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New results on the migrativity properties for overlap (grouping) functions and uninorms

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Abstract

Functional equations involving aggregation functions play an important role in fuzzy sets and fuzzy logic theory. That the migrativity equation as a kind of restricted general associative equation have been proven to be useful in a wide range of fields like decision making, aggregation of information, image processing and so on. In the literature, the already existing results concerning the migrativity equation between overlap (grouping) functions and uninorms are based on the assumption that uninorms belong to one of the most studied classes. In this study we will explore it involving uninorms in a more general setting. To be specific, we investigate the migrativity properties between overlap (grouping) functions and uninorms in the case when uninorms have continuous underlying operators. We will show along the paper that many new solutions to the equation are characterized from this new point of view.

Keywords: Migrativity equation, uninorms, overlap functions, grouping functions.

1 Introduction

In many applications like image processing or decision making, a usual task is the assignment of an object to one of the possible classes that make up the problem. For this purpose, we start from a membership degree (preference or support degree) of the object to each one of the classes. The representation and aggregation of these preferences has been usually used in the literature by taking advantage of t-norms and t-conorms. But nonetheless, as a matter of fact, the associativity of t-norms and t-conorms is not forcefully needed in many situations, for example when the problem consists of only two classes. From this argument Bustince et al. introduced overlap functions [3] and grouping functions [5]. Thereafter, these two functions have been widely used to measure the degree in which an object is simultaneously supported by both classes (overlap), and the degree in which the object is supported by the combination of the two classes (grouping).

The categories of overlap and grouping functions are richer than the classes of t-norms and t-conorms in many aspects. For instance, there is only one idempotent t-norm and t-conorm respectively. On the contrary, there exist uncountable number of idempotent overlap functions and grouping functions. Therefore, it is not surprising that these two kinds of functions have attracted the efforts of many researchers not only from the point of view of their applications, but also from the purely theoretical perspective as well. See, for example, the reference [13] focused on their application to image thresholding and [7] devoted to investigating the constructions of overlap functions by means of additive generators.

Analyzing some interesting properties of overlap and grouping functions is one of the most studied topics. In almost all the cases, the interest of each property originates from its specific applications in corresponding fields, and its study is usually reduced to the solution of a functional equation. Thus, properties like homogeneity, idempotency, existence of generators, distributive law, and so on have been paid extensive and close attention, see [3], [5], [6] and the references therein. There are also a lot of additional properties that have been studied in detail by other authors, and one of the most extensive lists of such properties can be found in [1].

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Among these additional properties, α-migrativity pointed by Durante and Sarkoci [8] has attracted the interest of many researchers. This property has several important applications, especially in image processing, decision making processes and aggregation of information. Specifically:

(i) In image processing, this property expresses the invariance of a given property under a proportional rescaling of some part of the image [20].

(ii) In decision making processes, when a repeated, partial information from different times, places or circumstances need to be amalgamated into a global summary [4].

(iii) In aggregation of information, it has proved to be particularly interesting whenever one has to aggregate partial information coming from sources with meaningful differences (information about recent events or places close to one another should in general not be treated similarly as information about events at distant moments in time or at remote locations). See [2] for more details.

Given $\alpha \in [0,1]$ and a mapping $F : [0,1]^2 \to [0,1]$, this property is described as

$$F(\alpha x, y) = F(x, \alpha y) \qquad \text{for all } x, y \in [0,1]. \tag{1}$$

It has been studied for t-norms, t-subnorms, copulas, semicopulas and so on. Note that in (1) the product αx can be replaced by any t-norm T_0, obtaining in this way the property for t-norms called (α, T_0)-migrativity, that can be written as

$$T(T_0(\alpha, x), y) = T(x, T_0(\alpha, y)) \qquad \text{for all } x, y \in [0,1]. \tag{2}$$

being T_0 a t-norm and $\alpha \in [0,1]$. One of the most influential and impressive results involving this generalization is probably given out by Fodor et al [9] and [10]. By dualization, a similar definition can be given for t-conorms as it was pointed out in [17]. Moreover, this study has been extended to uninorms ((α, U_0)-migrativity) with the same neutral element in [18]. In recent years, Qin and Su [24, 26, 27, 28] presented α-migrative uninorms over a fixed uninorm with different neutral elements and characterized all solutions of the migrativity equation for some possible combinations.

On the one hand, we have stated above that overlap and grouping functions, as two new special binary aggregation functions, have had a fast development latterly. In particular, there are many discussions focusing on some related properties for overlap and grouping functions such as homogeneity, existence of generators and so on from the theoretical point of view and those two binary aggregation functions can be regarded, respectively. On the other hand, it has been stated that migrativity among some peculiar classes of binary aggregation functions, as a meaningful and hot research area, has many important applications and has been continuously studied in many recent literature. However, in all previous studies, the migrativity between overlap (grouping) functions and uninorms has been developed only in the case when uninorms belong to one of the four most known classes [23], [31] and [32]. Therefore, as a supplement of this topic from the theoretical point of view, in this paper, firstly, we investigate the α-migritivity of uninorms with continuous underlying operators over overlap and grouping functions and call them (α, O)-migrativity and (α, G)-migrativity uninorms, respectively. Precisely, for all $x, y \in [0,1]$ and $\alpha \in [0,1]$, we deal with the following two equations

$$U(O(\alpha, x), y) = U(x, O(\alpha, y)), \tag{3}$$

and

$$U(G(\alpha, x), y) = U(x, G(\alpha, y)), \tag{4}$$

where U is a uninorm with continuous underlying operators, O is an overlap function and G is a grouping function.

In addition, analogous to the case for (α, O)-migrativity and (α, G)-migrativity of uninorms, we study the migrativity of overlap and grouping functions over uninorms with continuous underlying operators and call them (α, U)-migrativity for overlap and grouping functions, respectively. Precisely, for all $x, y \in [0,1]$ and $\alpha \in [0,1]$, we deal with the following two equations

$$O(U(\alpha, x), y) = O(x, U(\alpha, y)), \tag{5}$$

and

$$G(U(\alpha, x), y) = G(x, U(\alpha, y)), \tag{6}$$

where O is an overlap function, G is a grouping function and U is a uninorm with continuous underlying operators.

In order to explain the results obtained concerning Eq.(3) and Eq.(4) more clearly, we give a brief comparison with other results which were correspondingly discussed in [23]. Then, we will see that many new solutions appear from this new point of view that were not included in the previous approaches. Moreover, the results obtained involving Eq.(5) and Eq.(6) also generalized the corresponding ones in [32] and [31] under certain conditions. In this sense, we think that this work improves the already known studies with the help of these new solutions.

This article is organized as follows. In Section 2, we will establish the necessary notations and recall some basic definitions and properties considered later, specially on uninorms with continuous underlying operators. In Section 3 and 4, we review some fundamental facts concerning the migrativity, and then devote to studying the migrativity equation between overlap (grouping) functions and uninorms with continuous underlying operators. In Section 5, we draw a conclusion and make some advices for future works.

2 Preliminaries

We assume that the readers are familiar with the basic facts concerning t-norms and t-conorms. The fundamental definitions and results involving them can be found in [11] and [14]. Therefore, we just recall some facts about overlap functions, grouping functions and uninorms.

Definition 2.1. *A binary function $O\colon [0,1]^2 \to [0,1]$ is said to be an overlap function if it satisfies the following conditions:*
(i) O is commutative.
(ii) $O(x,y)=0$ iff $xy=0$.
(iii) $O(x,y)=1$ iff $xy=1$.
(iv) O is increasing.
(v) O is continuous.

There exist many examples of overlap functions, such as any continuous positive t-norm. Moreover, $O(x,y)=x^2y^2$, or, more generally, $O(x,y)=x^py^p$ with $p>1$ is an example of overlap functions that is neither associative nor has 1 as its neutral element. It should be noted that if an overlap function has neutral element, then this element is necessarily equal to 1.

Definition 2.2. *A binary function $G\colon [0,1]^2 \to [0,1]$ is said to be a grouping function if it satisfies the following conditions:*
(i) G is commutative.
(ii) $G(x,y)=0$ iff $x=y=0$.
(iii) $G(x,y)=1$ iff $x=1$ or $y=1$.
(iv) G is increasing.
(v) G is continuous.

As for a given overlap function, we can obtain a grouping function by using N-dual. Thus, it is obvious that any continuous positive t-conorm is a grouping function. Moreover, given an overlap function $O(x,y)=x^py^p$ $(p>1)$, the function $G(x,y)=O_N(x,y)=1-(1-x)^p(1-y)^p$ is a grouping function that is neither associative nor has 0 as neutral element. It should also be noted that if a grouping function has neutral element, then this element is necessarily equal to 0. Now we introduce the basic facts concerning uninorms and take a look at the recent overview of the known classes. Please see [19] for more details.

Definition 2.3. [11, 30] *A mapping $U\colon [0,1]^2 \to [0,1]$ is called* uninorm *if it is commutative, associative, non-decreasing and there exists $e\in[0,1]$ called a* neutral element *such that $U(x,e)=x$ for all $x\in[0,1]$.*

A uninorm with neutral element $e=1$ is a t-norm and a uninorm with neutral element $e=0$ is a t-conorm. For any $e\in(0,1)$ the operation works as a t-norm in the square $[0,e]^2$, and as a t-conorm in $[e,1]^2$, and its values are between minimum and maximum in the set of points $A(e)$ given by $A(e)=[0,e)\times(e,1]\cup(e,1]\times[0,e)$.

We will denote a uninorm with neutral element e and underlying t-norm T and underlying t-conorm S by $U=\langle T,e,S\rangle$. And the set of all uninorms with continuous underlying operators is denoted by $\mathcal{COU}$. Any uninorm U satisfies that $U(0,1)=U(1,0)\in\{0,1\}$ and it is called *conjunctive* when $U(0,1)=0$ and *disjunctive* when $U(0,1)=1$. A uninorm with neutral element e is *proper* if $0<e<1$. Throughout this paper, we exclusively consider uninorm with neutral element e strictly between 0 and 1. The most studied classes of uninorms with continuous underlying operators are:

- Representable uninorms, those that have additive generators. They were firstly introduced in [11] and then they were characterized as those uninorms that are continuous in $[0,1]\setminus\{(0,1),(1,0)\}$ in [25].

