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On matrix games with 2-tuple intuitionistic fuzzy linguistic payoffs

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Abstract

In real-world decision-making problems, experts often prefer to express their views, regarding problem parameters, in a natural language rather than precise numerical form. Linguistic representation models have been widely used to solve many decision-making problems with qualitative information. Game theory has been found successful applications in a wide range of areas. This paper presents an extensive study of matrix games with qualitative payoffs. The work uses 2-tuple intuitionistic fuzzy linguistic values (2-TIFLVs) to represent the payoffs of the matrix game. We develop the mathematical formulation and concepts of the solutions for matrix games with payoffs represented by 2-TIFLVs. Paper also shows that matrix games with payoffs of 2-TIFLVs have solutions that can be obtained by transforming the matrix game in a pair of linear/nonlinear programming problems. Finally, a real-life numerical is given to illustrate the developed method.

Keywords: Matrix game, linguistic variables, 2-tuple linguistic model, 2-tuple intuitionistic fuzzy linguistic set, nonlinear optimization.

1 Introduction

The intuitionistic fuzzy set (IFS), introduced by Atanassov [4], is a useful tool to describe the uncertain and vague concepts more precisely. An IFS is characterized by the membership degree and the nonmembership degree, respectively, which provide us additional flexibility to handle the uncertain data in real-life decision-making problems. The IFS has received more and more attention from researchers working in different disciplines since its appearance. The applications of IFSs have been reported in a wide range of areas, including decision making [47, 48, 49, 50, 52], medical diagnosis [13, 31, 43], clustering analysis [22, 46], pattern recognition [18], image segmentation [3, 12, 33] and time-series analysis [19, 28].

Game theory is an important research area of operation research, which mainly concerns with competitive and skillful interaction between the rational, intelligent decision-makers. So, how to make decisions in competitive situations is an interesting and relevant research problem. After the pioneering work of Neumann and Morgenstern [55], game theory has been used successfully in economics, social sciences, business model, management sciences, and engineering [20, 29, 38]. Numerous studies have been reported in the literature associated with matrix games with crisp/precise payoffs. However, in many real-world situations, the payoffs of the matrix game are not known precisely due to the presence of uncertainty or insufficient information about the problem data. To resolve this issue, fuzzy numbers [14] were used in matrix games to represent imprecise or uncertain payoffs. There have been several studies in the literature discussing the problem of fuzzy games. Firstly, fuzzy games were studied by Aubin[6] and Butnariu [9]. Campos [10] proposed some solution methods for matrix games with fuzzy payoffs. Sakawa and Nishizaki [37] studied zero-sum fuzzy matrix games with multi-objectives. Maeda [30] defined matrix equilibrium to fuzzy matrix games by using the fuzzy max order. In 2004, Bector et al. [7] developed a fuzzy linear programming duality based approach to solve matrix games with fuzzy goals. Later, Bector et al.[8] proved the duality result for linear programming problems with

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fuzzy parameters and utilized them to solve zero-sum matrix games with fuzzy payoffs. Cevikel and Ahlatcioglu [11] developed two models for studying two-person zero-sum matrix games with fuzzy payoffs and fuzzy goals. They also proved that the fuzzy relation approach and the max-min solution approach are equivalent. Further, by using the duality theory of the fuzzy relation approach, Vidyottama et al. [53] presented a generalized model to study matrix games with fuzzy goals and fuzzy payoffs. Jana and Roy [17] examined the solution of matrix games with generalized trapezoidal fuzzy payoffs.

The matrix game with payoff represented by IFS was initially studied by Atanassov [5]. Li and Nan [25] developed a nonlinear programming method for solving matrix games with payoffs represented by IFSs. Nan and Li [34] computed the average index of an intuitionistic fuzzy value to study matrix games with payoffs of triangular intuitionistic fuzzy numbers (TIFNs). Li [24] solved matrix games with interval-valued intuitionistic fuzzy payoffs by using linear/nonlinear programming approaches. Aggarwal et al. [1, 2] extended the work of Bector et al. [7] and Vijay et al.[54] to study intuitionistic fuzzy linear programming duality and discussed its application in zero-sum matrix game. Xia [60] proposed a generalized method to solve matrix games with interval-valued intuitionistic fuzzy payoffs based on generalized aggregation operators. Khan et al. [21] given an approach to address matrix games with intuitionistic fuzzy goals by resolving hesitancy degrees from each goal. Recently, Naqvi et al. [35] were designed a solution procedure for intuitionistic fuzzy matrix games by using Tanaka and Asais approach.

In many real-life decision-making problems, the experts may think of using the linguistic variables to express their assessment information rather than numerical ones [15]. For this, Zadeh [64, 63] proposed the concept of a linguistic variable (LV) and applied it to fuzzy reasoning. After that, many researchers have paid great attention to solve decision-making problems with LVs [23, 42, 45, 51, 57, 61, 66, 67]. In order to avoid the loss and distortion of information in the linguistic information process, Herrera et al. [16] proposed a 2-tuple linguistic representation model, which is a useful tool for handling decision-making problems with qualitative information. Since then, the 2-tuple linguistic representation model has been widely studied by researchers working in different application areas [26, 27, 32, 36, 41, 56, 58]. In 2014, Zhang [65] introduced the notion of the linguistic intuitionistic fuzzy set (LIFS) in which the membership degree and nonmembership degree are expressed by the linguistic values. Thus, by considering the fact that the LIFS unifies the LV and IFS in a single formulation, so it can easily communicate the qualitative, as well as the quantitative aspects. Further, Verma [44] generalized the theory of LIFSs by introducing linguistic trapezoidal fuzzy intuitionistic fuzzy sets (LTFIFSs) and discussed their application in multiple attribute group decision making. In 2019, Wei et al. [59] proposed the concept of 2-tuple intuitionistic fuzzy linguistic set (2-TIFLS) in which both the degrees of membership are represented in terms of 2-tuple linguistic values. It is a hybrid information representation model that integrates the advantages of 2-tuple linguistic variables with IFSs. He also developed some arithmetic and geometric aggregation operators for aggregating 2-tuple intuitionistic fuzzy linguistic values (2-TIFLVs) and utilized them to solve enterprise resource planning (ERP) system selection problem.

Recently, Singh et al. [40] studied matrix games with 2-tuple linguistic payoffs to consider the qualitative aspects related to matrix games. Besides, Singh and Gupta [39] developed a solution approach for matrix games with interval-valued 2-tuple linguistic information. It is worth mentioning that the 2-TIFLVs are very efficient tools for representing qualitative information by using member and nonmembership degrees in real-world decision-making problems. The 2-TIFLVs provide more flexibility to represent realistic evaluation information that is more close to the human cognition. It also generalizes the notion of LV and 2-TLV. Therefore, it will be very helpful to represent the qualitative payoffs of a matrix game in terms of 2-TIFLVs instead of 2-TLVs. To the best of our knowledge, there is no study of matrix games in which the payoffs are represented by 2-TIFLVs. Therefore, the main objective of this work is to study matrix games with payoff values as 2-TIFLNs. To do so, first, we present the mathematical formulation of the matrix games with payoffs represented by 2-TIFLVs, and the concepts of their solutions are defined. It is shown that solving such a game is equivalent to solve a pair of nonlinear bi-objective programming problems. Besides, we obtain the optimal mixed strategies and optimal values corresponding to Player I and Player II, respectively, with the expected value of the game. Finally, we give a numerical example to illustrate the solution procedure of developed method.

The remaining work is organized as follows: Section 2 presents some preliminary results on LVs, 2-TLVs, 2-TIFLSs, and the conventional matrix game. Section 3 formulates the matrix games with payoffs represented by 2-TIFLVs, and concepts of their solutions are defined. It is also proved that matrix games with payoffs of 2-TIFLVs have solutions, which can be obtained by solving a pair of nonlinear bi-objective programming models. In Section 4 a real-life numerical example is given to illustrate the solution steps of the presented approach, and the work is concluded with some potential future directions for research in Section 5.

2 Preliminaries

In this section, some basic definition and results with regard to LVs, 2-TLVs, 2-TIFLSs and the conventional matrix game are presented, which will be used for further development.

Definition 2.1. [4] *Let M be the finite universe of discourse. An IFS $\tilde{A}$ in M is given by*

$$\tilde{A} = \{\langle m, \zeta_{\tilde{A}}(m), \kappa_{\tilde{A}}(m)\rangle \mid m \in M\},$$

where $\zeta_{\tilde{A}} : M \to [0,1]$ and $\kappa_{\tilde{A}} : M \to [0,1]$ are the degree of membership (DM) and degree of non-membership (DNM) of an element $m \in M$, respectively, to the set $\tilde{A}$ such that $0 \leq \zeta_{\tilde{A}}(m) + \kappa_{\tilde{A}}(m) \leq 1$. The degree of hesitation or intuitionistic index of $m \in M$ to $\tilde{A}$ is defined by $\pi_{\tilde{A}}(m) = 1 - \zeta_{\tilde{A}}(m) - \kappa_{\tilde{A}}(m)$; $m \in M$.
For convenience, Xu and Yager [62] called the pair $\langle \zeta_{\tilde{A}}(m), \kappa_{\tilde{A}}(m)\rangle$ an intuitionistic fuzzy value (IFV) and simply denoted by $\varrho = \langle \zeta_\varrho, \kappa_\varrho \rangle$, where $\zeta_\varrho \geq 0$, $\kappa_\varrho \geq 0$ and $0 \leq \zeta_\varrho + \kappa_\varrho \leq 1$.

In many real-world situations, a lot of information to make a decision is qualitative in nature that can not be described by using numerical quantities. Therefore, in order to deal with such type of information, Zadeh [64, 63] introduced the fuzzy linguistic approach. It represents qualitative aspects in term of linguistic values by means of LVs. Herrera and Martinez [16] defined the LVs as follows.

2.1 Linguistic variable

Definition 2.2. [16] *Let $\mathbb{S} = \{\mathfrak{s}_d \mid d = 0, 1, \ldots, t\}$ be a totally ordered discrete linguistic term set (LTS) with the odd cardinality, where t is a positive integer. Any level $\mathfrak{s}_d$ represents a possible value for a linguistic variable. The LTS $\mathbb{S}$ should satisfy the following properties:*

(i) Order relation: If $\mathfrak{s}_i \leq \mathfrak{s}_j \Leftrightarrow i \leq j$
(ii) Negation operator: $neg\,(\mathfrak{s}_d) = \mathfrak{s}_{t-d}$
(iii) Maximum operator: $\max(\mathfrak{s}_i, \mathfrak{s}_j) = \mathfrak{s}_i \Leftrightarrow i \geq j$
(iv) Minimum operator: $\min(\mathfrak{s}_i, \mathfrak{s}_j) = \mathfrak{s}_i \Leftrightarrow i \leq j$.

For example, a set of nine linguistic terms can be defined as:

$$\mathbb{S} = \left\{ \begin{array}{l} \mathfrak{s}_0 = \text{extremely dissatisfied (ED)}, \mathfrak{s}_1 = \text{very dissatisfied (VD)}, \mathfrak{s}_2 = \text{moderately dissatisfied (MD)}, \\ \mathfrak{s}_3 = \text{ slightly dissatisfied (SD)}, \mathfrak{s}_4 = \text{neither satisfied nor dissatisfied (NSND)}, \mathfrak{s}_5 = \text{slightly satisfied(SS)}, \\ \mathfrak{s}_6 = \text{moderately satisfied (MS)}, \mathfrak{s}_7 = \text{very satisfied (VS)}, \mathfrak{s}_8 = \text{extremely satisfied (ES)} \end{array} \right\}.$$

To preserve all the given information, Xu [61] extend the discrete term set $\mathbb{S}$ to a continuous term set $\mathbb{S}_{[0,t]} = \{\mathfrak{s}_d | \mathfrak{s}_0 \leq \mathfrak{s}_d \leq \mathfrak{s}_t, d \in [0,t]\}$, whose elements also meet all the characteristics as defined above. If $\mathfrak{s}_d \in \mathbb{S}$, then $\mathfrak{s}_d$ is called the original linguistic term (OLT), otherwise, $\mathfrak{s}_d$ is known as the virtual linguistic term (VLT).

