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Original paper

Construction methods for triangular norms and triangular conorms on appropriate bounded lattices

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Abstract

In this study, new methods to construct triangular norms and triangular conorms on appropriate bounded lattices are introduced. Some illustrative examples are given for clarity. Also, the relation between introduced methods and some other approaches is investigated. Finally, it is shown that the introduced construction methods can be generalized by induction to a modified ordinal sum for triangular norms and triangular conorms on appropriate bounded lattices. And some illustrative examples are given.

Keywords: Bounded lattice, t-norms, t-conorms, ordinal sum.

1 Introduction and motivation

Triangular norms (t-norms) and triangular conorms (t-conorms) were introduced by Schweizer and Sklar [22] in 1963, within the framework of probabilistic metric spaces. More specifically, they are based on a notion used by Menger [18] in order to extend the triangle inequality in the definition of metric spaces towards probabilistic metric spaces. In fuzzy set theory, they were introduced for the first time by Alsina, Trillas and Valverde [1], and Prade [19], who used them for the definition of new classes of fuzzy union and intersection operators. Taking into account their probabilistic origins, it is hardly surprising that they have been defined on the unit interval $[0, 1]$. Upon inspection, we find however that none of their defining properties are typical only of the unit interval $[0, 1]$. These properties can therefore easily be generalized to define certain classes of operators on bounded lattices.

When the unit interval $[0, 1]$ is considered, the notion of ordinal sum in Clifford's sense [7] is a tool normally used to construct new t-norms if the operator considered is a t-norm, since the unit interval $[0, 1]$ together with a t-norm forms a semigroup. Since t-norms and t-conorms on the unit interval $[0, 1]$ are special semigroups, it follows that ordinal sums, as an important tool in the construction of operations, can be used to construct new t-norms and t-conorms. In 2002, Klement et al. [15] investigated t-norms as ordinal sums of semigroups in the sense of Clifford [7]. By use of the Cliffords definition [7], the ordinal sum of t-norms on bounded lattices was introduced by Saminger [20] in 2006. Saminger provided a necessary and sufficient condition for an ordinal sum function yielding again a t-norm on some bounded lattice, whereas the function is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. In 2008, Saminger-Platz et al. [21] discussed the extension of t-norms on bounded lattices. They used bounded sublattices as carriers for arbitrary summand t-norms. In 2010, Medina [16] provided a characterization of ordinal sums being t-norms on bounded lattices by ordinal sums of drastic t-norms. Moreover, in 2012, Medina [17] provided a characterization when an ordinal sum of t-norms is a t-norm on some bounded lattice. In 2015, Ertuğrul et al. [12] modified the ordinal sum construction for t-norms and t-conorms on bounded lattices.

In this paper, new ordinal sum constructions of t-norms and t-conorms on an arbitrary bounded lattice are introduced by using the existence of t-norms on $[0, a]$ and t-conorms on $[a, 1]$, respectively.

This paper is organized as follows. In Section 2, some basic notions are shortly presented. In Section 3, new classes of ordinal sums of t-norms and t-conorms with t-norms and t-conorms, respectively, on appropriate bounded lattices

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are introduced, where $a \in L \setminus \{0,1\}$, t-norms act on $[0,a]$, and t-conorms act on $[a,1]$, respectively. In order to well understand the constructed t-norm T and t-conorm S, some illustrative examples are given. Then, the relation between introduced methods and some other approaches is investigated. In Section 4, by using the recursive methods, ordinal sum constructions of t-norms and t-conorms on appropriate bounded lattices are proposed. And some illustrative examples are provided.

2 Preliminaries

A lattice [2, 3, 6, 8, 9, 11, 13] is a partially ordered set $(L,\leq)$ in which each two element subset $\{x,y\}$ has an infimum, denoted as $x\wedge y$, and a supremum, denoted as $x \vee y$. A bounded lattice $(L,\leq,0,1)$ is a lattice that has the bottom and top elements written as 0 and 1, respectively.

Given a bounded lattice $(L,\leq,0,1)$ and $a,b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$. We denote the set of elements which are incomparable with a by I_a. So $I_a = \{x \in L \mid x \parallel a\}$.

Given a bounded lattice $(L,\leq,0,1)$ and $a,b \in L$, $a \leq b$, a subinterval $[a,b]$ of L is defined as

$$[a,b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, $[a,b) = \{x \in L \mid a \leq x < b\}$, $(a,b] = \{x \in L \mid a < x \leq b\}$ and $(a,b) = \{x \in L \mid a < x < b\}$.

Definition 2.1. [5, 10, 20] *Let $(L,\leq,0,1)$ be a bounded lattice. A t-norm T (a t-conorm S) is a binary operation on L which is commutative, associative, increasing with respect to both variables and it satisfies $T(x,1)=x$ ($S(x,0)=x$) for all $x \in L$.*

Theorem 2.2. [12] *Let $(L,\leq,0,1)$ be a bounded lattice and $a \in L\setminus\{0,1\}$. If V_T is a t-norm on $[a,1]$ and W_T is a t-conorm on $[0,a]$, then the functions $T^*: L^2 \to L$ and $S^*: L^2 \to L$ are a t-norm and a t-conorm on L, respectively, where*

$$T^*(x,y) = \begin{cases} x\wedge y & \text{if } x=1 \ y=1 \ , \\ V_T(x,y) & \text{if } x,y \in [a,1) \ , \\ x \wedge y \wedge a & \text{otherwise.} \end{cases} \quad \text{and} \quad S^*(x,y) = \begin{cases} x\vee y & \text{if } x=0 \ y=0 \ , \\ W_T(x,y) & \text{if } x,y \in (0,a] \ , \\ x \vee y \vee a & \text{otherwise.} \end{cases}$$

Theorem 2.3. [4] *Let $(L,\leq,0,1)$ be a bounded lattice and $a \in L\setminus\{0,1\}$. Given t-norm V on $[0,a]$ and t-conorm W on $[a,1]$.*
i) If $x \parallel y$ for all $x \in I_a$ and $y \in (0,a]$, then the function $T^\sim : L^2 \to L$ defined as follows is a t-norm on L

$$T^\sim(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [0,a]^2 \ , \\ 0 & \text{if } (x,y) \in [0,a) \times I_a \cup I_a \times [0,a) \cup I_a \times I_a \\ & \quad \cup [a,1) \times I_a \cup I_a \times [a,1) \ , \\ x \wedge y & \text{otherwise.} \end{cases}$$

ii) If $x \parallel y$ for all $x \in I_a$ and $y \in [a,1)$, then the function $S^\sim : L^2 \to L$ defined as follows is a t-conorm on L

$$S^\sim(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [a,1]^2 \ , \\ 1 & \text{if } (x,y) \in (a,1] \times I_a \cup I_a \times (a,1] \cup I_a \times I_a \\ & \quad \cup (0,a] \times I_a \cup I_a \times (0,a] \ , \\ x \vee y & \text{otherwise.} \end{cases}$$

3 Some methods to obtain t-norms and t-conorms on appropriate bounded lattices

In this section, ordinal sums of t-norms and t-conorms on appropriate bounded lattices are constructed, where $a \in L \setminus \{0,1\}$, V is t-norm on $[0,a]$ and W is t-conorm on $[a,1]$, respectively. The relation between introduced methods in Theorem 3.5 and Theorem 3.14 and other methods proposed in [4] and [12] is investigated. Also, some illustrative examples are given.