- Uninorms continuous in $(0,1)^2$, which have two possible structures denoted by $U \equiv \langle T_1, \lambda, T_2, \mu, (R,e)\rangle_{cos,min}$ or $U \equiv \langle S_1, \nu, S_2, \omega, (R,e)\rangle_{cos,max}$, that were characterized in [12]. It is clear that this kind of uninorms includes representable uninorms.
- Idempotent uninorms, those such that $U(x,x) = x$ for all $x \in [0,1]$. Their characterization was given in [16].

Definition 2.4. [11] *Consider $e \in (0,1)$. A binary operation $U : [0,1]^2 \to [0,1]$ is a* representable uninorm *if there exist a continuous strictly increasing function $h : [0,1] \to [-\infty,+\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that*

$$U(x,y) = h^{-1}(h(x) + h(y)) \tag{7}$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and $U(0,1) = U(1,0) \in \{0,1\}$. The function h is usually called an additive generator *of U.*

Note that for any representable uninorm, the underlying operators T and S must be strict. In what follows we recall several properties of uninorms with continuous underlying operators that will be cited in sequential sections.

Lemma 2.5. (Lemma 2 in [21]) *Let $U \in \mathcal{COU}$. If $a \in [0,1]$ is an idempotent element of U, then $U(a,x) \in \{a,x\}$ for all $x \in [0,1]$.*

Definition 2.6. [21] *For $U \in \mathcal{COU}$ and each $x \in [0,1]$, we define a function $u_x : [0,1] \to [0,1]$ by $u_x(z) = U(x,z)$ for any $z \in [0,1]$.*

Lemma 2.7. (Lemma 3 in [21]) *Let $U \in \mathcal{COU}$ and $x \in [0,1]$. The function $U(x,\cdot)$ is continuous if and only if one of the following conditions is satisfied:*
(i) $U(x,1) < e$.
(ii) $U(x,0) > e$.
(iii) $e \in Ran(U(x,\cdot))$.

Lemma 2.8. (Lemma 3.11 and Corollary 3.12 in [22]) *Let $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$. Then the following statements are true.*
(i) If $U(x_0,y_0) = x_0$ for some $(x_0,y_0) \in [0,e) \times (e,1]$, then $U(x,y) = x$ for all $(x,y) \in [0,x_0] \times [e,y_0]$.
(ii) If $U(x_0,y_0) = y_0$ for some $(x_0,y_0) \in [0,e) \times (e,1]$, then $U(x,y) = y$ for all $(x,y) \in [x_0,e] \times [y_0,1]$.

For the sake of convenience, in this paper, for $x \in [0,1]$ and nonnegative integer n, we define the following symbol:

$$x_U^{(n)} = \begin{cases} e & \text{if } n = 0, \\ x & \text{if } n = 1, \\ U(x, x_U^{(n-1)}) & \text{if } n > 1. \end{cases}$$

Lemma 2.9. (Lemma 3.1 in [15]) *Let $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$. Then there exists some $(x_0,y_0) \in A(e), x_0 < y_0$ such that $U(x_0,y_0) = e$ if and only if U restricted on region $[a',b']^2$ is a representable uninorm R, i.e., $U(x,y) = a' + (b'-a')R(\frac{x-a'}{b'-a'}, \frac{y-a'}{b'-a'})$ for all $(x,y) \in [a',b']^2$, where $a' = \lim_{n\to\infty}(x_0)_U^{(n)}$ and $b' = \lim_{n\to\infty}(y_0)_U^{(n)}$.*

3 New results on the migrativity properties for $U \in \mathcal{COU}$ over overlap and grouping functions

The α-migrativity of uninorms over overlap functions and grouping functions has been investigated in [23]. However, authors considered the case that uninorms belong to one of the most studied classes. Therefore, as a theoretical continuation and development of this topic, in this section, we will deal with a more general case—U have just continuous underlying operators. And then, we will show that many new solutions appear from this new perspective.

3.1 Migrativity for $U \in \mathcal{COU}$ over overlap functions

To obtain the main results in this subsection, firstly, we need to review the migrative property involving overlap functions in the most general setting. Now, let us start from the definition of this property.

Definition 3.1. (Definition 3.1 in [23]) *Consider $\alpha \in [0,1]$ and a given overlap function O. A uninorm U is said to be α-migrative over O ((α, O)-migrative, for short) if*

$$U(O(\alpha,x),y) = U(x,O(\alpha,y)) \quad \text{for all } x,y \in [0,1]. \tag{8}$$

Proposition 3.2. (Proposition 3.1 in [23]) *For a given overlap function O, a uninorms U is $(0,O)$-migrative if and only if U is conjunctive.*

Proposition 3.3. (Proposition 3.4 in [23]) *Let α in $(0,1]$, O is a given overlap function and U is a uninorm with neutral element $e \in [0,1]$. Then the following statements are equivalent:*
(i) U is (α,O)-migrative.
(ii) $O(\alpha,x) = U(O(\alpha,e),x)$ for all $x \in [0,1]$.

In Proposition 3.3 from [23], it has been proved that if U with neutral element $e \in [0,1]$ is (α,O)-migrative for $\alpha \in (0,1)$, then U must be conjunctive. As a consequence of Proposition 3.2, the following discussions, we only need to consider the case when $\alpha \in (0,1]$. We begin with the situation for $\alpha = 1$. Since in the case of $O(1,e) = e$ we obtain U is $(1,O)$-migrative if and only if $O(1,x) = x$ for all $x \in [0,1]$. Therefore, in the following we consider that $O(1,e) \neq e$.

Proposition 3.4. *Suppose that O is a given overlap function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$ such that $O(1,e) \neq e$. Then the following items are equivalent:*
(i) U is $(1,O)$-migrative.
(ii) There exist values a,b satisfying the conditions that $e \in (a,b)$, $O(1,e) \in (a,b)$, and a representable uninorm R such that $U(x,y) = a + (b-a)R(\frac{x-a}{b-a}, \frac{y-a}{b-a})$ for all $(x,y) \in [a,b]^2$ and

$$O(1,y) = \begin{cases} y & \text{if } y \in [0,a] \cup [b,1], \\ a + (b-a)R(\frac{O(1,e)-a}{b-a}, \frac{y-a}{b-a}) & \text{if } y \in [a,b]. \end{cases}$$

Proof. ($\Rightarrow$) From Proposition 3.3 it holds that $O(1,x) = U(O(1,e),x)$ for all $x \in [0,1]$. Thus, we obtain $U(O(1,e),x)$ is continuous for $x \in [0,1]$ because $O(1,x)$ is continuous for $x \in [0,1]$. Now, it is sufficient to consider two cases $O(1,e) \in (0,e)$ and $O(1,e) \in (e,1)$. We only consider the case when $O(1,e) \in (0,e)$ since the proof is similar when $O(1,e) \in (e,1)$. According to Lemma 2.7 we have $U(O(1,e),1) = O(1,e)$ or $e \in Ran(U(O(1,e),\cdot))$. Now, we claim that $U(O(1,e),1) = O(1,e)$ is impossible. Otherwise, then we obtain $O(1,1) = U(O(1,e),1) = O(1,e)$, which contradicts the fact that $O(x,y) = 1$ if and only if $x = y = 1$. Therefore, it must be $e \in Ran(U(O(1,e),\cdot))$, that is, there exists $y_1 \in (e,1)$ such that $U(O(1,e),y_1) = e$. Letting $a = \lim_{n\to\infty} O(1,e)_U^{(n)}$ and $b = \lim_{n\to\infty} (y_1)_U^{(n)}$, then according to Lemma 2.9 and U with continuous underlying operators, we obtain that both a and b are idempotent elements of U and U restricted on region $[a,b]^2$ is a representable uninorm R, that is, $U(x,y) = a + (b-a)R(\frac{x-a}{b-a}, \frac{y-a}{b-a})$ for all $(x,y) \in [a,b]^2$. Further, it holds from $U(O(1,e),a) = a$, $U(O(1,e),b) = b$ and Lemma 2.8 that $U(O(1,e),x) = x$ for $x \in [0,a] \cup [b,1]$. Therefore, by the fact that $O(1,x) = U(O(1,e),x)$ for all $x \in [0,1]$ we obtain $O(1,y) = y$ for all $y \in [0,a] \cup [b,1]$ and $O(1,y) = a + (b-a)R(\frac{O(1,e)-a}{b-a}, \frac{y-a}{b-a})$ for all $y \in [a,b]$.

$\Leftarrow$ Conversely, according to Proposition 3.3 we only need to check the fact that $O(1,x) = U(O(1,e),x)$ is valid for all $x \in [0,1]$. We have the following two cases.

If $x \in [0,a] \cup [b,1]$, then we have $x = O(1,x) = U(O(1,e),x) = x$ by using the fact that U restricted on region $[a,b]^2$ is a representable uninorm and Lemma 2.8.

If $x \in [a,b]$, then it is obvious that $O(1,x) = U(O(1,e),x)$ by $O(1,x) = a + (b-a)R(\frac{O(1,e)-a}{b-a}, \frac{x-a}{b-a})$ for all $x \in [a,b]$. □

Proposition 3.5. *Let $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$ and O is a given overlap function. If for a given α in $(0,1)$, U is (α,O)-migrative, then the following statements are true:*
(i) $O(\alpha,1) = O(\alpha,e)$.
(ii) $O(\alpha,1) < e$.

