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Original paper

FGM generated archimedean copulas with concave multiplicative generators

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Abstract

The Farlie-Gumble-Morgenstren (FGM) family and archimedean family are the most popular parametric families of copulas. In the present paper, we propose an extension of archimedean copulas with concave multiplicative generators in the style of FGM family. In particular, our method allows the modelling of higher positive dependence than the other FGM extensions in the literature. The construction and characteristics of the proposed model along with some examples of parametric subfamilies are provided. A numerical study is used to illustrate the methodology.

Keywords: Archimedean copulas, FGM family, dependence, dependence measures.

1 Introduction

Copulas are functions that join or couple univariate marginal distribution functions and the corresponding joint distribution function. For detail on copulas, their properties and applications in probability and statistics see, [25] and [31]. Copulas are interesting not only in probability and statistics, but also in many other fields such as probabilistic metric spaces. Associative copulas are special continuous triangular norms (t-norms) and hence they are applied in several domains where t-norms play a role; see, e.g,[1], [4], [21], [34] and [36]. A popular parametric families of copulas is the FGM family defined by

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v), \quad u,v \in [0,1]^2, \quad \theta \in [-1,1],$$

due to [14], [20] and [29]. A main drawback of this family is that it cannot be used to model the large dependences because of its limited range of Pearson correlation $\rho \in [-1/3, 1/3]$. To overcome this drawback, several extensions of this family have been proposed and studied in the literature. [33] introduced an extension which is re-discovered in [32]. Unfortunately, characterization of admissible parameters is not tractable to obtain closed-form bounds on correlation coefficient. [22] proposed another extension with a maximal correlation coefficient of $\rho = 0.434$, [23] suggested two extensions with $\rho \in [-0.333, 0.375]$ and $\rho \in [-0.333, 0.391]$, [2] defined a generalization with $\rho \in [-0.75, 1]$, [8] introduced an asymmetric generalization copula family with $\rho \in [-0.5, 0.43]$ and [24] established an extension that its Spearman's rho and Kendall's tau are odd functions of the parameter and their values in $[-1, 1]$. To read more, see [3], [28] and a survey on FGM copula, its properties and generalizations could be found in [7].

In this paper, the proposed model is interesting since it provides an extension of FGM copula and generalizes the Archimedean family of copulas in an easy manner to construct bivariate distributions with a variety of dependence structures. Transformation of copulas by distortion functions (certain increasing functions on [0,1]) considered in several papers; see, e.g., [13], [17], [30] and [11]. In particular, a very interesting class of copulas is the Archimedean one that due to their analytically tractable forms and good properties are used in various fields, such as actuarial science,

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quantitative finance and biostatistics. Also, It is interesting to consider the generalization of Archimedean copula family to make the copulas more flexible. [12] presented a generalization of two-dimensional Archimedean copula with two generators.

[35]extended their copula structure to multivariate case. [27] established the connections between Archimedean copulas, ℓ_1-norm symmetric distributions and Williamson d-transforms. [5] discussed convergence of linear approximation of *Williamson n-transform* generators proposed by [27] and illustrated it by several examples.

The proposed structure is given in Section 2. Some dependency measures such as Kendall's τ, Spearman's ρ, Blomqvist's β and Gini's γ are described in Section 3. The dependence properties of the proposed structure is discussed in Section 4. Upper and lower tail dependence coefficients are given in Section 5. The dependence parameters estimation procedures via the maximum likelihood and moment estimators are provided in Section 6. A numerical study that contains a Monte Carlo simulation for comparing the performances of the maximum likelihood and moment estimators via MSE and also an analysis of real bivariate data set for illustrating the applications of the newly introduced copula are mentioned in Section 7. Finally, Section 8, concludes the results.

2 The proposed model

In this section based on a distorted FGM copula, we construct a new generalization of the Archimedean family of copulas.

Definition 2.1. *A function $g : [0, 1] \to [0, 1]$ is a distortion function if*

i) $g(1) = 1$,

ii) g *is a continuous and strictly increasing function.*

Let $g^{[-1]}(.)$ be the pseudo-inverse of g, defined by $g^{[-1]}(t) = g^{-1}(t)$, if $g(0) \le t \le 1$ and $g^{[-1]}(t) = 0$, if $0 \le t \le g(0)$. We note that $g^{[-1]}$ is increasing and strictly increasing on $[g(0), 1]$, with $g^{[-1]}(g(t)) = t$, and $g(g^{[-1]}(t) = \max(g(0), t)$, for $t \in [0, 1]$. If g is a concave distortion function, then for a given copula C, the function $C_g(u, v) = g^{[-1]}(C(g(u), g(v)))$ is always a copula; (see, Theorem 2.6 in [30]). So the distorted FGM copula is given by

$$C_g(u, v) = g^{[-1]}\{g(u)g(v)\big(1 + \theta(1 - g(u))(1 - g(v))\big)\}. \quad (1)$$

Let $\phi(t) = -\log(g(t))$, then $\phi(t) : [0, 1] \to [0, \infty]$ is a convex and decreasing function with $\phi(1) = 0$. Thus, $\phi(.)$ is a generator of an Archimedean copula and for all $u, v \in [0, 1]$, we have

$$C_\phi(u, v) = \phi^{[-1]}\{\phi(u) + \phi(v) - \log\big(1 + \theta(1 - e^{-\phi(u)})(1 - e^{-\phi(v)})\big)\}. \quad (2)$$

So, we obtain that $C_g = C_\phi$ and equation 1 can be seen as a generalization of the Archimedean copulas defined by multiplicative generator. For $\theta = 0$, we achieve

$$C_\phi(u, v) = \phi^{[-1]}\{\phi(u) + \phi(v)\},$$

which is the well-known Archimedean family of copulas. For $\phi(t) = -\log(t)$, (2) reduces to

$$C_\phi(u, v) = uv + \theta uv(1 - u)(1 - v).$$

Thus, the structure (2) is a generalization of the Archimedean family of copulas, which contains FGM copula as a special case.

Remark 2.2. *As we mentioned, the family of copulas defined by (2) generalizes both Archimedean and FGM copulas, but, in general, copulas generated from the proposed family are neither Archimedean nor FGM copulas. We called the family of copulas defined by (2) as the Archimedean-FGM generated family of copulas and used the notation AFGM for this family.*

Note that if $g(0) = 0$ or equivalently, $\phi(0) = \infty$. In this case, ϕ is the generator of a strict Archimedean copula and $\phi^{[-1]} = \phi^{-1}$; see, e.g, [31]. We called the copulas constructed from (2) by a strict generator, as the *strict AFGM* family. For convenience, we consider only the strict AFGM families. In Table 1, shows the AFGM copulas constructed by strict generators that change the dependence structure of the distorted copula dramatically.

The conditional distribution and copula density of C_ϕ is given by

$$P\{V \le v|U = u\} \quad = \quad \frac{\phi'(u)\left[1+\theta(1-2e^{-\phi(u)})(1-e^{-\phi(v)})\right]}{\phi'\left(C_\phi(u,v)\right)\left[1+\theta(1-e^{-\phi(u)})(1-e^{-\phi(v)})\right]}. \tag{3}$$

$$\begin{aligned} c_\phi(u,v) \quad &= -\frac{\phi'(u)\phi'(v)}{\phi'\left(C_\phi(u,v)\right)}\left[1+\theta(1-e^{-\phi(u)})(1-e^{-\phi(v)})\right] \times \Big\{\theta e^{-\phi(u)-\phi(v)} \\ &\phi'^2(C_\phi(u,v)) + \phi''(C_\phi(u,v))\left[1+\theta(1-e^{-\phi(u)})(1-2e^{-\phi(v)})\right]\Big\}. \end{aligned} \tag{4}$$

respectively. In the following, we provide the admissible range of θ for the model in (2). Since the density function (4) needs to be nonnegative, the condition $\lim_{u,v\to 0} c_\phi(u,v) \ge 0$ on $[0,1]^2$ requires $\theta \ge -1$. On the other hand, the condition $\lim_{u\to 0, v\to 1} c_\phi(u,v) = \lim_{u\to 1, v\to 0} c_\phi(u,v) \ge 0$ implies $\theta \le 1$ and $\lim_{u,v\to 1} C_\phi(u,v) \ge 0$ requires $\theta \ge \frac{-\phi''(1)}{\phi'^2(1)}$. Thus, the admissible range of θ is given by

$$\max\Big(-1, \frac{-\phi''(1)}{\phi'^2(1)}\Big) \le \theta \le 1.$$

Table 1 shows four examples of AFGM families with specific generator functions. For all these copulas, $-1 \le \theta \le 1$ is admissible range for θ.