The following definition of an ordinal sum of t-norms defined on subintervals of a bounded lattice $(L,\leq,0,1)$ has been extracted from [20], which generalizes the methods given in [14] on subintervals of $[0,1]$.

Definition 3.1. [20] *Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let V be a t-norm on $[a, b]$. Then $T : L^2 \to L$ defined by*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,b]^2 , \\ x \wedge y & \text{otherwise.} \end{cases} \tag{1}$$

is an ordinal sum $(< a, b, V >)$ of V on L.

Definition 3.2. [20] *Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a, b]$ of L. Let W be a t-conorm on $[a, b]$. Then $S : L^2 \to L$ defined by*

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [a,b]^2 , \\ x \vee y & \text{otherwise.} \end{cases} \tag{2}$$

is an ordinal sum $(< a, b, W >)$ of W on L.

However, the operation T (resp. S) given by Formula (1) (resp. Formula (2)) need not be a t-norm (resp. t-conorm), in general. Observe that condition ensuring that T (resp. S) given by (1) ((2)) is a t-norm (t-conorm) on L are given in Saminger's paper [20]. If L is a chain, then this T (S) is a t-norm (t-conorm) for any $[a, b] \subseteq L$.

Example 3.3. *Consider the lattice $(L_1 = \{0_{L_1}, k, a, n, p, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ given in Figure 1 and define the t-norm $V : [0_{L_1}, a]^2 \to [0_{L_1}, a]$ as follows.*

$$V(x,y) = \begin{cases} x \wedge y & \text{if } a \in \{x,y\} , \\ 0_{L_1} & \text{otherwise.} \end{cases}$$

Then, using Formula (1), the operation T on L_1 defined by Table 1 is not a t-norm.

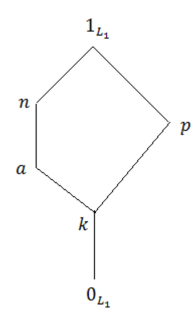

Figure 1: The lattice L_1

Table 1: The function T on L_1

T	0_{L_1}	k	a	n	p	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
k	0_{L_1}	0_{L_1}	k	k	k	k
a	0_{L_1}	k	a	a	k	a
n	0_{L_1}	k	a	n	k	n
p	0_{L_1}	k	k	k	p	p
1_{L_1}	0_{L_1}	k	a	n	p	1_{L_1}

Indeed, the function T does not satisfy associativity, because $T(a, T(p, k)) = T(a, k) = k > 0_{L_1} = T(k, k) = T(T(a, p), k)$. So, we obtain that T is not a t-norm on L_1.

Example 3.4. *Consider the lattice* $(L_2 = \{0_{L_2}, b, a, c, d, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ *given in Figure 2 and define the t-conorm* $W : [a, 1_{L_2}]^2 \to [a, 1_{L_2}]$ *as follows.*

$$W(x,y) = \begin{cases} x \vee y & \text{if } a \in \{x, y\}\ , \\ 1_{L_2} & \text{otherwise.} \end{cases}$$

Then, using Formula (2), the operation S *on* L_2 *defined by Table 2 is not a t-conorm.*

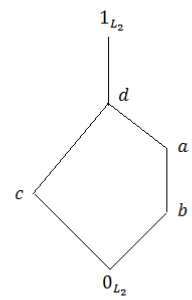

Figure 2: The lattice L_2

Table 2: The function S on L_2

S	0_{L_2}	b	a	c	d	1_{L_2}
0_{L_2}	0_{L_2}	b	a	c	d	1_{L_2}
b	b	b	a	d	d	1_{L_2}
a	a	a	a	d	d	1_{L_2}
c	c	d	d	c	d	1_{L_2}
d	d	d	d	d	1_{L_2}	1_{L_2}
1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}	1_{L_2}

Indeed, the function S *does not satisfy associativity, because*

$$S(a, S(c, d)) = S(a, d) = d < 1_{L_2} = S(d, d) = S(S(a, c), d).$$

So, we obtain that S *is not a t-conorm on* L.

Next, we construct ordinal sums of t-norms and t-conorms on an arbitrary bounded lattice in Theorem 3.5 and Theorem 3.14, respectively, where $a \in L \setminus \{0, 1\}$, V is a t-norm on $[0, a]$ and W is a t-conorm on $[a, 1]$, respectively.

Theorem 3.5. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $a \in L \setminus \{0, 1\}$. *If* $x \parallel y$ *for all* $x \in I_a$ *and* $y \in (0, a]$, *then the function* $T : L^2 \to L$ *defined as follows is a t-norm on* L, *where* V *is a t-norm on* $[0, a]$.

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [0,a]^2\ , \\ 0 & \text{if } (x,y) \in [0,a) \times I_a \cup I_a \times [0,a) \cup I_a \times I_a \cup [a,1) \times I_a \cup I_a \times [a,1)\ , \\ a & \text{if } (x,y) \in [a,1)^2\ , \\ x \wedge y & \text{otherwise.} \end{cases}$$

Proof. We have $T(x, 1) = x \wedge 1 = x$. So, the fact that $1 \in L$ is a neutral element of T. It is easy to see commutativity of T.

i) Monotonicity: We prove that if $x \leq y$, then $T(x, z) \leq T(y, z)$ for all $z \in L$. The proof can be split into all possible cases. If $z = 1$, then we have that $T(x, z) = T(x, 1) = x \leq y = T(y, 1) = T(y, z)$ for all $x, y \in L$. If $x = 0$, then we have that $T(0, z) = 0 \leq T(y, z)$ for all $y, z \in L$. If $z = 0$, then we have that $T(x, 0) = 0 = T(y, 0)$ for all $x, y \in L$. If $y = 0$, then it is clear that monotonicity is held.

1. $x \in (0, a)$

1.1 $y \in (0, a)$

1.1.1. $z \in (0, a)$

$$T(x, z) = V(x, z) \leq V(y, z) = T(y, z)$$

1.1.2. $z \in [a, 1)$

$$T(x, z) = x \wedge z \leq y \wedge z = T(y, z)$$

1.1.3. $z \in I_a$

$$T(x, z) = 0 = T(y, z)$$

1.2. $y \in [a, 1)$

1.2.1. $z \in (0, a)$

$$T(x, z) = V(x, z) \leq z = y \wedge z = T(y, z)$$

1.2.2. $z \in [a, 1)$

$$T(x, z) = x < a = T(y, z)$$

1.2.3. $z \in I_a$

$$T(x, z) = 0 = T(y, z)$$

1.3. $y \in I_a$. Since $x \in (0, a)$ and $y \in I_a$, then it holds $x \parallel y$. So, it can not be the case $y \in I_a$.

