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Sensitivity and strong sensitivity on induced dynamical systems

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Abstract

Given a metric space X, we consider the family of all normal upper semicontinuous fuzzy sets on X, denoted by $\mathcal{F}(X)$, and a discrete dynamical system (X, f). In this paper, we study when $(\mathcal{F}(X), \widehat{f})$ is (strongly) sensitive, where $\widehat{f}$ is the Zadeh's extension of f and $\mathcal{F}(X)$ is equipped with different metrics: The uniform metric, the Skorokhod metric, the sendograph metric and the endograph metric. We prove that the sensitivity in the induced dynamical system $(\mathcal{K}(X), \overline{f})$ is equivalent to the sensitivity in $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ with respect to the uniform metric, the Skorokhod metric and the sendograph metric. We also show that the following conditions are equivalent:

a) (X, f) is strongly sensitive;

b) $(\mathcal{F}(X), \widehat{f})$ is strongly sensitive, where $\mathcal{F}(X)$ is equipped with the uniform metric;

c) $(\mathcal{F}(X), \widehat{f})$ is strongly sensitive, where $\mathcal{F}(X)$ is equipped with the Skorokhod metric;

d) $(\mathcal{F}(X), \widehat{f})$ is strongly sensitive, where $\mathcal{F}(X)$ is equipped with the sendograph metric.

Keywords: Fuzzy set, Skorokhod metric, uniform metric, endograph metric, sendograph metric, Zadeh's extension, sensitive properties.

1 Introduction

In this paper, (X, d) denotes a metric space and $\mathcal{F}(X)$ is the family of all normal upper semicontinuous fuzzy sets on (X, d). We consider $\mathcal{F}(X)$ with different metrics: The uniform metric, the endograph metric, the sendograph metric and the Skorokhod metric; for more information related with these metrics, we refer to [8, 9, 10, 13, 14, 16], among others.

A discrete dynamical system is a pair (X, f) where X is a metric space and $f : X \to X$ is a continuous function. Collective dynamical properties of a dynamical system (X, f) can be explained through the study of its induced set-valued system $(\mathcal{K}(X), \overline{f})$, where $\mathcal{K}(X)$ is the hyperspace of non-empty compact subsets of X endowed with the Hausdorff metric and $\overline{f} \colon \mathcal{K}(X) \to \mathcal{K}(X)$ is the map induced by f. We refer the reader to the papers [2], [16], [19] and [21]. It is easy to see that if $(\mathcal{K}(X), \overline{f})$ is *sensitive on initial conditions* (*sensitive*, for short), then (X, f) is sensitive, but the converse is false [21, Example 2.11]. Abraham, Biau and Cadre studied dynamical properties on probability spaces in [1] and introduced the concept of *strong sensitivity*. This property is stronger than sensitive dependence on initial conditions. In contrast with the sensitivity, it was proved in [21, Proposition 2.8] that (X, f) is strongly sensitive if and only if $(\mathcal{K}(X), \overline{f})$ is strongly sensitive.

In the 1970s, Zadeh introduced a concept called *Zadeh's extension* (see [3, 8, 16, 19], etc.). It is known that if $f \colon X \to X$ is continuous, then $\widehat{f} \colon \mathcal{F}(X) \to \mathcal{F}(X)$ is continuous, where $\mathcal{F}(X)$ is equipped with one of the following metrics: the uniform metric, the endograph metric, the sendograph metric and the Skorokhod metric (see [10, 14, 18]).

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The latter fact permits us to define a *fuzzified* dynamical system $(\mathcal{F}(X), \widehat{f})$, where $\widehat{f}$ is the Zadeh's extension of f. It is natural to study the interaction of dynamical properties between dynamical systems (X, f), $(\mathcal{K}(X), \overline{f})$ and $(\mathcal{F}(X), \widehat{f})$ (see [4, 16, 20], etc.). For instance, the sensitivity has been studied in [16], [17], [20] and [24]. In particular, $(\mathcal{K}(X), \overline{f})$ is sensitive if and only if $(\mathcal{F}(X), \widehat{f})$ is sensitive with respect to the uniform metric d_∞ [24, Corollary 4.1].

The aim of this paper is to study the interaction of sensitivity and strong sensitivity between (X, f) and $(\mathcal{F}(X), \widehat{f})$. Here we consider different metrics on $\mathcal{F}(X)$. As expected, the induced dynamical system $(\mathcal{K}(X), \overline{f})$ is relevant in this work. In Section 3, we mainly prove that (X, f) is strongly sensitive if and only if $(\mathcal{F}(X), \widehat{f})$ is strongly sensitive with respect to the uniform metric, the Skorokhod metric and the sendograph metric (see Corollary 3.8). Note that this result extends [21, Proposition 2.8] mentioned previously. In Corollary 3.12, we prove that the sensitivity in the induced dynamical system $(\mathcal{K}(X), \overline{f})$ is equivalent to the sensitivity in $\widehat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ with respect to the uniform metric, the Skorokhod metric and the sendograph metric. In particular, we obtain [24, Corollary 4.1].

In the final part of this work, in Section 4, some facts about periodic density are studied. In Theorem 4.1, we show that if a dynamical system (X, f) has a countable dense subset of periodic points, then $(\mathcal{F}_0(X), \widehat{f})$ contains a countable dense subset of periodic points. Therefore, $(\mathcal{F}(X), \widehat{f})$ contains a countable dense subset of periodic points, where $\mathcal{F}(X)$ has the endograph metric or the sendograph metric.

2 Preliminaries

A fuzzy set u on the space X is a function $u\colon X \to I$, where I denotes the closed unit interval $[0, 1]$. The *levels* of u are defined as follows: $u_\alpha = \{x \in X : u(x) \geq \alpha\}$ for each $\alpha \in (0, 1]$, and the *support* of u, denoted by u_0, is the set $\overline{\{x \in X : u(x) > 0\}}$. Let us note that $u_0 = \overline{\bigcup\{u_\alpha : \alpha \in (0, 1]\}}$. Let $\mathcal{F}(X)$ be the family of all normal fuzzy sets on X, i.e., all upper semicontinuous fuzzy sets $u\colon X \to I$ such that u_0 is compact and u_1 is non-empty.