Table 1: Some examples of AFGM families

Name	$\phi(t)$	C_ϕ
C_{ϕ_1}	$-\log(t^{\frac{1}{\alpha}})$; $\alpha \ge 1$	$uv\left[1+\theta(1-u^{1/\alpha})(1-v^{1/\alpha})\right]^\alpha$
C_{ϕ_2}	$-\log(\frac{\exp(\alpha t)-1}{\exp(\alpha)-1})$; $\alpha \le 0$	$\frac{1}{\alpha}\log\left[1+\frac{(e^{\alpha u}-1)(e^{\alpha v}-1)}{e^\alpha-1}\left(1+\theta\frac{(e^\alpha-e^{\alpha u})(e^\alpha-e^{\alpha v})}{(e^r-1)^2}\right)\right]$
C_{ϕ_3}	$-\log(1-(1-t)^\alpha)$; $\alpha \ge 1$	$1-\left[1-(1-\bar{u}^r)(1-\bar{v}^r)(1+\theta\bar{u}^r\bar{v}^r)\right]^{1/r}$; $\bar{u}=1-u$; $\bar{v}=1-v$
C_{ϕ_4}	$(1-t^{1/\alpha})^\alpha$; $\alpha > 0$	$\Big\{1-\Big[(1-u^{1/\alpha})^\alpha+(1-v^{1/\alpha})^\alpha-$ $\log\Big\{1+\theta(1-e^{-(1-u^{1/\alpha})^\alpha})(1-e^{-(1-v^{1/\alpha})^\alpha})\Big\}^{1/\alpha}\ \alpha$

The copula C_{ϕ_1} is a special case of the generalized FGM model studied by [6]. For $\alpha = 1$, we have the FGM and the case $\theta \to 0$ gives the independence copula. The copula C_{ϕ_2} is two parameters extension of the Frank copula. The copula C_{ϕ_3} is connected to the copula studied by [10].

The copula C_{ϕ_4} is an extension of the Archimedean family (4.2.15) in [31], discussed by [15]. The corresponding

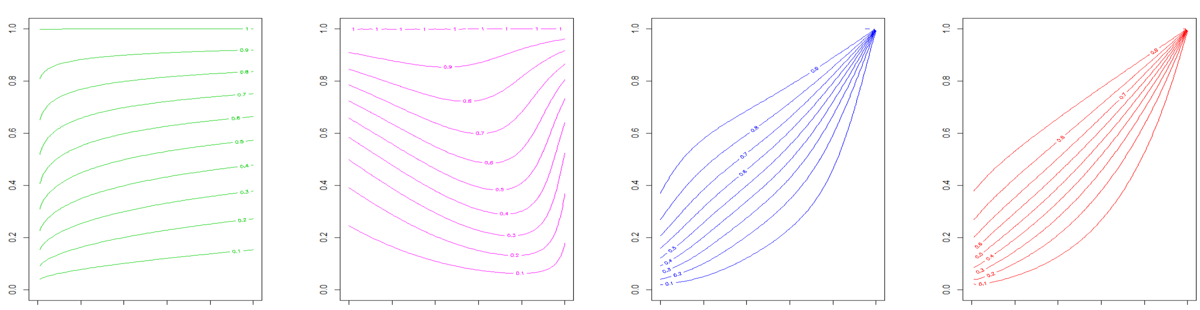

Figure 1: Contour plots of C_{ϕ_1} copula (first),C_{ϕ_2} copula (second),C_{ϕ_3} copula (third) and C_{ϕ_4} copula (forth) from left.

contour plots of the copulas C_{ϕ_i}, $i = 1, 2, 3, 4$, given in Table 1 are appeared in Fig. 1. It seems that, compared with FGM copula, these families allow weak and strong positive dependence. For instance, it seems that, the C_{ϕ_1} and C_{ϕ_2} have weaker positive dependence, while C_{ϕ_3} and C_{ϕ_4} support stronger positive dependence than the usual FGM family and heavy upper tail depenence.

3 Dependence structure

According to Nelsen (2006), let X and Y be two random variables with joint distribution and density functions H, h and associated copula C. Two random variables X and Y are said to be positively quadrant dependent denoted by PQD if and only if $C(u,v) \geq uv$ for all $u, v \in I$.

The random variable Y is left tail decreasing in X shown as $LTD(Y|X)$ if and only if $C(u,v)/u$ is decreasing in u, for all v,(or $\frac{\partial C(u,v)}{\partial u} \leq \frac{C(u,v)}{u}$, for all u). The random variable Y is said to be right tail increasing in X briefly, $RTI(Y|X)$ if and only if $\frac{v-C(u,v)}{1-u}$ is non-increasing in u, (or $\frac{\partial C(u,v)}{\partial u} \geq \frac{v-C(u,v)}{1-u}$, for all u). The random variable Y is stochastically increasing in X denoted by $SI(Y|X)$ if and only if $\frac{\partial C(u,v)}{\partial u}$ is non-decreasing in u for all v.

The random vector (X,Y) is said to be left corner set decreasing ($LCSD(X,Y)$) if and only if $C(u,v)$ is total positive of order two ($TP2$), that is,

$$C(u_1,v_1)C(u_2,v_2) \geq C(u_1,v_2)C(u_2,v_1),$$

for all $u_1 \leq u_2, v_1 \leq v_2$. The concepts of negative dependence such as $NQD(X,Y)$, $LTI(Y|X)$, $RTD(Y|X)$, $SD(Y|X)$, $RR2$ and $LCSI(X,Y)$, can be defined similarly with the reverse inequality signs.

Remark 3.1. *Two random variables X and Y having an AFGM copula, are exchangeable, so the concepts $LTD(Y|X)$ and $LTD(X|Y)$ are equivalent and both will be expressed as LTD. Similarly, we have the result for RTI and SI.*

The following propositions present some results for the proposed model using defined generators in Table 1 that are proved by simple computations..

Proposition 3.2. *Suppose that a random vector (X,Y) has the associated AFGM copula C_ϕ defined by (2), for all $u, v \in [0,1]$;*

(i) (X,Y) *is PQD (NQD) if* $\{1\text{-}\exp\{\phi(u)\}\}\{1-\exp\{\phi(v)\}\}\theta \geq (\leq) \exp\{\phi(u)+\phi(v)-\phi(uv)\}-1$

(ii) C_ϕ *is LTD(Y—X) if* $u\,\phi'(u)\big[1+\theta(1-\exp\{-\phi(v)\})(1-2\exp\{-\phi(u)\})\big] \leq C_\phi(u,v)\phi'(C_\phi(u,v))\big[1+\theta(1-\exp\{-\phi(v)\})(1-\exp\{-\phi(u)\})\big]$

(iii) C_ϕ *is* $RTI(Y|X)$ *if* $u\,\{v-C_\phi(u,v)\}\phi'(C_\phi(u,v))\big[1+\theta(1-\exp\{-\phi(v)\})(1-\exp\{-\phi(u)\})\big] \leq (1-u)\phi'(u)\big[1+\theta(1-\exp\{-\phi(v)\})(1-2\exp\{-\phi(u)\})\big]$

(iv) $SI(Y|X)$ *if* $\phi'(u)\big[1+\theta(1-\exp\{-\phi(v)\})(1-2\exp\{-\phi(u)\})\big]\Big\{\theta\phi'(u)\phi'^2(C_\phi(u,v))\exp\{-\phi(u)\}(1-\exp\{-\phi(v)\})+\phi'(u)\phi''(C_\phi(u,v))\big[1+\theta(1-\exp\{-\phi(v)\})(1-2\exp\{-\phi(u)\})\big]\Big\} \leq \phi'^2(C_\phi(u,v))\big[1+\theta(1-\exp\{-\phi(v)\})(1-\exp\{-\phi(u)\})\big]\Big\{\phi''(u)\big[1+\theta(1-\exp\{-\phi(v)\})(1-2\exp\{-\phi(u)\})\big]+2\theta\phi'^2(u)\exp\{-\phi(u)\}(1-\exp\{-\phi(v)\})\Big\}$

It can easily be shown that, under the condition of Proposition 3.2, a random vector (X,Y) distributed as C_{ϕ_i} for $i=1,2,3,4$ defined in Table 1 has the following dependence structure for different values of θ and α.