Proof. The proof is omitted because by using $U(1,x) \in \{1,x\}$ for all $x \in [0,1]$ we obtain that its proof is exactly the same to that of Proposition 3.5 in [23]. □

Theorem 3.6. *Suppose that $\alpha \in (0,1)$, O is a given overlap function and $U = \langle T,e,S \rangle \in \mathcal{COU}$ with neutral element $e \in (0,1)$. Then the following statements are true:*

(i) If $O(\alpha,1)$ is an idempotent element of U, then the following statements are equivalent:

(a) U is (α,O)-migrative;

(b) There exist a continuous t-norm T_1, a conjunctive uninorm U' such that

$$U(x,y)=\begin{cases} O(\alpha,1)T_1(\frac{x}{O(\alpha,1)},\frac{y}{O(\alpha,1)}) & \text{if } (x,y)\in[0,O(\alpha,1)]^2,\\ O(\alpha,1)+(1-O(\alpha,1))U'(\frac{x-O(\alpha,1)}{1-O(\alpha,1)},\frac{y-O(\alpha,1)}{1-O(\alpha,1)}) & \text{if } (x,y)\in[O(\alpha,1),1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{9}$$

and

$$O(\alpha,y)=\begin{cases} y & \text{if } y\in[0,O(\alpha,1)],\\ O(\alpha,1) & \text{if } y\in[O(\alpha,1),1].\end{cases} \tag{10}$$

(ii) If $O(\alpha,1)$ is not an idempotent element of U, then the following statements are equivalent:

(a) U is (α,O)-migrative;

(b) There exist values a_1,b_1 with $O(\alpha,1)\in(a_1,b_1)$, two continuous t-norms T_0, T_1, a uninorm U' such that

$$U(x,y)=\begin{cases} a_1T_0(\frac{x}{a_1},\frac{y}{a_1}) & \text{if } (x,y)\in[0,a_1]^2,\\ a_1+(b_1-a_1)T_1(\frac{x-a_1}{b_1-a_1},\frac{y-a_1}{b_1-a_1}) & \text{if } (x,y)\in[a_1,b_1]^2,\\ b_1+(1-b_1)U'(\frac{x-b_1}{1-b_1},\frac{y-b_1}{1-b_1}) & \text{if } (x,y)\in[b_1,1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases} \tag{11}$$

and

$$O(\alpha,y)=\begin{cases} y & \text{if } y\in[0,a_1],\\ a_1+(b_1-a_1)T_1(\frac{O(\alpha,1)-a_1}{b_1-a_1},\frac{y-a_1}{b_1-a_1}) & \text{if } y\in[a_1,b_1],\\ O(\alpha,1) & \text{if } y\in[b_1,1].\end{cases} \tag{12}$$

Moreover, if $a_1=0$, then T_1 is a strict t-norm.

Proof. $\Rightarrow$ From Proposition 3.5 and the monotonicity of O we obtain $O(\alpha,y)=O(\alpha,1)$ for all $y\in[e,1]$. To continue the procedure, next we consider the following two subcases: (1) $O(\alpha,1)$ is an idempotent element of U; (2) $O(\alpha,1)$ is not an idempotent element of U.

If $O(\alpha,1)$ is an idempotent element of U, then by the already result $O(\alpha,1)<e$ in Proposition 3.5 we know that there must be an ordinal sum of the form $T=(<0,\frac{O(\alpha,1)}{e},T_1>,<\frac{O(\alpha,1)}{e},1,T_2>)$, where T_1 and T_2 are continuous t-norms. Therefore, from $U(O(\alpha,1),1)=O(\alpha,1)$ and Lemma 2.8 (i) it establishes that the structure of U is given by Eq.(9). Further, according to $O(\alpha,x)=U(O(\alpha,e),x)$ for all $x\in[0,1]$ and $O(\alpha,1)$ is an idempotent element of U we obtain $O(\alpha,y)$ is represented as Eq.(10).

If $O(\alpha,1)$ is not an idempotent element of U, then we know from the already result $O(\alpha,1)<e$ in Proposition 3.5 that $\frac{O(\alpha,1)}{e}$ is also not an idempotent element of T, which implies that there must be an ordinal sum of the form $T=(\cdots,<\frac{a_1}{e},\frac{b_1}{e},T_1>,\cdots)$, where $O(\alpha,1)\in(a_1,b_1)$, T_1 is a continuous and Archimedean t-norm. Next, we prove $U(x,y)=\min\{x,y\}$ for any $(x,y)\in(a_1,b_1)\times[e,1]\cup[e,1]\times(a_1,b_1)$. Firstly, let us prove that $U(x,1)=x$ for all $x\in(O(\alpha,1),b_1)$. On the contrary, suppose that there exists some x_0 in this region such that $U(x_0,1)=1$, then we have $U(U(x_0,x_0),1)=1$. By using a regular recursion we obtain that for any nonnegative integer n it holds that $U((x_0)_U^n,1)=1$. Hence, there exists some positive integer n_0 such that $(x_0)_U^{(n_0)}=\beta<O(\alpha,1)$ and $U((x_0)_U^{(n_0)},1)=1$, which contradicts the fact that $U(O(\alpha,1),1)=O(\alpha,1)<1$. Therefore, it must be that $U(x,1)=x$ for all $x\in(O(\alpha,1),b_1)$ and then by Lemma 2.8 (i) implies that $U(x,y)=\min\{x,y\}$ for all $(x,y)\in[0,b_1)\times[e,1]\cup[e,1]\times[0,b_1)$. Thus, U is presented as Eq.(11).

Moreover, if $a_1=0$ and T_1 is nilpotent, then $\frac{O(\alpha,1)}{e}$ is a nilpotent element of T, that is , there exists some $\beta\in(0,b_1)$ such that $U(O(\alpha,1),\beta)=0$. Then from U is (α,O)-migrative we obtain $O(\alpha,\beta)=0$, but this contradicts to the fact that $O(x,y)=0$ implies $xy=0$. Therefore, if $a_1=0$, then it implies that T_1 is a strict t-norm.

Finally, from $O(\alpha,x)=U(O(\alpha,e),x)$ for all $x\in[0,1]$ one can easy obtain that $O(\alpha,y)=y$ for $y\in[0,a_1]$, $O(\alpha,y)=a_1+(b_1-a_1)T_1(\frac{O(\alpha,1)-a_1}{b_1-a_1},\frac{y-a_1}{b_1-a_1})$ for $y\in[a_1,b_1]$ and $O(\alpha,y)=O(\alpha,1)$ for all $y\in[b_1,1]$.

$\Leftarrow$ Conversely, it is routine to check that U is (α,O)-migrative. □

3.2 Migrativity for $U\in\mathcal{COU}$ over grouping functions

Next, we deal with the migrativity properties of U over a grouping function G. Since the arguments are completely similar, the results will be listed in what follows without proofs.

Definition 3.7. (Definition 5.1 in [23]) *Consider $\alpha \in [0,1]$ and a given grouping function G. A uninorm U is said to be α-migrative over G ((α, G)-migrative, for short) if*

$$U(G(\alpha,x),y) = U(x, G(\alpha,y)) \qquad \text{for all } x,y \in [0,1]. \tag{13}$$

Proposition 3.8. (Proposition 5.1 in [23]) *For a given grouping function G, a uninorm U is $(1,G)$-migrative if and only if U is disjunctive.*

Proposition 3.9. (Proposition 5.4 in [23]) *Suppose that α in $[0,1)$, G is a given grouping function and U is a uninorm with neutral element $e \in [0,1]$. Then the following statements are equivalent:*
(i) U is (α, G)-migrative.
(ii) $G(\alpha, x) = U(G(\alpha, e), x)$ for all $x \in [0,1]$.

In Proposition 3.9 from [23], it has been proved that if U with neutral element $e \in [0,1]$ is (α, O)-migrative for $\alpha \in (0,1)$, then U must be disjunctive. As a consequence of Proposition 3.8, the following discussions, we only need to consider the case when $\alpha \in [0,1)$. We begin with the situation for $\alpha = 0$. Since in the case of $G(0,e) = e$ we obtain U is $(0,G)$-migrative if and only if $G(0,x) = x$ for all $x \in [0,1]$. Therefore, the following we consider that $G(0,e) \neq e$.

Proposition 3.10. *Suppose that G is a given grouping function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$ such that $G(0,e) \neq e$. Then the following items are equivalent:*
(i) U is $(0,G)$-migrative.
(ii) There exist values a, b satisfying the conditions that $e \in (a,b)$, $G(0,e) \in (a,b)$, and a representable uninorm R such that $U(x,y) = a + (b-a)R(\frac{x-a}{b-a}, \frac{y-a}{b-a})$ for all $(x,y) \in [a,b]^2$ and

$$G(0,y) = \begin{cases} y & \text{if } y \in [0,a] \cup [b,1], \\ a + (b-a)R(\frac{G(0,e)-a}{b-a}, \frac{y-a}{b-a}) & \text{if } y \in [a,b]. \end{cases}$$

Proof. The proof is analogous to that of Proposition 3.4. □

Proposition 3.11. *Let $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$ and G is a given grouping function. If for a given α in $(0,1)$, U is (α, G)-migrative, then the following statements are true:*
(i) $G(\alpha, 0) = G(\alpha, e)$.
(ii) $G(\alpha, 0) > e$.