If $f\colon X \to X$ is a function, the *Zadeh's extension* of f is denoted by $\widehat{f}\colon \mathcal{F}(X) \to \mathcal{F}(X)$ and defined as follows:

$$\widehat{f}(u)(x) = \begin{cases} \sup\{u(z) : z \in f^{-1}(x)\}, & f^{-1}(x) \neq \emptyset \\ 0, & f^{-1}(x) = \emptyset \end{cases}.$$

Some papers related with the Zadeh's extension are [3, 8, 16, 19].

The next two propositions will be helpful, the first of them tells us that the Zadeh's extension sends α-level of u to α-level of $\widehat{f}(u)$.

Proposition 2.1. [10] *Let X be a Hausdorff space. If $f\colon X \to X$ is a continuous function, then $[\widehat{f}(u)]_\alpha = f(u_\alpha)$ for each $u \in \mathcal{F}(X)$ and $\alpha \in I$.*

Proposition 2.2. [20] *If $f\colon X \to X$ is a continuous function, then $\left(\widehat{f}\right)^n = \widehat{f^n}$ for each $n \in \mathbb{N}$.*

Due to Proposition 2.2, in the sequel we will put $\widehat{f^n}$ instead of $\left(\widehat{f}\right)^n$.

Consider a metric space (X, d). For $x \in X$ and $\epsilon > 0$, the symbol $B(x, \epsilon)$ denotes the open ball (respect to d) with center at x and radius ϵ. Let A, B be non-empty closed subsets of X. The *Hausdorff distance* between A and B is defined as follows:

$$d_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}.$$

Clearly, if A and B are bounded, then $d_H(A, B) < \infty$. Denote by $\mathcal{K}(X)$ the space of all non-empty compact subsets of (X, d) equipped with the *Hausdorff metric* d_H. We have the next auxiliar result.

Proposition 2.3. *Let (X, d) be a metric space. If $A, B, C, F, G \in \mathcal{K}(X)$, then we have the following:*

i) *If $A \subseteq B \subseteq C$, then $d_H(A, B) \leq d_H(A, C)$ and $d_H(B, C) \leq d_H(A, C)$.*

ii) *If $d_H(A, F) \leq \epsilon$ and $d_H(B, G) \leq \epsilon$, then $d_H(A \cup B, F \cup G) \leq \epsilon$.*

Proof. i) Put $H(A, B) = \sup\{d(a, B) : a \in A\}$. Then we have that

$$d_H(A, B) = \max\{H(A, B), H(B, A)\}.$$

Since $A \subset B \subset C$, we conclude that $H(A,B) = H(A,C) = H(B,C) = 0$, $H(C,B) \leq H(C,A)$ and $H(B,A) \leq H(C,A)$. Hence $d_H(A,B) \leq d_H(A,C)$ and $d_H(B,C) \leq d_H(A,C)$.

ii) Clearly, $H(A \cup B, F \cup G) \leq \max\{H(A,F), H(B,G)\} \leq \epsilon$. Similarly,

$$H(F \cup G, A \cup B) \leq \max\{H(F,A), H(G,B)\} \leq \epsilon.$$

This finishes the proof. □

Given a non-empty subset $A \subseteq X$, we denote by $\chi_A \colon X \to I$ the characteristic function of A. It is easy to see that $\chi_A \in \mathcal{F}(X)$, whenever $A \in \mathcal{K}(X)$, because it is an upper semicontinuous function. For the one-point set $\{x\}$, we put χ_x instead of $\chi_{\{x\}}$. The following result appears in [10] and it shows that the Zadeh's extension $\widehat{f}$ sends $\mathcal{K}(X)$ into itself.

Proposition 2.4. *Let f be a continuous function from (X,d) into itself. Then $\widehat{f}(\chi_K) = \chi_{f(K)}$ for each $K \in \mathcal{K}(X)$.*

2.1 The uniform metric on $\mathcal{F}(X)$

It is well known that the function $d_\infty \colon \mathcal{F}(X) \times \mathcal{F}(X) \to [0,\infty)$ defined by $d_\infty(u,v) = \sup\{d_H(u_\alpha, v_\alpha) : \alpha \in I\}$ is a metric on $\mathcal{F}(X)$. We say that d_∞ is the *uniform metric* on $\mathcal{F}(X)$. For $u \in \mathcal{F}(X)$ and ϵ, the symbol $B_\infty(u,\epsilon)$ denotes the open ball (respect to d_∞) with center at u and radius ϵ. The metric space $(\mathcal{F}(X), d_\infty)$ will be denoted by $\mathcal{F}_\infty(X)$. We can compute d_∞ as follows.

Proposition 2.5. [25] *Consider a metric space (X,d). If $u,v \in \mathcal{F}(X)$, then $d_\infty(u,v) = \sup\{d_H(u_\alpha, v_\alpha) : \alpha \in (0,1]\}$.*

The following result will be helpful in the sequel.

Lemma 2.6. *If (X,d) is a metric space, $B \in \mathcal{K}(X)$ and $u \in \mathcal{F}(X)$, then $d_\infty(u,\chi_B) = \max\{d_H(u_0,B), d_H(u_1,B)\}$.*

2.2 The Skorokhod metric on $\mathcal{F}(X)$

Denote by T the family of all strictly increasing homeomorphisms from I onto itself. Given a metric space (X,d), $u \in \mathcal{F}(X)$ and $t \in T$, we put $tu = t \circ u$. According to [10], the *Skorokhod metric* is defined on $\mathcal{F}(X)$ as follows:

$$d_0(u,v) = \inf\{\epsilon : \exists t \in T \text{ such that } \sup_{\alpha \in I} |t(\alpha) - \alpha| \leq \epsilon \text{ and } d_\infty(u, tv) \leq \epsilon\}.$$

Clearly, $d_0(u,v) \leq d_\infty(u,v)$ for each $u,v \in \mathcal{F}(X)$. Hence the topology τ_0 induced by d_0 is weaker than the topology τ_∞ induced by d_∞, i.e., $\tau_0 \subseteq \tau_\infty$. For $u \in \mathcal{F}(X)$ and $\epsilon > 0$, the symbol $B_0(u,\epsilon)$ denotes the open ball, in $\mathcal{F}_0(X) = (\mathcal{F}(X), d_0)$, with center at u and radius ϵ.