2.2 2-Tuple linguistic representation model

Herrera and Martínez [16] developed 2-tuple linguistic representation model based on the concept of symbolic translation. It represents the linguistic assessment information by means of a 2-tuple $(\mathfrak{s}_d, \alpha)$, where $\mathfrak{s}_d$ is a LT from predefined LTS $\mathbb{S}$ whose semantics is provided by a fuzzy membership function. On the other hand α is a numerical value (symbolic translation), that indicates the translation of the fuzzy membership function which represents the closest linguistic term $\mathfrak{s}_d \in \{\mathfrak{s}_0, \mathfrak{s}_1, \ldots, \mathfrak{s}_t\}$. The value of α is defined as

$$\alpha = \begin{cases} [-0.5, 0.5)\,, & \text{if } \mathfrak{s}_d \in \{\mathfrak{s}_1, \ldots, \mathfrak{s}_{t-1}\}\,, \\ [0, 0.5)\,, & \text{if } \mathfrak{s}_d = \mathfrak{s}_0, \\ [-0.5, 0)\,, & \text{if } \mathfrak{s}_d = \mathfrak{s}_t. \end{cases}$$

Definition 2.3. [16] *Let $(\mathfrak{s}_d, \alpha)$ and $(\mathfrak{s}_e, \beta)$ be two 2-tuple linguistic values (2-TLVs). Then the lexicographic ordering between two 2-TLVs is defined as*
(i) If $d < e$, then $(\mathfrak{s}_d, \alpha)$ is smaller than $(\mathfrak{s}_e, \beta)$.
(ii) If $d = e$, then

(a) if $\alpha = \beta$, then $(\mathfrak{s}_d, \alpha) = (\mathfrak{s}_e, \beta)$ and represent the same information,
(b) if $\alpha < \beta$, then $(\mathfrak{s}_d, \alpha) < (\mathfrak{s}_e, \beta)$,
(c) if $\alpha > \beta$, then $(\mathfrak{s}_d, \alpha) > (\mathfrak{s}_e, \beta)$.

Definition 2.4. [16] *Let* $\mathbb{S} = \{\mathfrak{s}_d \,|\, d = 0, 1, \ldots, t\}$ *be a LTS and* $\eta \in [0, t]$ *be a real number in the interval* $[0, t]$ *which denotes the symbolic aggregation result. The 2-TLV that expresses the equivalent information with* η *can be obtained by the following function*

$$\Delta : [0, t] \to \mathbb{S} \times [-0.5, 0.5),$$

where

$$\Delta(\eta) = (\mathfrak{s}_d, \alpha), \text{ such that} \begin{cases} d = round(\eta) \\ \alpha = \eta - d, \qquad \alpha \in [-0.5, 0.5), \end{cases}$$

and $round(\cdot)$ *is the usual round operation that assigns the closest integer number* $d = \{0, 1, \ldots, t\}$ *to* η.

Definition 2.5. [16] *Let* $\mathbb{S} = \{\mathfrak{s}_d \,|\, d = 0, 1, \ldots, t\}$ *be a LTS and* $(\mathfrak{s}_d, \alpha)$ *be a 2-TLV. Then, there is a function* Δ^{-1}, *defined by*

$$\Delta^{-1} : \mathbb{S} \times [-0.5, 0.5) \to [0, t], \;\; \Delta^{-1}(\mathfrak{s}_d, \alpha) = (d + \alpha) = \eta.$$

which converts a 2-TLV into an equivalent numerical value $\eta \in [0, t]$.

Definition 2.6. [16] *Let* $(\mathfrak{s}_d, \alpha)$ *be a 2-TLV and* $\mathbb{S} = \{\mathfrak{s}_d \,|\, d = 0, 1, \ldots, t\}$. *The negation operator for a 2-tuple is defined as follows :*

$$neg\left((\mathfrak{s}_d, \alpha)\right) = \Delta\left(t - \Delta^{-1}(\mathfrak{s}_d, \alpha)\right).$$

Remark 2.7. *Based on Definitions 2.4 and 2.5, we can see that a linguistic term* $\mathfrak{s}_d \in \mathbb{S}$ *can be expressed by the 2-tuple* $(\mathfrak{s}_d, 0)$.

2.3 2-tuple intuitionistic fuzzy linguistic set

To better deal with qualitative information in real-world decision environment, Zhang et al [65] introduced the notion of the linguistic intuitionistic fuzzy set (LIFS) by unifying the concept of LVs with IFSs. A LIFS takes its membership and nonmembership values by means of linguistic terms.

Definition 2.8. [65] *Let* M *be a finite universal set and* $\mathbb{S}_{[0,t]} = \{\mathfrak{s}_d \mid \mathfrak{s}_0 \leq \mathfrak{s}_d \leq \mathfrak{s}_t\}$ *be a continuous LTS. A LIFS set* $\hat{A}$ *in* M *is given by*

$$\hat{A} = \left\{\langle m, \mathfrak{s}_{\zeta_{\hat{A}}(m)}, \mathfrak{s}_{\kappa_{\hat{A}}(m)}\rangle \mid m \in M\right\},$$

where $\mathfrak{s}_{\zeta_{\hat{A}}(m)}, \mathfrak{s}_{\kappa_{\hat{A}}(m)} \in \mathbb{S}_{[0,t]}$ *represent the DM and DNM of the element* $m \in M$ *to* $\hat{A}$, *respectively, such that* $0 \leq \zeta_{\hat{A}}(m) + \kappa_{\hat{A}}(m) \leq t$. *The intuitionistic index of* $m \in M$ *to the set* $\hat{A}$ *is obtained by* $\mathfrak{s}_{\pi_{\hat{A}}(m)} = \mathfrak{s}_{t - \zeta_{\hat{A}}(m) - \kappa_{\hat{A}}(m)}$. *For a given element* $m \in M$, *the pair* $\langle(\mathfrak{s}_{\zeta_{\hat{A}}(m)}, \mathfrak{s}_{\kappa_{\hat{A}}(m)})\rangle$ *is called a linguistic intuitionistic fuzzy number (LIFN), which can be simply written as* $\gamma = \langle \mathfrak{s}_{\zeta_\gamma}, \mathfrak{s}_{\kappa_\gamma}\rangle$.

Definition 2.9. [65] *Let* $\gamma = \left\langle \mathfrak{s}_{\zeta_\gamma}, \mathfrak{s}_{\kappa_\gamma}\right\rangle$, $\gamma_1 = \left\langle \mathfrak{s}_{\zeta_{\gamma_1}}, \mathfrak{s}_{\kappa_{\gamma_1}}\right\rangle$ *and* $\gamma_2 = \left\langle \mathfrak{s}_{\zeta_{\gamma_2}}, \mathfrak{s}_{\kappa_{\gamma_2}}\right\rangle$ *be three LIFNs, then*

(i) $\gamma_1 \precsim \gamma_2$ *if* $\mathfrak{s}_{\zeta_{\gamma_1}} \leq \mathfrak{s}_{\zeta_{\gamma_2}}$ *and* $\mathfrak{s}_{\kappa_{\gamma_1}} \geq \mathfrak{s}_{\kappa_{\gamma_2}}$;

(ii) $\gamma_1 = \gamma_2$ *if and only if* $\mathfrak{s}_{\zeta_{\gamma_1}} = \mathfrak{s}_{\zeta_{\gamma_2}}$ *and* $\mathfrak{s}_{\kappa_{\gamma_1}} = \mathfrak{s}_{\kappa_{\gamma_2}}$;

(iii) $\gamma^C = \left\langle \mathfrak{s}_{\kappa_\gamma}, \mathfrak{s}_{\zeta_\gamma}\right\rangle$;

(iv) $\gamma_1 \cup \gamma_2 = \left\langle \max\left(s_{\zeta_{\gamma_1}}, s_{\zeta_{\gamma_2}}\right), \min\left(s_{\kappa_{\gamma_1}}, s_{\kappa_{\gamma_2}}\right)\right\rangle$;

(v) $\gamma_1 \cap \gamma_2 = \left\langle \min\left(s_{\zeta_{\gamma_1}}, s_{\zeta_{\gamma_2}}\right), \max\left(s_{\kappa_{\gamma_1}}, s_{\kappa_{\gamma_2}}\right)\right\rangle$.

Recently, Wei [59] extended the idea of LIFS and proposed the 2-tuple intuitionistic fuzzy linguistic set (2-TIFLS) on the basis of 2-TL information processing model [16].

Definition 2.10. [59] *Let* M *be a finite universal set and* $\mathbb{S} = \{\mathfrak{s}_d \,|\, d = 0, 1, \ldots, t\}$ *be a LTS. A 2-TIFLS set* A *in* M *can be define as*

$$A = \left\{\langle m, \left(\mathfrak{s}_{\zeta_A(m)}, \alpha\right), \left(\mathfrak{s}_{\kappa_A(m)}, \beta\right)\rangle \mid m \in M\right\},$$

where $\left(\mathfrak{s}_{\zeta_A(m)}, \alpha\right)$ *and* $\left(\mathfrak{s}_{\kappa_A(m)}, \beta\right)$ *are 2-TLVs, respectively, denote the DM and DNM of the element* $m \in M$ *to* A, *such that* $0 \leq \Delta^{-1}\left(\mathfrak{s}_{\zeta_A(m)}, \alpha\right) + \Delta^{-1}\left(\mathfrak{s}_{\kappa_A(m)}, \beta\right) \leq t$. *Note that here* $\mathfrak{s}_{\zeta_A(m)}, \mathfrak{s}_{\kappa_A(m)}$ *are linguistic terms from predefined LTS* $\mathbb{S}$ *and* α, β *are the value of symbolic translation, and* $\alpha, \beta \in [-0.5, 0.5)$.
For convenience, Wei [59] called the pair $\left(\left(\mathfrak{s}_{\zeta_A(m)}, \alpha\right), \left(\mathfrak{s}_{\kappa_A(m)}, \beta\right)\right)$ *a 2-tuple intuitionistic fuzzy linguistic value (2-TIFLV) and simply represented by* $\wp = \left\langle\left(\mathfrak{s}_{\zeta_\wp}, \alpha\right), \left(\mathfrak{s}_{\kappa_\wp}, \beta\right)\right\rangle$ *with* $0 \leq \Delta^{-1}(\mathfrak{s}_{\zeta_\wp}, \alpha) + \Delta^{-1}(\mathfrak{s}_{\kappa_\wp}, \beta) \leq t$. *Let* Ω *denote the set of all 2-TIFLVs defined in the LTS* $\mathbb{S}$.

Remark 2.11. *Similar to LIFV, the 2-TIFLV* $\wp = \left\langle\left(\mathfrak{s}_{\zeta_\wp}, \alpha\right), \left(\mathfrak{s}_{\kappa_\wp}, \beta\right)\right\rangle$ *can be transformed into an interval-valued 2-tuple linguistic value* $\breve{\wp} = \left[\left(\mathfrak{s}_{\zeta_{\breve{\wp}}}, \alpha\right), \Delta\left(t - \Delta^{-1}\left(\mathfrak{s}_{\kappa_{\breve{\wp}}}, \beta\right)\right)\right] = \left[\left(\mathfrak{s}_{\zeta_{\breve{\wp}}}, \breve{\alpha}\right), \left(\mathfrak{s}_{\kappa_{\breve{\wp}}}, \breve{\beta}\right)\right]$. *Further, if* $\mathfrak{s}_{\zeta_{\breve{\wp}}} = \mathfrak{s}_{\kappa_{\breve{\wp}}}$ *and* $\breve{\alpha} = \breve{\beta}$, *then we get 2-TLV.*

Definition 2.12. [59] *Let $\mathbb{S}$ be a LTS and $(\xi,\eta)\in[0,t]$ be an ordered numerical pair representing the aggregation results of linguistic symbolic. Then the function Δ is used to obtain the 2-TIFL information equivalent to (ξ,η) as follows*

$$\Delta:[0,t]\times[0,t]\to(\mathbb{S}\times[-0.5,0.5))\times(\mathbb{S}\times[-0.5,0.5)),$$

such that

$$\Delta(\xi,\eta)=(\Delta(\xi),\Delta(\eta))=\left((\mathfrak{s}_{\zeta_\wp},\alpha),(\mathfrak{s}_{\kappa_\wp},\beta)\right),$$

where Δ is the translation function given in Definition 2.4 which gives $\zeta_\wp=round(\xi)$, $\kappa_\wp=round(\eta)$, $\alpha=\xi-\zeta_\wp$, $\beta=\eta-\kappa_\wp$, and $\alpha,\beta\in[-0.5,0.5)$. The $round(\cdot)$ is the usual round operation.