Proof. It can be proved in a similar way as Proposition 3.5. □

Theorem 3.12. *Suppose that $\alpha \in (0,1)$, G is a given grouping function and $U = \langle T, e, S\rangle \in \mathcal{COU}$ with neutral element $e \in (0,1)$. Then the following statements are true:*

(i) If $G(\alpha,0)$ is an idempotent element of U, then the following statements are equivalent:

(a) U is (α, G)-migrative;

(b) There exist a continuous t-conorm S_1, a disjunctive uninorm U' such that

$$U(x,y) = \begin{cases} G(\alpha,0)U'(\frac{x}{G(\alpha,0)}, \frac{y}{G(\alpha,0)}) & \text{if } (x,y) \in [0, G(\alpha,0)]^2, \\ G(\alpha,0) + (1-G(\alpha,0))S_1(\frac{x-G(\alpha,0)}{1-G(\alpha,0)}, \frac{y-G(\alpha,0)}{1-G(\alpha,0)}) & \text{if } (x,y) \in [G(\alpha,0),1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

and

$$G(\alpha,y) = \begin{cases} G(\alpha,0) & \text{if } y \in [0, G(\alpha,0)], \\ y & \text{if } y \in [G(\alpha,0),1]. \end{cases}$$

(ii) If $G(\alpha,0)$ is not an idempotent element of U, then the following statements are equivalent:

(a) U is (α, G)-migrative;

(b) There exist value a_1, b_1 with $G(\alpha,0) \in (a_1,b_1)$, two continuous t-conorms S_1, S_2, and a uninorm U' such that

$$U(x,y) = \begin{cases} a_1U'(\frac{x}{a_1}, \frac{y}{a_1}) & \text{if } (x,y) \in [0,a_1]^2, \\ a_1 + (b_1-a_1)S_1(\frac{x-a_1}{b_1-a_1}, \frac{y-a_1}{b_1-a_1}) & \text{if } (x,y) \in [a_1,b_1]^2, \\ b_1 + (1-b_1)S_2(\frac{x-b_1}{1-b_1}, \frac{y-b_1}{1-b_1}) & \text{if } (x,y) \in [b_1,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

and

$$G(\alpha,y)=\begin{cases} G(\alpha,0) & \text{if } y\in[0,a_1],\\ a_1+(b_1-a_1)S_1(\frac{G(\alpha,0)-a_1}{b_1-a_1},\frac{y-a_1}{b_1-a_1}) & \text{if } y\in[a_1,b_1],\\ y & \text{if } y\in[b_1,1].\end{cases}$$

Moreover, if $b_1=1$*, then* S_1 *is a strict t-conorm.*

Proof. By using Lemma 2.8 (ii) it can be proved in a similar way as Theorem 3.6. □

3.3 A comparison with the results of migrativity equation concerning $U\in\mathcal{COU}$ over overlap (grouping) functions

In this subsection, in order to explain the results obtained in this section more clearly, we give a brief comparison with other results concerning migrativity equation involving uninorms with continuous underlying operators over overlap (grouping) functions, which were correspondingly discussed in [23].

Proposition 3.13. (Proposition 4.11 in [23]) *Let* O *be a given overlap function and* $U=\langle(R,e),\nu,S_1,\omega,S_2\rangle_{cos,max}$ *with* $O(1,e)\geq\nu$*. Then* U *is not* $(1,O)$*-migrative.*

Proposition 3.14. (Proposition 4.14 in [23]) *Let* O *be a given overlap function and* $U=\langle T_1,\lambda,T_2,\mu,(R,e)\rangle_{cos,min}$ *with* $O(1,e)\leq\mu$*. Then* U *is not* $(1,O)$*-migrative.*

Proposition 3.15. (Proposition 4.7 in [23]) *Suppose that, for a given* $\alpha\in(0,1)$ *and an overlap function* O *and* $U=\langle h,e\rangle_{rep}$ *a conjunctive representable uninorm with neutral element* $e\in(0,1)$*. Then* U *is not* (α,O)*-migrative.*

Proposition 3.16. (Proposition 4.18 in [23]) *Suppose that* $\alpha\in(0,1)$*,* O *is a given overlap function and* $U=\langle T_1,\lambda,T_2,\mu,(R,e)\rangle_{cos,min}$ *with* $O(\alpha,1)>\lambda$*. Then* U *is not* (α,O)*-migrative.*

Proposition 3.17. (Proposition 4.19 in [23]) *Suppose that* $\alpha\in(0,1)$*,* O *is a given overlap function and* $U=\langle T_1,\lambda,T_2,\mu,(R,e)\rangle_{cos,min}$ *with* $O(\alpha,1)=\lambda$*. Then the following statements are equivalent:*
(i) U *is* (α,O)*-migrative.*
(ii) $U(\lambda,1)=\lambda$ *and* $O(\alpha,x)=\min(\lambda,x)$ *for all* $x\in[0,1]$*.*

The results of Proposition 3.13 and Proposition 3.14 are compatible with the result of Proposition 3.4. Indeed, in Proposition 3.13, if U is $(1,O)$-migrative then according to Proposition 3.4 we know the value of $O(1,e)$ is strictly bounded by 0 and ν, which contradicts the condition that $O(1,e)\geq\nu$. The analysis and comparison of Proposition 3.14 are the same. The results of Proposition 3.15 and Proposition 3.16 are consistent with the result of Theorem 3.6. Indeed, in Proposition 3.15 and Proposition 3.16, if U is (α,O)-migrative, then according to Theorem 3.6 we obtain that the value of U restricted on region $(x,y)\in[0,O(\alpha,1)]\times[e,1]$ is given by minimum, which is a contradiction. As for Proposition 3.17, obviously, it is compatible with the conclusion of Theorem 3.6 (i). On the other hand, as for $\alpha\in(0,1)$, it should be emphasized that the result of Proposition 4.10 from [23] that a conjunctive idempotent uninorm $U\equiv\langle g,e\rangle_{id}$ with neutral element $e\in(0,1)$ never α-migrative a given overlap function O is incorrect. We have the following counterexample.

Example 3.18. *Let a binary operator* $U:[0,1]^2\to[0,1]$ *is given by*

$$U(x,y)=\begin{cases}\max(x,y) & \text{if } (x,y)\in[0.5,1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases}$$

and a binary operator $O:[0,1]^2\to[0,1]$ *is given by*

$$O(x,y)=\begin{cases}0.3T_1(\frac{x}{0.3},\frac{y}{0.3}) & \text{if } (x,y)\in[0,0.3]^2,\\ 0.3+(1-0.3)T_2(\frac{x-0.3}{1-0.3},\frac{y-0.3}{1-0.3}) & \text{if } (x,y)\in[0.3,1]^2,\\ \min(x,y) & \text{otherwise,}\end{cases}$$

where T_1 *and* T_2 *are strict t-norm and continuous t-norm respectively. Clearly,* U *is an idempotent uninorm with neutral element* 0.5 *and* O *is an overlap function. Now, we claim that* U *is* $(0.3,O)$*-migrative. According to Proposition 3.3 we only need to check the fact that* $O(0.3,x)=U(O(0.3,0.5),x)$ *is established for all* $x\in[0,1]$*. Indeed, for any* $x\in[0,1]$*, we have* $\min(0.3,x)=O(0.3,x)=U(O(0.3,0.5),x)=U(0.3,x)=\min(0.3,x)$*. Therefore, the fact that* U *is* $(0.3,O)$*-migrative is true. As for the results concerning a conjunctive idempotent uninorm* $U=\langle g,e\rangle_{id}$ *with neutral element* $e\in(0,1)$ *is* (α,O)*-migrative, one can refer to Theorem 3.6.*

Proposition 3.19. (Proposition 5.19 in [23]) *Let G be a grouping function and $U = \langle (R,e), \nu, S_1, \omega, S_2\rangle_{cos,max}$ with $G(0,e) \geq \nu$. Then U is not $(0,G)$-migrative.*

Proposition 3.20. (Proposition 5.22 in [23]) *Let G be a grouping function and $U = \langle T_1, \lambda, T_2, \mu, (R,e)\rangle_{cos,min}$ with $G(0,e) \leq \mu$. Then U is not $(0,G)$-migrative.*

Proposition 3.21. (See Proposition 5.15 in [23]) *Suppose that, for a given $\alpha \in (0,1)$ and a grouping function G and $U = \langle h, e\rangle_{rep}$ a disjunctive representable uninorm with neutral element $e \in (0,1)$. Then U is not (α, G)-migrative.*

Proposition 3.22. (See Proposition 5.26 in [23]) *Suppose that $\alpha \in (0,1)$, G is a given grouping function and $U = \langle (R,e), \nu, S_1, \omega, S_2\rangle_{cos,max}$ with $G(\alpha,0) < \omega$. Then U is not a (α,G)-migrative uninorm.*

Proposition 3.23. (See Proposition 5.27 in [23]) *Suppose that $\alpha \in (0,1)$, G is a given grouping function and $U = \langle (R,e), \nu, S_1, \omega, S_2\rangle_{cos,max}$ with $G(\alpha,0) = \omega$. Then the following statements are equivalent:*
(i) U is (α,G)-migrative.
(ii) $U(\omega,0) = \omega$ and $G(\alpha,x) = \max(\omega,x)$ for all $x \in [0,1]$.

The results of Proposition 3.19 and Proposition 3.20 are compatible with the result of Proposition 3.10. Indeed, in Proposition 3.19, if U is $(0,G)$-migrative then according to Proposition 3.10 we know that the value of $G(0,e)$ is strictly bounded by 0 and ν, which contradicts the condition that $G(0,e) \geq \nu$. The analysis and comparison of Proposition 3.20 are the same. The results of Proposition 3.21 and Proposition 3.22 are consistent with the result of Theorem 3.12. Indeed, in Proposition 3.21 and Proposition 3.22, if U is (α,G)-migrative, then by Theorem 3.12 we obtain that the value of U restricted on region $(x,y) \in [0,e] \times [G(\alpha,0),1]$ is given by maximum, which is a contradiction. As for Proposition 3.23, obviously, it is compatible with the conclusion of Theorem 3.12 (i). On the other hand, as for $\alpha \in (0,1)$, it should be emphasized that the result of Proposition 5.18 from [23] that a disjunctive idempotent uninorm $U \equiv \langle g, e\rangle_{id}$ with neutral element $e \in (0,1)$ never α-migrative a given grouping function G is incorrect. We have the following counterexample.

