$, hence one has

$$\sum_{n=0}^{\infty} D_0[u_{n+1}(t)\ominus_{gH} u_n(t),\hat{0}] \leq \varepsilon E_{\alpha,1}\left((\psi^*/(1-L))(t-a)^\alpha\right), \tag{42}$$

which yields the series $u_0(t)+\sum\limits_{n=0}^{\infty}[u_{n+1}(t)\ominus_{gH} u_n(t)]$ is uniformly convergent on J concerning the metric D_0. Next, we assume

$$u^*(t) = u_0(t) + \sum_{n=0}^{\infty}[u_{n+1}(t)\ominus_{gH} u_n(t)], \quad \forall t\in J. \tag{43}$$

Then,

$$u_k(t) = u_0(t) + \sum_{n=0}^{k}[u_{n+1}(t)\ominus_{gH} u_n(t)], \quad \forall t\in J \tag{44}$$

is the k^{th} partial of the series (43). By (43) and (44), we observe that

$$\lim_{k\to\infty} D_0[u_k(t), u^*(t)] = \lim_{k\to\infty} D_0[u_k(t) \ominus_{gH} u^*(t), \hat{0}] = 0. \tag{45}$$

Put $u(t) = u^*(t)$. We will verify u is a solution of the fuzzy fractional integral equation as follows:

$$u(t) \ominus_{gH} v_0 = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds, \quad t \in J, \tag{46}$$

where $g \in C(J, E)$ with $g(t) = f(t, u(t), g(t)), t \in J$. Denote $(G_\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds$. By using (38), we get

$$\begin{aligned} D_0[u(t) \ominus_{gH} v_0, G_\alpha(t)] &= D_0[u(t) \ominus_{gH} v_0 \ominus_{gH} (u_k(t) \ominus_{gH} v_0), (G_\alpha u)(t) \ominus_{gH} (G_\alpha u_{k-1})(t)] \\ &\le D_0[u(t) \ominus_{gH} u_k(t), \hat{0}] + D_0[(G_\alpha u)(t) \ominus_{gH} (G_\alpha u_{k-1})(t), \hat{0}] \\ &\le D_0[u^*(t), u_k(t)] + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0[g(s), g_{k-1}(s)]ds. \end{aligned} \tag{47}$$

Otherwise, by (33) one obtains

$$D_0[g(t), g_{k-1}(t)] = D_0[f(t, u(t), g(t)), f(t, u_{k-1}(t), g_{k-1}(t))] \le \psi(t) D_0[u(t), u_{k-1}(t)] + L D_0[g(t), g_{k-1}(t)].$$

This yields that $D_0[g(t), g_{k-1}(t)] \le (\psi^*/(1-L)) D_0[u^*(t), u_{k-1}(t)]$. Hence, by (45) we deduce that $\lim_{k\to\infty} D_0[g(t), g_{k-1}(t)] = 0$. Therefore, the right hand side of (47) tends to 0 as $k \to \infty$. This yields (46) is a solution of (27) provided that $u_0 = v_0$. Furthermore, it follows from (42) and (43) that the following estimate is satisfied:

$$D_0[u(t), v(t)] \le \varepsilon E_{\alpha,1}\left((\psi^*/(1-L))(b-a)^\alpha\right), \quad \forall t \in J.$$

So, by Definition 4.2-(i), we infer the problem (27) has Ulam-Hyers stability. Furthermore, if we let $K_f(\varepsilon) = \varepsilon E_{\alpha,1}\left((\psi^*/(1-L))(t-a)^\alpha\right)$, we notice $K_f(0) = 0$. Then, according to Definition 4.2-(ii), we conclude the problem (27) has the generalized Ulam-Hyers stability.

- Prove the assertion (B): Based on the sequence (38) with $u_0(t) = v(t)$, and with the same way as in the proof of the assertion (A), we prove by the principle of mathematical induction that, for $n \ge 1$,

$$D_0[u_n(t), u_{n-1}(t)] \le \varepsilon \left(C_\varphi \frac{\psi^*}{1-L}\right)^{n-1} \varphi(t). \tag{48}$$

Indeed, for $n = 1$, from (37) and by (38), since $u_0(t) = v(t)$ one has

$$D_0[u_1(t), u_0(t)] \le \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0[g_0(s), g(s)]ds + D_0[\delta_2(t), \hat{0}] \le \varepsilon\varphi(t).$$

Assuming that (48) is true for $n = k \in \mathbb{N}$ and by (33), one gets

$$\begin{aligned} D_0[u_{k+1}(t), u_k(t)] &\le \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0[g_k(s), g_{k-1}(s)]ds \le \frac{\psi^*}{(1-L)\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D_0[u_k(s), u_{k-1}(s)]ds \\ &\le \frac{\varepsilon}{C_\varphi} \left(C_\varphi \frac{\psi^*}{1-L}\right)^k \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \varphi(s)ds \le \varepsilon \left(C_\varphi \frac{\psi^*}{1-L}\right)^k \varphi(t). \end{aligned}$$