Definition 2.13. [59] *Let $\mathbb{S}$ be a LTS and $\left\langle(\mathfrak{s}_{\zeta_\wp},\alpha),(\mathfrak{s}_{\kappa_\wp},\beta)\right\rangle$ be a 2-TIFLV. There is always a function Δ^{-1}, which can convert the 2-TIFLV $\left\langle(\mathfrak{s}_{\zeta_\wp},\alpha),(\mathfrak{s}_{\kappa_\wp},\beta)\right\rangle$ to $(\xi,\eta)\in[0,t]$ with the equivalent information, shown as follows*

$$\Delta^{-1}:(\mathbb{S}\times[-0.5,0.5))\times(\mathbb{S}\times[-0.5,0.5))\to[0,t]\times[0,t],$$

such that

$$\Delta^{-1}\left\langle(\mathfrak{s}_{\zeta_\wp},\alpha),(\mathfrak{s}_{\kappa_\wp},\beta)\right\rangle=\left(\Delta^{-1}(\mathfrak{s}_{\zeta_\wp},\alpha),\Delta^{-1}(\mathfrak{s}_{\kappa_\wp},\beta)\right)=(\zeta_\wp+\alpha,\kappa_\wp+\beta)=(\xi,\eta).$$

Definition 2.14. [59] *Let $\wp=\left\langle(\mathfrak{s}_{\zeta_\wp},\alpha),(\mathfrak{s}_{\kappa_\wp},\beta)\right\rangle$ be a 2-TIFLV, the score value $\Delta^{-1}(S(\wp))$ and the accuracy value $\Delta^{-1}(A(\wp))$ of $\wp$ can be defined as*

$$\Delta^{-1}(S(\wp))=\left(\frac{t+\Delta^{-1}(\mathfrak{s}_{\zeta_\wp},\alpha)-\Delta^{-1}(\mathfrak{s}_{\kappa_\wp},\beta)}{2}\right);\quad \Delta^{-1}S(\wp)\in[0,t]$$

and

$$\Delta^{-1}(A(\wp))=\left(\Delta^{-1}(\mathfrak{s}_{\zeta_\wp},\alpha)+\Delta^{-1}(\mathfrak{s}_{\kappa_\wp},\beta)\right);\quad \Delta^{-1}A(\wp)\in[0,t].$$

Based on these values, an order relation between two 2-TIFLVs $\wp_1=\left\langle(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1),(\mathfrak{s}_{\kappa_{\wp_1}},\beta_1)\right\rangle$ and $\wp_2=\left\langle(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2),(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)\right\rangle$, is defined as follows:

(i) If $\Delta^{-1}(S(\wp_1))>\Delta^{-1}(S(\wp_2))$ then, $\wp_1\succ\wp_2$;
(ii) If $\Delta^{-1}(S(\wp_1))=\Delta^{-1}(S(\wp_2))$, then

(a) $\Delta^{-1}(A(\wp_1))>\Delta^{-1}(A(\wp_2))$, then $\wp_1\succ\wp_2$; *(b)* $\Delta^{-1}(A(\wp_1))=\Delta^{-1}(A(\wp_2))$, then $\wp_1=\wp_2$.

Definition 2.15. *Let $\wp=\left\langle(\mathfrak{s}_{\zeta_\wp},\alpha),(\mathfrak{s}_{\kappa_\wp},\beta)\right\rangle$, $\wp_1=\left\langle(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1),(\mathfrak{s}_{\kappa_{\wp_1}},\beta_1)\right\rangle$ and $\wp_2=\left\langle(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2),(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)\right\rangle\in\Omega$, then operational laws for 2-TIFLVs analogous to Definition 2.9 can be defined as follows:*

(i) $\wp_1\precsim\wp_2$ if $\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1)\leq\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2)$ *and* $\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_1}},\beta_1)\geq\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)$;
(ii) $\wp_1=\wp_2$ if and only if $\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1)=\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2)$ and $\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_1}},\beta_1)=\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)$;
(iii) $\wp^C=\left\langle(\mathfrak{s}_{\kappa_\wp},\beta),(\mathfrak{s}_{\zeta_\wp},\alpha)\right\rangle$;
(iv) $\wp_1\cup\wp_2=\left\langle\Delta\left(\max\left(\Delta^{-1}\left(s_{\zeta_{\wp_1}},\alpha_1\right),\Delta^{-1}\left(s_{\zeta_{\wp_2}},\alpha_2\right)\right)\right),\Delta\left(\min\left(\Delta^{-1}\left(s_{\kappa_{\wp_1}},\beta_1\right),\Delta^{-1}\left(s_{\kappa_{\wp_2}},\beta_{\wp_2}\right)\right)\right)\right\rangle$;
(v) $\wp_1\cap\wp_2=\left\langle\Delta\left(\min\left(\Delta^{-1}\left(s_{\zeta_{\wp_1}},\alpha_1\right),\Delta^{-1}\left(s_{\zeta_{\wp_2}},\alpha_2\right)\right)\right),\Delta\left(\max\left(\Delta^{-1}\left(s_{\kappa_{\wp_1}},\beta_1\right),\Delta^{-1}\left(s_{\kappa_{\wp_2}},\beta_{\wp_2}\right)\right)\right)\right\rangle$.

Definition 2.16. [59] *Let $\wp=\left\langle(\mathfrak{s}_{\zeta_\wp},\alpha),(\mathfrak{s}_{\kappa_\wp},\beta)\right\rangle$, $\wp_1=\left\langle(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1),(\mathfrak{s}_{\kappa_{\wp_1}},\beta_1)\right\rangle$ and $\wp_2=\left\langle(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2),(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)\right\rangle\in\Omega$, then some algebraic operations are defined as follows:*

(i) $\wp_1\oplus\wp_2=\left\langle\Delta(t(\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1)}{t}+\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2)}{t}-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1)}{t}.\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2)}{t})),\Delta(t(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_1}},\beta_1)}{t}.\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)}{t}))\right\rangle$;
(ii) $\wp_1\otimes\wp_2=\left\langle\Delta(t(\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_1}},\alpha_1)}{t}.\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_2}},\alpha_2)}{t})),\Delta(t(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_1}},\alpha_1)}{t}+\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)}{t}-\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_1}},\beta_1)}{t}.\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_2}},\beta_2)}{t}))\right\rangle$;
(iii) $\theta\wp=\left\langle\Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\wp},\alpha)}{t}\right)^\theta\right)\right),\Delta\left(t\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\wp},\beta)}{t}\right)^\theta\right)\right\rangle$;
(iv) $\wp^\theta=\left\langle\Delta\left(t\left(\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\wp},\alpha)}{t}\right)^\theta\right),\Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\wp},\beta)}{t}\right)^\theta\right)\right)\right\rangle$.

Definition 2.17. [59] *Let $\wp_i=\left\langle(\mathfrak{s}_{\zeta_{\wp_i}},\alpha_i),(\mathfrak{s}_{\kappa_{\wp_i}},\beta_i)\right\rangle,(i=1,2,\dots,n)$ be a collection of n 2-TIFLVs and $2-TIFLWA:\Omega^n\to\Omega$, if*

$$2-TIFLWA(\wp_1,\wp_2,\dots,\wp_n)=\left\langle\Delta\left(t\left(1-\prod_{i=1}^{n}\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_i}},\alpha_i)}{t}\right)^{w_i}\right)\right),\Delta\left(t\left(\prod_{i=1}^{n}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_i}},\beta_i)}{t}\right)^{w_i}\right)\right)\right\rangle,\qquad(1)$$

then 2-TIFLWA is called the 2-tuple intuitionistic fuzzy linguistic weighted averaging (2-TIFLWA) operator, where $w=(w_1,w_2,\dots,w_n)^T$ is a weight vector of $\wp_i$ with $w_i>0$ and $\sum_{i=1}^{n}w_i=1$.

2.4 Two person zero sum matrix game in crisp environment

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. A matrix game is composed of two players, each of them can choose a finite number of strategies that are beneficial to them. Assume that $S^m = (\varphi_1, \varphi_2, \dots, \varphi_m)$ and $S^n = (\chi_1, \chi_2, \dots, \chi_n)$ are sets of pure strategies for Player I and Player II, respectively. A payoff matrix for Player I is represented by $A = [a_{ij}]_{m\times n}$, where a_{ij} are real numbers. Let $e^T = (1, 1, \dots, 1)$ be a vector of ones whose dimension is specified as per the specific context. The vectors $x = (x_1, x_2, \dots, x_m)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ are known as the mixed strategies of Player I and Player II,respectively, which satisfy $\sum_{i=1}^m x_i = 1, x_i \geq 0$ and $\sum_{j=1}^n y_j = 1, y_j \geq 0$. Here, $x_i\,(i = 1, 2, \dots, m)$ denotes the probability of Player I choosing strategy $\varphi_i \in S^m$ and y_j is the probability of Player II selecting strategy $\chi_j \in S^n$. The sets of all mixed strategies for Player I and Player II are denoted by

$$X = \left\{(x_1, x_2, \dots, x_m) : x_i \geq 0, i = 1, 2, \dots, m; e^T x = 1\right\}; \;\; Y = \left\{(y_1, y_2, \dots, y_n) : y_j \geq 0, j = 1, 2, \dots, n; e^T y = 1\right\}.$$

A two person zero sum matrix game G can be expressed by the following expression $G = (X, Y, A)$. If Player I plays the mixed strategy $x \in X$ and Player II plays the mixed strategy $y \in Y$, then expected payoff of Player I can be calculated by $E(x, y) = x^T Ay = \sum_{j=1}^n \sum_{i=1}^m x_i a_{ij} y_j$. Since the game is a zero sum game, the expected payoffs for Player II is $-x^T Ay$.

The Player I (the maximizing player) and Player II (the minimizing player) will select their strategies according to the maximin and minimax principles, respectively, then

$$v^- \text{ (the lower value of the game)} = \max_{x\in X} \min_{y\in Y} \left\{x^T Ay\right\}, \;\; v^+ \text{ (the upper value of the game)} = \min_{y\in Y} \max_{x\in X} \left\{x^T Ay\right\}.$$

It is well known that $v^+ \geq v^-$ and the game G has value if and only if $v^+ = v^-$. A triplet $(x^*, y^*, v^*) \in X \times Y \times \mathbb{R}$ is called a solution of the game G if

$$x^{*T} Ay \geq v^* \quad \forall\, y \in Y, \quad x^T Ay^* \leq v^* \quad \forall\, x \in X.$$

Here x^* and y^* are called the optimal strategy for Players I and II, respectively and v^* is called the value of the game G and $v^+ = v^- = v^*$.

3 Matrix games with 2-tuple linguistic intuitionistic fuzzy payoffs

3.1 Proposed model

A matrix game with payoffs represented by 2-TIFLVs can be expressed as $2 - \mathfrak{TLIFG} = \left(S^m, X, S^n, Y, \mathbb{S}, \widetilde{\mathcal{M}}\right)$, where S^m and S^n are strategy sets discussed above, $\mathbb{S} = \{\mathfrak{s}_d \,|\, d = 0, 1, \dots, t\}$ be a predefined LTS with odd cardinality, $\widetilde{\mathcal{M}}$ is the 2-TLIF payoff matrix of Player I against Player II, X and Y are mixed strategies spaces for Players I and II. The payoff matrix $\widetilde{\mathcal{M}} = (\wp_{ij})_{m\times n}$ for Player I is formed in such a way that if Player I chooses a pure strategy $\varphi_i \in S^m$ and Player II chooses a pure strategy $\chi_j \in S^n$, then the Player I gains a payoff $\wp_{ij} = \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij}\right), \left(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij}\right)\right\rangle$ with the conditions $\mathfrak{s}_{\zeta_{\wp_{ij}}}, \mathfrak{s}_{\kappa_{\wp_{ij}}} \in \mathbb{S}$, $\alpha_{ij}, \beta_{ij} \in [-0.5, 0.5)$ and $0 \leq \Delta^{-1}\left(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij}\right) + \Delta^{-1}\left(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij}\right) \leq t\ \forall i, j$. On the other hand, the Player II gains a payoff $\wp^C_{ij} = \left\langle \left(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij}\right), \left(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij}\right)\right\rangle$. The payoff matrix of Player I is represented as

$$\widetilde{\mathcal{M}} = (\wp_{ij})_{m\times n} = \begin{array}{c} \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_m \end{array} \begin{array}{c} \begin{array}{cccc} \chi_1 & \chi_2 & \cdots & \chi_n \end{array} \\ \begin{pmatrix} \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{11}}}, \alpha_{11}\right), \left(\mathfrak{s}_{\kappa_{\wp_{11}}}, \beta_{11}\right)\right\rangle & \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{12}}}, \alpha_{12}\right), \left(\mathfrak{s}_{\kappa_{\wp_{12}}}, \beta_{12}\right)\right\rangle & \cdots & \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{1n}}}, \alpha_{1n}\right), \left(\mathfrak{s}_{\kappa_{\wp_{1n}}}, \beta_{1n}\right)\right\rangle \\ \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{21}}}, \alpha_{21}\right), \left(\mathfrak{s}_{\kappa_{\wp_{21}}}, \beta_{21}\right)\right\rangle & \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{22}}}, \alpha_{22}\right), \left(\mathfrak{s}_{\kappa_{\wp_{22}}}, \beta_{22}\right)\right\rangle & \cdots & \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{2n}}}, \alpha_{2n}\right), \left(\mathfrak{s}_{\kappa_{\wp_{2n}}}, \beta_{2n}\right)\right\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{m1}}}, \alpha_{m1}\right), \left(\mathfrak{s}_{\kappa_{\wp_{m1}}}, \beta_{m1}\right)\right\rangle & \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{m2}}}, \alpha_{m2}\right), \left(\mathfrak{s}_{\kappa_{\wp_{m2}}}, \beta_{m2}\right)\right\rangle & \cdots & \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{mn}}}, \alpha_{mn}\right), \left(\mathfrak{s}_{\kappa_{\wp_{mn}}}, \beta_{mn}\right)\right\rangle \end{pmatrix} \end{array}.$$

Let $x = (x_1, x_2, \dots, x_m) \in X$ and $y = (y_1, y_2, \dots, y_m) \in Y$ be a pair of mixed strategies for Player I and Player II, respectively, then the expected payoff of Player I can be obtained by using the 2-TIFLWA operator mentioned in Eq.