Proposition 3.3. *Suppose that a random vector (X,Y) has the associated AFGM copula C_{ϕ_i} defined by (2), where ϕ_i, $i=1,2,3,4$, are given in Table 1. The following results hold:*

(i) C_{ϕ_1} *is SI when* $0 \leq \theta \leq 1$ *and SD when* $-1 \leq \theta \leq 0$ *for all* α;

(ii) C_{ϕ_2} *is SI when* $-0.3 \leq \theta \leq 1$ *for all* α;

(iii) C_{ϕ_3} *is SI when* $-0.2 \leq \theta \leq 1$ *for all* α;

(iv) C_{ϕ_4} *for* $\alpha = 1$ *is SI when* $0 \leq \theta \leq 1$ *and SD when* $-1 \leq \theta \leq 0$. *for* $1 \leq \alpha \leq 2$ *is SD and for* $\alpha \geq 2$ *is SI when* $-1 \leq \theta \leq 1$.

4 Some measures of dependence

In this section, we study the dependence behaviour of AFGM copulas in terms of the measures of concordance such as Kendall's τ, Spearman's ρ, Gini's γ and Blomqvist's β defined by

$$\tau = \int_{[0,1]^2} C(u,v)dC(u,v) - 1, \tag{5}$$

$$\rho = 12\int_{[0,1]^2} C(u,v)dudv - 3, \tag{6}$$

$$\gamma = 4\left[\int_0^1 C(u,1-u)\,du - \int_0^1 [u - C(u,u)]\,du\right], \tag{7}$$

and

$$\beta = 4C(\frac{1}{2},\frac{1}{2}) - 1, \tag{8}$$

for a given copula C. See [31] for more details. We conclude the following propositions, by using the above definition and simple computation.

Proposition 4.1. *Consider the copulas C_{ϕ_i} for $i = 1,2,3,4$ given in Table 1;*

a) If (X,Y) is a random vector with corresponding copula C_{ϕ_1}, then

$$\tau_{\phi_1} = 1 - 4\alpha^2 \int_0^1\int_0^1 \frac{\big[1+\theta(1-u)(1-2v)\big]\big[1+\theta(1-v)(1-2u)\big]}{[1+\theta(1-u)(1-v)]^{2-2\alpha}(uv)^{1-2\alpha}\,du\,dv,}$$

$$\rho_{\phi_1} = 12\alpha^2\int_0^1\int_0^1 \frac{[1+\theta(1-u)(1-v)]^{\alpha}}{uv}\,du\,dv - 3,$$

$$\gamma_{\phi_1} = 4\{\int_0^1 u(1-u)\big[1+\theta(1-u^{\frac{1}{\alpha}})(1-(1-u)^{\frac{1}{\alpha}})\big]^{\alpha}\,du + \int_0^1 u^2\big[1+\theta(1-u^{\frac{1}{\alpha}})^2\big]^{\alpha}\,du - \frac{1}{2}\},$$

and

$$\beta_{\phi_1} = \big[1+\theta(1-(0.5)^{\frac{1}{\alpha}})^2\big]^{\alpha} - 1.$$

b) If (X,Y) is a random vector with corresponding copula C_{ϕ_2}, then

$$\tau_{\phi_2} = 1 - 4\frac{e^{\alpha}-1}{\alpha^2}\int_0^1\int_0^1 \frac{uv\big[1+\theta(1-u)(1-2v)\big]\big[1+\theta(1-v)(1-2u)\big]}{\big[1+uv(e^{\alpha}-1)(1+\theta(1-u)(1-v))\big]^2}\,du\,dv,$$

$$\rho_{\phi_2} = 12\frac{(e^{\alpha}-1)^2}{\alpha^3}\int_0^1\int_0^1 \frac{\log\big(1+uv(e^{\alpha}-1)(1+\theta(1-u)(1-v))\big)}{\big[1+(e^{\alpha}-1)u\big]\big[1+(e^{\alpha}-1)v\big]}\,du\,dv - 3,$$

$$\gamma_{\phi_2} = \frac{4}{r}\Big[\int_0^1 \log\big(1+\frac{(e^{\alpha u}-1)(e^{\alpha(1-u)}-1)}{(e^{\alpha}-1)}(1+\theta\frac{(e^{\alpha}-e^{\alpha u})(e^{\alpha}-e^{\alpha(1-u)})}{(e^{\alpha}-1)^2})\big)\,du + \int_0^1 \log\big(1+\frac{(e^{\alpha u}-1)^2}{(e^{\alpha}-1)}(1+\theta\frac{(e^{\alpha}-e^{\alpha u})^2}{(e^{\alpha}-1)^2})\big)\,du - \frac{r}{2}\Big],$$

and

$$\beta_{\phi_2} = \frac{4}{\alpha}\log\big[1+\frac{(e^{\frac{\alpha}{2}}-1)^2}{e^{\alpha}-1}\big(1+\theta\frac{(e^{\alpha}-e^{\frac{\alpha}{2}})^2}{(e^{\alpha}-1)^2}\big)\big].$$

c) If (X,Y) is a random vector with corresponding copula C_{ϕ_3}, then

$$\tau_{\phi_3} = 1 - 4\int_0^1\int_0^1 \frac{uv[1+\theta(1-u)(1-2v)][1+\theta(1-v)(1-2u)]}{\alpha^2\big[1-uv(1+\theta(1-u)(1-v))\big]^{2-\frac{2}{\alpha}}}\,du\,dv,$$

$$\rho_{\phi_3} = \frac{12}{\alpha^2}\int_0^1\int_0^1 \frac{1-[1-uv(1+\theta(1-u)(1-v))]^{\frac{1}{\alpha}}}{[(1-u)(1-v)]^{1-\frac{1}{\alpha}}}\,du\,dv - 3,$$

$$\gamma_{\phi_3} = 4\Big[\frac{3}{2} - \int_0^1 \big[1-(1-(1-u)^{\alpha})(1-u^{\alpha}) - \int_0^1 \big[1-(1-(1-u)^{\alpha})^2(1+\theta(1-u)^{\alpha})^2\big]^{\frac{1}{\alpha}}\,du\Big],$$

and

$$\beta_{\phi_3} = 3 - 4\big[1-(1-0.5^{\alpha})^2(1+\theta 0.5^{2\alpha})\big]^{\frac{1}{\alpha}}.$$

d) If (X,Y) is a random vector with corresponding copula C_{ϕ_4}, then

$$\tau_{\phi_4} = 1 - 4\int_0^1\int_0^1 \frac{[1+\theta(1-u)(1-2v)][1+\theta(1-v)(1-2u)]\,du\,dv}{(1+\theta(1-u)(1-v))\big[1-(-\log(u)-\log(v)-\log(1+\theta(1-u)(1-v)))^{\frac{1}{\alpha}}\big]^{1-\alpha}},$$

$$\rho_{\phi_4} = 12\int_0^1\int_0^1 \frac{\big[1-(-\log(u)-\log(v)-\log(1+\theta(1-u)(1-v)))^{\frac{1}{\alpha}}\big]^{\alpha}}{\big[\big(1-(-\log(u))^{\frac{1}{\alpha}}\big)\big(1-(-\log(v))^{\frac{1}{\alpha}}\big)\big]^{1-\alpha}}\,du\,dv - 3,$$

$$\begin{aligned}\gamma_{\phi_4} = 4\Big[&\int_0^1 \big[1-[(1-u^{\frac{1}{\alpha}})^{\alpha} + (1-(1-u)^{\frac{1}{\alpha}})^{\alpha}\\ &-\log\big(1+\theta(1-\exp\{-(1-u^{\frac{1}{\alpha}})^{\alpha}\})(1-\exp\{-(1-(1-u)^{\frac{1}{\alpha}})^{\alpha}\})\big)\big]^{\frac{1}{\alpha}}\big]^{\alpha}\,du\\ &+\int_0^1 \big[1-[2(1-u^{\frac{1}{\alpha}})^{\alpha} - \log\big(1+\theta(1-\exp\{-(1-u^{\frac{1}{\alpha}})^{\alpha}\})^2\big)\big]^{\frac{1}{\alpha}}\big]^{\alpha}\,du - \frac{1}{2}\Big],\end{aligned}$$

and

$$\beta_{\phi_4} = 4\Big[1-\big[2(1-0.5^{\frac{1}{\alpha}})^{\alpha} - \log\big(1+\theta(1-\exp\{-(1-0.5^{\frac{1}{\alpha}})^{\alpha}\})^2\big)\big]^{\frac{1}{\alpha}}\Big]^{\alpha} - 1.$$