1.4. $y = 1$

1.4.1. $z \in (0, a)$

$$T(x, z) = V(x, z) \leq z = T(1, z)$$

1.4.2. $z \in [a, 1)$

$$T(x, z) = x < a \leq z = T(1, z)$$

1.4.3. $z \in I_a$

$$T(x, z) = 0 \leq T(1, z)$$

2. $x \in [a, 1)$. Then, it must be the case that $y \in [a, 1]$.

2.1 $y \in [a, 1)$

2.1.1. $z \in (0, a)$

$$T(x, z) = z = T(y, z)$$

2.1.2. $z \in [a, 1)$

$$T(x, z) = a = T(y, z)$$

2.1.3. $z \in I_a$

$$T(x, z) = 0 = T(y, z)$$

2.1 $y = 1$

2.1.1. $z \in (0, a)$

$$T(x, z) = z = T(1, z)$$

2.1.2. $z \in [a, 1)$

$$T(x, z) = a \leq z = T(1, z)$$

2.1.3. $z \in I_a$

$$T(x, z) = 0 \leq z = T(1, z)$$

3. $x \in I_a$. Then, it must be the case that $y \in I_a$ or $y \in (a, 1]$.

3.1 $y \in I_a$

3.1.1. $z \in (0, a)$ or $z \in I_a$ or $z \in [a, 1)$

$$T(x, z) = 0 = T(y, z)$$

3.2. $y \in (a, 1)$

3.2.1. $z \in (0, a)$

$$T(x, z) = 0 \leq z = T(y, z)$$

3.2.2. $z \in [a,1)$.

$$T(x,z) = 0 \le a = T(y,z)$$

3.2.3. $z \in I_a$

$$T(x,z) = 0 = T(y,z)$$

3.3 $y = 1$

3.3.1. $z \in (0,a)$ or $z \in [a,1)$ or $z \in I_a$

$$T(x,z) = 0 \le z = T(1,z)$$

4. $x = 1$. Then, since $y = 1$, it is clear that $T(x,z) = T(y,z)$ for all $z \in L$.

ii) Associativity: We need to prove that $T(x,T(y,z)) = T(T(x,y),z)$ for all $x,y,z \in L$. If at least one of x,y,z in L is 1, then it is obvious. So, the proof is split into all possible cases.

1. $x \in [0,a)$

1.1 $y \in [0,a)$

1.1.1. $z \in [0,a)$

$$T(x,T(y,z)) = T(x,V(y,z)) = V(x,V(y,z)) = V(V(x,y),z) = T(T(x,y),z)$$

1.1.2. $z \in [a,1)$

$$T(x,T(y,z)) = T(x,y) = V(x,y) = T(V(x,y),z) = T(T(x,y),z)$$

1.1.3. $z \in I_a$

$$T(x,T(y,z)) = T(x,0) = 0 = T(V(x,y),z) = T(T(x,y),z)$$

1.2. $y \in [a,1)$

1.2.1. $z \in [0,a)$

$$T(x,T(y,z)) = T(x,z) = T(T(x,y),z)$$

1.2.2. $z \in [a,1)$

$$T(x,T(y,z)) = T(x,a) = x = T(x,z) = T(T(x,y),z)$$

1.2.3. $z \in I_a$

$$T(x,T(y,z)) = T(x,0) = 0 = T(x,z) = T(T(x,y),z)$$

1.3. $y \in I_a$

1.3.1. $z \in [0,a)$ or $z \in I_a$

$$T(x,T(y,z)) = T(x,0) = 0 = T(0,z) = T(T(x,y),z)$$

1.3.2. $z \in [a,1)$

$$T(x,T(y,z)) = T(x,0) = 0 = T(x,z) = T(T(x,y),z)$$

2. $x \in [a,1)$

2.1 $y \in [0,a)$

2.1.1. $z \in [0,a)$

$$T(x,T(y,z)) = T(x,V(y,z)) = V(y,z) = T(y,z) = T(T(x,y),z)$$

2.1.2. $z \in [a,1)$

$$T(x,T(y,z)) = T(x,y) = y = T(y,z) = T(T(x,y),z)$$

2.1.3. $z \in I_a$

$$T(x,T(y,z)) = T(x,0) = 0 = T(y,z) = T(T(x,y),z)$$

2.2. $y \in [a,1)$

2.2.1. $z \in [0,a)$

$$T(x,T(y,z)) = T(x,z) = z = T(a,z) = T(T(x,y),z)$$

2.2.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, a) = a = T(a, z) = T(T(x, y), z)$$

2.2.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(a, z) = T(T(x, y), z)$$

2.3. $y \in I_a$

2.3.1. $z \in [0, a)$ or $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(y, z) = T(T(x, y), z)$$

2.3.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(0, z) = T(T(x, y), z)$$

3. $x \in I_a$

3.1 $y \in [0, a)$

3.1.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, V(y, z)) = 0 = T(0, z) = T(T(x, y), z)$$

3.1.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, y) = 0 = T(0, z) = T(T(x, y), z)$$

3.1.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(0, z) = T(T(x, y), z)$$

3.2. $y \in [a, 1)$

3.2.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, z) = 0 = T(0, z) = T(T(x, y), z)$$

3.2.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, a) = 0 = T(0, z) = T(T(x, y), z)$$

3.2.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(0, z) = T(T(x, y), z)$$

3.3. $y \in I_a$

3.3.1. $z \in [0, a)$ or $z \in I_a$ or $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, 0) = 0 = T(0, z) = T(T(x, y), z)$$

So, we have the fact that T is a t-norm on L. □

Corollary 3.6. *If we take $V = T_\wedge$ on $[0, a]$ given in Theorem 3.5, then we obtain the following t-norm on L.*

$$T(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, a) \times I_a \cup I_a \times [0, a) \cup I_a \times I_a \cup [a, 1) \times I_a \cup I_a \times [a, 1) , \\ a & \text{if } (x, y) \in [a, 1)^2 , \\ x \wedge y & \text{otherwise.} \end{cases}$$

In the following, we provide two lattices L_1 and L_2 which satisfies and does not satisfy the constraint of Theorem 3.5, respectively.

Example 3.7. *The lattice $(L_3 = \{0_{L_3}, m, q, p, s, a, t, r, k, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ in Figure 3 satisfies the constraint of Theorem 3.5 (for element $a \in L_3$). That is, $x \parallel y$ for all $x \in I_a$ and $y \in (0_{L_3}, a]$. Consider the t-norm $V : [0_{L_3}, a]^2 \to [0_{L_3}, a]$ as follows:*

$$V(x, y) = \begin{cases} x \wedge y & \text{if } a \in \{x, y\} , \\ 0_{L_3} & \text{otherwise.} \end{cases}$$

Then, by using Theorem 3.5, the function T on L_3 defined by Table 3 is a t-norm.