The next propositions are immediate consequences from definition of Skorokhod metric.

Proposition 2.7. *Let (X,d) be a metric space. For $u,w \in \mathcal{F}(X)$ we have the following:*

i) $d_0(u,w) \leq d_\infty(u,w)$.

ii) $d_H(u_\alpha, w_\alpha) \leq d_\infty(u,w)$ *for each* $\alpha \in I$.

iii) $d_H(u_\alpha, w_\alpha) \leq d_0(u,w)$ *for* $\alpha \in \{0,1\}$.

Proposition 2.8. *Let (X,d) be a metric space. If $K \in \mathcal{K}(X)$ and $u \in \mathcal{F}(X)$, then $d_0(u,\chi_K) = d_\infty(u,\chi_K)$.*

As a consequence of Lemma 2.6 and Proposition 2.8, we have the following.

Corollary 2.9. *Let (X,d) be a metric space. If $K \in \mathcal{K}(X)$ and $u \in \mathcal{F}(X)$, then*

$$d_0(u,\chi_K) = d_\infty(u,\chi_K) = \max\{d_H(u_0,K), d_H(u_1,K)\}.$$

Corollary 2.10. *Let (X,d) be a metric space. If $x \in X$ and $u \in \mathcal{F}(X)$, then*

$$d_0(u,\chi_x) = d_\infty(u,\chi_x) = d_H(u_0,\{x\}) = \sup_{y \in u_0} d(x,y).$$

2.3 The endograph and sendograph metrics

Consider a metric space (X,d). Take $u \in \mathcal{F}(X)$. We define the *endograph* of u as follows:

$$end(u) = \{(x,\alpha) \in X \times I : u(x) \geq \alpha\}.$$

Similarly, the *sendograph* of u is defined by $send(u) = end(u) \cap (u_0 \times I)$.

The *endograph metric* d_E on $\mathcal{F}(X)$ is defined as the Hausdorff distance (respect to $X \times I$) between $end(u)$ and $end(v)$ for each $u,v \in \mathcal{F}(X)$. The *sendograph metric* d_S on $\mathcal{F}(X)$ is the Hausdorff metric (on $\mathcal{K}(X \times I)$) between the non-empty compact subsets $send(u)$ and $send(v)$ for every $u,v \in \mathcal{F}(X)$ (see [15]). It is known that $d_E \leq d_S \leq d_\infty$ (see [5] and [26]). The metric spaces $(\mathcal{F}(X), d_E)$ and $(\mathcal{F}(X), d_S)$ will be denoted by $\mathcal{F}_E(X)$ and $\mathcal{F}_S(X)$, respectively. We have the following lower bound for d_S.

Proposition 2.11. [26] *Consider a metric space (X,d). Then $d_S(u,v) \geq d_H(u_0,v_0)$ for each $u,v \in \mathcal{F}(X)$.*

Proposition 2.12. *Let (X,d) be a metric space. If $x \in X$ and $u \in \mathcal{F}(X)$, then $d_S(u,\chi_x) = d_H(u_0,\{x\})$.*

Proof. We have that $d_S \leq d_\infty$. Then $d_S(u,\chi_x) \leq d_\infty(u,\chi_x) = d_H(u_0,\{x\})$. By Proposition 2.11, $d_H(u_0,\{x\}) \leq d_S(u,\chi_x)$. This finishes the proof. □

Corollary 2.13. *Let (X,d) be a metric space. If $x \in X$ and $u \in \mathcal{F}(X)$, then*

$$d_S(u,\chi_x) = d_0(u,\chi_x) = d_\infty(u,\chi_x) = d_H(u_0,\{x\}).$$

The following example shows that Corollary 2.13 is false for the endograph metric.

Example 2.14. Consider the interval $X = [0,2]$ with its usual metric. Note that for each $x,y \in X$, we have that $d_E(\chi_x,\chi_y) = \min\{1, |x-y|\}$. Therefore, $d_E(\chi_0,\chi_2) = 1 < d_H(\{0\},\{2\})$.

Consider a metric space (X,d). Let τ_0, τ_E and τ_S be the topologies induced by d_0, d_E and d_S, respectively. We have the following result.

Proposition 2.15. [9] *Consider a metric space (X,d). Then $\tau_E \subseteq \tau_0$ and $\tau_S \subseteq \tau_0$.*

3 Strong Sensitivity

Let (X,d) be a metric space. A dynamical system (X,f) is *sensitive on initial conditions* (*sensitive*, for short) if there exists $\delta > 0$ such that for each $x \in X$ and any $\epsilon > 0$ there exist $y \in X$ and $n \in \mathbb{N}$ satisfying $d(x,y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$.

Theorem 3.1. *Let (X,d) be a metric space. Suppose that ρ is a metric on $\mathcal{F}(X)$ such that $\rho(\chi_x,u) = d_\infty(\chi_x,u)$ for each $x \in X$ and $u \in \mathcal{F}(X)$. If $\widehat{f}\colon (\mathcal{F}(X),\rho) \to (\mathcal{F}(X),\rho)$ is sensitive, then $f\colon (X,d) \to (X,d)$ is sensitive.*