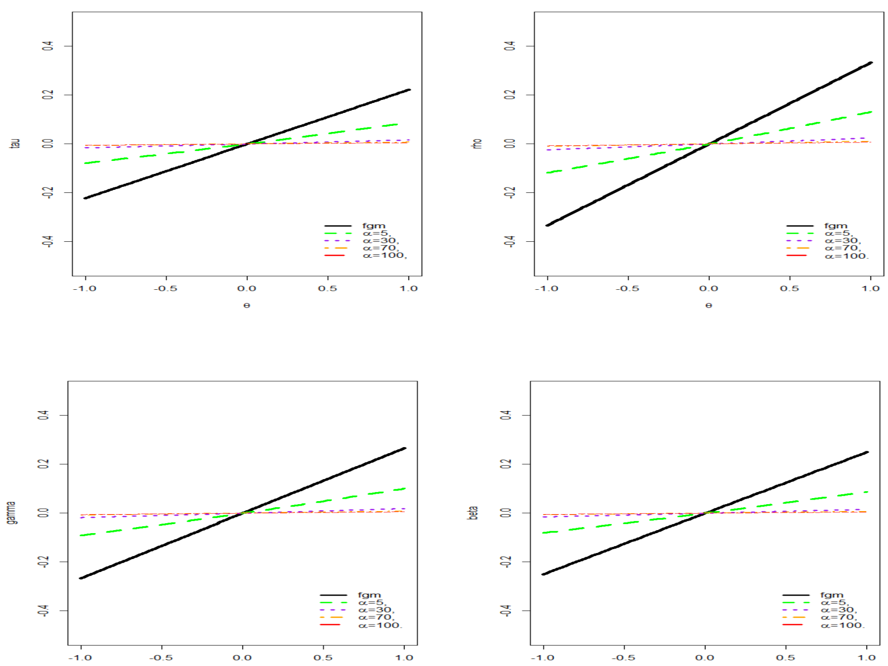

Figure 2: Behaviour of the dependence measures for C_{ϕ_1} against different values of α and θ.

Figures 2 - 5 compare the values of Kendall's τ, Spearman's ρ, Blomqvist's β and Gini's γ for C_{ϕ_i}, $i = 1,2,3,4$, using $\alpha = 5, 30, 70$ and 100 with FGM copula. For C_{ϕ_1}, the values of these measures are smaller than those of the usual FGM copula (Fig. 2). They are also increasing in θ and decreasing in α. These measures for C_{ϕ_2}, C_{ϕ_3} and C_{ϕ_4} are bigger than those one of the usual FGM copula. Also, they are increasing in the parameters θ and α.

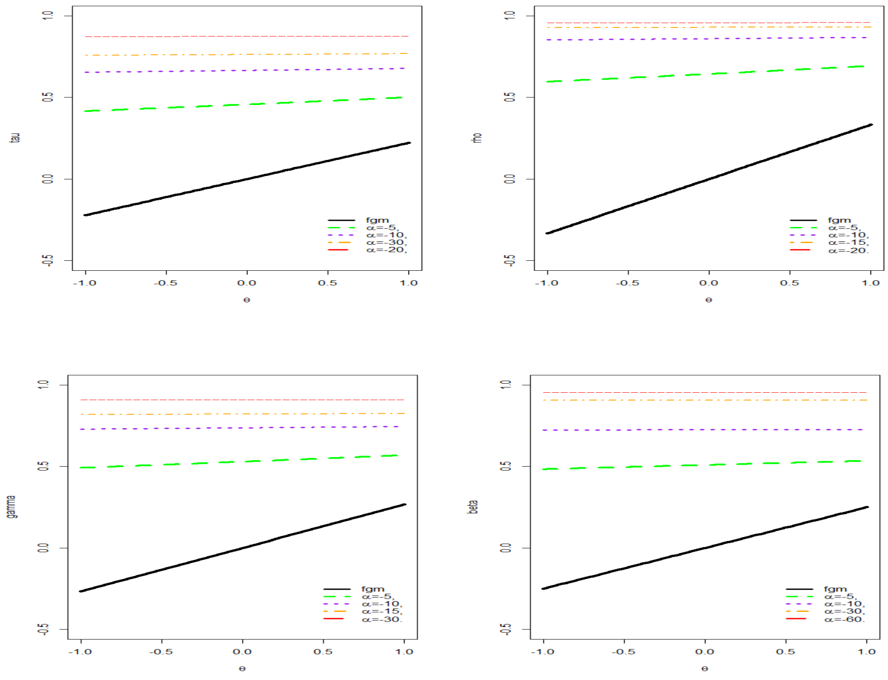

Figure 3: Behaviour of the dependence measures for C_{ϕ_2} against different values of α and θ.

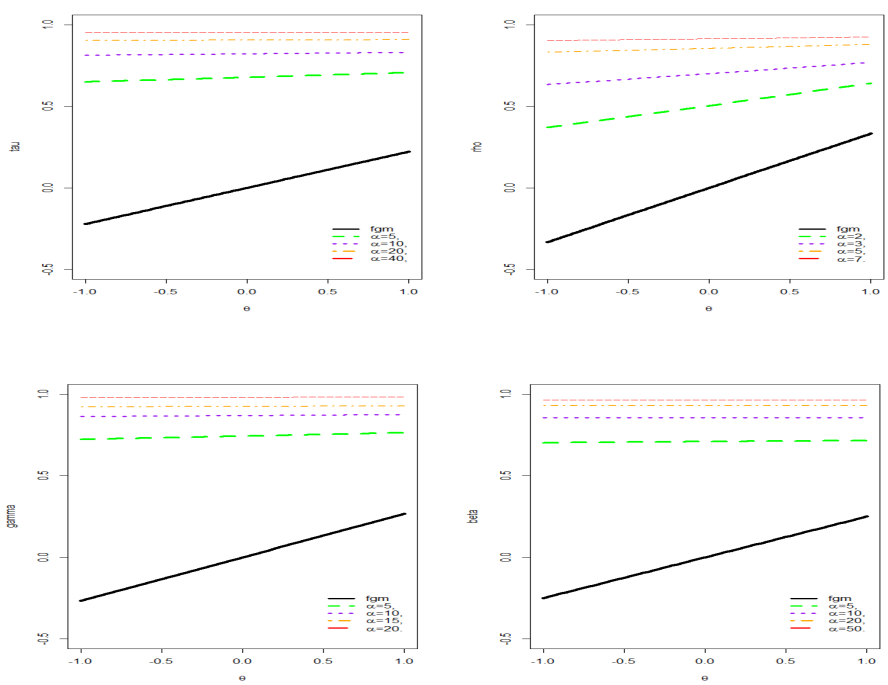

Figure 4: Behaviour of the dependence measures for C_{ϕ_3} against different values of α and θ.

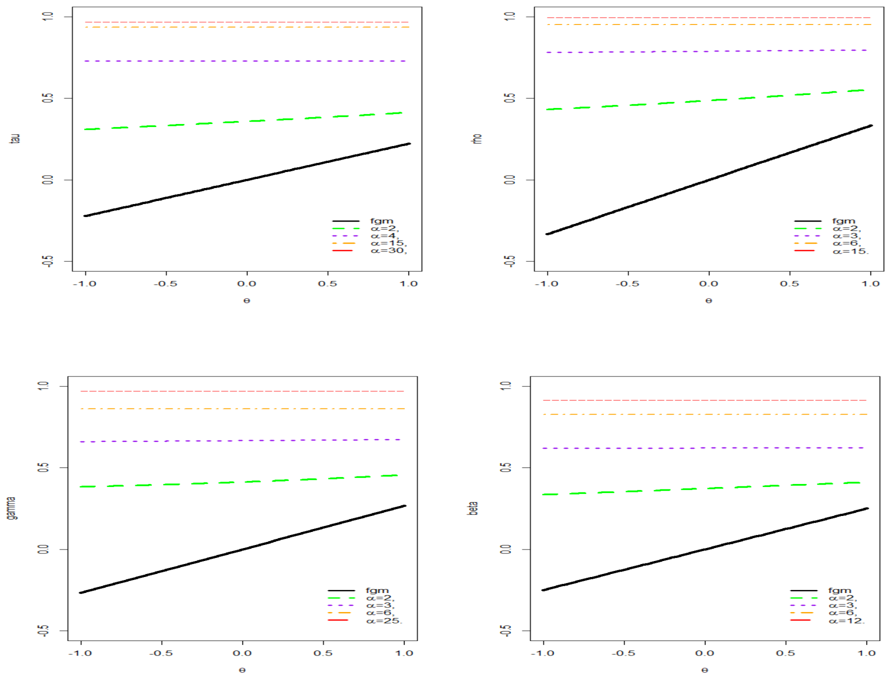

Figure 5: Behaviour of the dependence measures for C_{ϕ_4} against different values of α and θ.