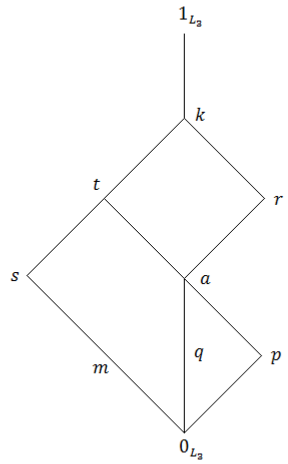

Figure 3: The lattice L_3

Table 3: The t-norm T on L_3

T	0_{L_3}	m	q	p	s	a	t	r	k	1_{L_3}
0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}
m	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	m
q	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	q	q	q	q	q
p	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	p	p	p	p	p
s	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	0_{L_3}	s
a	0_{L_3}	0_{L_3}	q	p	0_{L_3}	a	a	a	a	a
t	0_{L_3}	0_{L_3}	q	p	0_{L_3}	a	a	a	a	t
r	0_{L_3}	0_{L_3}	q	p	0_{L_3}	a	a	a	a	r
k	0_{L_3}	0_{L_3}	q	p	0_{L_3}	a	a	a	a	k
1_{L_3}	0_{L_3}	m	q	p	s	a	t	r	k	1_{L_3}

Table 4: The function T on L_4

T	0_{L_4}	m	q	p	s	a	t	r	k	1_{L_4}
0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}
m	0_{L_4}	m	0_{L_4}	0_{L_4}	0_{L_4}	m	m	m	m	m
q	0_{L_4}	0_{L_4}	q	0_{L_4}	0_{L_4}	q	q	q	q	q
p	0_{L_4}	0_{L_4}	0_{L_4}	p	0_{L_4}	p	p	p	p	p
s	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	0_{L_4}	s
a	0_{L_4}	m	q	p	0_{L_4}	a	a	a	a	a
t	0_{L_4}	m	q	p	0_{L_4}	a	a	a	a	t
r	0_{L_4}	m	q	p	0_{L_4}	a	a	a	a	r
k	0_{L_4}	m	q	p	0_{L_4}	a	a	a	a	k
1_{L_4}	0_{L_4}	m	q	p	s	a	t	r	k	1_{L_4}

Example 3.8. *The lattice* $(L_4 = \{0_{L_4}, m, q, p, s, a, t, r, k, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ *in Figure 4 does not satisfy (for* $a \in L_4$*) constraint of Theorem 3.5. That is, there is the element* $m \in L_4$ *such that* $m < s$ *for* $s \in I_a$ *and* $m \in (0_{L_4}, a)$*. Consider the t-norm* $V : [0_{L_4}, a]^2 \to [0_{L_4}, a]$*,* $V(x, y) = x \wedge y$*.*

The function T *on* L_4 *defined by Table 4 is not a t-norm. Indeed, it does not satisfy monotonicity. Clearly,* $m < s$ *and* $T(m, m) = m \nleq 0_{L_4} = T(s, m)$*.*

Proposition 3.9. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice with* $a \in L \setminus \{0, 1\}$ *such that for all* $x \in I_a$ *and* $y \in (0, a]$ *it holds* $x \parallel y$*. Suppose* T*,* T^* *and* $T^\sim$ *are the t-norms on* L *defined as in Theorem 3.5, Theorem 2.2 and Theorem 2.3 with*

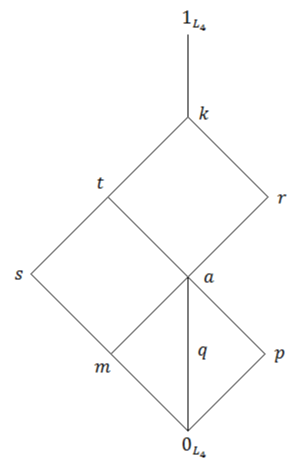

Figure 4: The lattice L_4

underlying t-norms V and V_T on $[0,a]$ and $[a,1]$, respectively. Then,

(i) $T \leq T^*$.

(ii) $T \leq T^{\sim}$.

Proof. (i) Let V and V_T be arbitrary t-norms on $[0,a]$ and $[a,1]$, respectively.

Let $(x,y) \in [0,a]^2$. Then, we have $T(x,y) = V(x,y) \leq x \wedge y = T^*(x,y)$.

Let $(x,y) \in [0,a) \times I_a \cup I_a \times [0,a)$. Then, we have $T(x,y) = 0 \leq x \wedge y = T^*(x,y)$.

Let $(x,y) \in [a,1) \times I_a$. Then, we have $T(x,y) = 0 \leq y \wedge a = T^*(x,y)$.

Let $(x,y) \in I_a \times [a,1)$. Then, we have $T(x,y) = 0 \leq x \wedge a = T^*(x,y)$.

Let $(x,y) \in I_a \times I_a$. Then, we have $T(x,y) = 0 \leq x \wedge y \wedge a = T^*(x,y)$.

Let $(x,y) \in [a,1)^2$. Then, we have $T(x,y) = a \leq V_T(x,y) = T^*(x,y)$.

For all other possible conditions, we have $T(x,y) = x \wedge y = T^*(x,y)$.

(ii) Since T and $T^{\sim}$ have the same t-norm V on $[0,a]$, we need only to consider the cases such as on $[0,a] \times [a,1] \cup [a,1] \times [0,a]$, $[0,a) \times I_a$, $I_a \times [0,a)$, $I_a \times I_a$, $[a,1]^2$, $[a,1) \times I_a$ and $I_a \times [a,1)$.

Let $(x,y) \in [0,a) \times I_a \cup I_a \times [0,a) \cup [a,1) \times I_a \cup I_a \times [a,1) \cup I_a \times I_a$. Then, we have $T(x,y) = 0 = T^{\sim}(x,y)$.

Let $(x,y) \in [a,1)^2$. Then, we have $T(x,y) = a < x \wedge y = T^{\sim}(x,y)$.

For all other possible conditions, we have $T(x,y) = x \wedge y = T^{\sim}(x,y)$. □

Remark 3.10. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L \setminus \{0,1\}$ such that for all $x \in I_a$ and $y \in (0,a]$ it holds $x \parallel y$. Suppose T and T^* are the t-norms on L defined as in Theorem 3.5 and Theorem 2.2 with underlying t-norms V and V_T on $[0,a]$ and $[a,1]$, respectively. One can wonder if the t-norms T and T^* can coincide on any bounded lattice. To illustrate this question we shall give the following example.*

Example 3.11. *Consider the lattice $(L_5 = \{0_{L_5}, c, d, a, e, f, 1_{L_5}\}, \leq, 0_{L_5}, 1_{L_5})$ which is depicted by Hasse diagram in Figure 5. Consider the t-norm V on $[0_{L_5}, a]$, $V(x,y) = x \wedge y$ and the t-norm V_T on $[a, 1_{L_5}]$ defined as follows:*

$$V(x,y) = \begin{cases} x \wedge y & \text{if } 1_{L_5} \in \{x,y\}, \\ a & \text{otherwise.} \end{cases}$$

By using Theorem 3.5 and Theorem 2.2 define the corresponding t-norms T and $T^\star$ as given in Table 5.