This prove (48) is satisfied, $\forall n \ge 1$. Then, from the hypothesis $\lambda := C_\varphi \frac{\psi^*}{1-L} \in (0,1)$ and from (48) we have that

$$\sum_{n=0}^{\infty} D_0[u_{n+1}(t), u_n(t)] \le \varepsilon\varphi(t) \sum_{n=0}^{\infty} \left(C_\varphi \frac{\psi^*}{1-L}\right)^n \le \frac{\varepsilon}{1-\lambda}\varphi(t). \tag{49}$$

Because the function $\varphi(t)$ is continuous on J, it is bounded. Hence, it follows from (49) that the series $u_0(t) + \sum_{n=0}^{\infty} [u_{n+1}(t) \ominus_{gH} u_n(t)]$ is uniformly convergent on J concerning D_0. So, we put

$$u(t) = u_0(t) + \sum_{n=0}^{\infty} [u_{n+1}(t) \ominus_{gH} u_n(t)]. \tag{50}$$

With the same manner as in the proof of (A), one also get the estimate as follows: $D_0[u(t), v(t)] \leq \varepsilon \frac{1}{1-\lambda}\varphi(t)$. By Definition 4.2-(iii), we deduce the problem (27) has Ulam-Hyers-Rassias stability. Furthermore, if we let $\varepsilon = 1$ and $K_{f,\varphi} = \frac{1}{1-\lambda}$, then the problem (27) is generalized Ulam-Hyers-Rassias stable basing on Definition 4.2-(iv). □

Example 4.5. Let $\alpha \in (0,1), \lambda \in [-1,1]\backslash\{0\}$. We assume that the fuzzy function v is $d-$monotone on J and satisfies the implicit fuzzy fractional integral inequality as follows:

$$D_0\Big[v(t) \ominus_{gH} v_0, \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\big(\lambda v(s) + \frac{1}{2}{}^C\mathcal{D}^\alpha_{a^+}v(s)\big)ds\Big] \leq \varepsilon E_{\alpha,1}(8(t-a)^\alpha), \tag{51}$$

where $t \in J$, $v_0 \in E$ and $\varepsilon > 0$. By setting $f(t, v, {}^C\mathcal{D}^\alpha_{a^+}v(t)) = \lambda v(t) + \frac{1}{2}{}^C\mathcal{D}^\alpha_{a^+}v(t)$ and $\varphi(t) = E_{\alpha,1}(8(t-a)^\alpha)$, one notices that $\psi^* = |\lambda|$, $L = 1/2$. With the same manner as in (19), we also get

$$\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} E_{\alpha,1}(8(s-a)^\alpha)ds \leq \frac{1}{8}E_{\alpha,1}(8(t-a)^\alpha).$$

Then, by taking $C_\varphi = 1/8$ one has that $C := \frac{C_\varphi \psi^*}{(1-L)} = (\lambda/4) \in (0,1)$. Next, in this example, by basing on the definition of $\ominus_{gH}$ we will examine two cases as follows:
Case 1. We assume that $\lambda \in (0,1]$ and the fuzzy function v is $d-$increasing which satisfies the implicit fuzzy fractional integral inequality as follows:

$$D_0\Big[v(t) \ominus v_0, \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\big(\lambda v(s) + \frac{1}{2}{}^C\mathcal{D}^\alpha_{a^+}v(s)\big)ds\Big] \leq \varepsilon E_{\alpha,1}(8(t-a)^\alpha). \tag{52}$$

We notice the hypotheses of Theorem 4.4 hold, so there exists a unique solution u with the initial condition $u_0 = v_0$ such that

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\big(\lambda u(s) + \frac{1}{2}{}^C\mathcal{D}^\alpha_{a^+}u(s)\big)ds \tag{53}$$

and the following estimate holds

$$D_0[v(t), u(t)] \leq \frac{\varepsilon}{1-C}E_{\alpha,1}(8(t-a)^\alpha), \quad \forall t \in J. \tag{54}$$

In addition, it follows from Theorem 2.10 and Remark 2.3 that Eq. (53) becomes

$$u(t) = u_0 + \frac{2\lambda}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}u(s)ds. \tag{55}$$

Then, by using the method of successive approximation given Theorem 4.4, one can get the following exact form of solution to Eq. (55):

$$u(t) = u_0E_{\alpha,1}(2\lambda(t-a)^\alpha), \quad t \in J.$$

The graph of solution is shown in Fig. 3 . In our numerical simulation, the parameters are given by: $J = (0,1]$, $\lambda = 1/2, u_0 = (-1/2, 0, 1/2), \alpha = 0.75$.