(1) as

$$E(x,y) = x^T \tilde{\mathcal{M}} y = \left\langle \Delta\left(t\left(1-\prod_{j=1}^{n}\prod_{i=1}^{m}\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^{x_i y_j}\right)\right), \Delta\left(t\left(\prod_{j=1}^{n}\prod_{i=1}^{m}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{x_i y_j}\right)\right)\right\rangle$$
$$= \left\langle \left(\mathfrak{s}_{\zeta_{\wp_{xy}}},\alpha_{xy}\right), \left(\mathfrak{s}_{\kappa_{\wp_{xy}}},\beta_{xy}\right)\right\rangle. \quad (2)$$

Assume that Players I and II select their strategies according to the maximin and minimax principles, respectively. If there exists a pair of mix strategies $(x^{\circledast}, y^{\circledast})$ $(x^{\circledast} \in X, y^{\circledast} \in Y)$ so that they satisfy the following equality

$$x^{\circledast T} \tilde{\mathcal{M}} y^{\circledast} = \max_{x\in X}\min_{y\in Y}\left\{x^T \tilde{\mathcal{M}} y\right\} = \min_{y\in Y}\max_{x\in X}\left\{x^T \tilde{\mathcal{M}} y\right\}, \quad (3)$$

in the sense of the operations given in Definition 13, then $x^{\circledast}$ and $y^{\circledast}$ are known as optimal strategies for Players I and II, respectively, and $x^{\circledast T}\tilde{\mathcal{M}} y^{\circledast}$ is called the value of the matrix game with 2-TIFLVs. It is worth mentioning that $(x^{\circledast}, y^{\circledast})$ is called a saddle point of the matrix game $\tilde{\mathcal{M}}$. Usually, there do not exist optimal strategies $x^{\circledast}$ and $y^{\circledast}$ for Players I and II because the DM and DNM of 2-TLIFVs are usually conflicting one another and the ranking order defined by the Definition 2.14 is a partial order.
In fact, according to Eq. (2) and Definition 2.14, the optimization problem $x^T \tilde{\mathcal{M}} y$ may be regarded as 2-tuple linguistic bi-objective optimization programming problems, where one objective function is the 2-tuple linguistic function as follows:

$$\left(\mathfrak{s}_{\zeta_{\wp_{xy}}},\alpha_{xy}\right) = \Delta\left(t\left(1-\prod_{j=1}^{n}\prod_{i=1}^{m}\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^{x_i y_j}\right)\right),$$

and the another is the 2-tuple linguistic function

$$\left(\mathfrak{s}_{\kappa_{\wp_{xy}}},\beta_{xy}\right) = \Delta\left(t\left(\prod_{j=1}^{n}\prod_{i=1}^{m}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{x_i y_j}\right)\right).$$

Therefore, the concept of solutions of the matrix game $\tilde{\mathcal{M}}$ with payoffs represented by 2-TIFLVs can be defined similarly as Pareto optimal solutions of multiobjective decision making.

Definition 3.1. *Let $\hat{\wp}$ and $\hat{\hat{\wp}}$ be two 2-TIFLVs. If there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that $\bar{x}^T \tilde{\mathcal{M}} y \succsim \hat{\wp}$ and $x^T \tilde{\mathcal{M}} \bar{y} \precsim \hat{\hat{\wp}}$ for any $x \in X$, $y \in Y$, then $(\bar{x}, \bar{y}, \hat{\wp}, \hat{\hat{\wp}})$ is called the feasible solution of the matrix game $\tilde{\mathcal{M}}$ with payoffs represented by 2-TIFLVs, $\hat{\wp}$ and $\hat{\hat{\wp}}$ are known as the feasible values for the Player I and II, respectively, and $\bar{x}$ and $\bar{y}$ are called feasible strategies for the Players I and II, respectively.*

Definition 3.2. *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the sets of all feasible values for the Players I and II, respectively and there exist $\wp^* \in \mathcal{Z}_1$ and $\wp^{**} \in \mathcal{Z}_2$. If there do not exist $\hat{\wp} \in \mathcal{Z}_1$ and $\hat{\hat{\wp}} \in \mathcal{Z}_2$ such that $\hat{\wp} \succsim \wp^*$and $\hat{\hat{\wp}} \precsim \wp^{**}$, with $\hat{\wp} \neq \wp^*$and $\hat{\hat{\wp}} \neq \wp^{**}$, then $(x^*, y^*, \wp^*, \wp^{**})$ is called the solution of the matrix game $\tilde{\mathcal{M}}$ with 2-TLIFV payoffs. Here, x^* is called a maximin strategy for Player I and y^* is called a minimax strategy for Player II; $\wp^*$ and $\wp^{**}$ are called the values of the matrix game $\tilde{\mathcal{M}}$ with 2-TLIFV payoffs for Players I and II, respectively.*

3.2 Mathematical programming approach for the solution of the game

The minimum expected gain Θ for Player I can be given as:

$$\Theta = \left\langle (\mathfrak{s}_{\zeta_\Theta}, \dot{\alpha}), \left(\mathfrak{s}_{\kappa_\Theta}, \dot{\beta}\right)\right\rangle = \left\langle \min_{y\in Y}\left(\mathfrak{s}_{\zeta_{\wp_{xy}}},\alpha_{xy}\right), \max_{y\in Y}\left(\mathfrak{s}_{\kappa_{\wp_{xy}}},\beta_{xy}\right)\right\rangle.$$

Note that here Θ is the function of x only. Then, to maximize the value of Θ, the Player I will select a mixed strategy $x^* \in X$, i.e.,

$$\Theta^* = \left\langle (\mathfrak{s}_{\zeta_{\Theta^*}}, \hat{\alpha}), \left(\mathfrak{s}_{\kappa_{\Theta^*}}, \hat{\beta}\right)\right\rangle = \left\langle \max_{x\in X}\min_{y\in Y}\left(\mathfrak{s}_{\zeta_{\wp_{xy}}},\alpha_{xy}\right), \min_{x\in X}\max_{y\in Y}\left(\mathfrak{s}_{\kappa_{\wp_{xy}}},\beta_{xy}\right)\right\rangle,$$

where Θ^* is the minimum gain for Player I and the mixed strategy x^* is called the maximin strategy for Player I.

The maximum expected loss Ψ for Player II is represented as:

$$\Psi = \left\langle (\mathfrak{s}_{\zeta_\Psi}, \ddot{\alpha}), \left(\mathfrak{s}_{\kappa_\Psi}, \ddot{\beta}\right)\right\rangle = \left\langle \max_{x\in X}\left(\mathfrak{s}_{\zeta_{\wp_{xy}}},\alpha_{xy}\right), \min_{x\in X}\left(\mathfrak{s}_{\kappa_{\wp_{xy}}},\beta_{xy}\right)\right\rangle.$$

Since Ψ is the function of y only, then, to minimize the value of Ψ, the Player II should choose a mixed strategy $y^* \in Y$, i.e.,

$$\Psi^* = \left\langle \left(\mathfrak{s}_{\zeta_{\Psi^*}}, \hat{\hat{\alpha}}\right), \left(\mathfrak{s}_{\kappa_{\Psi^*}}, \hat{\hat{\beta}}\right) \right\rangle = \left\langle \min_{y\in Y}\max_{x\in X}\left(\mathfrak{s}_{\zeta_{\wp xy}}, \alpha_{xy}\right), \max_{y\in Y}\min_{x\in X}\left(\mathfrak{s}_{\kappa_{\wp xy}}, \beta_{xy}\right) \right\rangle,$$

where Ψ^* is the maximum loss for Player II and the mixed strategy y^* is called the minimax strategy for Player II.

Theorem 3.3. *Θ^* and Ψ^* are 2-TIFLVs and $\Theta^* \precsim \Psi^*$.*

Proof. From Eq. (2), it is clear that $E(x,y)$ is a 2-TLIFV. Then

$$\mathfrak{s}_{\zeta_{\wp xy}}, \mathfrak{s}_{\kappa_{\wp xy}} \in \mathbb{S},\ \alpha_{xy}, \beta_{xy} \in [-0.5, 0.5),\ \text{such that } 0 \le \Delta^{-1}\left(\mathfrak{s}_{\zeta_{\wp xy}}, \alpha_{xy}\right) + \Delta^{-1}\left(\mathfrak{s}_{\kappa_{\wp xy}}, \beta_{xy}\right) \le t.$$

Let $\Delta^{-1}\left(\mathfrak{s}_{\zeta_{\wp xy}}, \alpha_{xy}\right) = \zeta_{\wp xy} + \alpha_{xy} = \xi_{xy}$ and $\Delta^{-1}\left(\mathfrak{s}_{\kappa_{\wp xy}}, \beta_{xy}\right) = \left(\kappa_{\wp xy} + \beta_{xy}\right) = \eta_{xy}$, then

$$0 \le \xi_{xy} + \eta_{xy} \le t. \tag{4}$$

Using Definition 2.15, we get

$$\max_{x\in X}\min_{y\in Y}\left(\mathfrak{s}_{\zeta_{\wp xy}}, \alpha_{xy}\right) = \Delta\left(\max_{x\in X}\min_{y\in Y}\left(\Delta^{-1}\left(\mathfrak{s}_{\zeta_{\wp xy}}, \alpha_{xy}\right)\right)\right) = \Delta\left(\max_{x\in X}\min_{y\in Y}\left(\xi_{xy}\right)\right) = \left(\mathfrak{s}_{\zeta_{\Theta^*}}, \hat{\alpha}\right),$$
$$\min_{x\in X}\max_{y\in Y}\left(\mathfrak{s}_{\kappa_{\wp xy}}, \alpha_{xy}\right) = \Delta\left(\min_{x\in X}\max_{y\in Y}\left(\Delta^{-1}\left(\mathfrak{s}_{\kappa_{\wp xy}}, \alpha_{xy}\right)\right)\right) = \Delta\left(\min_{x\in X}\max_{y\in Y}\left(\eta_{xy}\right)\right) = \left(\mathfrak{s}_{\kappa_{\Theta^*}}, \hat{\beta}\right),$$

such that $\mathfrak{s}_{\zeta_{\Theta^*}}, \mathfrak{s}_{\kappa_{\Theta^*}} \in \mathbb{S},\ \hat{\alpha}, \hat{\beta} \in [-0.5, 0.5)$. In order to prove that $\Theta^* = \left\langle \left(\mathfrak{s}_{\zeta_{\Theta^*}}, \hat{\alpha}\right), \left(\mathfrak{s}_{\kappa_{\Theta^*}}, \hat{\beta}\right) \right\rangle$ is a 2-TLIFV, we have to show that

$$0 \le \Delta^{-1}\left(\mathfrak{s}_{\zeta_{\Theta^*}}, \hat{\alpha}\right) + \Delta^{-1}\left(\mathfrak{s}_{\kappa_{\Theta^*}}, \hat{\beta}\right) \le t \Leftrightarrow 0 \le \max_{x\in X}\min_{y\in Y}\left(\xi_{xy}\right) + \min_{x\in X}\max_{y\in Y}\left(\eta_{xy}\right) \le t.$$

Using Eq. (4), it can be easily obtained that

$$0 \le \xi_{xy} + \min_{y\in Y}\left(\eta_{xy}\right) \le \xi_{xy} + \max_{y\in Y}\left(\eta_{xy}\right) \le t.$$

Thus, we have

$$0 \le \min_{y\in Y}\left(\xi_{xy}\right) + \min_{y\in Y}\left(\eta_{xy}\right) \le \xi_{xy} + \max_{y\in Y}\left(\eta_{xy}\right) \le t.$$

It follows that

$$0 \le \min_{y\in Y}\left(\xi_{xy}\right) + \min_{x\in X}\min_{y\in Y}\left(\eta_{xy}\right) \le \min_{y\in Y}\left(\xi_{xy}\right) + \min_{x\in X}\max_{y\in Y}\left(\eta_{xy}\right) \le \xi_{xy} + \min_{x\in X}\max_{y\in Y}\left(\eta_{xy}\right) \le t.$$

Therefore, we have

$$0 \le \max_{x\in X}\min_{y\in Y}\left(\xi_{xy}\right) + \min_{x\in X}\min_{y\in Y}\left(\eta_{xy}\right) \le \max_{x\in X}\min_{y\in Y}\left(\xi_{xy}\right) + \min_{x\in X}\max_{y\in Y}\left(\eta_{xy}\right) \le \max_{x\in X}\left(\xi_{xy}\right) + \min_{x\in X}\max_{y\in Y}\left(\eta_{xy}\right) \le t,$$

which gives

$$0 \le \max_{x\in X}\min_{y\in Y}\left(\xi_{xy}\right) + \min_{x\in X}\max_{y\in Y}\left(\eta_{xy}\right) \le t. \tag{5}$$

Hence Θ^* is a 2-TLIFV. Similarly, we can also prove that Ψ^* is a 2-TLIFV.