According to Table 5, it is clear that the t-norm T coincides with the t-norm T^* on the bounded lattice L_5.

Remark 3.12. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L \setminus \{0,1\}$ such that for all $x \in I_a$ and $y \in (0,a]$ it holds $x \parallel y$. Suppose T and $T^{\sim}$ are the t-norms on L defined as in Theorem 3.5 and Theorem 2.3 with underlying t-norms V on $[0,a]$. One can wonder if the t-norms T and $T^{\sim}$ can coincide on any bounded lattice. To illustrate this question we shall give the following example.*

Example 3.13. *Consider the lattice $(L_6 = \{0_{L_6}, p, t, k, a, 1_{L_6}\}, \leq, 0_{L_6}, 1_{L_6})$ which is depicted by Hasse diagram in Figure 6. Consider the t-norm V on $[0_{L_6}, a]$, $V(x,y) = x \wedge y$.*

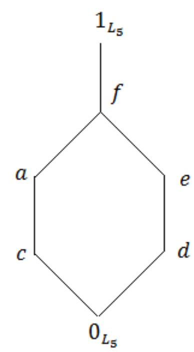

Figure 5: The lattice L_5

Table 5: The t-norm $T = T^*$ on L_5

$T = T^*$	0_{L_5}	c	d	a	e	f	1_{L_5}
0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}
c	0_{L_5}	c	0_{L_5}	c	0_{L_5}	c	c
d	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	d
a	0_{L_5}	c	0_{L_5}	a	0_{L_5}	a	a
e	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	0_{L_5}	e
f	0_{L_5}	c	0_{L_5}	a	0_{L_5}	a	f
1_{L_5}	0_{L_5}	c	d	a	e	f	1_{L_5}

Table 6: The t-norm $T = T^{\sim}$ on L_6

$T = T^{\sim}$	0_{L_6}	p	t	k	a	1_{L_6}
0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}	0_{L_6}
p	0_{L_6}	p	0_{L_6}	0_{L_6}	p	p
t	0_{L_6}	0_{L_6}	t	0_{L_6}	t	t
k	0_{L_6}	0_{L_6}	0_{L_6}	k	k	k
a	0_{L_6}	p	t	k	a	a
1_{L_6}	0_{L_6}	p	t	k	a	1_{L_6}

By using Theorem 3.5 and Theorem 2.3 define the corresponding t-norms T and $T^{\sim}$ as given in Table 6. According to Table 6, it is clear that the t-norm T coincides with the t-norm $T^{\sim}$ on the bounded lattice L_6.

Next, we present ordinal sum construction of t-conorms on arbitrary bounded lattice L with underlying t-conorm W on $[a, 1]$. We omit the proof of the next Theorem due to its similarity to the proof of Theorem 3.5.

Theorem 3.14. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. If $x \parallel y$ for all $x \in I_a$ and $y \in [a, 1)$, then the function $S : L^2 \to L$ defined as follows is a t-conorm on L, where W is a t-conorm on $[a, 1]$.*

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [a,1]^2 , \\ 1 & \text{if } (x,y) \in (a,1] \times I_a \cup I_a \times (a,1] \cup (0,a] \times I_a \cup I_a \times (0,a] \cup I_a \times I_a , \\ a & \text{if } (x,y) \in (0,a]^2 , \\ x \vee y & \text{otherwise.} \end{cases}$$

Corollary 3.15. *If we take $W = S_\vee$ on $[a, 1]$ given in Theorem 3.14, then we obtain the following t-conorm on L.*

$$S(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (a,1] \times I_a \cup I_a \times (a,1] \cup (0,a] \times I_a \cup I_a \times (0,a] \cup I_a \times I_a , \\ a & \text{if } (x,y) \in (0,a]^2 , \\ x \vee y & \text{otherwise.} \end{cases}$$

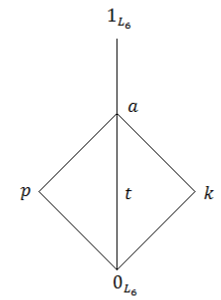

Figure 6: The lattice L_6

In the following, we provide two lattices L_7 and L_8 which satisfies and does not satisfy the constraint of Theorem 3.14, respectively.

Example 3.16. *The lattice* $(L_7 = \{0_{L_7}, b, c, d, a, e, n, m, f, 1_{L_7}\}, \leq, 0_{L_7}, 1_{L_7})$ *in Figure 7 satisfies the constraint of Theorem 3.14. That is,* $x \parallel y$ *for all* $x \in I_a$ *and* $y \in [a, 1_{L_7})$*. Consider the t-conorm* $W : [a, 1_{L_7}]^2 \to [a, 1_{L_7}]$ *as follows:*

$$W(x,y) = \begin{cases} x \vee y & if\ a \in \{x,y\} \\ 1_{L_7} & otherwise. \end{cases},$$

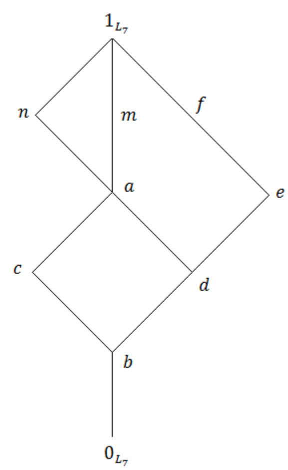

Figure 7: The lattice L_7

Then, by using Theorem 3.14, the function S *on* L_7 *defined by Table 7 is a t-conorm.*

Example 3.17. *The lattice* $(L_8 = \{0_{L_8}, b, c, d, a, e, n, m, f, 1_{L_8}\}, \leq, 0_{L_8}, 1_{L_8})$ *in Figure 8 does not satisfy the constraint of Theorem 3.14. Indeed, there is an element* $f \in L_8$ *such that* $e < f$ *for* $e \in I_a$ *and* $f \in (a, 1_{L_8})$*. Consider the t-conorm* $W : [a, 1_{L_8}]^2 \to [a, 1_{L_8}]$, $W(x,y) = x \vee y$.

The function S *on* L_8 *defined by Table 8 is not a t-conorm. It does not satisfy monotonicity. Clearly* $e < f$, $S(e,f) = 1_{L_8} \not\leq f = S(f,f)$.

Proposition 3.18. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice with* $a \in L \setminus \{0,1\}$ *such that for all* $x \in I_a$ *and* $y \in [a, 1)$ *it holds* $x \parallel y$*. Suppose* S, S^* *and* $S^\sim$ *are the t-conorms on* L *defined as in Theorem 3.14, Theorem 2.2 and Theorem 2.3 with underlying t-conorms* W *and* W_T *on* $[a, 1]$ *and* $[0, a]$*, respectively. Then,*

(i) $S^* \leq S$.

(ii) $S^\sim \leq S$.