5 Tail dependence

Let X and Y be two continuous random variables with distribution functions F and G, respectively. The upper (lower) tail dependence coefficient λ_U (λ_L) of (X, Y) is defined as

$$\lambda_U = \lim_{t \to 1^-} P\big[Y > G^{-1}(t) | X > F^{-1}(t)\big]$$

and

$$\lambda_L = \lim_{t \to 0^+} P\big[Y \leq G^{-1}(t) | X \leq F^{-1}(t)\big],$$

provided that the above limits exist.

Proposition 5.1. *Let (X, Y) be a random vector with corresponding copula C_{ϕ_i}, $i = 1, 2, 3, 4$, given in Table 1. Then, $\lambda_L(C_{\phi_i}) = \lambda_U(C_{\phi_i}) = 0$, for $i = 1, 2$, $\lambda_L(C_{\phi_i}) = 0$, $i = 3, 4$, and $\lambda_U(C_{\phi_3}) = \lambda_U(C_{\phi_4}) = 2 - 2^{1/\alpha}$.*

Proof. As known (see, e.g., [25]) tail dependence coefficients λ_U and λ_L of a pair (X, Y), are computed by their associated copula C in view of

$$\lambda_L = \lim_{t \to 0^+} \frac{C(t,t)}{t}, \quad \textit{and} \quad \lambda_U = \lim_{t \to 1^-} \frac{1 - 2t + C(t,t)}{1 - t}.$$

So, using the direct calculations, we obtain the desired results. □

Note that for usual FGM copula C, $\lambda_U(C) = \lambda_L(C) = 0$. Thus, the proposed method can produce a new model with upper tail dependence.

6 Estimation of parameters

In this section, we consider the estimation of the dependence parameters θ and α, using maximum likelihood and moment methods.

6.1 Maximum likelihood approach

Consider a data set of observations (x_i, y_i); $i = 1, ..., n$ with uniform marginal distributions $u = F(x)$ and $v = G(y)$ and a dependence structure C_{ϕ_i} copulas defined in Table 1. The log likelihood functions of each copula c_{ϕ_i}; $i = 1, 2, 3, 4$ are obtained bellow that denoted by ℓ_1, ℓ_2, ℓ_3 and ℓ_4, respectively:

$$\ell_1(\theta,\alpha) = (\alpha-2)\sum_{i=1}^{n}\ln\Big(1+\theta(1-u_i^{1/\alpha})(1-v_i^{1/\alpha})\Big) + \sum_{i=1}^{n}\ln\Big(1+\theta\beta_i+\theta^2(1-v_i^{1/\alpha})(1-2u_i^{1/\alpha})\lambda_i\Big), \tag{9}$$

where

$$\beta_i = (1-u_i^{1/\alpha})(1-v_i^{1/\alpha}) + (1-2u_i^{1/\alpha})(1-v_i^{1/\alpha}) - \frac{v_i^{1/\alpha}}{\alpha}(1-2u_i^{1/\alpha}) - (1-\frac{1}{\alpha})v_i u_i^{\frac{1}{\alpha}-1}(1-v_i^{1/\alpha})$$

and

$$\lambda_i = (1-u_i^{1/\alpha})(1-v_i^{1/\alpha}) - \frac{v_i^{1/\alpha}}{\alpha}(1-u_i^{1/\alpha}) - (1-\frac{1}{\alpha})v_i u_i^{\frac{1}{\alpha}-1}(1-v_i^{1/\alpha}).$$

$$\ell_2(\theta,\alpha) = \sum_{i=1}^{n}\ln\Big(\frac{\alpha^3 e^{\alpha u_i+\alpha v_i}}{e^\alpha-1} + \theta\frac{\alpha^3 e^{\alpha u_i+\alpha v_i}}{(e^\alpha-1)^3}\gamma_i\Big) - \sum_{i=1}^{n}\ln\Big(\alpha^2\big\{1+\frac{(e^{\alpha u_i}-1)(e^{\alpha v_i}-1)}{e^\alpha-1}\big(1+\theta\frac{(e^\alpha-e^{\alpha u_i})(e^r-e^{\alpha v_i})}{(e^\alpha-1)^2}\big)\big\}^2\Big), \tag{10}$$

where

$$\gamma_i = \frac{1}{e^r-1}\big\{(e^\alpha-e^{\alpha u_i})(e^\alpha-e^{\alpha v_i})(e^{\alpha u_i}-1)(e^{\alpha v_i}-1)\big\} + (1-2e^{\alpha u_i}-e^\alpha)(1-2e^{\alpha v_i}-e^\alpha) + \frac{1}{e^\alpha-1}\big\{(e^{\alpha u_i}-1)(e^{\alpha v_i}-1)\big\}$$
$$\times\big\{(1-2e^{\alpha u_i}-e^\alpha)(1-2e^{\alpha v_i)}e^\alpha) - (e^\alpha-e^{\alpha v_i})(1-2e^{\alpha u_i}-e^\alpha) - (e^\alpha-e^{\alpha u_i})(1-2e^{\alpha v_i}-e^\alpha)\big\}.$$

$$\begin{aligned}\ell_3(\theta,\alpha) = &\sum_{i=1}^{n}\ln(\delta_i^{\frac{1}{\alpha}-2}\bar{u}_i^{\alpha-1}) + \sum_{i=1}^{n}\ln\Big(\alpha\bar{v}_i^{\alpha-1}\big(1+\theta\bar{v}_i^{\alpha}(2\bar{u}_i^{\alpha}-1)\big) + \theta\bar{v}_i^{\alpha-1}(1-\bar{v}_i^{\alpha})\\ &(\alpha-2\bar{u}_i^{\alpha})\delta_i + (\frac{1}{\alpha}-1)(1-\bar{u}_i^{\alpha})(1-\bar{v}_i^{\alpha})\big(1+\theta\bar{v}_i^{\alpha}(2\bar{u}_i^{\alpha}-1)\big)\delta_i\\ &+\bar{v}_i^{\alpha-1}\big(\theta\bar{u}_i^{\alpha}(\bar{v}_i^{\alpha}-\alpha\bar{v}_i^{\alpha}+\alpha)-1\big)\Big),\end{aligned} \tag{11}$$

where $\delta_i = 1-(1-\bar{u}_i^{\alpha})(1-\bar{v}_i^{\alpha})(1+\theta\bar{u}_i^{\alpha}\bar{v}_i^{\alpha})$.

$$\begin{aligned}\ell_4(\theta,\alpha) = &\sum_{i=1}^{n}\ln(\eta_i^{\frac{1}{\alpha}-2}(1-\eta_i^{\frac{1}{\alpha}})^{\alpha-2}) + \sum_{i=1}^{n}\ln(u_i^{\frac{1}{\alpha}-1}v_i^{\frac{1}{\alpha}-1}(1-u_i^{\frac{1}{\alpha}})^{\alpha-1}(1-v_i^{\frac{1}{\alpha}})^{\alpha-1})\\ &+\sum_{i=1}^{n}\ln\Big((\frac{1}{\alpha}-1)\frac{\theta\exp(-(1-v_i^{\frac{1}{\alpha}})^\alpha)\big(1-\exp\{-(1-u_i^{\frac{1}{\alpha}})^\alpha\}\big)}{1+\theta\big(1-\exp\{-(1-u_i^{\frac{1}{\alpha}})^\alpha\}\big)\big(1-\exp(-(1-v_i^{\frac{1}{\alpha}})^\alpha)\big)}\\ &\times\big\{1+\frac{\exp\{-(1-u_i^{\frac{1}{\alpha}})^\alpha\}\big(1-\exp\{-(1-v_i^{\frac{1}{\alpha}})^\alpha\}\big)}{1+\theta\big(1-\exp\{-(1-u_i^{\frac{1}{\alpha}})^\alpha\}\big)\big(1-\exp\{-(1-v_i^{\frac{1}{\alpha}})^\alpha\}\big)}\big\}\\ &\quad+\big\{\frac{\theta\big(1-\exp(-(1-u_i^{\frac{1}{\alpha}})^\alpha)\big)\big(1-\exp(-(1-v_i^{\frac{1}{\alpha}})^\alpha)\big)}{1+\theta\big(1-\exp(-(1-u_i^{\frac{1}{\alpha}})^\alpha)\big)\big(1-\exp(-(1-v_i^{\frac{1}{\alpha}})^\alpha)\big)}\big\}\\ &\quad\times\exp\{-(1-u_i^{\frac{1}{\alpha}})^\alpha-(1-v_i^{\frac{1}{\alpha}})^\alpha\}\Big),\end{aligned} \tag{12}$$

where $\eta_i = (1-u_i^{\frac{1}{\alpha}})^\alpha + (1-v_i^{\frac{1}{\alpha}})^\alpha - \ln\Big(1+\theta(1-\exp\{-(1-u_i^{\frac{1}{\alpha}})^\alpha\})(1-\exp\{-(1-v_i^{\frac{1}{\alpha}})^\alpha\})\Big)$. Considering that there is not a closed form to obtain analytically maximum for (9), (10), (11) and (12), we used the numerical method according to the optim function in R software, with the CG and Nelder-Mead method to obtain the MLE of parameters.