Proof. Since $\widehat{f}$ is sensitive on initial conditions, then there exists a constant $\delta > 0$ such that for every $u \in \mathcal{F}(X)$ and every $\epsilon > 0$ there exist $v \in \mathcal{F}(X)$ and $n \in \mathbb{N}$ such that $\rho(u,v) < \epsilon$ and $\rho(\widehat{f^n}(u), \widehat{f^n}(v)) \geq \delta$. We claim that the same δ helps us to conclude that f is sensitive. Indeed, take $x \in X$ and $\epsilon > 0$. Consider $\chi_x \in \mathcal{F}(X)$. Then there exist $v \in \mathcal{F}(X)$ and $n \in \mathbb{N}$ such that $\rho(\chi_x,v) < \epsilon$ and $\rho(\widehat{f^n}(\chi_x), \widehat{f^n}(v)) > \delta$. By Proposition 2.4, we have that $\widehat{f^n}(\chi_x) = \chi_{f^n(x)}$. The latter facts and our hypothesis imply that $d_\infty(\chi_x,v) < \epsilon$ and $d_\infty(\chi_{f^n(x)}, \widehat{f^n}(v)) > \delta$. It follows from Proposition 2.1 that

$$\delta < d_\infty(\chi_{f^n(x)}, \widehat{f^n}(v)) = \sup_{\alpha \in I} d_H([\chi_{f^n(x)}]_\alpha, [\widehat{f^n}(v)]_\alpha) = \sup_{\alpha \in I} d_H(\{f^n(x)\}, f^n(v_\alpha)) = \sup_{y \in v_0} d(f^n(x), f^n(y)).$$

Therefore, there exists $y_0 \in v_0$ such that $d(f^n(x), f^n(y_0)) > \delta$. On the other hand, $d_\infty(\chi_x,v) < \epsilon$. By Corollary 2.9, we have that $d_H(\{x\},v_0) \leq d_\infty(\chi_x,v) < \epsilon$. Therefore, $d(x,y_0) < \epsilon$ and $d(f^n(x), f^n(y_0)) > \delta$ which implies that $f\colon (X,d) \to (X,d)$ is sensitive. □

Corollary 2.13 and Theorem 3.1 imply the following.

Corollary 3.2. *Let X be a metric space and $f\colon X \to X$ a continuous function. Suppose that one of the following conditions is satisfied:*

i) $(\mathcal{F}_\infty(X), \widehat{f})$ *is sensitive;*

ii) $(\mathcal{F}_0(X), \widehat{f})$ *is sensitive;*

iii) $(\mathcal{F}_S(X), \widehat{f})$ *is sensitive.*

Then (X, f) *is sensitive.*

Condition i) of previous corollary was proved in [20, Theorem 4]. Also, condition iii) is probably known. In Example 3.13, we show that the converse of Corollary 3.10 is false.

According to [1], a dynamical system (X, f) is *strongly sensitive* if we can find $\delta > 0$ such that for each $x \in X$ and $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there is $y \in X$ with $d(x, y) < \epsilon$ and $d(f^n(x), f^n(y)) > \delta$. It is easy to see that strong sensitivity implies sensitivity on initial conditions, but the converse is false (see Example 3.14).

We can argue as in Theorem 3.1 to show the following result.

Proposition 3.3. *Let X be a metric space and $f\colon X \to X$ a continuous function. Suppose that one of the following conditions is satisfied:*

i) $(\mathcal{F}_\infty(X), \widehat{f})$ *is strongly sensitive;*

ii) $(\mathcal{F}_0(X), \widehat{f})$ *is strongly sensitive;*

iii) $(\mathcal{F}_S(X), \widehat{f})$ *is strongly sensitive.*

Then (X, f) *is strongly sensitive.*

The next result permits us to understand the statement of Lemma 3.5.

Proposition 3.4. [10] *Let (X, d) be a metric space and $u \in \mathcal{F}(X)$. If $L\colon [0, 1] \to (\mathcal{K}(X), d_H)$ is the function defined by $L(\alpha) = u_\alpha$ for all $\alpha \in [0, 1]$, then:*

i) *L is left continuous on $(0, 1]$;*

ii) $\lim_{\lambda \to \alpha^+} L(\lambda) = \overline{\bigcup_{\beta > \alpha} u_\beta}$ *and* $\lim_{\lambda \to \alpha^+} L(\lambda) \subset u_\alpha$ *for each* $\alpha \in [0, 1)$;

iii) *L is right continuous at* 0.

If $u \in \mathcal{F}(X)$, we define $u_{\alpha^+} = \lim_{\lambda \to \alpha^+} L(\lambda)$. Note that $u_{\alpha^+} \in \mathcal{K}(X)$. In [10] we can found the following result.

Lemma 3.5. *Suppose that (X, d) is a metric space. For any $u \in \mathcal{F}(X)$ and $\epsilon > 0$ there exist numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ such that $d_H(u_{\alpha_k^+}, u_{\alpha_{k+1}}) < \epsilon$ for $k = 0, 1, \ldots, n-1$.*

Let (X, d) be a metric space. Take a closed subset A of X and $\epsilon > 0$, we put $N(A, \epsilon) = \bigcup_{x \in A} B(x, \epsilon)$. It is known that for every two closed subsets A and B in X the following equality holds:

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subseteq N(B, \epsilon) \text{ and } B \subseteq N(A, \epsilon)\}.$$

The following result plays an important role in the proof of Theorem 3.7.

Lemma 3.6. *Let (X, d) be a metric space, D a non-empty finite subset of X and $u \in \mathcal{F}(X)$ such that $d_H(u_0, D) < \epsilon/4$. Then there exists $w \in \mathcal{F}(X)$ such that $d_\infty(u, w) < \epsilon$ and $w_0 = D$.*

Proof. By Lemma 3.5, we can choose numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = 1$ such that $d_H(u_{\alpha_k^+}, u_{\alpha_{k+1}}) < \frac{\epsilon}{4}$ for $k = 0, 1, \ldots, m-1$. Since $u_{\alpha_k} \subseteq u_0$ for each $k = 1, \ldots, m$, the set $A_k = D \cap N(u_{\alpha_k}, \epsilon/4)$ is non-empty and we have the following:

$$d_H(u_{\alpha_k}, A_k) < \frac{\epsilon}{4}. \tag{1}$$

It is easy to see that $A_1 = D$. Define $\chi_\emptyset = 0$, $B_m = A_m$ and $B_k = A_k \setminus (\bigcup\{A_i : i = k+1, \ldots, m\})$ for each $k = 1, 2, \ldots, m-1$.

Observe that $w = \Sigma_{k=1}^m \alpha_k \chi_{B_k} \in \mathcal{F}(X)$, because it is an upper semicontinuous map from X to I such that $w_1 = A_m \neq \emptyset$ and $w_0 = \bigcup\{B_k : k = 1, 2, \ldots, m\} = D$.