For any $x \in X$ and $y \in Y$, we have

$$\min_{y\in Y}\left(\xi_{xy}\right) \le \left(\xi_{xy}\right) \le \max_{x\in X}\left(\xi_{xy}\right).$$

Hence

$$\min_{y\in Y}\left(\xi_{xy}\right) \le \min_{y\in Y}\max_{x\in X}\left(\xi_{xy}\right).$$

Therefore, we get

$$\max_{x\in X}\min_{y\in Y}\left(\xi_{xy}\right) \le \min_{y\in Y}\max_{x\in X}\left(\xi_{xy}\right) \Rightarrow \xi_{\Theta^*} \le \xi_{\Psi^*} \Leftrightarrow \Delta^{-1}\left(\mathfrak{s}_{\zeta_{\Theta^*}}, \hat{\alpha}\right) \le \Delta^{-1}\left(\mathfrak{s}_{\zeta_{\Psi^*}}, \hat{\hat{\alpha}}\right). \tag{6}$$

On the other hand, for any $x \in X$ and $y \in Y$, we have

$$\max_{y\in Y}\left(\eta_{xy}\right) \ge \left(\eta_{xy}\right) \ge \min_{x\in X}\left(\eta_{xy}\right).$$

Hence

$$\max_{y\in Y}\left(\eta_{xy}\right) \ge \max_{y\in Y}\min_{x\in X}\left(\eta_{xy}\right).$$

Therefore, we obtain

$$\min_{x\in X}\max_{y\in Y}(\eta_{xy}) \geq \max_{y\in Y}\min_{x\in X}(\eta_{xy}) \Rightarrow \eta_{\Theta^*} \geq \eta_{\Psi^*} \Leftrightarrow \Delta^{-1}\left(\mathfrak{s}_{\zeta_\Theta}, \hat{\beta}\right) \geq \Delta^{-1}\left(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\beta}}\right). \quad (7)$$

Using Definition 2.15 with inequalities given in Eqs. (6) and (7), we can conclude that

$$\Theta^* \precsim \Psi^*.$$

This completes the proof. □

The maximin strategy x^* and the minimum gain $\Theta^* = \left\langle (\mathfrak{s}_{\zeta_{\Theta^*}}, \hat{\alpha}), \left(\mathfrak{s}_{\kappa_{\Theta^*}}, \hat{\beta}\right)\right\rangle$ for Player I can be obtained by solving the following nonlinear 2-tuple linguistic bi-objective optimization model:

(**MOD 1**) $$\max\{(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha})\}, \ \min\left\{(\mathfrak{s}_{\kappa_\Theta}, \hat{\beta})\right\}$$

$$s.t.\begin{cases} \Delta\left(t\left(1-\prod_{j=1}^n\prod_{i=1}^m\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^{x_iy_j}\right)\right) \geq (\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha}), & \text{for any } y\in Y \\ \Delta\left(t\left(\prod_{j=1}^n\prod_{i=1}^m\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{x_iy_j}\right)\right) \leq (\mathfrak{s}_{\kappa_\Theta}, \hat{\beta}), & \text{for any } y\in Y \\ \mathfrak{s}_{\zeta_\Theta}, \mathfrak{s}_{\kappa_\Theta} \in \mathbb{S}, \ \hat{\alpha}, \hat{\beta} \in [-0.5, 0.5), \ 0 \leq \Delta^{-1}(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha}) + \Delta^{-1}(\mathfrak{s}_{\kappa_\Theta}, \hat{\beta}) \leq t \\ x_i \geq 0, \ \sum_{i=1}^m x_i = 1, \ i = 1, 2, \ldots, m. \end{cases} \quad (8)$$

where

$$\left\{(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha}) = \min_{y\in Y}\left(\Delta\left(t\left(1-\prod_{j=1}^n\prod_{i=1}^m\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^{x_iy_j}\right)\right)\right), \left(\mathfrak{s}_{\kappa_\Theta}, \hat{\beta}\right) = \max_{y\in Y}\left(\Delta\left(t\left(\prod_{j=1}^n\prod_{i=1}^m\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{x_iy_j}\right)\right)\right).\right.$$

According to the Definition 2.15, we get

$$\begin{cases} \Delta\left(t\left(1-\prod_{j=1}^n\prod_{i=1}^m\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^{x_iy_j}\right)\right) \geq (\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha}) \\ \Delta\left(t\left(\prod_{j=1}^n\prod_{i=1}^m\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{x_iy_j}\right)\right) \leq (\mathfrak{s}_{\kappa_\Theta}, \hat{\beta}), \end{cases} \Leftrightarrow \begin{cases} \prod_{j=1}^n\prod_{i=1}^m\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^{x_iy_j} \leq \left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right) \\ \prod_{j=1}^n\prod_{i=1}^m\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{x_iy_j} \leq \frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}. \end{cases}$$

which are equivalent to the following inequalities:

$$\sum_{j=1}^n\sum_{i=1}^m x_iy_j \ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right) \leq \ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right), \sum_{j=1}^n\sum_{i=1}^m x_iy_j \ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right) \leq \frac{\Delta^{-1}\left(\mathfrak{s}_{\kappa_\Theta},\hat{\beta}\right)}{t}. \quad (9)$$

Since Δ is a strictly monotonically increasing function, then we have

$$\max\{(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha})\} \Leftrightarrow \min\left\{\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right)\right\} \ \text{ and } \ \min\left\{(\mathfrak{s}_{\kappa_\Theta}, \hat{\beta})\right\} \Leftrightarrow \min\left\{\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right)\right\}. \quad (10)$$

Using the results obtained in Eqs. (9) and (10), (MOD 1) can be transformed into the following optimization model

(**MOD 2**) $$\min\left\{\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right)\right\}$$

$$s.t.\begin{cases} \sum_{j=1}^n\sum_{i=1}^m\left[\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)\right]x_iy_j \\ \qquad \leq \theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right); \text{ for any } y\in Y \\ \mathfrak{s}_{\zeta_\Theta}, \mathfrak{s}_{\kappa_\Theta} \in \mathbb{S}, \ \hat{\alpha}, \hat{\beta} \in [-0.5, 0.5), \ 0 \leq \Delta^{-1}(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha}) + \Delta^{-1}(\mathfrak{s}_{\kappa_\Theta}, \hat{\beta}) \leq t \\ x_i \geq 0, \ \sum_{i=1}^m x_i = 1, \ i = 1, 2, \ldots, m, \end{cases} \quad (11)$$

except for $(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha}) = (\mathfrak{s}_t, 0)$, $\left(\mathfrak{s}_{\kappa_\Theta}, \hat{\beta}\right) = (\mathfrak{s}_0, 0)$, $\left(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij}\right) = (\mathfrak{s}_t, 0)$, $\left(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij}\right) = (\mathfrak{s}_0, 0)$ and the $\theta \in [0, 1]$ is a weight, which can be decided by Players.

Further, assume that $\mathfrak{J} = \theta \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta}, \hat{\beta})}{t}\right)$, then (MOD 2) becomes

(**MOD 3**) $\min\{\mathfrak{J}\}$

$$s.t. \begin{cases} \sum_{j=1}^{n}\sum_{i=1}^{m}\left[\theta \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right)\right] x_i y_j \leq \mathfrak{J}, \quad \text{for any } y \in Y \\ \mathfrak{J} \leq 0, x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1, \ i = 1, 2, \ldots, m, \end{cases} \tag{12}$$

except for $\left(\mathfrak{s}_{\zeta_{ij}}, \alpha_{ij}\right) = (\mathfrak{s}_t, 0)$, $\left(\mathfrak{s}_{\kappa_{ij}}, \beta_{ij}\right) = (\mathfrak{s}_0, 0)$.

Since Y is a finite and compact convex set, so, we can consider only the extreme points of the set. Therefore, (MOD-3) can be modified as follows:

(**MOD 4**) $\min\{\mathfrak{J}\}$

$$s.t. \begin{cases} \sum_{i=1}^{m}\left[\theta \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right)\right] x_i \leq \mathfrak{J}, \quad j = 1, 2, \ldots, n \\ \mathfrak{J} \leq 0, x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1, \ i = 1, 2, \ldots, m. \end{cases} \tag{13}$$

Similarly, the minimax strategy y^* and the minimum loss $\Psi^* = \left\langle \left(\mathfrak{s}_{\zeta_{\Psi^*}}, \hat{\hat{\alpha}}\right), \left(\mathfrak{s}_{\kappa_{\Psi^*}}, \hat{\hat{\beta}}\right) \right\rangle$ for Player II can be generated by solving the following nonlinear bi-objective programming model given by:

(**MOD 5**) $\min\left\{(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})\right\}, \max\left\{(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}})\right\}$

$$s.t. \begin{cases} \Delta\left(t\left(1 - \prod_{j=1}^{n}\prod_{i=1}^{m}\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right)^{x_i y_j}\right)\right) \leq (\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}}), \quad \text{for any } x \in X \\ \Delta\left(t\left(\prod_{j=1}^{n}\prod_{i=1}^{m}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right)^{x_i y_j}\right)\right) \geq (\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}}), \quad \text{for any } x \in X \\ \mathfrak{s}_{\zeta_\Psi}, \mathfrak{s}_{\kappa_\Psi} \in \mathbb{S}, \ \hat{\hat{\alpha}}, \hat{\hat{\beta}} \in [-0.5, 0.5), 0 \leq \Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}}) + \Delta^{-1}(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}}) \leq t \\ y_j \geq 0, \ \sum_{i=1}^{n} y_j = 1, \ j = 1, 2, \ldots, n. \end{cases} \tag{14}$$

where

$$\left\{\left(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}}\right) = \max_{x \in X}\left(\Delta\left(t\left(1 - \prod_{j=1}^{n}\prod_{i=1}^{m}\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right)^{x_i y_j}\right)\right)\right), \left(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}}\right) = \min_{x \in X}\left(\Delta\left(t\left(\prod_{j=1}^{n}\prod_{i=1}^{m}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right)^{x_i y_j}\right)\right)\right).\right.$$

Based on Definition 2.15, we obtain

$$\begin{cases} \Delta\left(t\left(1 - \prod_{j=1}^{n}\prod_{i=1}^{m}\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right)^{x_i y_j}\right)\right) \leq (\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}}) \\ \Delta\left(t\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right)^{x_i y_j}\right)\right) \geq (\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}}), \end{cases} \Leftrightarrow \begin{cases} \prod_{j=1}^{n}\prod_{i=1}^{m}\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right)^{x_i y_j} \geq \left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})}{t}\right) \\ \left(\prod_{j=1}^{n}\prod_{i=1}^{m}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right)^{x_i y_j}\right) \geq \frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}})}{t}. \end{cases}$$

which are equivalent to the following inequalities:

$$\sum_{j=1}^{n}\sum_{i=1}^{m} x_i y_j \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right) \geq \ln\left(1 - \frac{\Delta^{-1}\left(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}}\right)}{t}\right), \ \sum_{j=1}^{n}\sum_{i=1}^{m} x_i y_j \ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right) \geq \ln\left(\frac{\Delta^{-1}\left(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}}\right)}{t}\right). \tag{15}$$

Since Δ is a strictly monotonically increasing function, then we have

$$\min\left\{(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})\right\} \Leftrightarrow \max\left\{\ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})}{t}\right)\right\} \text{ and } \max\left\{(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}})\right\} \Leftrightarrow \max\left\{\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}})}{t}\right)\right\}. \tag{16}$$

Based on the results obtained in Eqs. (15) and (16), (MOD 5) can be written as:

(**MOD 6**) $\max\left\{\theta \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}})}{t}\right)\right\}$

$$s.t. \begin{cases} \sum_{j=1}^{n}\sum_{i=1}^{m}\left[\theta \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij})}{t}\right)\right] x_i y_j \\ \quad \geq \theta \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})}{t}\right) + (1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}})}{t}\right), \ \text{for any } x \in X \\ \mathfrak{s}_{\zeta_\Psi}, \mathfrak{s}_{\kappa_\Psi} \in \mathbb{S}, \ \hat{\hat{\alpha}}, \hat{\hat{\beta}} \in [-0.5, 0.5), \ 0 \leq \Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}}) + \Delta^{-1}(\mathfrak{s}_{\kappa_\Psi}, \hat{\hat{\beta}}) \leq t \\ y_j \geq 0, \ \sum_{j=1}^{n} y_j = 1, \ j = 1, 2, \ldots, n. \end{cases} \tag{17}$$

except for $\left(\mathfrak{s}_{\zeta_\Psi},\hat{\hat{\alpha}}\right)=(\mathfrak{s}_t,0)$, $\left(\mathfrak{s}_{\kappa_\Psi},\hat{\hat{\beta}}\right)=(\mathfrak{s}_0,0)$, $\left(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij}\right)=(\mathfrak{s}_t,0)$, $\left(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij}\right)=(\mathfrak{s}_0,0)$ and $\theta\in[0,1]$ is a weight, which is decided by Players.