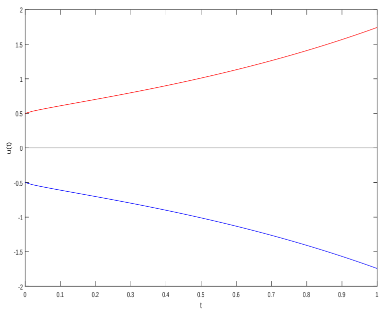

Figure 3: The graph of $u(t)$ in Case 1.

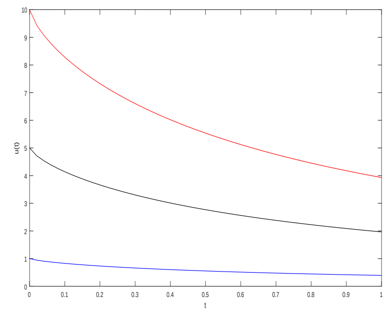

Figure 4: The graph of $u(t)$ in Case 2.

Case 2. Assume that $\lambda \in [-1, 0)$ and v is $d-$decreasing which satisfies the implicit fuzzy fractional integral inequality as follows:

$$D_0\Big[(-1)(v_0 \ominus v(t)), \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\big(\lambda v(s) + \frac{1}{2}{}^C\mathcal{D}^{\alpha}_{a+}v(s)\big)ds\Big] \leq \varepsilon E_{\alpha,1}(8(t-a)^{\alpha}). \tag{56}$$

We notice the conditions of Theorem 4.4 are satisfied, hence there is a unique solution u with the initial condition $u_0 = v_0$ such that

$$u(t) = u_0 \ominus (-1)\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\big(\lambda u(s) + \frac{1}{2}{}^C\mathcal{D}^{\alpha}_{a+}u(s)\big)ds \tag{57}$$

and we also get the estimate (54). In addition, it follows from Theorem 2.10 and Remark 2.3 that Eq. (53) becomes

$$u(t) = u_0 \ominus \frac{(-2)\lambda}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds. \tag{58}$$

Then, according to the successive approximation method, one also gets the following exact form: $u(t) = u_0E_{\alpha,1}(2\lambda(t-a)^{\alpha}), \quad t \in J$. The graph of solution is given by Fig. 4 . In our numerical simulation, the parameters are given by: $J = (0, 1]$, $\lambda = -1/2, u_0 = (1, 5, 10), \alpha = 0.75$.

5 Conclusion and future works

The investigation of the basic theories and applications of FFIEs and FFDEs has raised in the past decade, and a large number of researchers have introduced essential results in this new area. The results of this paper have introduced a standard framework to study the existence, uniqueness, and stability of solutions for two new classes of implicit FFIEs, which plays an essential role in considering the qualitative theories of implicit FFDEs in the future. Generally, we cannot obtain analytical solutions to most implicit FFIEs and FFDEs. Therefore, it is really necessary to construct the approaches of Ulam-Hyers and Ulam-Hyers-Rassias stability for the given problems because it is not demanded to obtain exact solutions when the problems are stable in the above sense. So, the results about the Ulam-Hyers-Rassias stability in this paper are a reliable method for approximately solving the given fuzzy problem since if the problem is Ulam-Hyers stable, there exists an almost exact solution.

We observe that if the functions f in the problems (6) and (27) turn into the crisp functions, the assertions given by Theorem 3.5 and Theorem 4.4 are still satisfied in the case of classical implicit fractional integral equations without the fuzzy concepts. However, by the special structure in the space of fuzzy sets, in this work, we implicitly assume that the existence of Hukuhara difference must be imposed under the forms of the implicit fractional integral equation (9) and (29) to investigate the Ulam-Hyers stability of the given problems. This will cause a major problem in the studies of qualitative theory of FFDEs since it is quite difficult to determine the existence of Hukuhra difference in the given problems. So, it is vital to propose the new concepts of fractional integral and fractional derivative in the fuzzy setting to overcome the drawbacks proposed by approaches based on the Hukuhara difference. In the present, some useful approaches such as the generalized difference between fuzzy sets [11, 18], and granular difference [38]. Therefore, future researches might consider constructing the concepts of Ulam-Hyers stability of the FFDEs using the approaches proposed by [11, 18, 38].

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On the stability for implicit uncertain fractional integral equations with fuzzy concept

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پایداری برای معادلات انتگرال کسری نامعلوم ضمنی با مفهوم فازی

چکیده. هدف این مقاله تأیید نتایج وجود و یکتایی دو نوع از معادلات انتگرال کسری نامعلوم ضمنی با مفهوم فازی از طریق قضیه نقطه ثابت و روش تقریب متوالی است. علاوه‌برآن، نتایج پایداری مختلف Ulam- Hyers و Ulam- Hyers-Rassias دو نوع از معادلات انتگرال کسری فازی ضمنی را نیز ارائه می‌دهیم. در آخر مثال‌هایی جهت روشن شدن نتایج اصلی ارائه شده‌است.