Table 2: The mean and empirical MSE of Simultaneous estimators of θ and α with MM and ML methods.

Weak ($\theta = 0.2$)			$\alpha = 6(-6)$				$\alpha = 10(-10)$			
			ML		MM		ML		MM	
	n		$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
C_{g_1}	10	mean	0.26404	5.97669	0.48742	6.38137	0.21603	9.99251	0.46618	10.51191
		MSE	1.03865	0.01448	1.16875	2.91582	0.25264	0.00343	1.85144	4.97834
	20	mean	0.25418	5.97664	0.44919	6.38219	0.21938	9.99261	0.39782	10.50646
		MSE	0.93563	0.01267	0.85933	2.46642	0.28450	0.00284	1.63854	4.25508
	30	mean	0.25161	5.97686	0.39825	6.35561	0.21967	9.99281	0.41611	10.48579
		MSE	0.90496	0.01854	0.80849	2.27840	0.26778	0.00245	1.21934	3.75265
	50	mean	0.24553	5.97587	0.39211	6.31813	0.21708	9.99303	0.43285	10.44737
		MSE	0.82579	0.01229	0.54917	1.67549	**0.25608**	**0.00204**	1.10018	3.30529
C_{g_2}	10	mean	0.25224	-5.96809	0.11765	-5.63796	0.23652	-9.99225	-0.00313	-9.05949
		MSE	0.30554	0.13015	0.68945	5.02109	0.18452	0.05077	3.19146	30.37487
	20	mean	0.24879	-5.97362	0.19715	-5.95086	0.22873	-9.98719	0.17733	-9.86057
		MSE	0.30020	0.12945	0.47549	1.02464	0.17428	0.05080	0.39710	5.79004
	30	mean	0.23997	-5.97107	0.20217	-5.98557	0.22975	-9.98983	0.19233	-9.96896
		MSE	0.29249	0.12815	0.05075	0.55736	0.17011	0.05042	0.10693	1.42373
	50	mean	0.24011	-5.97419	0.19640	-6.00201	0.22656	-9.99014	0.19841	-9.99886
		MSE	0.28243	0.12732	0.07432	0.21232	0.15817	0.04977	**0.01429**	**0.01280**
C_{g_3}	10	mean	0.24326	5.96349	0.19402	5.93548	0.22784	10.16994	0.19779	9.31692
		MSE	0.29673	0.17046	0.00120	0.16742	0.04373	10.14184	0.00131	3.95443
	20	mean	0.23697	5.95883	0.20043	5.99533	0.22479	10.17994	0.19941	9.94642
		MSE	0.28214	0.17002	0.00059	0.01167	0.04208	10.03709	0.00006	0.30357
	30	mean	0.24016	5.95755	0.19967	6.00042	0.22507	10.17472	0.19987	9.99954
		MSE	0.28716	0.17026	0.00002	0.00026	0.04158	9.92499	0.00001	0.00026
	50	mean	0.23258	5.95850	0.20001	6.00005	0.22447	10.13817	0.20000	9.99997
		MSE	0.27502	0.16991	**0.00000**	**0.00000**	0.04131	9.70721	0.00000	0.00000
C_{g_4}	10	mean	0.48970	6.13406	0.96473	6.16428	0.27669	8.88778	0.19915	10.02753
		MSE	0.25753	13.33067	0.02709	3.02859	0.03743	34.65521	0.00882	0.61295
	20	mean	0.45708	6.07756	0.99583	6.04327	0.24467	8.99607	0.19945	10.01753
		MSE	0.23418	12.80431	0.01503	0.43038	0.02642	30.43291	0.00764	0.47981
	30	mean	0.40041	5.99227	0.20660	6.03696	0.19511	8.85716	0.19998	10.00112
		MSE	0.19191	12.23728	0.05723	0.39129	0.00937	25.02407	0.00002	0.00158
	50	mean	0.21426	5.81534	0.19996	5.99992	0.20022	9.62913	0.19988	9.99903
		MSE	0.04234	9.71225	**0.0000**	**0.00000**	0.00006	7.32432	0.00001	0.00112

6.2 Moments approach

Consider the observations of (x_i, y_i); $i = 1, ..., n$ from a pair (X, Y), with the copula C_ϕ. We can estimate Kendall's τ and Spearman's ρ using the corresponding sample version τ_n and ρ_n, respectively. By solving the equations $\tau = \tau_n$ and $\rho = \rho_n$ simultaneously, the moment estimates (MM) are obtained.

7 Numerical study

For illustrating the methodology, in continue, we consider a simulation study and real data for comparing the performance of subfamilies defined in Tabel 1.

7.1 Monte Carlo simulation and comparison

A simulation study is carried out to compare the performances of two methods of estimating the parameters θ and α, in terms of the corresponding means and mean squared errors (MSE). For computing the estimates, we generate data from C_ϕ using an algorithm proposed by [26]. Indeed, we wrote an algorithm in R software as follows:

(i) Given value $\theta \in [-1, 1]$ and $\alpha \in [0, \infty)$.

(ii) Generate two random sample u and t form $U(0, 1)$.

(iii) Compute $v = C_u^{-1}(t)$ and insert u and t from the previous step. where $C_u(v) = P\{V \le v | U = u\}$ defined by

$$C_u(v) = \frac{\partial}{\partial u} C_\phi(u, v) = \frac{\phi'(u)\big\{1 + \theta(1 -^{-\phi(v)})(1 - 2e^{-\phi(u)})\big\}}{\phi'(C_\phi(u, v))\big\{1 + \theta(1 -^{-\phi(v)})(1 -^{-\phi(u)})\big\}}. \quad (13)$$

According to the above algorithm, we simulate 1000 independent samples from each $C_{\phi_i}; i = 1, 2, 3, 4$ using different values of dependence parameters $\theta \in \{0.2, 0.5, 0.9\}$ and $\alpha \in \{6, 10\}$ and four sample sizes $n \in \{10, 20, 30, 50\}$. We assumed that, the weak dependence is achieved when $\theta = 0.2$, moderate for $\theta = 0.5$ and strong for $\theta = 0.9$. The result are given in Tables 2-4 . According to these tables, we conclude the following result:

Table 3: The mean and empirical MSE of simultaneous estimators of θ and α with MM and ML methods.

Modereate ($\theta = 0.5$)			$\alpha = 6(-6)$				$\alpha = 10(-10)$			
			ML		MM		ML		MM	
	n		$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
C_{ϕ_1}	10	mean	0.60432	5.96998	0.70893	6.25113	0.52151	9.99237	0.68633	10.48291
		MSE	1.66011	0.02997	1.02561	2.23524	0.44669	0.00686	1.84131	4.24203
	20	mean	0.58726	5.97125	0.68884	6.30130	0.52343	9.99105	0.74763	10.44668
		MSE	1.49226	0.02715	0.79308	2.15848	0.40431	0.00548	1.51042	3.74724
	30	mean	0.60432	5.96998	0.71158	6.30444	0.51957	9.99080	0.74839	10.60303
		MSE	1.66011	0.02997	1.36220	0.77555	2.03616	0.00744	3.85212	0.29027
	50	mean	0.56531	5.97298	0.67129	6.40018	0.51871	9.99105	0.68540	10.33272
		MSE	1.25001	0.02672	0.55521	1.88359	**0.33914**	**0.00632**	1.12302	3.23991
C_{ϕ_2}	10	mean	0.55119	-5.97105	0.45453	-5.61687	0.53486	-9.98563	0.26664	-10.71287
		MSE	0.30693	0.11981	5.44196	14.64038	0.19486	0.04851	1.97319	25.22584
	20	mean	0.54728	-5.97462	0.48611	-5.95711	0.53515	-9.98790	0.47969	-9.91534
		MSE	0.30367	0.11886	0.14506	0.57914	0.19076	0.04961	0.20289	3.54957
	30	mean	0.54520	-5.97506	0.49658	-5.99326	0.53696	-9.98523	0.49421	-9.96708
		MSE	0.30167	0.11832	0.02234	0.07836	0.19201	0.04867	0.08545	1.28943
	50	mean	0.54315	-5.97992	0.50082	-5.99811	0.53265	-9.98701	0.49935	-9.99604
		MSE	0.31138	0.11693	**0.00587**	**0.01777**	0.17739	0.04871	0.00163	0.05138
C_{ϕ_3}	10	mean	0.54758	5.95775	0.51764	5.70486	0.48063	10.39208	0.50856	9.28888
		MSE	0.31332	0.16215	0.01456	1.34213	0.03474	6.74536	0.00759	0.46542
	20	mean	0.53888	5.96124	0.50427	5.93301	0.47702	10.35389	0.50011	9.96974
		MSE	0.29613	0.15892	0.00458	0.30229	0.03462	6.64664	0.00001	0.23714
	30	mean	0.53525	5.95589	0.49964	5.99117	0.48006	10.33768	0.50002	9.98905
		MSE	0.28669	0.15639	0.00008	0.03602	0.03393	6.51285	0.00000	0.08305
	50	mean	0.53862	5.96664	0.50000	5.99556	0.47926	10.30915	0.50157	9.99348
		MSE	0.29432	0.15340	**0.00000**	**0.02366**	0.03392	6.47033	0.00297	0.05054
C_{ϕ_4}	10	mean	0.28051	6.05621	0.99001	6.21734	0.44614	8.83195	0.49755	10.01151
		MSE	0.07179	13.67985	0.22970	4.59363	0.02087	34.84758	0.00123	0.07216
	20	mean	0.28749	6.13612	0.99748	5.99576	0.45814	8.88747	0.49884	10.01991
		MSE	0.07129	13.23934	0.00011	0.00682	0.01577	30.82371	0.00288	0.61165
	30	mean	0.30437	5.98313	0.99916	6.00039	0.47963	9.03701	0.49869	10.01899
		MSE	0.06680	12.45861	0.00000	0.00019	0.00616	24.70646	0.00285	0.61067
	50	mean	0.37969	5.81662	0.50001	5.99999	0.49994	9.51583	0.50000	9.99999
		MSE	0.04436	9.65601	**0.00000**	**0.00000**	0.00000	7.42465	0.00000	0.00000