Let $\mathfrak{T}=\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi},\hat{\hat{\alpha}})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi},\hat{\hat{\beta}})}{t}\right)$, then (MOD 6) becomes

(**MOD 7**) $\max\{\mathfrak{T}\}$

$$s.t.\begin{cases}\sum_{j=1}^n\sum_{i=1}^m\left[\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)\right]x_iy_j\geq\mathfrak{T}, & \text{for any } x\in X\\ \mathfrak{T}\leq 0, y_j\geq 0,\ \sum_{j=1}^n y_j=1,\ j=1,2,\ldots,n.\end{cases}\tag{18}$$

except for $\left(\mathfrak{s}_{\zeta_{ij}},\alpha_{ij}\right)=(\mathfrak{s}_t,0)$, $\left(\mathfrak{s}_{\kappa_{ij}},\beta_{ij}\right)=(\mathfrak{s}_0,0)$.
Since X is a finite and compact convex set, so, we can consider only the extreme points of the set X. Therefore, (MOD-7) is converted to the following:

(**MOD 8**) $\max\{\mathfrak{T}\}$

$$s.t.\begin{cases}\sum_{j=1}^n\left[\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)\right]y_j\geq\mathfrak{T}, & i=1,2,\ldots,m\\ \mathfrak{T}\leq 0, y_j\geq 0,\ \sum_{j=1}^n y_j=1,\ j=1,2,\ldots,n.\end{cases}\tag{19}$$

Theorem 3.4. *For any $\theta\in[0,1]$, the matrix game $\tilde{\mathcal{M}}$ with payoffs represented by 2-TIFLVs always has a solution $(x^*,y^*,x^{*T}\tilde{\mathcal{M}}y^*)$.*

Theorem 3.5. *$\mathfrak{J}$ and $\mathfrak{T}$ are monotonic and nondecreasing functions of $\theta\in[0,1]$.*

Proof. Since $\mathfrak{J}=\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right)$ with $\mathfrak{s}_{\zeta_\Theta},\mathfrak{s}_{\kappa_\Theta}\in\mathbb{S}$, and $\hat{\alpha},\hat{\beta}\in[-0.5,0.5)$ such that $0\leq\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})+\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})\leq t$. Then, the partial derivative of $\mathfrak{J}$ with respect to θ is given by:

$$\frac{d\mathfrak{J}}{d\theta}=\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right)-\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right)=\ln\left(\frac{t-\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}\right).\tag{20}$$

Since $\left(\frac{t-\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}\right)\geq 1$, except for $\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\beta})=0$. Hence $\ln\left(\frac{t-\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}\right)\geq 0\Rightarrow\frac{d\mathfrak{J}}{d\theta}\geq 0$, which shows that the $\mathfrak{J}$ is a monotonic and nondecreasing functions of $\theta\in[0,1]$. Similarly, we can prove that $\mathfrak{T}$ is a monotonic and nondecreasing functions of $\theta\in[0,1]$. □

Theorem 3.6. *Let us assume that $(x^*,\mathfrak{J}^*)$ and $(y^*,\mathfrak{T}^*)$ are optimal solutions of (MOD 4) and (MOD 8) with $0<\theta<1$, respectively. Then, (x^*,Θ^*) and (y^*,Ψ^*) are Pareto optimal/noninferior solutions of (MOD 1) and (MOD 5), where $\Theta^*=\left\langle(\mathfrak{s}_{\zeta_{\Theta^*}},\hat{\alpha}),\left(\mathfrak{s}_{\kappa_{\Theta^*}},\hat{\beta}\right)\right\rangle$ and $\Psi^*=\left\langle(\mathfrak{s}_{\zeta_{\Psi^*}},\hat{\alpha}),\left(\mathfrak{s}_{\kappa_{\Psi^*}},\hat{\hat{\beta}}\right)\right\rangle$ are the 2-TIFLVs, respectively.*

Proof. Let us suppose (x^*,Θ^*) is not Pareto optimal/noninferior solution of (MOD 1), then there exists an optimal solution $(x^\bullet,\Theta^\bullet)$, where $x^\bullet\in X$ and $\Theta^\bullet=\langle(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\breve{\alpha}),(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\breve{\beta})\rangle$ such that

$$s.t.\begin{cases}\Delta\left(t\left(1-\prod_{j=1}^n\prod_{i=1}^m\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^{x_i^\bullet y_j}\right)\right)\geq(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\breve{\alpha}), & \text{for any } y\in Y\\ \Delta\left(t\left(\prod_{j=1}^n\prod_{i=1}^m\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{x_i^\bullet y_j}\right)\right)\leq(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\breve{\beta}), & \text{for any } y\in Y\\ \mathfrak{s}_{\zeta_{\Theta^\bullet}},\mathfrak{s}_{\kappa_{\Theta^\bullet}}\in\mathbb{S},\ \breve{\alpha},\breve{\beta}\in[-0.5,0.5),\ 0\leq\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\breve{\alpha})+\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\breve{\beta})\leq t\\ x_i^\bullet\geq 0,\ \sum_{i=1}^m x_i^\bullet=1,\ i=1,2,\ldots,m.\end{cases}\tag{21}$$

$\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\breve{\alpha})\geq\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^*}},\hat{\alpha})$, $\Delta^{-1}\left(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\breve{\beta}\right)\leq\Delta^{-1}\left(\mathfrak{s}_{\kappa_{\Theta^*}},\hat{\beta}\right)$, and at least one of which is strictly valid.
As $0<\theta<1$, then we have

$$s.t.\begin{cases}\sum_{j=1}^n\sum_{i=1}^m\left[\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)\right]x_i^\bullet y_j\\ \quad\leq\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\breve{\alpha})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\breve{\beta})}{t}\right), \text{ for any } y\in Y\\ \mathfrak{s}_{\zeta_{\Theta_\bullet}},\mathfrak{s}_{\kappa_{\Theta_\bullet}}\in\mathbb{S},\ \breve{\alpha},\breve{\beta}\in[-0.5,0.5),\ 0\leq\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\breve{\alpha})+\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\breve{\beta})\leq t\\ x_i^\bullet\geq 0,\ \sum_{i=1}^m x_i^\bullet=1,\ i=1,2,\ldots,m.\end{cases}\tag{22}$$

and

$$\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\check{\alpha})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\check{\beta})}{t}\right)<\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^*}},\hat{\alpha})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^*}},\hat{\beta})}{t}\right). \quad (23)$$

Since Y is a finite and compact convex set, it makes sense to consider only the extreme points of the set Y, thus, Eq. (22) is becomes:

$$s.t.\begin{cases}\sum_{i=1}^m\left[\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)\right]x_i^\bullet\\ \quad\le\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\check{\alpha})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\check{\beta})}{t}\right),\ j=1,2,\ldots,n\\ \mathfrak{s}_{\zeta_{\Theta_\bullet}},\mathfrak{s}_{\kappa_{\Theta_\bullet}}\in\mathbb{S},\ \check{\alpha},\check{\beta}\in[-0.5,0.5),\ 0\le\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\check{\alpha})+\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\check{\beta})\le t\\ x_i^\bullet\ge0,\ \sum_{i=1}^m x_i^\bullet=1,\ i=1,2,\ldots,m.\end{cases} \quad (24)$$

Let $\mathfrak{J}_1^\bullet=\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\Theta^\bullet}},\check{\alpha})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\Theta^\bullet}},\check{\beta})}{t}\right)$, then

$$s.t.\begin{cases}\sum_{i=1}^m\left[\theta\ln\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)+(1-\theta)\ln\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)\right]x_i^\bullet\le\mathfrak{J}_1^\bullet,\ j=1,2,\ldots,n\\ \mathfrak{J}_1^\bullet\le0,x_i^\bullet\ge0,\ \sum_{i=1}^m x_i^\bullet=1,\ i=1,2,\ldots,m.\end{cases} \quad (25)$$

It shows that $(x^\bullet,\mathfrak{J}^\bullet)$ is an optimal solution of (MOD 4), then Eq. (25) gives $\mathfrak{J}^\bullet<\mathfrak{J}^*$. Therefore, there exists a contradiction with the fact that $(x^*,\mathfrak{J}^*)$ is the optimal solution of (MOD 4). Hence, (x^*,Θ^*) is a Pareto optimal/noninferior solution of (MOD-1). Similarly, we can prove that (y^*,Ψ^*) is a Pareto optimal/noninferior solution of (MOD-5). ☐

It is important to note that if $\Delta^{-1}\left(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij}\right)=t$ and $\Delta^{-1}\left(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij}\right)=0$, then $\ln\left(1-\frac{\Delta^{-1}\left(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij}\right)}{t}\right)\to-\infty$ and $\ln\left(\frac{\Delta^{-1}\left(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij}\right)}{t}\right)\to-\infty$. Hence, the (MOD 4) and (MOD 8) have no sense. Thus, we can rewrite the (MOD 4) and (MOD 8) as the following nonlinear-programming models:

(**MOD 9**) $\quad\min\left\{\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right)^{(1-\theta)}\right\}$

$$s.t.\begin{cases}\prod_{i=1}^m\left[\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{(1-\theta)}\right]^{x_i}\le\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right)^{(1-\theta)},\quad j=1,2,\ldots,n\\ \mathfrak{s}_{\zeta_\Theta},\mathfrak{s}_{\kappa_\Theta}\in\mathbb{S},\ \hat{\alpha},\hat{\beta}\in[-0.5,0.5),\ 0\le\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})+\Delta^{-1}\left(\mathfrak{s}_{\kappa_\Theta},\hat{\beta}\right)\le1,\\ x_i\ge0,\ \sum_{i=1}^m x_i=1,\ i=1,2,\ldots,m.\end{cases} \quad (26)$$

and

(**MOD 10**) $\quad\max\left\{\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi},\hat{\hat{\alpha}})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi},\hat{\hat{\beta}})}{t}\right)^{(1-\theta)}\right\}$

$$s.t.\begin{cases}\prod_{j=1}^n\left[\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{(1-\theta)}\right]^{x_i}\ge\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi},\hat{\hat{\alpha}})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi},\hat{\hat{\beta}})}{t}\right)^{(1-\theta)},\quad i=1,2,\ldots,m\\ \mathfrak{s}_{\zeta_\Psi},\mathfrak{s}_{\kappa_\Psi}\in\mathbb{S},\ \hat{\hat{\alpha}},\hat{\hat{\beta}}\in[-0.5,0.5),\ 0\le\Delta^{-1}\left(\mathfrak{s}_{\zeta_\Psi},\hat{\hat{\alpha}}\right)+\Delta^{-1}\left(\mathfrak{s}_{\kappa_\Psi},\hat{\hat{\beta}}\right)\le1\\ y_j\ge0,\ \sum_{j=1}^n x_i=1,\ j=1,2,\ldots,n.\end{cases} \quad (27)$$

Let $\mathfrak{J}_1=\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta},\hat{\alpha})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Theta},\hat{\beta})}{t}\right)^{(1-\theta)}$ and $\mathfrak{T}_1=\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi},\hat{\hat{\alpha}})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_\Psi},\hat{\hat{\beta}})}{t}\right)^{(1-\theta)}$, the (MOD 9) and (MOD 10) are reduced into the following nonlinear-programming models given as:

(**MOD 11**) $\quad\min\{\mathfrak{J}_1\}$

$$s.t.\begin{cases}\prod_{i=1}^m\left[\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}},\alpha_{ij})}{t}\right)^\theta\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\wp_{ij}}},\beta_{ij})}{t}\right)^{(1-\theta)}\right]^{x_i}\le\mathfrak{J}_1,\quad j=1,2,\ldots,n\\ 0\le\mathfrak{J}_1\le1,x_i\ge0,\ \sum_{i=1}^m x_i=1,\ i=1,2,\ldots,m.\end{cases} \quad (28)$$

and
(**MOD 12**) $\max\{\mathfrak{T}_1\}$

$$s.t. \begin{cases} \prod_{i=1}^{m}\left[\left(1-\frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\varphi_{ij}}},\alpha_{ij})}{t}\right)^{\theta}\left(\frac{\Delta^{-1}(\mathfrak{s}_{\kappa_{\varphi_{ij}}},\beta_{ij})}{t}\right)^{(1-\theta)}\right]^{y_j} \geq \mathfrak{T}_1, \qquad i=1,2,\ldots,m \\ 0\leq \mathfrak{T}_1\leq 1, y_j\geq 0,\ \sum_{j=1}^{n} y_j=1,\ j=1,2,\ldots,n. \end{cases} \tag{29}$$

From Eqs. (13), (19), (28), and (29), we get $\mathfrak{J}_1^*=\mathfrak{T}_1^*$ and $\mathfrak{J}_1^*=e^{\mathfrak{J}^*}, \mathfrak{T}_1^*=e^{\mathfrak{T}^*}$, where $(x^*,\mathfrak{J}^*)$ and $(y^*,\mathfrak{T}^*)$ are the optimal solutions of (MOD 4) and (MOD 8) and $(x^*,\mathfrak{J}_1^*)$ and $(y^\star,\Im_2^\star)$ are the optimal solutions of (MOD 11) and (MOD 12) , receptively.
In the next section, a real-life numerical example is considered to demonstrate the solution steps.