Remark 3.19. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice with* $a \in L \setminus \{0,1\}$ *such that for all* $x \in I_a$ *and* $y \in [a, 1)$ *it holds* $x \parallel y$*. Suppose* S *and* S^* *are the t-conorms on* L *defined as in Theorem 3.14 and Theorem 2.2 with underlying t-conorms*

Table 7: The t-conorm S on L_7

S	0_{L_7}	b	c	d	a	e	n	m	f	1_{L_7}
0_{L_7}	0_{L_7}	b	c	d	a	e	n	m	f	1_{L_7}
b	b	a	a	a	a	1_{L_7}	n	m	1_{L_7}	1_{L_7}
c	c	a	a	a	a	1_{L_7}	n	m	1_{L_7}	1_{L_7}
d	d	a	a	a	a	1_{L_7}	n	m	1_{L_7}	1_{L_7}
a	a	a	a	a	a	1_{L_7}	n	m	1_{L_7}	1_{L_7}
e	e	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}
n	n	n	n	n	n	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}
m	m	m	m	m	m	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}
f	f	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}
1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}	1_{L_7}

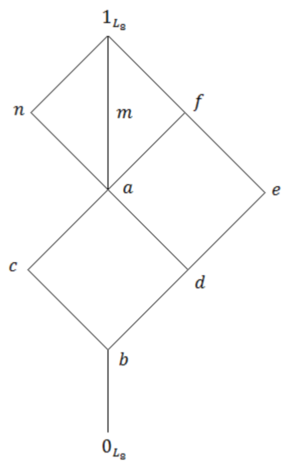

Figure 8: The lattice L_8

Table 8: The function S on L_8

S	0_{L_8}	b	c	d	a	e	n	m	f	1_{L_8}
0_{L_8}	0_{L_8}	b	c	d	a	e	n	m	f	1_{L_8}
b	b	a	a	a	a	1_{L_8}	n	m	f	1_{L_8}
c	c	a	a	a	a	1_{L_8}	n	m	f	1_{L_8}
d	d	a	a	a	a	1_{L_8}	n	m	f	1_{L_8}
a	a	a	a	a	a	1_{L_8}	n	m	f	1_{L_8}
e	e	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}
n	n	n	n	n	n	1_{L_8}	n	1_{L_8}	1_{L_8}	1_{L_8}
m	m	m	m	m	m	1_{L_8}	1_{L_8}	m	1_{L_8}	1_{L_8}
f	f	f	f	f	f	1_{L_8}	1_{L_8}	1_{L_8}	f	1_{L_8}
1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}	1_{L_8}

W and W_S on $[a, 1]$ and $[0, a]$, respectively. One can wonder if the t-conorms S and S^* can coincide on any bounded lattice. To illustrate this question we shall give the following example.

Example 3.20. *Consider the lattice $(L_9 = \{0_{L_9}, t, k, a, s, p, 1_{L_9}\}, \leq, 0_{L_9}, 1_{L_9})$ which is depicted by Hasse diagram in Figure 9. Consider the t-conorm W on $[a, 1_{L_9}]$ $W(x, y) = x \vee y$ and the t-conorm W_S on $[0_{L_9}, a]$ defined as follows:*

$$W(x,y) = \begin{cases} x \vee y & \text{if } 0_{L_9} \in \{x, y\} \\ a & \text{otherwise.} \end{cases},$$

By using Theorem 3.14 and Theorem 2.2 define the corresponding t-conorms S and $S^\star$ as given in Table 9.

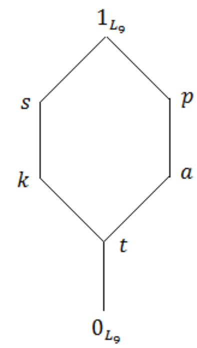

Figure 9: The lattice L_9

Table 9: The t-norm $S = S^*$ on L_9

$S = S^*$	0_{L_9}	t	k	a	s	p	1_{L_9}
0_{L_9}	0_{L_9}	t	k	a	s	p	1_{L_9}
t	t	a	1_{L_9}	a	1_{L_9}	p	1_{L_9}
k	k	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}
a	a	a	1_{L_9}	a	1_{L_9}	p	1_{L_9}
s	s	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}
p	p	p	1_{L_9}	p	1_{L_9}	p	1_{L_9}
1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}	1_{L_9}

According to Table 9, it is clear that the t-conorm S coincides with the t-conorm S^* on the bounded lattice L_9.

Remark 3.21. *Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L \setminus \{0, 1\}$ such that for all $x \in I_a$ and $y \in [a, 1)$ it holds $x \parallel y$. Suppose S and $S^\sim$ are the t-conorms on L defined as in Theorem 3.14 and Theorem 2.3 with underlying t-conorms W on $[a, 1]$. One can wonder if the t-conorms S and $S^\sim$ can coincide on any bounded lattice. To illustrate this question we shall give the following example.*

Example 3.22. *Consider the lattice $(L_{10} = \{0_{10}, a, q, m, n, 1_{10}\}, \leq, 0_{10}, 1_{10})$ which is depicted by Hasse diagram in Figure 10. Consider the t-conorm W on $[a, 1_{10}]$, $W(x, y) = x \vee y$.*

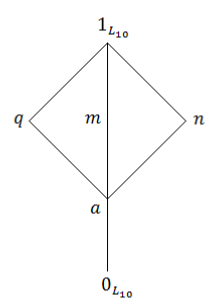

Figure 10: The lattice L_{10}

By using Theorem 3.14 and Theorem 2.3 define the corresponding t-conorms S and $S^\sim$ as given in Table 10.
According to Table 10, it is clear that the t-conorm S coincides with the t-conorm S^* on the bounded lattice L_{10}.

Table 10: The t-norm $S = S^{\sim}$ on L_{10}

$S = S^{\sim}$	0_{10}	a	q	m	n	1_{10}
0_{10}	0_{10}	a	q	m	n	1_{10}
a	a	a	q	m	n	1_{10}
q	q	q	q	1_{10}	1_{10}	1_{10}
m	m	m	1_{10}	m	1_{10}	1_{10}
n	n	n	1_{10}	1_{10}	n	1_{10}
1_{10}	1_{10}	1_{10}	1_{10}	1_{10}	1_{10}	1_{10}

4 Modified ordinal sum constructions of t-norms and t-conorms on appropriate bounded lattices

From [12], we know that new t-norms and t-conorms on bounded lattices can be obtained using recursion. In this section, based on the approaches of constructing t-norms and t-conorms proposed in Section 3, we introduce a new ordinal sum constructions of t-norms and t-conorms on an arbitrary bounded lattice L using recursion.