We claim that $d_\infty(w,u) < \epsilon$. Note that $w_\alpha = A_{k+1} \cup A_{k+2} \cup \cdots \cup A_m$ for each $\alpha \in (\alpha_k, \alpha_{k+1}]$ and $k = 0,1,2,\ldots,m-1$. Then take any $\alpha \in (\alpha_k, \alpha_{k+1}]$ and observe that

$$d_H(u_\alpha, w_\alpha) \le d_H(u_\alpha, u_{\alpha_{k+1}}) + d_H(u_{\alpha_{k+1}}, w_\alpha). \tag{2}$$

Since $u_{\alpha_{k+1}} \subset u_\alpha \subset u_{\alpha_k^+}$ and $d_H(u_{\alpha_k^+}, u_{\alpha_{k+1}}) < \frac{\epsilon}{4}$, Proposition 2.3 implies the following:

$$d_H(u_\alpha, u_{\alpha_{k+1}}) < \frac{\epsilon}{4}. \tag{3}$$

Recall that $u_0 \supseteq u_{\alpha_1} \supseteq \cdots \supseteq u_{\alpha_m} = u_1$. We have that $d_H(u_{\alpha_{k+1}}, w_\alpha) = d_H(u_{\alpha_{k+1}}, A_{k+1} \cup \cdots \cup A_m)$. Inequality (1) and Proposition 2.3 imply that

$$d_H(u_{\alpha_{k+1}}, w_\alpha) < \frac{\epsilon}{4}. \tag{4}$$

Putting inequalities (3) and (4) in inequality (2), we conclude that $d_H(u_\alpha, w_\alpha) < \frac{\epsilon}{2}$ for each $\alpha \in (0,1]$. Hence Proposition 2.5 implies that $d_\infty(u,w) = \sup\{d_H(u_\alpha, w_\alpha) : \alpha \in (0,1]\} \le \frac{\epsilon}{2} < \epsilon$. □

We now are ready to present the main result of this paper.

Theorem 3.7. *Let (X,d) be a metric space. If $f : X \to X$ is a strongly sensitive continuous map, then we have the following:*

a) $\widehat{f} : \mathcal{F}_\infty(X) \to \mathcal{F}_\infty(X)$ *is strongly sensitive;*

b) $\widehat{f} : \mathcal{F}_0(X) \to \mathcal{F}_0(X)$ *is strongly sensitive;*

c) $\widehat{f} : \mathcal{F}_S(X) \to \mathcal{F}_S(X)$ *is strongly sensitive.*

Proof. Suppose that $f : X \to X$ is strongly sensitive. By [21, Proposition 2.8], the induced map $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ is strongly sensitive. Take $\lambda > 0$ which witnesses this fact. Take $u \in \mathcal{F}(X)$ and $\epsilon > 0$. Observe that $u_0 \in \mathcal{K}(X)$, hence there exists $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$ there is $D_n \in \mathcal{K}(X)$ with $d_H(u_0, D_n) < \frac{\epsilon}{4}$ and $d_H(\overline{f}^n(u_0), \overline{f}^n(D_n)) > \lambda$. Suppose from now that n and D_n are fixed. We can assume, as in the proof of [21, Proposition 2.8], that D_n is finite. By Lemma 3.6, there exists $w \in \mathcal{F}(X)$ such that $d_\infty(u,w) < \epsilon$ and $w_0 = D_n$. Hence $d_0(u,w) < \epsilon$. Notice that propositions 2.1 and 2.2 imply that $[\widehat{f^n}(u)]_0 = f^n(u_0)$. It is a well known fact that $\overline{f}^n(A) = f^n(A)$ for each compact $A \subset X$ (see for example [2]), whence $[\widehat{f^n}(u)]_0 = f^n(u_0) = \overline{f}^n(u_0)$ and $[\widehat{f^n}(w)]_0 = f^n(w_0) = \overline{f}^n(w_0) = \overline{f}^n(D_n)$ which imply that $d_H\left([\widehat{f^n}(u)]_0, [\widehat{f^n}(w)]_0\right) = d_H\left(\overline{f}^n(u_0), \overline{f}^n(D_n)\right)$. From Proposition 2.7, we conclude the following:

$$d_\infty\left(\widehat{f^n}(u), \widehat{f^n}(w)\right) \ge d_0\left(\widehat{f^n}(u), \widehat{f^n}(w)\right) \ge d_H\left([\widehat{f^n}(u)]_0, [\widehat{f^n}(w)]_0\right) = d_H\left(\overline{f}^n(u_0), \overline{f}^n(D_n)\right) > \lambda.$$

Therefore, $(\mathcal{F}_\infty(X), \widehat{f})$ and $(\mathcal{F}_0(X), \widehat{f})$ are strongly sensitive. We have thus proved a) and b).

Using a similar argument and Proposition 2.11, we conclude that the dynamical system $(\mathcal{F}_S(X), \widehat{f})$ is strongly sensitive. □

We don't know if Theorem 3.7 remains true for the endograph metric.

Let (X,d) be a metric space. Suppose that the dynamical system (X,f) is strongly sensitive. Is $(\mathcal{F}_E(X), \widehat{f})$ strongly sensitive?

Proposition 3.3 and Theorem 3.7 imply the following.

Corollary 3.8. *Let (X,d) be a metric space. If $f : X \to X$ is a continuous map, then the following conditions are equivalent:*

i) $f : X \to X$ *is strongly sensitive;*

ii) $\widehat{f} : \mathcal{F}_\infty(X) \to \mathcal{F}_\infty(X)$ *is strongly sensitive;*

iii) $\widehat{f} : \mathcal{F}_0(X) \to \mathcal{F}_0(X)$ *is strongly sensitive;*

iv) $\widehat{f} : \mathcal{F}_S(X) \to \mathcal{F}_S(X)$ *is strongly sensitive.*

The next example tells us that sensitive maps on I have very well behavior and this fact is reflected on hyperspaces of fuzzy sets.