Example 3.24. *Let a binary operator $U : [0,1]^2 \to [0,1]$ is given by*

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,0.5]^2, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

and a binary operator $G : [0,1]^2 \to [0,1]$ is given by

$$G(x,y) = \begin{cases} 0.7S_1(\frac{x}{0.3}, \frac{y}{0.3}) & \text{if } (x,y) \in [0,0.7]^2, \\ 0.7 + (1-0.7)S_2(\frac{x-0.7}{1-0.7}, \frac{y-0.7}{1-0.7}) & \text{if } (x,y) \in [0.7,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases}$$

where S_1 and S_2 are continuous t-conorm and a strict t-conorm respectively. Clearly, U is an idempotent uninorm with neutral element 0.5 and G is a grouping function. Then by using Proposition 3.9 one can easily obtain that U is $(0.7,G)$-migrative. As for the results concerning a disjunctive idempotent uninorm $U = \langle g, e\rangle_{id}$ with neutral element $e \in (0,1)$ is (α,G)-migrative, one can refer to Theorem 3.12.

4 New results on the migrativity properties for overlap and grouping functions over $U \in \mathcal{COU}$

Although the migrativity for overlap and grouping functions over U has been investigated in [31] and [32]. Unfortunately, the results are obtained under the assumptions that uninorms lie in anyone of the most studied classes [31] or $\alpha \in \{0, e, 1\}$ in [32]. Therefore, as a theoretical continuation and development of this topic, next, we will deal with the migrativity for overlap and grouping functions over $U \in \mathcal{COU}$, and we will show that many new solutions appear from this new point.

4.1 Migrativity properties for overlap functions over $U \in \mathcal{COU}$

Definition 4.1. (See Definition 3.1 in [32]) *Consider $\alpha \in [0,1]$ and a given uninorm U. An overlap function O is said to be α-migrative over U or (α,U)-migrative if*

$$O(U(\alpha,x),y) = O(x,U(\alpha,y)) \quad \text{for all } x,y \in [0,1]. \tag{14}$$

Proposition 4.2. (See Proposition 3.6 in [32]) *Suppose that O is a given overlap function and U is a uninorm with neutral element $e \in (0,1)$. If O is 0-migrative over U if and only if U is conjunctive.*

Proposition 4.3. (See Remark 3.12 in [31]) *Suppose that O is a given overlap function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$. Then O is not 1-migrative over U.*

As a consequence of Proposition 4.2 and Proposition 4.3, in the following discussions, we only need to consider the case when $\alpha \in (0,1)$ and we will develop the investigation under the condition that $U(\alpha,\cdot)$ is continuous. Meanwhile, the following discussions are sufficient to consider two cases that α is an idempotent element of U and α is not an idempotent element of U. First, let us discuss the case that α is an idempotent element of U.

Proposition 4.4. *Suppose that O is a given overlap function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$, $\alpha \in (0,e)$ is an idempotent element of U such that $U(\alpha,\cdot)$ is continuous. Then O is α-migrative over U if and only if the following statements are true:*

(i) $O(x,y) = O(\min\{x,y\},\alpha)$ for all $0 \le \min\{x,y\} \le \alpha \le \max\{x,y\} \le 1$.

(ii) $U(x,y) = \min\{x,y\}$ for all $(x,y) \in [0,\alpha]\times[\alpha,1] \cup [\alpha,1]\times[0,\alpha]$.

Proof. ($\Rightarrow$) According to Lemma 2.7 and that $U(\alpha,\cdot)$ is continuous we obtain that $U(\alpha,1)=\alpha$ or $e \in Ran(U(\alpha,\cdot))$. Indeed, from Lemma 2.5 and that α is an idempotent element of U we have $U(\alpha,x) \in \{\alpha,x\}$ that for all $x \in [0,1]$, which implies $e \in Ran(U(\alpha,\cdot))$ is impossible. Thus, only $U(\alpha,1)=\alpha$ is established. By using Lemma 2.8 we have $U(x,y)=\min\{x,y\}$ for all $(x,y) \in [0,\alpha]\times[\alpha,1] \cup [\alpha,1]\times[0,\alpha]$. Taking $x \in [0,\alpha]$ and $y=1$ in Eq.(14), it holds that $O(x,1)=O(U(\alpha,x),1)=O(x,U(\alpha,1))=O(x,\alpha)$. Therefore, $O(x,y)=O(\min\{x,y\},\alpha)$ for all $0 \le \min\{x,y\} \le \alpha \le \max\{x,y\} \le 1$ by the monotonicity and commutativity of O.

($\Leftarrow$) We need to check that the migrativity equation is valid under the conditions above. Without loss of generality, we suppose that $x \le y$.

If $x,y \in [0,\alpha]$, we have $U(\alpha,x)=x$ and $U(\alpha,y)=y$ for $x \in [0,\alpha]$, which means $O(x,U(\alpha,y))=O(y,U(\alpha,x))$.

If $x,y \in [\alpha,1]$, we have $O(x,U(\alpha,y))=O(x,\alpha)$ and $O(y,U(\alpha,x))=O(y,\alpha)$. Then from $O(x,1)=O(x,\alpha)$ for all $x \in [0,\alpha]$ we obtain $O(x,\alpha)=O(y,\alpha)$.

If $x \in [0,\alpha]$ and $y \in [\alpha,1]$, we have $O(y,U(\alpha,x))=O(x,y)$ and $O(x,U(\alpha,y))=O(x,\alpha)$. Similarly, from $O(x,1)=O(x,\alpha)$ for all $x \in [0,\alpha]$ we obtain $O(x,y)=O(x,\alpha)$ is valid. □

Proposition 4.5. *Suppose that O is a given overlap function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$, $\alpha \in (e,1)$ is an idempotent element of U such that $U(\alpha,\cdot)$ is continuous. Then O is not α-migrative over U.*

Proof. On the contrary, suppose that O is α-migrative over U, then similar to the proof of Proposition 4.4 it establishes that $U(\alpha,0)=\alpha$. Taking $x=1, y=0$ in Eq.(14) we have $O(1,U(\alpha,0))=O(0,U(\alpha,1))$, that is, $O(1,\alpha)=O(0,1)=0$. A contradiction to the fact that $O(x,y)=0$ implies $x=0$ or $y=0$. □

Now, we study the case that α is not an idempotent element of U. To do this, we need the following lemma.

Lemma 4.6. *Consider O is a given overlap function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$, $\alpha \in (e,1)$ is not an idempotent element of U such that $U(\alpha,\cdot)$ is continuous. If O is α-migrative over U, then $e \in Ran(U(\alpha,\cdot))$.*

Proof. According to Lemma 2.7 we know that $U(\alpha,0)=\alpha$ or $e \in Ran(U(\alpha,\cdot))$. Assume that $U(\alpha,0)=\alpha$, then taking $y=0$ and $x \in (0,1)$ in Eq.(14), we have $O(x,\alpha)=O(x,U(\alpha,0))=O(0,U(\alpha,x))=0$, which contradicts the fact that $O(x,y)=0$ implies that $xy=0$. □

Proposition 4.7. *Suppose that O is a given overlap function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$. If $\alpha \in (0,e)$ is not an idempotent element of U and $e \in Ran(U(\alpha,\cdot))$. Then O is α-migrative over U if and only if there exist $a,b \in [0,1]$ that are idempotent elements of U satisfying $0<a<\alpha<e<b<1$ such that the following statements are true:*

(i) $O(x,y)=O(b,b)$ for all $x,y \in [a,b]$.

(ii) $O(x,a)=O(x,b)$ for all $x \in [0,a]\cup[b,1]$.

(iii) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y) = a+(b-a)R(\frac{x-a}{b-a},\frac{y-a}{b-a})$.

Proof. ($\Rightarrow$) According to Lemma 2.9 we know that $e \in Ran(U(\alpha,\cdot))$ implies that there exist idempotent elements $a,b \in [0,1]$ satisfying $a<\alpha<e<b$ and a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y)=a+(b-a)R(\frac{x-a}{b-a},\frac{y-a}{b-a})$.

Again, setting $x \in [0,a]$ and $y \in [a,b)$ in Eq.(14), since O is α-migrative over U we have $O(y,x) = O(y,U(\alpha,x)) = O(x,U(\alpha,y)) = O(U(\alpha,x),U(\alpha,y)) = O(x,U(\alpha_U^{(2)},y))$. By induction, we get $O(y,x) = O(x,U(\lim_{n\to\infty}\alpha_U^{(n)},y))$. Then the continuity of O implies that $O(x,y) = O(x,U(a,y)) = O(x,a)$ for all $y \in [a,b)$. Thus, we obtain $O(x,a) = O(x,b)$ for all $x \in [0,a]$ as a consequence of the continuity of O. Similarly, setting $x \in [a,b)$ and $y \in [b,1]$ in Eq.(14) we obtain $O(a,y) = O(b,y)$ for all $y \in [b,1]$. Therefore, item (ii) has been proved. Further, by $O(a,a) = O(b,b)$ we know that item (i) is valid. As a result, $a > 0$ and $b < 1$.

($\Leftarrow$) Let us check the migrativity equation is valid under the conditions above. Without loss of generality, we suppose that $x \leq y$.

If $x,y \in [0,a]$, then we have $U(\alpha,x) = x$ and $U(\alpha,y) = y$, which means that $O(x,U(\alpha,y)) = O(y,U(\alpha,x))$.

If $x \in [0,a]$ and $y \in [a,b]$, we have $O(y,U(\alpha,x)) = O(y,x)$ and from $O(x,a) = O(x,b)$ for all $x \in [0,a]$ and $U(\alpha,y) \in [a,b]$ we obtain $O(x,y) = O(x,U(\alpha,y))$.

If $x \in [0,a]$ and $y \in [b,1]$, we have $O(y,U(\alpha,x)) = O(y,x)$ and $O(x,U(\alpha,y)) = O(x,y)$.

If $x,y \in [a,b]$, from $U(\alpha,x),U(\alpha,y) \in [a,b]$ it holds that $O(y,U(\alpha,x)) = O(x,U(\alpha,y)) = O(b,b)$.

If $x \in [a,b]$ and $y \in [b,1]$, we have $O(x,U(\alpha,y)) = O(x,y)$ and from $O(x,a) = O(x,b)$ for all $x \in [b,1]$ and $U(\alpha,x) \in [a,b]$ it establishes that that $O(y,U(\alpha,x)) = O(y,x)$.