(*i*) **Results for C_{ϕ_1} copula**

For estimating θ, we have overestimation in both method for all levels $\theta \in \{0.2, 0.5, 0.9\}$, but the moment estimator is over estimated more than the ML. The estimator obtained using the moment approach is overestimated the value of α, but in maximum likelihood method the tendency is opposite. Generally the maximum likelihood estimator have lowest MSE, regardless of sample size. For both methods, by increasing n, the $\hat{\theta}$ and $\hat{\alpha}$ approaching to their true values, while the values of MSE for moment estimator have faster descending trend. By increasing the values of α, the overestimation of θ in both methods decreases and also the MSE of the maximum likelihood method for two parameters has a decreasing trend.

(*ii*) **Results for C_{ϕ_2} copula**

In estimation θ, we have overestimation in maximum likelihood method and underestimation in moment method, for all levels $\theta \in \{0.2, 0.5, 0.9\}$, but the moment estimator is better. Both of them have overestimated for the values of α. The maximum likelihood method have lowest values of MSE, while by increasing the values of n, the values of MSE for moment estimator have faster descending trend, such that for $n = 50$, the moment estimator is the best. By increasing the values of α, the overestimation of θ for ML method decreases and also the MSE of both methods for two parameters have a decreasing trend.

(*iii*) **Results for C_{ϕ_3} copula**

For $\theta \in \{0.2, 0.5\}$, in estimating θ, it was overestimated by maximum likelihood method and underestimated by moment method and the moment method is the best estimator for θ. The results indicates that for $\alpha = 6$, in both methods the parameters θ underestimated and for $\alpha = 10$, the maximum likelihood method overestimated, while the moment method is the best estimator for α. For $\theta = 0.9$, parameter θ, overestimated by maximum likelihood method and parameter α, for $\alpha = 6$, in both methods underestimated and for $\alpha = 10$, we have overestimated of parameter α and also the MM method is the best estimator for θ and α.

(*iv*) **Results for C_{ϕ_4} copula**

In estimating θ, for $\theta = 0.2$, there is overestimation except for $\alpha = 10$ by moment estimators, while for $\theta \in \{0.5, 0.9\}$, there is underestimation except for $\alpha = 6$ by moment estimators.

However the moment method have lowest values of MSE, especially in large sample size. Except $\alpha = 10$, in estimating α by maximum likelihood method, we have overestimation and again the moment estimator is the best.

Table 4: The mean and empirical MSE of simultaneous estimators of θ and α with MM and ML methods.

High ($\theta = 0.9$)	n		$\alpha = 6(-6)$				$\alpha = 10(-10)$			
			ML		MM		ML		MM	
			$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$
C_{ϕ_1}	10	mean	1.02169	5.97155	1.14448	6.36986	0.93732	9.98904	1.01487	10.47806
		MSE	2.75018	0.06061	0.99957	4.53776	0.82773	0.01504	1.90073	4.48425
	20	mean	1.01997	5.97103	1.03577	6.12274	0.93469	9.98938	1.04347	10.42369
		MSE	2.67339	0.05415	0.74268	2.07005	0.76493	0.01298	1.47408	3.99158
	30	mean	1.03775	5.96684	1.00318	6.30328	0.92891	9.98839	1.08408	10.43576
		MSE	2.67564	0.05078	0.67712	1.91462	0.52358	0.01362	1.23642	3.40478
	50	mean	1.02228	5.96899	1.01839	6.24447	0.92042	9.99012	1.06199	10.39331
		MSE	2.27971	0.04383	0.60633	1.85525	**0.51813**	**0.00916**	1.02150	3.10901
C_{ϕ_2}	10	mean	0.95191	-5.96993	0.80302	-5.70384	0.93356	-9.98393	0.73031	-9.28138
		MSE	0.34679	0.11116	1.57953	5.68458	0.19709	0.04807	1.49211	17.23095
	20	mean	0.9554	-5.97930	0.88979	-5.96744	0.93425	-9.98933	0.87689	-9.93536
		MSE	0.33626	0.10829	0.06277	0.25860	0.19679	0.04723	0.39394	2.34840
	30	mean	0.95542	-5.97403	0.89817	-5.99051	0.93295	-9.98689	0.90370	-9.95538
		MSE	0.32170	0.10931	0.01749	0.10658	0.19441	0.04706	0.10214	3.81477
	50	mean	0.93855	-5.97064	0.89979	-5.99833	0.92849	-9.99115	0.89914	-9.99347
		MSE	0.29166	0.10896	**0.00041**	**0.01399**	0.17760	0.04735	0.00622	0.08481
C_{ϕ_3}	10	mean	0.94005	5.96086	1.10818	5.51781	0.83272	10.77251	1.10668	8.24629
		MSE	0.30596	0.15875	0.50371	1.92349	0.03508	8.08328	0.35243	13.39162
	20	mean	0.94079	5.96066	0.92522	5.91832	0.83332	10.76881	0.92633	9.79178
		MSE	0.30425	0.15511	0.06004	0.31950	0.03490	8.04952	0.040486	1.63095
	30	mean	0.93519	5.95403	0.89147	5.91182	0.83054	10.73749	0.90569	9.96339
		MSE	0.30292	0.15620	0.01167	0.01444	0.03632	7.74049	0.00905	0.26806
	50	mean	0.93792	5.95628	0.90000	6.0000	0.83091	10.70401	0.89977	9.99192
		MSE	0.29941	0.15404	**0.00000**	**0.00000**	0.03559	7.35414	0.00000	0.07753
C_{ϕ_4}	10	mean	0.56218	6.10961	0.90182	6.01560	0.78637	8.96150	0.89981	10.00012
		MSE	0.18643	13.67094	0.00307	0.13934	0.09419	34.23899	0.00030	0.02079
	20	mean	0.57253	6.09474	0.99833	6.00082	0.79349	8.88410	0.89889	9.99565
		MSE	0.18613	13.18878	0.00019	0.00079	0.08281	31.11451	0.00096	0.03226
	30	mean	0.59004	6.06764	0.89999	5.99999	0.81790	8.98011	0.90003	10.00023
		MSE	0.18233	12.40933	0.00000	0.00000	0.05479	24.97387	0.00000	0.00036
	50	mean	0.68083	5.71914	0.90000	6.00000	0.89972	9.64121	0.90003	10.00037
		MSE	0.14985	9.77891	**0.00000**	**0.00000**	0.00011	7.29716	0.00000	0.00016

The above results, indicate that for the copula C_{ϕ_1}, the ML estimator is better than MM estimator and we can see the lowest MSE of ML estimator for $\alpha = 10$, $n = 50$ and for all levels of $\theta \in \{0.2, 0.5, 0.9\}$. For the others copulas, the MM estimator is the best. For the copula C_{ϕ_2}, the cases $\theta = 0.2, \alpha = 10, n = 50$ and $\theta \in \{0.5, 0.9\}, \alpha = 6, n = 50$, have the lowest values of MSE for MM estimator and eventually for C_{ϕ_3} and C_{ϕ_4}, we have lowest MSE for MM estimator in $\alpha = 6$, $n = 50$ and for all levels of $\theta \in \{0.2, 0.5, 0.9\}$.