Example 3.9. If the continuous function $f : I \to I$ is sensitive, then $(\mathcal{F}_\infty(I), \widehat{f})$, $(\mathcal{F}_\mathrm{S}(I), \widehat{f})$ and $(\mathcal{F}_0(I), \widehat{f})$ are strongly sensitive.

Proof. From [23, Theorem 2] it follows that f is strongly sensitive. Therefore, Corollary 3.8 implies that $(\mathcal{F}_\infty(I), \widehat{f})$, $(\mathcal{F}_\mathrm{S}(I), \widehat{f})$ and $(\mathcal{F}_0(I), \widehat{f})$ are also strongly sensitive. □

We can argue as in the proof of Theorem 3.7 to show the following result.

Theorem 3.10. *Let (X, d) be a metric space. If $f : X \to X$ is a continuous map such that $\overline{f} : (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ is sensitive, then the following conditions are satisfied:*

i) $\widehat{f} : \mathcal{F}_\infty(X) \to \mathcal{F}_\infty(X)$ *is sensitive;*

ii) $\widehat{f} : \mathcal{F}_0(X) \to \mathcal{F}_0(X)$ *is sensitive;*

iii) $\widehat{f} : \mathcal{F}_S(X) \to \mathcal{F}_S(X)$ *is sensitive.*

We present the converse of conditions i) and ii) in Theorem 3.10. We don't know if Theorem 3.10 is valid for the endograph metric.

Let (X, d) be a metric space and $f\colon X \to X$ a continuous function. Suppose that the dynamical system $(\mathcal{K}(X), \overline{f})$ is sensitive. Is $(\mathcal{F}_E(X), \widehat{f})$ sensitive?

Proposition 3.11. *Let X be a metric space and $f\colon X \to X$ a continuous function. Suppose that one of the following conditions is satisfied:*

i) $(\mathcal{F}_\infty(X), \widehat{f})$ *is sensitive;*

ii) $(\mathcal{F}_0(X), \widehat{f})$ *is sensitive;*

iii) $(\mathcal{F}_S(X), \widehat{f})$ *is sensitive.*

Then $\overline{f} : (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ *is sensitive.*

Proof. It is a consequence of Corollary 2.13. □

By Theorem 3.10 and Proposition 3.11, we have the following.

Corollary 3.12. *Let (X, d) be a metric space. If $f : X \to X$ is a continuous map, then the following conditions are equivalent:*

i) $\overline{f} : (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ *is sensitive;*

ii) $\widehat{f} : \mathcal{F}_\infty(X) \to \mathcal{F}_\infty(X)$ *is sensitive;*

iii) $\widehat{f} : \mathcal{F}_0(X) \to \mathcal{F}_0(X)$ *is sensitive;*

iv) $\widehat{f} : \mathcal{F}_S(X) \to \mathcal{F}_S(X)$ *is sensitive.*

The equivalence of i) and ii) in previous result appears in [24].

Example 3.13. According to [21, Example 2.11], there exists a metric space (X, d) and a sensitive dynamical system (X, f) such that $(\mathcal{K}(X), \overline{f})$ is not sensitive. Thus by Corollary 3.12, we have that $(\mathcal{F}_\infty(X), \widehat{f})$ and $(\mathcal{F}_0(X), \widehat{f})$ are not sensitive.

Example 3.14. Sharma constructed in [22] an example of a non strongly sensitive dynamical system (Ω, σ) such that $(\mathcal{K}(\Omega), \overline{\sigma})$ is sensitive. Observe that by Corollary 3.12 and Corollary 3.8, $(\mathcal{F}_\infty(\Omega), \hat{\sigma})$, $(\mathcal{F}_S(\Omega), \hat{\sigma})$ and $(\mathcal{F}_0(\Omega), \hat{\sigma})$ are sensitive, but they are not strongly sensitive.

4 Periodic density

We finish this work with some facts about periodic density.

Let X be a topological space. A dynamical system (X, f) has *periodic density* if the set of periodic points $\{x \in X : f^n(x) = x \text{ for some } n \in \mathbb{N}\}$ is dense in X. Since $(\mathcal{K}(X), d_H)$ and $\mathcal{F}_\infty(X)$ are topological spaces, because they are metric spaces, the periodic density is defined in the same way for induced dynamical systems $\overline{f}\colon (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ and $\widehat{f}\colon \mathcal{F}_\infty(X) \to \mathcal{F}_\infty(X)$. In Example 4.4 we will deal with periodic density of an induced map $\widehat{f}$. It is well known that if f has periodic density on (X, d), then the induced dynamical systems $\overline{f}\colon (\mathcal{K}(X), d_H) \to (\mathcal{K}(X), d_H)$ and $\widehat{f}\colon \mathcal{F}_\infty(X) \to \mathcal{F}_\infty(X)$ have periodic density (see [7, Corollary 3.2] and [20, Theorem 5], respectively). It is worth to mention that in [16, Lemma 1] it was proved that the set of piecewise constant fuzzy sets in $\mathcal{F}_\infty(X)$ is everywhere dense. A similar result can be obtained for $(\mathcal{F}_0(X), \widehat{f})$ as the following theorem shows.

Theorem 4.1. *Let (X, d) be a metric space. If f has periodic density on (X, d), then $\widehat{f}$ has periodic density on $\mathcal{F}_0(X)$ as well. Moreover, if there exists a countable dense subset of periodic points of (X, f), then there exists a countable dense subset of periodic points of $(\mathcal{F}_0(X), \widehat{f})$, whence $\mathcal{F}_0(X)$ is separable.*

Proof. As in the proof of [10, Theorem 4.12] if Y is a dense subset of (X, d), then the set $\mathcal{D}(Y)$ is dense in $\mathcal{F}_0(X)$, where $\mathcal{D}(Y)$ is the set of all elements of $\mathcal{F}(X)$ which can be expressed in the form $\Sigma_{i=1}^n d_i \chi_{A_i}$, where each A_n is non-empty, $\{A_i : 1 \leq i \leq n\}$ is a family of pairwise disjoints finite subsets of Y and $0 \leq d_1 < d_2 < \cdots < d_{n-1} < d_n = 1$ with $\{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{Q}$. Clearly, if every element of Y is a periodic point of f, then $\mathcal{D}(Y)$ consists of periodic points of $\widehat{f}$. If Y is countable, then it is easy to see that $\mathcal{D}(Y)$ is countable, whence $\mathcal{F}_0(X)$ is separable. This finishes the proof. □

Proposition 2.15 and Theorem 4.1 imply the following two results.