If $x,y \in [b,1]$, we have $O(y,U(\alpha,x)) = O(x,U(\alpha,y)) = U(y,x)$. □

Proposition 4.8. *Suppose that O is a given overlap function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$, $\alpha \in (e,1)$ is not an idempotent element of U and $e \in Ran(U(\alpha,\cdot))$. Then O is α-migrative over U if and only if there exist $a,b \in [0,1]$ that are idempotent elements of U satisfying $0 < a < e < \alpha < b < 1$ such that the following statements are true:*

(i) $O(x,y) = O(b,b)$ for all $x,y \in [a,b]$.

(ii) $O(x,a) = O(x,b)$ for all $x \in [0,a] \cup [b,1]$.

(iii) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y) = a + (b-a)R(\frac{x-a}{b-a},\frac{y-a}{b-a})$.

Proof. Completely similar to the corresponding proof of Proposition 4.7, therefore, it is omitted. □

Remark 4.9. *According to Proposition 4.7 and Proposition 4.8 one concludes that the following statements are true, which are compatible with the results of Remark 3.23 and Remark 3.30 in [31].*

(i) Suppose that U is a representable uninorm with neutral element $e \in (0,1)$ and $\alpha \in (0,e) \cup (e,1)$, then there is no overlap function O such that O is (α,U)-migrative.

(ii) Suppose that $U = \langle T_1,\lambda,T_2,\mu,(R,e)\rangle_{cos,min}$ and $\alpha \in (\mu,1)$, then there is no overlap function O such that O is (α,U)-migrative.

(iii) Suppose that $U = \langle (R,e),\nu,S_1,\omega,S_2\rangle_{cos,max}$ and $\alpha \in (0,\nu)$, then there is no overlap function O such that O is (α,U)-migrative.

Example 4.10. *Let a binary operator $U : [0,1]^2 \to [0,1]$ is given by*

$$U(x,y) = \begin{cases} 0.2T_1(\frac{x}{0.2},\frac{x}{0.2}) & \text{if } (x,y) \in [0,0.2]^2, \\ 0.2 + 0.5R(\frac{x-0.2}{0.7-0.2},\frac{y-0.2}{0.7-0.2}) & \text{if } (x,y) \in [0.2,0.7]^2, \\ 0.7 + 0.3S_1(\frac{x-0.7}{1-0.7},\frac{y-0.7}{1-0.7}) & \text{if } (x,y) \in [0.7,1]^2, \\ \max(x,y) & \text{if } (x,y) \in [0.2,0.7) \times (0.7,1] \cup (0.7,1] \times [0.2,0.7), \\ \min(x,y) & \text{otherwise,} \end{cases}$$

where R is a representable uninorm with neutral element 0.6, T_1 is a continuous t-norm and S_1 is a continuous t-conorm.
Let a binary operator $O : [0,1]^2 \to [0,1]$ is given by

$$O(x,y) = \begin{cases} 0.2 + T(\frac{x-0.7}{1-0.7},\frac{y-0.7}{1-0.7}) & \text{if } (x,y) \in [0.7,1]^2, \\ 0.2 & \text{if } (x,y) \in [0.2,0.7] \times [0.2,1] \cup [0.2,1] \times [0.2,0.7], \\ \min(x,y) & \text{otherwise,} \end{cases}$$

where T is a continuous t-norm. Then one can easily obtain that U is a uninorm with neutral element 0.5 and O is an overlap function. Thus, by Proposition 4.7 and Proposition 4.8, we conclude that O is α-migrative over U for each fixed $\alpha \in (0.2,0.5) \cup (0.5,0.7)$.

Now, we deal with the remaining case, that is, $\alpha \in (0,e)$ is not an idempotent element of U and $U(\alpha,1) = \alpha$. This case is more complex and we will explore it with the additional condition that 1 is the neutral element of O.

Proposition 4.11. *Suppose that O is a given overlap function with neutral element 1 and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$. If $\alpha \in (0,e)$ is not an idempotent element of U with $U(\alpha,1)=\alpha$, then O is α-migrative over U if and only if the following items hold.*

(i) O is α-migrative over O.

(ii) U is given by

$$U(x,y)=\begin{cases} aT_1(\frac{x}{a},\frac{y}{a}) & \text{if } (x,y)\in[0,a]^2,\\ a+(b-a)T_2(\frac{x-a}{b-a},\frac{y-a}{b-a}) & \text{if } (x,y)\in[a,b]^2,\\ b+(1-b)U'(\frac{x-b}{1-b},\frac{y-b}{1-b}) & \text{if } (x,y)\in[b,1]^2,\\ \min\{x,y\} & \text{otherwise,}\end{cases}$$

and

$$O(\alpha,x)=\begin{cases} x & \text{if } x\in[0,a],\\ a+(b-a)T_2(\frac{\alpha-a}{b-a},\frac{x-a}{b-a}) & \text{if } x\in[a,b],\\ \alpha & \text{otherwise,}\end{cases}$$

where T_1 and T_2 are continuous t-norm and continuous Archimedean t-norm respectively, U' is a uninorm, a is the biggest idempotent element of U in $[0,\alpha)$ and b is the smallest idempotent element of U in $(\alpha,e]$.

Proof. ($\Rightarrow$) Clearly, it holds from $\alpha\in(0,e)$ is not an idempotent element of U that there exist $a=\sup\{x\in[0,\alpha)|U(x,x)=x\}$ and $b=\inf\{x\in(\alpha,e]|U(x,x)=x\}$ that are idempotent elements of U such that U restricted on region $[a,b]^2$ is a continuous Archimedean t-norm. By a completely similar proof of that in Theorem 3.6, we obtain the structure of U. Taking $y=1$ and $x\in[0,1]$ in Eq.(14), by using $U(\alpha,1)=\alpha$ and 1 is the neutral element of O, one concludes that $O(x,\alpha)=O(x,U(\alpha,1))=O(1,U(\alpha,x))=U(x,\alpha)$ for all $x\in[0,1]$. Therefore, by the already known structure of U we obtain $O(\alpha,x)=x$ for $x\in[0,a]$, $O(\alpha,x)=a+(b-a)T_2(\frac{\alpha-a}{b-a},\frac{x-a}{b-a})$ for $x\in[a,b]$ and $O(\alpha,x)=\alpha$ for $x\in[b,1]$.

Finally, we prove the fact that O is (α,O)-migrative is true. Indeed, from O is α-migrative over U and the expression of $O(\alpha,x)$ we have $O(x,O(\alpha,y))=O(x,U(\alpha,y))=O(y,U(\alpha,x))=O(y,O(\alpha,x))$. That is, O is α-migrative over O.

($\Leftarrow$) Let us check the migrativity equation is valid under the conditions above. Without loss of generality, we suppose that $x\le y$.

If $x,y\in[0,a]$, we have $U(\alpha,x)=x$ and $U(\alpha,y)=y$, which means that $O(x,U(\alpha,y))=O(y,U(\alpha,x))$.

If $x\in[0,a]$ and $y\in[a,b]$, from the structure of U we have $O(y,U(\alpha,x))=O(y,x)$. From $O(\alpha,y)=a+(b-a)T_2(\frac{\alpha-a}{b-a},\frac{y-a}{b-a})$ for $y\in[a,b]$ we have $O(x,U(\alpha,y))=O(x,a+(b-a)T_2(\frac{\alpha-a}{b-a},\frac{y-a}{b-a}))=O(x,O(\alpha,y))$. Further, the fact O is α-migrative over O implies that $O(x,U(\alpha,y))=O(x,O(\alpha,y))=O(y,O(\alpha,x))=O(y,x)$.

If $x\in[0,a]$ and $y\in[b,1]$, from the structure of U we have $O(y,U(\alpha,x))=O(y,x)=O(y,O(\alpha,x))$ and $O(x,U(\alpha,y))=O(x,\alpha)=O(x,O(\alpha,y))$. Therefore, the fact that O is α-migrative over O implies that $O(x,U(\alpha,y))=O(y,U(\alpha,x))$.

If $x,y\in[a,b]$, according to $O(\alpha,x)=a+(b-a)T_2(\frac{\alpha-a}{b-a},\frac{x-a}{b-a})$ for $x\in[a,b]$ we have $O(x,U(\alpha,y))=O(x,a+(b-a)T_2(\frac{\alpha-a}{b-a},\frac{y-a}{b-a}))=O(x,O(\alpha,y))$. Similarly, we also have $O(y,U(\alpha,x))=O(y,O(\alpha,x))$. Therefore, the result is true due to O is α-migrative over O.

If $x\in[a,b]$ and $y\in[b,1]$, by a similar discussion we have $O(x,U(\alpha,y))=O(y,U(\alpha,x))=O(x,\alpha)$.

If $x,y\in[b,1]$, then $O(x,U(\alpha,y))=O(y,U(\alpha,x))=\alpha$. □

Example 4.12. *Let a binary operator $U:[0,1]^2\to[0,1]$ is given by*

$$U(x,y)=\begin{cases} 0.5+0.5S(\frac{x-0.5}{1-0.5},\frac{y-0.5}{1-0.5}) & \text{if } (x,y)\in[0.5,1]^2,\\ 0.2+0.2T(\frac{x-0.2}{0.4-0.2},\frac{y-0.2}{0.4-0.2}) & \text{if } (x,y)\in[0.2,0.4]^2,\\ \min(x,y) & \text{otherwise,}\end{cases}$$

and a binary operator $O:[0,1]^2\to[0,1]$ is given by

$$O(x,y)=\begin{cases} 0.2+0.2T(\frac{x-0.2}{0.4-0.2},\frac{y-0.2}{0.4-0.2}) & \text{if } (x,y)\in[0.2,0.4]^2,\\ \min(x,y) & \text{otherwise,}\end{cases}$$

where T and S are strict t-norm and continuous t-conorm respectively. Obviously, U is a uninorms with neutral element 0.5 and O is an overlap function. Consider $\alpha=0.3$, clearly, we have $U(\alpha,1)=\alpha$ and O satisfies the conditions in Proposition 4.11. Thus, we obtain O is 0.3-migrative over U.