Theorem 4.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \cdots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$. Let $x \parallel y$ for all $x \in I_{a_i}$ and $y \in (0, a_i]$ and let $V : [0, a_1]^2 \to [0, a_1]$ be a t-norm. Then, the function $T_n : L^2 \to L$ defined recursively as follows is a t-norm, where $T_1 = V$ and for $i \in \{2, \cdots, n\}$, the function $T_i : [0, a_i]^2 \to [0, a_i]$ is given by*

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [0,a_{i-1}]^2 , \\ 0 & \text{if } (x,y) \in [0,a_{i-1}) \times I_{a_{i-1}} \cup I_{a_{i-1}} \times [0,a_{i-1}) \\ & \cup I_{a_{i-1}} \times I_{a_{i-1}} \cup [a_{i-1},a_i) \times I_{a_{i-1}} \cup I_{a_{i-1}} \times [a_{i-1},a_i) \\ a_{i-1} & \text{if } (x,y) \in [a_{i-1},a_i)^2 \\ x \wedge y & \text{otherwise.} \end{cases} \tag{3}$$

Proof. The proof follows easily from Theorem 3.5 by induction and therefore it is omitted. □

Example 4.2. *Consider the lattice $(L_{11} = \{0_{L_{11}}, a_1, a_2, d, a_3, c, e, a_4, 1_{L_{11}}\}, \leq, 0_{L_{11}}, 1_{L_{11}})$ described in Figure 11 with the finite chain $0_{L_{11}} < a_1 < a_2 < a_3 < a_4 < 1_{L_{11}}$ in L_{11} and define the t-norm $V : [0_{L_{11}}, a_1]^2 \to [0_{L_{11}}, a_1]$ by $V = T_\wedge$. By using Theorem 4.1, where $V = T_1$, t-norms $T_2 : [0_{L_{11}}, a_2]^2 \to [0_{L_{11}}, a_2]$, $T_3 : [0_{L_{11}}, a_3]^2 \to [0_{L_{11}}, a_3]$, $T_4 : [0_{L_{11}}, a_4]^2 \to [0_{L_{11}}, a_4]$, $T_5 : {L_{11}}^2 \to L_{11}$ are defined in Tables 11-14.*

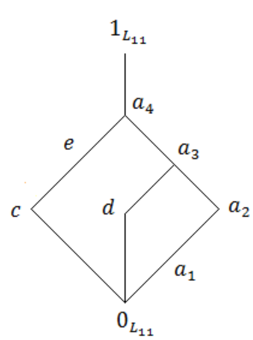

Figure 11: The lattice L_{11}

Remark 4.3. *By using the construction method in Theorem 3.5 taking $V = T : [0_{L_{11}}, a_4]^2 \to [0_{L_{11}}, a_4]$ $V = T_\wedge$, we define the t-norm T on L described in Figure 11 by Table 15. It is clear that T_5 is different from T. Because $T_5(c, e) = 0_{L_{11}} \neq c = T(c, e)$.*

Table 11: The t-norm T_2 on L_{11}

T_2	$0_{L_{11}}$	a_1	a_2
$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$
a_1	$0_{L_{11}}$	a_1	a_1
a_2	$0_{L_{11}}$	a_1	a_2

Table 12: The t-norm T_3 on L_{11}

T_3	$0_{L_{11}}$	a_1	a_2	d	a_3
$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$
a_1	$0_{L_{11}}$	a_1	a_1	$0_{L_{11}}$	a_1
a_2	$0_{L_{11}}$	a_1	a_2	$0_{L_{11}}$	a_2
d	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	d
a_3	$0_{L_{11}}$	a_1	a_2	d	a_3

Table 13: The t-norm T_4 on L_{11}

T_4	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4
$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$
a_1	$0_{L_{11}}$	a_1	a_1	$0_{L_{11}}$	a_1	$0_{L_{11}}$	$0_{L_{11}}$	a_1
a_2	$0_{L_{11}}$	a_1	a_2	$0_{L_{11}}$	a_2	$0_{L_{11}}$	$0_{L_{11}}$	a_2
d	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	d	$0_{L_{11}}$	$0_{L_{11}}$	d
a_3	$0_{L_{11}}$	a_1	a_2	d	a_3	$0_{L_{11}}$	$0_{L_{11}}$	a_3
c	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	c
e	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	e
a_4	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4

Table 14: The t-norm T_5 on L_{11}

T_5	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4	$1_{L_{11}}$
$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$
a_1	$0_{L_{11}}$	a_1	a_1	$0_{L_{11}}$	a_1	$0_{L_{11}}$	$0_{L_{11}}$	a_1	a_1
a_2	$0_{L_{11}}$	a_1	a_2	$0_{L_{11}}$	a_2	$0_{L_{11}}$	$0_{L_{11}}$	a_2	a_2
d	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	d	$0_{L_{11}}$	$0_{L_{11}}$	d	d
a_3	$0_{L_{11}}$	a_1	a_2	d	a_3	$0_{L_{11}}$	$0_{L_{11}}$	a_3	a_3
c	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	c	c
e	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	e	e
a_4	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4	a_4
$1_{L_{11}}$	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4	$1_{L_{11}}$

Theorem 4.4. *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, ..., a_n\}$ be a finite chain in L such that $1 = a_0 > a_1 > a_2 > ... > a_n = 0$. Let $x \parallel y$ for all $x \in I_{a_i}$ and $y \in [a_i, 1)$ and let $W : [a_1, 1]^2 \to [a_1, 1]$ be a t-conorm. Then, the function $S_n : L^2 \to L$ defined recursively as follows is a t-conorm, where $S_1 = W$ and for $i \in \{2, ..., n\}$, the function $S_i : [a_i, 1]^2 \to [a_i, 1]$ is given by*

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1},1]^2 , \\ 1 & \text{if } (x,y) \in (a_{i-1},1] \times I_{a_{i-1}} \cup I_{a_{i-1}} \times (a_{i-1},1] \\ & \cup I_{a_{i-1}} \times I_{a_{i-1}} \cup (a_i,a_{i-1}] \times I_{a_{i-1}} \cup I_{a_{i-1}} \times (a_i,a_{i-1}] \\ a_{i-1} & \text{if } (x,y) \in (a_{i-1},a_i]^2 \\ x \wedge y & \text{otherwise.} \end{cases} \quad (4)$$

Example 4.5. *Consider the lattice $(L_{12} = \{0_{L_{12}}, a_4, p, t, a_3, a_2, k, a_1, 1_{L_{12}}\}, \leq, 0_{L_{12}}, 1_{L_{12}})$ described in Figure 12 with*