4 Numerical illustration

Example 4.1. *Let us suppose two companies $\aleph_1$ and $\aleph_2$ produce three-dimensional (3D) printers. It is assumed that the demand amount of the 3D printers in the targeted market basically is fixed. It means the market share of one company is increased while the other is decreased. In order to achieve their goal, these two companies have to choose their marketing strategies very carefully. The company $\aleph_1$ has three options: (i) to reduce the price of their 3D printers (φ_1) (ii) to up-to-date the technology (φ_2) and (iii) to give some advertisements in print media and sell their 3D printers at a reasonable price (φ_3). The company $\aleph_2$ has three options as well: (i) to sell their 3D printers at a low price with free home delivery (χ_1) (ii) to give an advertisement on the broadcast media and sell at a reasonable price (χ_2) (iii) to up-to-date the technology (χ_3). Note that both the companies have limited amount of money, so, they can choose one option only. Thus, this competitive decision problem can be considered as a matrix game, i.e., the companies $\aleph_1$ and $\aleph_2$ are regarded as Players I and II, respectively. The payoffs correspond to how much market share a company can expect to receive will depend on the option as selected by the respective company. Due to very limited information and unpredictable nature of the market, the payoffs can not be quantified in numerics. A group of experts analyze the market situation and provide their assessments in the form of 2-TIFLV. After aggregating the experts opinions, the payoff matrix $\tilde{\mathcal{M}}$ for the company $\aleph_1$ is formed in the terms 2-TIFLVs defined on LTS*

$$\mathbb{S}=\left\{\begin{array}{l}\mathfrak{s}_0=\text{very very low (VVL)}, \mathfrak{s}_1=\text{very low (VL)}, \mathfrak{s}_2=\text{moderately low (ML)}, \mathfrak{s}_3=\text{ slightly low (SL)}, \mathfrak{s}_4=\text{average (Avg)},\\ \mathfrak{s}_5=\text{slightly high (SH)}, \mathfrak{s}_6=\text{moderately high (MH)}, \mathfrak{s}_7=\text{very high (VH)}, \mathfrak{s}_8=\text{ very very high (VVH)}\end{array}\right\},$$

as follows:

$$\tilde{\mathcal{M}}=\begin{array}{c} \\ \varphi_1\\ \varphi_2\\ \varphi_3\end{array}\begin{array}{c}\begin{array}{ccc}\chi_1 & \chi_2 & \chi_3\end{array}\\ \begin{pmatrix} \langle(\mathfrak{s}_7,0.28),(\mathfrak{s}_1,-0.40)\rangle & \langle(\mathfrak{s}_6,-0.20),(\mathfrak{s}_2,-0.10)\rangle & \langle\mathfrak{s}_1,\mathfrak{s}_2\rangle\\ \langle\mathfrak{s}_2,\mathfrak{s}_6\rangle & \langle(\mathfrak{s}_7,-0.30),(\mathfrak{s}_1,0.23)\rangle & \langle(\mathfrak{s}_6,0.10),(\mathfrak{s}_2,-0.30)\rangle\\ \langle(\mathfrak{s}_1,0.37),(\mathfrak{s}_2,0.20)\rangle & \langle\mathfrak{s}_1,\mathfrak{s}_7\rangle & \langle(\mathfrak{s}_7,-0.23),(\mathfrak{s}_1,0.15)\rangle \end{pmatrix}\end{array}.$$

Solution steps: Using the (MOD 4) and (MOD 8) given in Eqs. (13) and (19), the linear-programming models are obtained as follows:

$$\min\{\mathfrak{J}\}$$

$$s.t. \begin{cases} \left[\theta\ln\left(1-\frac{7.28}{8}\right)+(1-\theta)\ln\left(\frac{0.60}{8}\right)\right]x_1+\left[\theta\ln\left(1-\frac{5.80}{8}\right)+(1-\theta)\ln\left(\frac{1.90}{8}\right)\right]x_2 \\ \qquad +\left[\theta\ln\left(1-\frac{1.00}{8}\right)+(1-\theta)\ln\left(\frac{2.00}{8}\right)\right]x_3\leq \mathfrak{J}, \\ \left[\theta\ln\left(1-\frac{2.00}{8}\right)+(1-\theta)\ln\left(\frac{6.00}{8}\right)\right]x_1+\left[\theta\ln\left(1-\frac{6.70}{8}\right)+(1-\theta)\ln\left(\frac{0.77}{8}\right)\right]x_2 \\ \qquad +\left[\theta\ln\left(1-\frac{6.10}{8}\right)+(1-\theta)\ln\left(\frac{1.70}{8}\right)\right]x_3\leq \mathfrak{J}, \\ \left[\theta\ln\left(1-\frac{1.37}{8}\right)+(1-\theta)\ln\left(\frac{2.20}{8}\right)\right]x_1+\left[\theta\ln\left(1-\frac{1.00}{8}\right)+(1-\theta)\ln\left(\frac{7.00}{8}\right)\right]x_2 \\ \qquad +\left[\theta\ln\left(1-\frac{6.77}{8}\right)+(1-\theta)\ln\left(\frac{1.15}{8}\right)\right]x_3\leq \mathfrak{J}, \\ x_i\geq 0,\ i=1,2,3 \text{ and } x_1+x_2+x_3=1. \end{cases} \tag{30}$$

and

$$\max\{\mathfrak{T}\}$$

$$s.t. \begin{cases} \left[\theta\ln\left(1-\frac{7.28}{8}\right)+(1-\theta)\ln\left(\frac{0.60}{8}\right)\right]y_1+\left[\theta\ln\left(1-\frac{2.00}{8}\right)+(1-\theta)\ln\left(\frac{6.00}{8}\right)\right]y_2 \\ \qquad +\left[\theta\ln\left(1-\frac{1.37}{8}\right)+(1-\theta)\ln\left(\frac{2.20}{8}\right)\right]y_3\geq \mathfrak{T}, \\ \left[\theta\ln\left(1-\frac{5.80}{8}\right)+(1-\theta)\ln\left(\frac{1.90}{8}\right)\right]y_1+\left[\theta\ln\left(1-\frac{6.70}{8}\right)+(1-\theta)\ln\left(\frac{0.77}{8}\right)\right]y_2 \\ \qquad +\left[\theta\ln\left(1-\frac{1.00}{8}\right)+(1-\theta)\ln\left(\frac{7.00}{8}\right)\right]y_3\geq \mathfrak{T}, \\ \left[\theta\ln\left(1-\frac{1.00}{8}\right)+(1-\theta)\ln\left(\frac{2.00}{8}\right)\right]y_1+\left[\theta\ln\left(1-\frac{6.10}{8}\right)+(1-\theta)\ln\left(\frac{1.70}{8}\right)\right]y_2 \\ \qquad +\left[\theta\ln\left(1-\frac{6.77}{8}\right)+(1-\theta)\ln\left(\frac{1.15}{8}\right)\right]y_3\geq \mathfrak{T}, \\ y_j\geq 0,\ j=1,2,3 \text{ and } y_1+y_2+y_3=1. \end{cases} \tag{31}$$

Table 1: Optimal solutions of the models given in Eqs. (30) and (31) and the corresponding expected payoffs

θ	x^{*T}	$\mathfrak{J}^*$	y^{*T}	$\mathfrak{T}^*$	$E(x^*, y^*)$
0.1	(0.1504, 0.1844, 0.6652)	-1.4881	(0.4511, 0.3587, 0.1902)	-1.4881	$\langle (p_4, 0.2456), (p_2, 0.2040) \rangle$
0.2	(0.1778, 0.1869, 0.6352)	-1.4389	(0.4364, 0.3563, 0.2073)	-1.4389	$\langle (p_4, 0.4145), (p_2, 0.1340) \rangle$
0.3	(0.2028, 0.1868, 0.6104)	-1.3913	(0.4243, 0.3527, 0.2230)	-1.3913	$\langle (p_5, -0.4528), (p_2, 0.0735) \rangle$
0.4	(0.2259, 0.1842, 0.5899)	-1.3449	(0.4143, 0.3481, 0.2376)	-1.3449	$\langle (p_5, -0.3476), (p_2, 0.0209) \rangle$
0.5	(0.2479, 0.1792, 0.5730)	-1.2993	(0.4061, 0.3424, 0.2516)	-1.2993	$\langle (p_5, -0.0276), (p_2, -0.0276) \rangle$
0.6	(0.2691, 0.1718, 0.5592)	-1.2542	(0.3994, 0.3354, 0.2652)	-1.2542	$\langle (p_5, -0.1898), (p_2, -0.0720) \rangle$
0.7	(0.2899, 0.1620, 0.5481)	-1.2094	(0.3942, 0.3272, 0.2786)	-1.2094	$\langle (p_5, -0.1299), (p_2, -0.1146) \rangle$
0.8	(0.3109, 0.1495, 0.5396)	-1.1646	(0.3902, 0.3176, 0.2922)	-1.1646	$\langle (p_5, -0.0778), (p_2, -0.1572) \rangle$
0.9	(0.3322, 0.1343, 0.5335)	-1.1195	(0.3875, 0.3063, 0.3062)	-1.1195	$\langle (p_5, -0.0318), (p_2, -0.2008) \rangle$

Taking different values of $\theta \in (0,1)$, we can solve the optimization models given in Eqs. (30) and (31) by the existing methods for linear programming problem. The obtained optimal solutions and corresponding expected values of the models mentioned in Eqs. (30) and (31) are shown in Tables 1.
Further, corresponding to the models given in Eqs. (30) and (31), the non-linear programming models are obtained as follows:

$$\min\{\mathfrak{J}_1\}$$

$$s.t. \begin{cases} \left(\left(1-\frac{7.28}{8}\right)^{\theta}\left(\frac{0.60}{8}\right)^{(1-\theta)}\right)^{x_1}\left(\left(1-\frac{5.80}{8}\right)^{\theta}\left(\frac{1.90}{8}\right)^{(1-\theta)}\right)^{x_2}\left(\left(1-\frac{1.00}{8}\right)^{\theta}\left(\frac{2.00}{8}\right)^{(1-\theta)}\right)^{x_3} \leq \mathfrak{J}_1, \\ \left(\left(1-\frac{2.00}{8}\right)^{\theta}\left(\frac{6.00}{8}\right)^{(1-\theta)}\right)^{x_1}\left(\left(1-\frac{6.70}{8}\right)^{\theta}\left(\frac{0.77}{8}\right)^{(1-\theta)}\right)^{x_2}\left(\left(1-\frac{6.10}{8}\right)^{\theta}\left(\frac{1.70}{8}\right)^{(1-\theta)}\right)^{x_3} \leq \mathfrak{J}_1, \\ \left(\left(1-\frac{1.37}{8}\right)^{\theta}\left(\frac{2.20}{8}\right)^{(1-\theta)}\right)^{x_1}\left(\left(1-\frac{1.00}{8}\right)^{\theta}\left(\frac{7.00}{8}\right)^{(1-\theta)}\right)^{x_2}\left(\left(1-\frac{6.77}{8}\right)^{\theta}\left(\frac{1.15}{8}\right)^{(1-\theta)}\right)^{x_3} \leq \mathfrak{J}_1, \\ x_i \geq 0,\ i=1,2,3 \text{ and } x_1+x_2+x_3=1. \end{cases} \tag{32}$$

$$\max\{\mathfrak{T}_1\}$$

$$s.t. \begin{cases} \left(\left(1-\frac{7.28}{8}\right)^{\theta}\left(\frac{0.60}{8}\right)^{(1-\theta)}\right)^{y_1}\left(\left(1-\frac{2.00}{8}\right)^{\theta}\left(\frac{6.00}{8}\right)^{(1-\theta)}\right)^{y_2}\left(\left(1-\frac{1.37}{8}\right)^{\theta}\left(\frac{2.20}{8}\right)^{(1-\theta)}\right)^{y_3} \geq \mathfrak{T}_1, \\ \left(\left(1-\frac{5.80}{8}\right)^{\theta}\left(\frac{1.90}{8}\right)^{(1-\theta)}\right)^{y_1}\left(\left(1-\frac{6.70}{8}\right)^{\theta}\left(\frac{0.77}{8}\right)^{(1-\theta)}\right)^{y_2}\left(\left(1-\frac{1.00}{8}\right)^{\theta}\left(\frac{7.00}{8}\right)^{(1-\theta)}\right)^{y_3} \geq \mathfrak{T}_1, \\ \left(\left(1-\frac{1.00}{8}\right)^{\theta}\left(\frac{2.00}{8}\right)^{(1-\theta)}\right)^{y_1}\left(\left(1-\frac{6.10}{8}\right)^{\theta}\left(\frac{1.70}{8}\right)^{(1-\theta)}\right)^{y_2}\left(\left(1-\frac{6.77}{8}\right)^{\theta}\left(\frac{1.15}{8}\right)^{(1-\theta)}\right)^{y_3} \geq \mathfrak{T}_1, \\ y_j \geq 0,\ j=1,2,3 \text{ and } y_1+y_2+y_3=1. \end{cases} \tag{33}$$

We solve these models with the help of existing methods and summarize the results in Table 2.