The results obtained in this subsection, in general, we observe that the condition $U(\alpha,\cdot)$ is continuous cannot be omitted. In the following we provide a corresponding example to show that O is not α-migrative over U without this condition.

Example 4.13. *Let a binary operator $U:[0,1]^2\to[0,1]$ is given by*

$$U(x,y)=\begin{cases} 0.5T(\frac{x}{0.5},\frac{y}{0.5}) & \text{if } (x,y)\in[0,0.5]^2,\\ 0.5+(0.7-0.5)S_1(\frac{x-0.5}{0.7-0.5},\frac{y-0.5}{0.7-0.5}) & \text{if } (x,y)\in[0.5,0.7]^2,\\ 0.7+(1-0.7)S_2(\frac{x-0.7}{1-0.7},\frac{y-0.7}{1-0.7}) & \text{if } (x,y)\in[0.7,1]^2,\\ \min(x,y) & \text{if } (x,y)\in[0,0.5]\times[0.5,0.7]\cup[0.5,0.7]\times[0,0.5],\\ \max(x,y) & \text{otherwise,}\end{cases}$$

and a binary operator $O:[0,1]^2\to[0,1]$ is given by $O(x,y)=xy$ for all $x,y\in[0,1]$, where T is a continuous t-norm, S_1 and S_2 are continuous t-conorms. Obviously, U is a uninorm with neutral element 0.5 and O is an overlap function. Consider $\alpha=0.3$, $x=0.6$ and $y=0.8$, then we have $U(\alpha,\cdot)$ is not continuous. By a simple computation we have $O(x,U(\alpha,y))=O(x,y)=0.48$ and $O(U(\alpha,x),y)=O(\alpha,y)=0.24$. Thus, we obtain O is not 0.3-migrative over U.

Summing up all the results above, we have the following theorem.

Theorem 4.14. *Suppose that O is a given overlap function and $U\in\mathcal{COU}$ with neutral element $e\in(0,1)$ such that $U(\alpha,\cdot)$ is continuous. Then the following statements are true:*

(i) If $\alpha\in(0,e)$ is an idempotent element of U, then O is α-migrative over U if and only if the following statements are true:

- *(a) $O(x,y)=O(\min\{x,y\},\alpha)$ for all $0\le\min\{x,y\}\le\alpha\le\max\{x,y\}\le1$.*
- *(b) $U(x,y)=\min\{x,y\}$ for all $(x,y)\in[0,\alpha]\times[\alpha,1]\cup[\alpha,1]\times[0,\alpha]$.*

(ii) If $\alpha\in(e,1)$ is an idempotent element of U, then O is not α-migrative over U.

(iii) If $\alpha\in(e,1)$ is not an idempotent element of U, then the following statements are true:

- *(a) If $U(\alpha,0)=\alpha$, then O is not α-migrative over U.*
- *(b) If $e\in Ran(U(\alpha,\cdot))$, then O is α-migrative over U if and only if there exist $a,b\in[0,1]$ that are idempotent elements of U satisfying $0<a<e<\alpha<b<1$ such that the following statements are true:*
  - *(1) $O(x,y)=O(b,b)$ for all $x,y\in[a,b]$.*
  - *(2) $O(x,a)=O(x,b)$ for all $x\in[0,a]\cup[b,1]$.*
  - *(3) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y)=a+(b-a)R(\frac{x-a}{b-a},\frac{y-a}{b-a})$.*

(iv) If $\alpha\in(0,e)$ is not an idempotent element of U, then the following statements are true:

- *(a) If $e\in Ran(U(\alpha,\cdot))$, then O is α-migrative over U if and only if there exist $a,b\in[0,1]$ that are idempotent elements of U satisfying $0<a<\alpha<e<b<1$ such that the following statements are true:*
  - *(1) $O(x,y)=O(b,b)$ for all $x,y\in[a,b]$.*
  - *(2) $O(x,a)=O(x,b)$ for all $x\in[0,a]\cup[b,1]$.*
  - *(3) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y)=a+(b-a)R(\frac{x-a}{b-a},\frac{y-a}{b-a})$.*
- *(b) If $U(\alpha,1)=\alpha$ and O has neutral element 1, then O is α-migrative over U if and only if the following items hold.*
  - *(1) O is α-migrative over O.*
  - *(2) U is given by*

$$U(x,y)=\begin{cases} aT_1(\frac{x}{a},\frac{y}{a}) & \text{if } (x,y)\in[0,a]^2,\\ a+(b-a)T_2(\frac{x-a}{b-a},\frac{y-a}{b-a}) & \text{if } (x,y)\in[a,b]^2,\\ b+(1-b)U'(\frac{x-b}{1-b},\frac{y-b}{1-b}) & \text{if } (x,y)\in[b,1]^2,\\ \min\{x,y\} & \text{otherwise,}\end{cases}$$

and

$$O(\alpha,x)=\begin{cases} x & if\ x\in[0,a],\\ a+(b-a)T_2(\frac{\alpha-a}{b-a},\frac{x-a}{b-a}) & if\ x\in[a,b],\\ \alpha & otherwise,\end{cases}$$

where T_1 and T_2 are continuous t-norm and continuous Archimedean t-norm respectively, U' is a uninorm, a is the biggest idempotent element of U in $[0,\alpha)$ and b is the smallest idempotent element of U in $(\alpha,e]$.

4.2 Migrativity for grouping functions over $U\in\mathcal{COU}$

Next, we analyse the migrativity properties of a grouping function G over U. Since the arguments are completely the same, the results will be listed without proofs.

Definition 4.15. (See Definition 3.2 in [32]) *Consider $\alpha\in[0,1]$ and a given uninorm U. A grouping function G is said to be α-migrative over U or (α,U)-migrative if*

$$G(U(\alpha,x),y)=G(x,U(\alpha,y))\qquad for\ all\ x,y\in[0,1]. \tag{15}$$

Proposition 4.16. *Suppose that G is a given grouping function and U is a uninorm with neutral element $e\in(0,1)$. Then G is 1-migrative over U if and only if U is disjunctive.*

Proof. Assume that G is $(1,U)$-migrative, then one has that $G(U(0,1),e)=G(0,U(1,e))=G(0,1)=1$. Thus, one gets that $U(0,1)=1$. Therefore, U is disjunctive.

Conversely, suppose that U is disjunctive, then one has that $G(U(1,x),y)=G(x,U(1,y))=1$. Thus, G is $(1,U)$-migrative. □

Proposition 4.17. *Suppose that G is a given grouping function and $U\in\mathcal{COU}$ with neutral element $e\in(0,1)$. Then G is not 0-migrative over U.*

Proof. Suppose that G is 0-migrative over U, then taking $x\in(0,e)$ and $y=0$ in Eq.(15), we obtain $G(x,0)=G(x,U(0,0))=G(0,U(0,x))=G(0,0)=0$, which contradicts the fact that $G(x,y)=0$ if and only if $x=y=0$. □

For the following discussions, we consider $\alpha\in(0,1)$ and we will develop the investigation under the condition $U(\alpha,\cdot)$ is continuous. Meanwhile, the following discussions it is sufficient to consider the two cases that α is an idempotent element of U and α is not an idempotent element of U. First, let us discuss the case that α is an idempotent element of U.

Proposition 4.18. *Suppose that G is a given grouping function and $U\in\mathcal{COU}$ with neutral element $e\in(0,1)$, $\alpha\in(0,e)$ is an idempotent element of U such that $U(\alpha,\cdot)$ is continuous. Then G is not α-migrative over U.*

Proof. According to Lemma 2.7 and that $U(\alpha,\cdot)$ is continuous we obtain that $U(\alpha,1)=\alpha$ or $e\in Ran(U(\alpha,\cdot))$. Indeed, from Lemma 2.5 and α that is an idempotent element of U we have that $e\in Ran(U(\alpha,\cdot))$ is impossible. Thus, only $U(\alpha,1)=\alpha$ is established. On the contrary, suppose that G is α-migrative over U, then take $x=0,y=1$ in Eq.(15). As a result, we have $G(0,U(\alpha,1))=G(1,U(\alpha,0))$, that is, $G(0,\alpha)=G(1,0)=1$. A contradiction to the fact that $G(x,y)=1$ implies $x=1$ or $y=1$. □

Proposition 4.19. *Suppose that G is a given grouping function and $U\in\mathcal{COU}$ with neutral element $e\in(0,1)$, $\alpha\in(e,1)$ is an idempotent element of U such that $U(\alpha,\cdot)$ is continuous. Then G is α-migrative over U if and only if the following statements are true:*

(i) $G(x,y)=G(\max\{x,y\},0)$ for all $0\le\min\{x,y\}\le\alpha\le\max\{x,y\}\le1$.

(ii) $U(x,y)=\max\{x,y\}$ for all $(x,y)\in[0,\alpha]\times[\alpha,1]\cup[\alpha,1]\times[0,\alpha]$.

Proof. The proof is omitted because it is completely similar to that of Proposition 4.4. □

Now, we study the case that α is not an idempotent element of U. To do this, we need the following lemma.

Lemma 4.20. *Consider G is a given grouping function and $U\in\mathcal{COU}$ with neutral element $e\in(0,1)$, $\alpha\in(0,e)$ is not an idempotent element of U such that $U(\alpha,\cdot)$ is continuous. If G is α-migrative over U, then $e\in Ran(U(\alpha,\cdot))$.*

Proof. It can be proved in a similar way to Lemma 4.6. □

Proposition 4.21. *Suppose that G is a given grouping function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$, $\alpha \in (0,e)$ is not an idempotent element of U and $e \in Ran(U(\alpha,\cdot))$. Then G is α-migrative over U if and only if there exist $a,b \in [0,1]$ that are idempotent elements of U satisfying $0 < a < \alpha < e < b < 1$ such that the following statements are true:*

(i) $G(x,y) = G(a,a)$ for all $x,y \in [a,b]$.