7.2 Real data

For the application, we use our model to a real data set on the percentage of people under age 18 in poverty in 3109 US cities (Y) versus their per capita personal income (X) in the year 1995 ([9]).

Histogram of margins and summary statistics are given in Figure 6 and Table 5. To get a logical conclusion, the marginal distributions are estimated non-parametrically by their sample empirical distributions. Let R_{1i} and R_{2i} be the ranks of observations x_i and y_i respectively, in fact we convert the observation x_i and y_i to $u_i = \frac{R_{1i}}{n+1}$ and $v_i = \frac{R_{2i}}{n+1}$ for $i = 1, ..., n$ and formally test uncorrelatedness in terms of Kendall's tau and Spearman's rho between the two underlying variables. As expected from the graphical diagnostics, the extremely small p-value provides strong evidence against the null hypothesis that $\tau = 0$, as $\hat{\tau} = -0.45$ and $\hat{\rho} = -0.66$. Since, our new model supports strong positive dependence structure, we transform the observation v_i to $1 - v_i$.

Due to the result of uncorrelatedness tests ($\hat{\tau} = 0.45$ and $\hat{} = 0.66$) and Figure 6 that also shows a little tail dependence, the FGM copula is not suitable. Different distribution of observations below and above the main diagonal in scatter plot in data and also lake of symmetry with respect to the point $(1/2, 1/2)$ suggests that a model with radially asymmetry is suitable for data.

The result of exchangeability test used in [19] by statistics value 0.08903 and p-value= 0.00049 and also the test of radial symmetry studied in [18] by statistics value 0.11036 and p-value =0.01548 confirm the graphical diagnostics. Next, we compare the performance of C_ϕ copulas and that FGM copula.

Table 5: Summary statistics of income and percentage of people in poverty data

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
personal income (X)	6939	15856	17764	18353	20033	61594
percentage of people in poverty (Y)	2.20	14.20	19.80	21.61	27.40	65.40

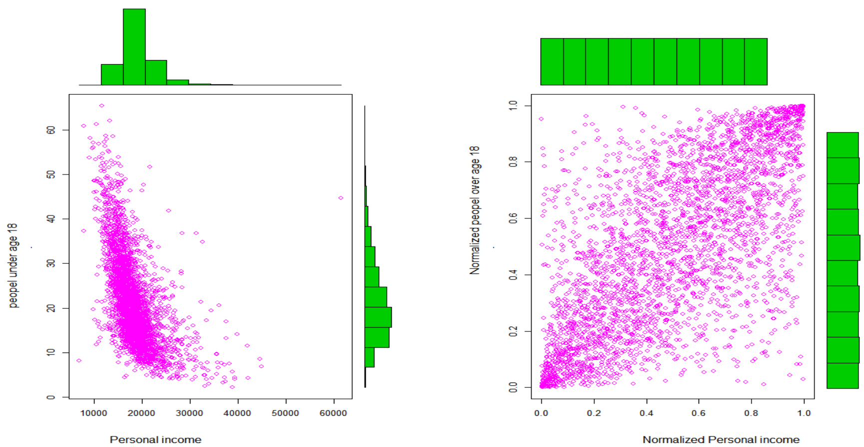

Figure 6: Scatter plot with histogram of margins for raw data (left) and for transformed data (right).

Table 6: The MLEs, SE, Log-liklihood valus of some AFGM copula with the results of Cramer-von Mises and Kolmogorov–Smirnov tests for C_{ϕ_i} i=1,2,3,4.

	C_{ϕ_1}	C_{ϕ_2}	C_{ϕ_3}	C_{ϕ_4}
Estimated parameters	$\hat{\theta}$=0.9±0.00057	$\hat{\theta}$= 0.05229 ±0.00021	$\hat{\theta}$=0.18003 ±3.45649$e-06$	$\hat{\theta}$= 0.14952 ±4.25626$e-06$
and SEs	$\hat{\alpha}$=1.0005 ±0.00505	$\hat{\alpha}$=-3.43048 ±0.00018	$\hat{\alpha}$=4.09999 ±0.00008	,$\hat{\alpha}$= 3.07957 ±0.00023
Log-liklihood	-2.35176	-2.36639	-2.36802	-2.36454
P-value(S_n)	0 (397.9894)	0.25 (388.3142)	0.986 (419.6749)	0.511 (435.5237)
P-value(T_n)	0.017 (0.98489)	0.078 (0.97366)	0.405 (0.97373)	0.065 (0.97278)

The ML estimators of dependence parameters of AFGM families given in Table 1 are given in Table 6. To assess which AFGM copula fits best, graphically, we compare contour plots of the parametric estimates with those of the empirical copula of the data presented in Figure 7.

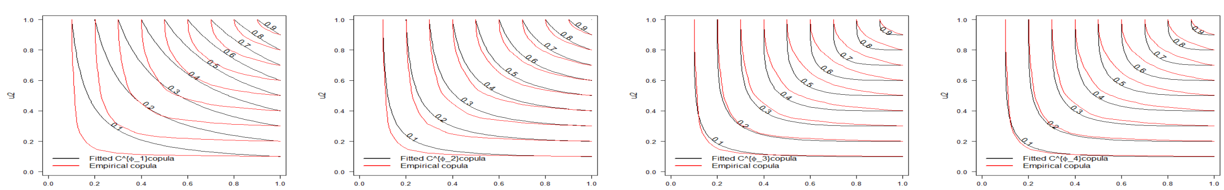

Figure 7: Contour plots of the fitted C_{ϕ_1}, C_{ϕ_2}, C_{ϕ_3} and C_{ϕ_4} copulas overlaid with the contours of the empirical copula for the data.

Figure 7 suggests that the C_{ϕ_1} copula is not good for data and C_{ϕ_3} fits better than the C_{ϕ_2} and C_{ϕ_4}. One can also perform a copula goodness-of-fit test. According to the goodness of fit procedures given in [16], two tests based on the empirical process $\mathbb{D} = \sqrt{n}(C_n - C_{\theta_n})$ are Cramer-von Mises and Kolmogorov-Smirnov tests with corresponding statistics:

$$S_n = \int_0^1 \int_0^1 \mathbb{D}^2(u,v)\, dC_n(u,v) \qquad and \qquad T_n = \sup_{(u,v)\in[0,1]^2} |\mathbb{D}(u,v)|,$$

respectively. Large values of these statistics lead to the rejection of $H_0 : C \in C_0$. According to [16] for computing p-values of these goodness-of-fit tests, we use parametric bootstrap method with sample size $B = 1000$. The results are summarized in Table 6. Thus, we may conclude that except C_{ϕ_1}, the other copulas are good for data and C_{ϕ_3} with dependence parameters $\hat{\theta} = 0.18, \hat{\alpha} = 3.46$ is the best one.

8 Conclusions

In this paper, we considered a bivariate copula structure which generalizes the Archimedean copulas with concave multiplicative generators in the style of FGM family. The dependency properties of the several subfamilies of this structure are studied in details. The proposed structure is quite flexible for modelling high positive dependence and heavy upper tail dependence. We discussed the estimation of the parameters using the maximum likelihood and moment methods. A real data set is analyzed. The result could be generalized to the multivariate dimensional case.

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FGM generated archimedean copulas with the concave multiplicative generators

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مفصل‌های ارشمیدسی تولید شده با خانواده FGM توسط مولدهای ضربی مقعر

چکیده. خانواده‌های فارلی-گامبل-مرگسترن (FGM) و ارشمیدسی از خانواده‌های معروف و پرکاربرد مفصل هستند. در این مقاله، تعمیمی از مفصل‌های ارشمیدسی با مولدهای ضربی مقعر براساس مفصل FGM ارائه می‌شود. مدل ارائه شده در این مقاله، امکان مدل‌سازی داده‌های دارای وابستگی بالاتر از آن چه که دیگر تعمیم‌های مفصل FGM اجازه می‌دهند، فراهم می‌سازد. روش ساخت و مشخصه‌سازی مدل پیشنهاد شده با ارائه مثال‌هایی از زیر خانواده‌هایی از این مدل توضیح داده شده‌است. مطالعات عددی برای تشریح و روشن‌سازی نتایج نظری انجام شده‌است.