Corollary 4.2. *Let (X, d) be a metric space. If f has periodic density on (X, d), then $\widehat{f}$ has periodic density on $\mathcal{F}_E(X)$ as well. Moreover, if there exists a countable dense subset of periodic points of (X, f), then there exists a countable dense subset of periodic points of $(\mathcal{F}_E(X), \widehat{f})$, whence $\mathcal{F}_E(X)$ is separable.*

Corollary 4.3. *Let (X, d) be a metric space. If f has periodic density on (X, d), then $\widehat{f}$ has periodic density on $\mathcal{F}_S(X)$ as well. Moreover, if there exists a countable dense subset of periodic points of (X, f), then there exists a countable dense subset of periodic points of $(\mathcal{F}_S(X), \widehat{f})$, whence $\mathcal{F}_S(X)$ is separable.*

Example 4.4. Suppose that $f : S^1 \to S^1$ is defined by $f(\mathrm{e}^{i\theta}) = \mathrm{e}^{2i\theta}$. Devaney proved in [6, Example 3.4] that f has periodic density, moreover S^1 has a countable dense subset containing only periodic points of f. Theorem 4.1, Corollary 4.2 and Corollary 4.3 imply that $(\mathcal{F}_0(S^1), \widehat{f})$, $(\mathcal{F}_E(S^1), \widehat{f})$ and $(\mathcal{F}_S(S^1), \widehat{f})$ have periodic density; moreover $\mathcal{F}_0(S^1)$, $\mathcal{F}_E(S^1)$ and $\mathcal{F}_S(S^1)$ have a countable dense subset of periodic points of $\widehat{f}$. By [20, Theorem 5], the dynamical system $(\mathcal{F}_\infty(S^1), \widehat{f})$ has periodic density, but it does not have a countable dense subset of periodic points (see Proposition 4.5). It is worth mentioning that each $u \in \mathcal{F}(X)$ is completely defined by the compact subsets $\{u_\alpha : \alpha \in I\} \subset \mathcal{K}(S^1)$ (see Proposition 3.4 and its converse in [10, Proposition 4.9]), hence $\widehat{f}(u)$ is represented by the compact sets $\{[\widehat{f}(u)]_\alpha = \overline{f}(u_\alpha) = f(u_\alpha) : \alpha \in I\} \subset \mathcal{K}(S^1)$.

It is well known that the hyperspace $(\mathcal{K}(X), d_H)$ is separable if and only if the metric space (X, d) is separable. According to [13, Proposition 3.3], the metric space $\mathcal{F}_\infty(\mathbb{R}^n)$ is not separable. The latter result is also cited in [12]. In [15], it is mentioned that if (X, d) is a locally compact metric space with more than one point, then $\mathcal{F}_\infty(X)$ is not separable. This fact is also cited in [24]. Surprisingly, the following proposition and all results mentioned in this paragraph can be proved by arguing as in the proof of [13, Proposition 3.3]. Here, we present such a proof for an arbitrary metric space (X, d).

Proposition 4.5. *If (X, d) is a metric space with more than one point, then $\mathcal{F}_\infty(X)$ is not separable.*

Proof. Take $a, b \in X$ two different points. For each $r \in (0, 1)$, we define the fuzzy set $u^r = \chi_a + r\chi_b$ and $\mathcal{A} = \{u^r : r \in (0, 1)\}$. Observe that for each $r \in (0, 1)$, we have that $B_\infty(u^r, \frac{\rho}{2}) \cap \mathcal{A} = \{u^r\}$, where $\rho = d(a, b)$. In fact, if $r, s \in (0, 1)$ and $s < r$, then $[u^s]_r = \{a\}$ and $[u^r]_r = \{a, b\}$. Thus $d_H([u^s]_r, [u^r]_r) = \rho$ and $d_\infty(u^r, u^s) = \rho$. We have proved that $\mathcal{A}$ is not separable, because it is a discrete space with cardinality $\mathfrak{c}$. Therefore, $\mathcal{F}_\infty(X)$ is not separable. □

According to [11], the metric space $\mathcal{F}_0(\mathbb{R})$ is separable. This result was extended in [12, Theorem 3.2]: the metric space $\mathcal{F}_0(\mathbb{R}^n)$ is separable. Then in [10] it was proved that a metric space (X, d) is separable if and only if $\mathcal{F}_0(X)$ is separable. The latter fact and Proposition 2.15 imply the following.

Theorem 4.6. *Let (X, d) be a metric space. Then the following conditions are equivalent:*

i) (X, d) *is separable;*

ii) $\mathcal{F}_0(X)$ *is separable;*

iii) $\mathcal{F}_E(X)$ *is separable;*

iv) $\mathcal{F}_S(X)$ *is separable.*

The conditions $i) \Leftrightarrow iii)$ and $i) \Leftrightarrow iv)$ are mentioned in [15] for locally compact metric spaces. Therefore, Theorem 4.6 generalizes the latter results obtained by Kupka in [15].

5 Conclusions

In this paper, we study sensitivity and strong sensitivity. The latter notion was introduced in [1]. We consider different metrics on $\mathcal{F}(X)$: the uniform metric, the endograph metric, the sendograph metric and the Skorokhod metric. We have extended a result in [21] on strong sensitivity. On sensitivity we obtain as a corollary a result in [24]. Therefore, the importance of this paper is in the study of sensitive properties on fuzzy hyperspaces and the extension of some known results.

For the continuation of this research, we propose two questions (see Problem 3 and Problem 3). We also include the following:

a) Suppose that (X, f) is a strongly sensitive discrete dynamical system. Is $(\mathcal{F}(X), \widehat{f})$ strongly sensitive?

b) Suppose that $(\mathcal{K}(X), \overline{f})$ is a sensitive discrete dynamical system. Is $(\mathcal{F}(X), \widehat{f})$ sensitive?

where $\mathcal{K}(X)$ is equipped with the Hausdorff metric and $\mathcal{F}(X)$ is equipped with a metric obtained from the metric on X.