(ii) $G(x,a) = G(x,b)$ for all $x \in [0,a] \cup [b,1]$.

(iii) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y) = a + (b-a)R(\frac{x-a}{b-a}, \frac{y-a}{b-a})$.

Proof. It can be proved in a similar way to Proposition 4.7. □

Proposition 4.22. *Suppose that G is a given grouping function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$. If $\alpha \in (e,1)$ is not an idempotent element of U and $e \in Ran(U(\alpha,\cdot))$. Then G is α-migrative over U if and only if there exist $a,b \in [0,1]$ that are idempotent elements of U satisfying $0 < a < e < \alpha < b < 1$ such that the following statements are true:*

(i) $G(x,y) = G(a,a)$ for all $x,y \in [a,b]$.

(ii) $G(x,a) = G(x,b)$ for all $x \in [0,a] \cup [b,1]$.

(iii) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y) = a + (b-a)R(\frac{x-a}{b-a}, \frac{y-a}{b-a})$.

Proof. It can be proved in a similar way to Proposition 4.8. □

Remark 4.23. *According to Proposition 4.21 and Proposition 4.22 one concludes that the following statements are true:*

(i)For $\alpha \in (0,e)\cup(e,1)$, a given grouping function G is not α-migrative over a representable uninorm R with neutral element $e \in (0,1)$.

(ii) Suppose that $U = \langle T_1, \lambda, T_2, \mu, (R,e)\rangle_{cos,min}$ and $\alpha \in (\mu,1)$, then there is no grouping function G such that G is (α,U)-migrative.

(iii) Suppose that $U = \langle (R,e), \nu, S_1, \omega, S_2\rangle_{cos,max}$ and $\alpha \in (0,\nu)$, then there is no grouping function G such that G is (α,U)-migrative.

Now, we deal with the remaining case, that is, $\alpha \in (e,1)$ is not an idempotent element of U and $U(\alpha,0) = \alpha$. This case is more complex and we will explore it with the additional condition that 0 is the neutral element of G.

Proposition 4.24. *Suppose that G is a given grouping function with neutral element 0 and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$. If $\alpha \in (e,1)$ is not an idempotent element of U with $U(\alpha,0) = \alpha$, then G is α-migrative over U if and only if the following statements are true:*

(i) G is α-migrative over G.

(ii) U is given by

$$U(x,y) = \begin{cases} aU'(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a]^2, \\ a + (b-a)S_1(\frac{x-a}{b-a}, \frac{y-a}{b-a}) & \text{if } (x,y) \in [a,b]^2, \\ b + (1-b)S_2(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & \text{if } (x,y) \in [b,1]^2, \\ \max\{x,y\} & \text{otherwise,} \end{cases}$$

and

$$G(\alpha,x) = \begin{cases} \alpha & \text{if } x \in [0,a], \\ a + (b-a)S_1(\frac{\alpha-a}{b-a}, \frac{x-a}{b-a}) & \text{if } x \in [a,b], \\ x & \text{otherwise,} \end{cases}$$

where S_1 and S_2 are continuous Archimedean t-conorm and continuous t-conorm respectively, U' is a uninorm, a is the biggest idempotent element of U in $[e,\alpha)$ and b is the smallest idempotent element of U in $(\alpha,1]$.

Proof. It can be proved in a similar way to Proposition 4.11. □

As for Proposition 4.21, Proposition 4.22 and Proposition 4.24, completely similar examples can be listed. Meanwhile, an analogical example can be found to illustrate the condition $U(\alpha,\cdot)$ is continuous considered in this subsection is necessary. Therefore, we don't exhibit it here.

Summing up all the results above, we have the following theorem.

Theorem 4.25. *Suppose that G is a given grouping function and $U \in \mathcal{COU}$ with neutral element $e \in (0,1)$ such that $U(\alpha,\cdot)$ is continuous, then the following statements are true:*

(i) If $\alpha \in (0,e)$ is an idempotent element of U, then G is not α-migrative over U.

(ii) If $\alpha \in (e,1)$ is an idempotent element of U, then G is α-migrative over U if and only if the following statements are true:

(a) $G(x,y) = G(\max\{x,y\},0)$ for all $0 \le \min\{x,y\} \le \alpha \le \max\{x,y\}$.

(b) $U(x,y) = \max\{x,y\}$ for all $(x,y) \in [0,\alpha]\times[\alpha,1] \cup [\alpha,1]\times[0,\alpha]$.

(iii) If $\alpha \in (0,e)$ is not an idempotent element of U, then the following statements are true.

(a) If $U(\alpha,1) = \alpha$, then G is not α-migrative over U.

(b) If $e \in Ran(U(\alpha,\cdot))$, then G is α-migrative over U if and only if there exist $a,b \in [0,1]$ that are idempotent elements of U satisfying $0 < a < \alpha < e < b < 1$ such that the following statements are true.

(1) $G(x,y) = G(a,a)$ for all $x,y \in [a,b]$.

(2) $G(x,a) = G(x,b)$ for all $x \in [0,a] \cup [b,1]$.

(3) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y) = a + (b-a)R(\frac{x-a}{b-a},\frac{y-a}{b-a})$.

(iv) If $\alpha \in (e,1)$ is not an idempotent element of U, then the following statements are true.

(a) If $e \in Ran(U(\alpha,\cdot))$, then G is α-migrative over U if and only if there exist $a,b \in [0,1]$ that are idempotent elements of U satisfying $0 < a < e < \alpha < b < 1$ such that the following statements are true.

(1) $G(x,y) = G(a,a)$ for all $x,y \in [a,b]$.

(2) $G(x,a) = G(x,b)$ for all $x \in [0,a] \cup [b,1]$.

(3) There exists a representable uninorm R such that U restricted on region $[a,b]^2$ is expressed by $U(x,y) = a + (b-a)R(\frac{x-a}{b-a},\frac{y-a}{b-a})$.

(b) If $U(\alpha,0) = \alpha$ and 0 is neutral element of G. Then G is α-migrative over U if and only if the following statements are true:

(1) G is α-migrative over G.

(2) U is given by

$$U(x,y) = \begin{cases} aU'(\frac{x}{a},\frac{y}{a}) & \text{if } (x,y) \in [0,a]^2, \\ a + (b-a)S_1(\frac{x-a}{b-a},\frac{y-a}{b-a}) & \text{if } (x,y) \in [a,b]^2, \\ b + (1-b)S_2(\frac{x-b}{1-b},\frac{y-b}{1-b}) & \text{if } (x,y) \in [b,1]^2, \\ \max\{x,y\} & \text{otherwise,} \end{cases}$$

and

$$G(\alpha,x) = \begin{cases} \alpha & \text{if } x \in [0,a], \\ a + (b-a)S_1(\frac{\alpha-a}{b-a},\frac{x-a}{b-a}) & \text{if } x \in [a,b], \\ x & \text{otherwise,} \end{cases}$$

where S_1 and S_2 are continuous Archimedean t-conorm and continuous t-conorm respectively, U' is a uninorm, a is the biggest idempotent element of U in $[e,\alpha)$ and b is the smallest idempotent element of U in $(\alpha,1]$.

5 Conclusions

On the one hand, overlap and grouping functions as two special kind of aggregation functions have been investigated in many recent works for applications in image processing, classification problems and decision making. On the other hand, migrativity as an important property has been discussed for several of the most studied classes of uninorms in [23], [24], [28]. In this paper, we mainly investigated the migrativity equation between overlap (grouping) functions and uninorms with continuous underlying operators. This work is a further study on the migrativity equations for overlap function and grouping functions and the main contributions of this paper are listed as follows.

(i) We discussed the migrativity for $U \in \mathcal{COU}$ over overlap and grouping functions. In order to explain the results more clearly, we gave a brief comparison with other results of that concerning migrativity equation involving uninorms with continuous underlying operators, which were correspondingly discussed in [23].

(ii) It's divided into two cases that α is an idempotent element of U and α is not an idempotent element of U to investigate the migrativity for overlap and grouping functions over $U \in \mathcal{COU}$. The results obtained in this part generalized the corresponding ones in [31] and [32] under certain conditions.

In addition, it is stated in Introduction that there are many recent literature which proposed the distributive laws concerning aggregation functions [29]. Therefore, as a future work, we intend to investigate the distributive laws for $U \in \mathcal{COU}$ over overlap and grouping functions.

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New results on the migrativity properties for overlap (grouping) functions and uninorms

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نتایج جدید در مورد خواص مهاجرت برای توابع (گروه‌بندی) هم‌پوشانی و هم‌نرم‌ها

چکیده. معادلات عملکردی، از جمله توابع تجمع نقش مهمی در مجموعه‌های فازی و تئوری منطق فازی ایفا می‌کنند. ثابت شده‌است که معادله مهاجرت به عنوان یک نوع معادله جمعی عمومی محدود، در طیف وسیعی از زمینه‌ها، مانند تصمیم‌گیری، جمع‌آوری اطلاعات، تصویرپردازی و غیره مفید می باشد. در متون، نتایج قبلی موجود درمورد معادله مهاجرت بین توابع (گروه‌بندی) هم‌پوشانی و هم‌نرم‌ها براساس این فرض است که هم‌نرم‌ها به یکی از بیشترین کلاس‌های مورد مطالعه تعلق دارد. در این تحقیق، ما آنچه را که شامل هم‌نرم‌ها در یک محیط کلی‌تر است، کشف خواهیم کرد. برای مشخص بودن، در مواردی که هم‌نرم‌ها دارای عملگرهای اساسی پیوسته می‌باشند، ویژگی‌های مهاجرت بین توابع (گروه‌بندی) هم‌پوشانی و هم‌نرم‌ها را بررسی می‌کنیم. در راستای این مقاله نشان خواهیم داد که بسیاری از راه حل‌های جدید معادله از این دیدگاه جدید مشخص می‌شوند.