Table 15: The t-norm T on L_{11}

T	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4	$1_{L_{11}}$
$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$
a_1	$0_{L_{11}}$	a_1	a_1	$0_{L_{11}}$	a_1	$0_{L_{11}}$	$0_{L_{11}}$	a_1	a_1
a_2	$0_{L_{11}}$	a_1	a_2	$0_{L_{11}}$	a_2	$0_{L_{11}}$	$0_{L_{11}}$	a_2	a_2
d	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	d	d	$0_{L_{11}}$	$0_{L_{11}}$	d	d
a_3	$0_{L_{11}}$	a_1	a_2	d	a_3	$0_{L_{11}}$	$0_{L_{11}}$	a_3	a_3
c	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	c	c	c	c
e	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	$0_{L_{11}}$	c	e	e	e
a_4	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4	a_4
$1_{L_{11}}$	$0_{L_{11}}$	a_1	a_2	d	a_3	c	e	a_4	$1_{L_{11}}$

the finite chain $1_{L_{12}} > a_1 > a_2 > a_3 > a_4 > 0_{L_{12}}$ *in* L_{12} *and define the t-conorm* $W : [a_1, 1_{L_{12}}]^2 \to [a_1, 1_{L_{12}}]$ *by* $W = S_\vee$. *By using Theorem 4.4, where* $W = S_1$, *t-conorms* $S_2 : [a_2, 1_{L_{12}}]^2 \to [a_2, 1_{L_{12}}]$, $S_3 : [a_3, 1_{L_{12}}]^2 \to [a_3, 1_{L_{12}}]$, $S_4 : [a_4, 1_{L_{12}}]^2 \to [a_4, 1_{L_{12}}]$, $S_5 : L_{12}^2 \to L_{12}$ *are defined in Tables 16-19.*

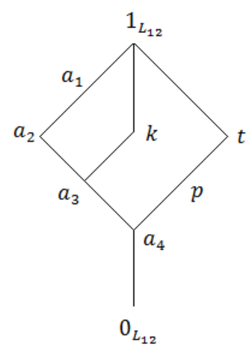

Figure 12: The lattice L_{12}

Table 16: The t-conorm S_2 on L_{12}

S_2	a_2	a_1	$1_{L_{12}}$
a_2	a_2	a_1	$1_{L_{12}}$
a_1	a_1	a_1	$1_{L_{12}}$
$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$

Table 17: The t-conorm S_3 on L_{12}

S_3	a_3	a_2	k	a_1	$1_{L_{12}}$
a_3	a_3	a_2	k	a_1	$1_{L_{12}}$
a_2	a_2	a_2	$1_{L_{12}}$	a_1	$1_{L_{12}}$
k	k	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
a_1	a_1	a_1	$1_{L_{12}}$	a_1	$1_{L_{12}}$
$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$

Remark 4.6. *By using the construction method in Theorem 3.14 taking* $W = S : [a_4, 1_{L_{12}}]^2 \to [a_4, 1_{L_{12}}]$ $W = S_\vee$, *we define the t-norm* S *on* L_{12} *described in Figure 12 by Table 20. It is clear that* S_5 *is different from* $S^\sim$. *Because* $S_5(p, t) = 1_{L_{12}} \neq t = S(p, t)$.

Table 18: The t-conorm S_4 on L_{12}

S_4	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
a_4	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
p	p	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
t	t	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
a_3	a_3	$1_{L_{12}}$	$1_{L_{12}}$	a_3	a_2	k	a_1	$1_{L_{12}}$
a_2	a_2	$1_{L_{12}}$	$1_{L_{12}}$	a_2	a_2	$1_{L_{12}}$	a_1	$1_{L_{12}}$
k	k	$1_{L_{12}}$	$1_{L_{12}}$	k	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
a_1	a_1	$1_{L_{12}}$	$1_{L_{12}}$	a_1	a_1	$1_{L_{12}}$	a_1	$1_{L_{12}}$
$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$

Table 19: The t-conorm S_5 on L_{12}

S_5	$0_{L_{12}}$	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
$0_{L_{12}}$	$0_{L_{12}}$	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
a_4	a_4	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
p	p	p	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
t	t	t	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
a_3	a_3	a_3	$1_{L_{12}}$	$1_{L_{12}}$	a_3	a_2	k	a_1	$1_{L_{12}}$
a_2	a_2	a_2	$1_{L_{12}}$	$1_{L_{12}}$	a_2	a_2	$1_{L_{12}}$	a_1	$1_{L_{12}}$
k	k	k	$1_{L_{12}}$	$1_{L_{12}}$	k	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
a_1	a_1	a_1	$1_{L_{12}}$	$1_{L_{12}}$	a_1	a_1	$1_{L_{12}}$	a_1	$1_{L_{12}}$
$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$

Table 20: The t-conorm S_6 on L_{12}

S	$0_{L_{12}}$	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
$0_{L_{12}}$	$0_{L_{12}}$	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
a_4	a_4	a_4	p	t	a_3	a_2	k	a_1	$1_{L_{12}}$
p	p	p	p	t	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
t	t	t	t	t	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$
a_3	a_3	a_3	$1_{L_{12}}$	$1_{L_{12}}$	a_3	a_2	k	a_1	$1_{L_{12}}$
a_2	a_2	a_2	$1_{L_{12}}$	$1_{L_{12}}$	a_2	a_2	$1_{L_{12}}$	a_1	$1_{L_{12}}$
k	k	k	$1_{L_{12}}$	$1_{L_{12}}$	k	$1_{L_{12}}$	k	$1_{L_{12}}$	$1_{L_{12}}$
a_1	a_1	a_1	$1_{L_{12}}$	$1_{L_{12}}$	a_1	a_1	$1_{L_{12}}$	a_1	$1_{L_{12}}$
$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$	$1_{L_{12}}$

5 Concluding remarks

Recently, the topic related to the construction of t-norms on bounded lattices by means of ordinal sums has been extensively studied. As dual operations of t-norms, t-conorms have been investigated simultaneously. This paper has further constructed the t-norms and t-conorms on bounded lattices from a mathematical viewpoint. In this aim, we have investigated and introduced new construction methods for building t-norms and t-conorms on appropriate bounded lattices with a constraint concerning the specific splitting point a. Based on this ordinal sum method, we have introduced a new class of t-norms T and t-conorms S on an arbitrary bounded lattice, respectively, by using the existence of a t-norm V on a sublattice $[0, a]$ and a t-conorm W on a sublattice $[a, 1]$ in Theorem 3.5 and Theorem 3.14. In order to well understand the constructed t-norm T and t-conorm S, we have given some illustrative examples. Also, we have investigated the relation between introduced methods and some other approaches. Finally, we have shown that the new construction methods can be generalized by induction to a modified ordinal sum for t-norms and t-conorms on arbitrary bounded lattice, respectively.

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Construction methods for triangular norms and triangular conorms on appropriate bounded lattices

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روش‌های ساخت نرم‌ها و هم‌نرم‌های مثلثی روی شبکه‌های کراندار مناسب

چکیده. در این مطالعه، روش‌های جدیدی جهت ساخت نرم‌ها و هم‌نرم‌های مثلثی روی شبکه‌های کراندار مناسب معرفی شده‌اند. برای درک بهتر مطالب، مثال‌های گویایی ارائه شده‌است همچنین، رابطه بین روش‌های معرفی شده و برخی دیگر از روش‌ها بررسی شده‌است. سرانجام، نشان داده شده‌است که روش‌های ساختاری معرفی شده می‌توانند به استقراء به یک حاصل‌جمع ترتیبی اصلاح شده برای نرم‌ها و هم‌نرم‌های مثلثی روی شبکه‌های کراندار مناسب تعمیم داده شوند. چند مثال گویا در این زمینه بیان شده‌است.