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References

[1] C. Abraham, G. Biau, B. Cadre, *Chaotic properties of mappings on a probability space*, Journal of Mathematical Analysis and Applications, **266** (2002), 420-431.

[2] J. Banks, *Chaos for induced hyperspace maps*, Chaos, Solitons and Fractals, **25** (2005), 681-685.

[3] L. C. Barros, R. C. Bassanezi, P. A. Tonelli, *On the continuity of Zadeh's extension*, Proc. 7th IFSA World Congress, Prague, **II** (1997), 3-8.

[4] R. C. Bassanezi, L. C. De Barros, P. A. Tonelli, *Attractors and asymptotic stability for fuzzy dynamical systems*, Fuzzy Sets and Systems, **113** (2000), 473-483.

[5] J. J. Buckley, A. Yan, *Fuzzy functional analysis (I): Basic concepts*, Fuzzy Sets and Systems, **115**(3) (2000), 393-402.

[6] R. Devaney, *An introduction to chaotic dynamical systems*, Addison-Wesley, New York, Second Edition, 1989.

[7] A. Fedeli, *On chaotic set-valued discrete dynamical systems*, Chaos, Solitons and Fractals, **23** (2005), 1381-1384.

[8] Y. H. Goo, J. S. Park, *On the continuity of the Zadeh extensions*, Journal of the Chungcheong Mathematical Society, **20**(4) (2007), 525-533.

[9] H. Huang, *Some properties of Skorokhod metric on fuzzy sets*, (2019), arXiv:1904.07489v4.

[10] D. Jardón, I. Sánchez, M. Sanchis, *Some questions about Zadeh's extension on metric spaces*, Fuzzy Sets and Systems, **379** (2020), 115-124.

[11] S. Y. Joo, Y. K. Kim, *The Skorokhod topology on space of fuzzy numbers*, Fuzzy Sets and Systems, **111** (2000), 497-501.

[12] S. Y. Joo, Y. K. Kim, *Topological properties on the space of fuzzy numbers*, Journal of Mathematical Analysis and Applications, **246** (2000), 576-590.

[13] E. P. Klement, M. L. Puri, D. A. Ralescu, *Limit theorems for fuzzy random variables*, Proceedings of the Royal Society of London. Series A, **407** (1986), 171-182.

[14] P. E. Kloeden, *Fuzzy dynamical systems*, Fuzzy Sets and Systems, **7** (1982), 275-296.

[15] J. Kupka, *On fuzzifications of discrete dynamical systems*, Information Sciences, **181** (2011), 2858-2872.

[16] J. Kupka, *On devaney chaotic induced fuzzy and set-valued dynamical systems*, Fuzzy Sets and Systems, **177** (2011), 34-44.

[17] C. Ma, P. Zhu, T. Lu, *Some chaotic properties of fuzzified dynamical systems*, Springer Plus, **640**(5) (2016), 1-7.

[18] H. Román-Flores, L. C. Barros, R. C. Bassanezi, *A note on Zadeh's extensions*, Fuzzy Sets and Systems, **117** (2001), 327-331.

[19] H. Román-Flores, Y. Chalco-Cano, *Robinson's chaos in set-valued discrete systems*, Chaos, Solitons and Fractals, **25** (2005), 33-42.

[20] H. Román-Flores, Y. Chalco-Cano, *Some chaotic properties of Zadeh's extensions*, Chaos, Solitons and Fractals, **35** (2008), 452-459.

[21] P. Sharma, A. Nagar, *Inducing sensitivity on hyperspaces*, Topology and Its Applications, **157** (2010), 2052-2058.

[22] P. Sharma, A. Nagar, *Dynamics of the induced shift maps*, (2017), arXiv:1702.02728v2.

[23] T. K. Subrahmonian Moothathu, *Stronger forms of sensitivity for dynamical systems*, Nonlinearity, **20** (2007), 2115-2126.

[24] X. Wu, G. Chen, *Sensitivity and transitivity of fuzzified dynamical systems*, Information Sciences, **396** (2017), 14-23.

[25] X. Wu, X. Zhang, G. Chen, *Answers to some questions about Zadeh's extension principle on metric spaces*, Fuzzy Sets and Systems, **387** (2020), 174-180.

[26] L. Zhang, *The topological structure of fuzzy sets with sendograph metric*, Topology and Its Applications, **160** (2013), 1501-1506.

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Sensitivity and strong sensitivity on induced dynamical systems

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حساسیت و حساسیت قوی روی سیستم‌های دینامیکی القایی

چکیده. برای یک فضای متریک X داده شده، خانواده تمام مجموعه‌های فازی نیمه‌پیوسته بالایی نرمال روی X را که با F(X) نشان داده می‌شوند، و یک سیستم دینامیکی گسسته (X, f) را در نظر می‌گیریم. در این مقاله، ما زمانی که $(F(X),\hat{f})$ (قویاً) حساس است مورد مطالعه قرار می‌دهیم، که در آن $\hat{f}$ توسیع زاده (Zadeh) از f است و F(X) با متریک‌های مختلفی به صورت زیر مجهز شده‌است: متریک یکنواخت، متریک اسکوروخود، متریک سندوگراف، متریک اندوگراف. نشان می‌دهیم که حساسیت در سیستم دینامیکی القایی $(K(X),\bar{f})$ معادل حساسیت در $\hat{f}:F(X)\to F(Y)$ نسبت به متریک یکنواخت، متریک اسکوروخود، و متریک سندوگراف می باشد. همچنین نشان می‌دهیم که شرایط زیر معادل هستند:

۱) (X,f) قویاً حساس است.

۲) $(F(X),\hat{f})$، که F(X) مجهز به متریک یکنواخت است، قویاً حساس است.

۳) $(F(X),\hat{f})$ که F(X) مجهز به متریک اسکوروخود است، قویاً حساس است.

۴) $(F(X),\hat{f})$ که F(X) مجهز به متریک سندوگراف است، قویاً حساس است.