Table 2: Optimal solutions of the models given in Eqs. (32) and (33) and the corresponding expected payoffs

θ	x^{*T}	$\mathfrak{J}_1^*$	y^{*T}	$\mathfrak{T}_1^*$	$E(x^*, y^*)$
0.1	(0.1504, 0.1844, 0.6652)	0.2258	(0.4511, 0.3587, 0.1902)	0.2258	$\langle (p_4, 0.2456), (p_2, 0.2040) \rangle$
0.2	(0.1778, 0.1869, 0.6352)	0.2372	(0.4364, 0.3563, 0.2073)	0.2372	$\langle (p_4, 0.4145), (p_2, 0.1340) \rangle$
0.3	(0.2028, 0.1868, 0.6104)	0.2488	(0.4243, 0.3527, 0.2230)	0.2488	$\langle (p_5, -0.4528), (p_2, 0.0735) \rangle$
0.4	(0.2259, 0.1842, 0.5899)	0.2606	(0.4143, 0.3481, 0.2376)	0.2606	$\langle (p_5, -0.3476), (p_2, 0.0209) \rangle$
0.5	(0.2479, 0.1792, 0.5730)	0.2727	(0.4061, 0.3424, 0.2516)	0.2727	$\langle (p_5, -0.0276), (p_2, -0.0276) \rangle$
0.6	(0.2691, 0.1718, 0.5592)	0.2853	(0.3994, 0.3354, 0.2652)	0.2853	$\langle (p_5, -0.1898), (p_2, -0.0720) \rangle$
0.7	(0.2899, 0.1620, 0.5481)	0.2984	(0.3942, 0.3272, 0.2786)	0.2984	$\langle (p_5, -0.1299), (p_2, -0.1146) \rangle$
0.8	(0.3109, 0.1495, 0.5396)	0.3121	(0.3902, 0.3176, 0.2922)	0.3121	$\langle (p_5, -0.0778), (p_2, -0.1572) \rangle$
0.9	(0.3322, 0.1343, 0.5335)	0.3265	(0.3875, 0.3063, 0.3062)	0.3265	$\langle (p_5, -0.0318), (p_2, -0.2008) \rangle$

The results summarized in the Tables 1 and 2 clearly indicate that as the value of parameter θ changes, different mixed strategies are obtained corresponding to the Players I and II, respectively. It is worth mentioning that the values of $\mathfrak{J}^*$, $\mathfrak{T}^*$, $\mathfrak{J}_1^*$ and $\mathfrak{T}_1^*$ are monotonic and nondecreasing with respect to θ. Also the maxmin strategies x^* and minmax strategies y^* obtained by the models mentioned in Eqs. (30) and (31) are, respectively, similar as those obtained by the models given in Eqs. (32) and (33), i.e., $\mathfrak{J}_1^* = e^{\mathfrak{J}^*}$ and $\mathfrak{T}_1^* = e^{\mathfrak{T}^*}$, with $\left(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij}\right) \neq (\mathfrak{s}_t, 0)$ and $\left(\mathfrak{s}_{\kappa_{\wp_{ij}}}, \beta_{ij}\right) \neq (\mathfrak{s}_0, 0)$ $(i, j = 1, 2, 3)$.

Further discussion

Recently, Singh et al. [39] studied a two-player constant-sum matrix game with 2-tuple linguistic payoffs. They developed a linguistic linear programming (LPP) approach to solve this class of games. It is interesting to note that we can also use our developed optimization models to solve matrix games with payoffs represented by 2-TLVs by considering only membership 2-TLVs as follows:

For Player I:
(**MOD 13**)

$$\min\{\Im_1\}$$
$$s.t. \begin{cases} \sum_{i=1}^m x_i \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right) \leq \Im_1, & j = 1, 2, \ldots, n \\ \Im_1 \leq 0, x_i \geq 0, \ \sum_{i=1}^m x_i = 1, \ i = 1, 2, \ldots, m. \end{cases} \tag{34}$$

For Player II:
(**MOD 14**)

$$\max\{\Im_2\}$$
$$s.t. \begin{cases} \sum_{j=1}^n y_j \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right) \geq \Im_2, & i = 1, 2, \ldots, m \\ \Im_2 \leq 0, y_j \geq 0, \ \sum_{j=1}^n y_j = 1, \ i = 1, 2, \ldots, n. \end{cases} \tag{35}$$

where $\Im_1 = \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha})}{t}\right)$ and $\Im_2 = \ln\left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})}{t}\right)$ denote the optimal values corresponding to the Player I and Player II, respectively. Note that (MOD 13) and (MOD 14) have no use if $\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij}) = t$ for any i, j. To resolve this issue, we can rewrite these models as follows:

(**MOD 15**)

$$\min\{\Im_3\}$$
$$s.t. \begin{cases} \prod_{i=1}^m \left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right)^{x_i} \leq \Im_3, & j = 1, 2, \ldots, n \\ 0 \leq \Im_3 \leq 1, x_i \geq 0, \ \sum_{i=1}^m x_i = 1, \ i = 1, 2, \ldots, m. \end{cases} \tag{36}$$

(**MOD 16**)

$$\max\{\Im_4\}$$
$$s.t. \begin{cases} \prod_{j=1}^n \left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right)^{y_j} \geq \Im_4, & i = 1, 2, \ldots, m \\ 0 \leq \Im_4 \leq 1, y_j \geq 0, \ \sum_{j=1}^n y_j = 1, \ i = 1, 2, \ldots, n. \end{cases} \tag{37}$$

where $\Im_3 = \left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Theta}, \hat{\alpha})}{t}\right)$ and $\Im_4 = \left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_\Psi}, \hat{\hat{\alpha}})}{t}\right)$ denote the optimal values corresponding to the Player I and Player II, respectively. Besides, The optimal value of the game can be calculated by using the following expression:

$$E(x, y) = x^T \tilde{A} y = \Delta\left(t\left(1 - \prod_{j=1}^n \prod_{i=1}^m \left(1 - \frac{\Delta^{-1}(\mathfrak{s}_{\zeta_{\wp_{ij}}}, \alpha_{ij})}{t}\right)^{x_i y_j}\right)\right). \tag{38}$$

To demonstrate the applicability of these models, we have been borrowed the example 2 from Singh et al.[39].

Example 4.2. [39] *Consider the constant-sum linguistic game with payoffs from the linguistic terms set* $\mathbb{S} = \left\{ \begin{array}{l} \mathfrak{s}_0 = \text{very poor (VP)}, \mathfrak{s}_1 = \text{Poor (P)}, \mathfrak{s}_2 = \text{Moderately Poor (MP)}, \mathfrak{s}_3 = \text{Fair (F)}, \\ \mathfrak{s}_4 = \text{Moderately Good (MG)}, \mathfrak{s}_5 = \text{Good (G)}, \mathfrak{s}_6 = \text{Very Good (VG)}, \end{array} \right\}$ *and payoff matrix*

$$\tilde{A} = \begin{array}{c} \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{array} \begin{array}{c} \begin{array}{ccccc} \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 \end{array} \\ \begin{pmatrix} (\mathfrak{s}_4, 0) & (\mathfrak{s}_4, 0) & (\mathfrak{s}_3, 0) & (\mathfrak{s}_3, 0) & (\mathfrak{s}_4, 0.3) \\ (\mathfrak{s}_4, 0) & (\mathfrak{s}_3, -0.4) & (\mathfrak{s}_6, -0.1) & (\mathfrak{s}_4, 0) & (\mathfrak{s}_0, 0) \\ (\mathfrak{s}_3, 0.4) & (\mathfrak{s}_6, 0) & (\mathfrak{s}_4, 0) & (\mathfrak{s}_4, 0) & (\mathfrak{s}_4, 0) \end{pmatrix} \end{array}.$$

Since $\Delta^{-1}(\mathfrak{s}_6, 0) = 6$, therefore (MOD 13) and (MOD 14) cannot be used to solve given problem. So, based on the (MOD 15) and (16), we get the following non-linear programming problems given as:

$$\min\{\Im_3\}$$

$$s.t.\begin{cases}\left(1-\frac{4.00}{6}\right)^{x_1}\left(1-\frac{4.00}{6}\right)^{x_2}\left(1-\frac{3.40}{6}\right)^{x_3}\leq \Im_3,\\ \left(1-\frac{4.00}{6}\right)^{x_1}\left(1-\frac{2.60}{6}\right)^{x_2}\left(1-\frac{6.00}{6}\right)^{x_3}\leq \Im_3,\\ \left(1-\frac{3.00}{6}\right)^{x_1}\left(1-\frac{5.90}{6}\right)^{x_2}\left(1-\frac{4.00}{6}\right)^{x_3}\leq \Im_3,\\ \left(1-\frac{3.00}{6}\right)^{x_1}\left(1-\frac{4.00}{6}\right)^{x_2}\left(1-\frac{4.00}{6}\right)^{x_3}\leq \Im_3,\\ \left(1-\frac{4.30}{6}\right)^{x_1}\left(1-\frac{0.00}{6}\right)^{x_2}\left(1-\frac{4.00}{6}\right)^{x_3}\leq \Im_3,\\ x_i\geq 0,\ i=1,2,3 \text{ and } x_1+x_2+x_3=1.\end{cases} \tag{39}$$

and

$$\max\{\Im_4\}$$

$$s.t.\begin{cases}\left(1-\frac{4.00}{6}\right)^{y_1}\left(1-\frac{4.00}{6}\right)^{y_2}\left(1-\frac{3.00}{6}\right)^{y_3}\left(1-\frac{3.00}{6}\right)^{y_4}\left(1-\frac{5.30}{6}\right)^{y_5}\geq \Im_4,\\ \left(1-\frac{4.00}{6}\right)^{y_1}\left(1-\frac{2.60}{6}\right)^{y_2}\left(1-\frac{5.90}{6}\right)^{y_3}\left(1-\frac{4.00}{6}\right)^{y_4}\left(1-\frac{0.00}{6}\right)^{y_5}\geq \Im_4,\\ \left(1-\frac{3.40}{6}\right)^{y_1}\left(1-\frac{6.00}{6}\right)^{y_2}\left(1-\frac{4.00}{6}\right)^{y_3}\left(1-\frac{4.00}{6}\right)^{y_4}\left(1-\frac{4.00}{6}\right)^{y_5}\geq \Im_4,\\ y_j\geq 0,\ j=1,2,3,4,5 \text{ and } y_1+y_2+y_3+y_4+y_5=1.\end{cases} \tag{40}$$

After solving these models by using existing methods, we obtain the optimal solution as presented in Table 3. Note that the obtained solution is close to the optimal solution as obtained by Singh et al. [39], which shows the effectiveness of the developed approach in solving matix games with payoffs represented by 2-TLVs.

Table 3: Optimal solutions of the models given in Eqs. (39) and (40) and the corresponding expected payoff

x^{*T}	$\Im_3^*$	y^{*T}	$\Im_4^*$	$E(x^*, y^*)$
$(0.3265, 0.1688, 0.5046)$	$(\mathfrak{s}_4, -0.2830)$	$(0.5046, 0.0000, 0.0000, 0.3708, 0.1205)$	$(\mathfrak{s}_4, -0.2830)$	$(\mathfrak{s}_4, -0.2830)$

5 Conclusions

In this paper, we have studied matrix games under qualitative information environment. Using the idea of 2-TIFLVs, matrix games with payoffs represented in terms of 2-TIFLVs have been formulated and studied in detail. We have defined maximin and minimax strategies for Players I and II under 2-TIFL environment. Paper has transformed the given matrix game problem in a pair of linear/nonlinear bi-objective programming models and solved them by using existing methods.The 2-TIFLWA operator has been used to calculate the expected value of the game. It has been shown that solving such a game is equivalent to solve a pair of nonlinear bi-objective programming problems. Finally, a real-life numerical example has been given to illustrate the solution procedure.
In future work, we will extend the developed models and methods to multi objective matrix games with payoffs represented by 2-TIFLVs. We will explore the matrix games by considering payoffs in terms of 2-tuple Pythagorean fuzzy linguistic values (2-TPFLVs), 2-tuple hesitant fuzzy linguistic values (2-THFLVs). Applications of matrix game with payoffs of 2-TIFLVs will also be studied in multiple attribute/criteria decision making problems.

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On matrix games with 2-tuple intuitionistic fuzzy linguistic payoffs

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بازی‌های ماتریسی با بازده‌های زبانی فازی شهودی ۲-تایی

چکیده. در مسائل تصمیم‌گیری دنیای واقعی متخصصان اغلب ترجیح می‌دهند نظرات خود در مورد پارامترهای مسئله را به جای شکل دقیق عددی به زبان طبیعی بیان کنند. مدل‌های نمایش زبانی برای حل بسیاری از مسائل تصمیم‌گیری، به طور گسترده‌ای با اطلاعات کیفی استفاده شده‌است. نظریه بازی‌ها در طیف گسترده‌ای از زمینه-ها، کاربردهای موفقیت‌آمیزی داشته‌اند. در این مقاله، یک مطالعه گسترده از بازی‌های ماتریسی با بازه کیفی ارائه شده‌است. این کار از مقادیر شهودی دوتایی (2-TIFLs) برای نمایش بازده بازی ماتریسی استفاده می‌کند. ما فرمول ریاضی و مفاهیم راه حل‌ها برای بازی‌های ماتریسی را با بازدهی که توسط 2-TIFLVs نشان داده می‌شود، توسعه می‌دهیم. همچنین، نشان می‌دهیم که بازی‌های ماتریسی با بازده‌ها توسط 2-TIFLs دارای راه حل‌هایی هستند که می‌توانند با تبدیل بازی ماتریسی در یک جفت مسئله برنامه‌ریزی خطی/غیرخطی بدست آیند. سرانجام، برای نشان دادن روش توسعه یافته، یک تصویر عددی از زندگی واقعی ارائه شده‌است.