Iranian Journal of Fuzzy Systems
Volume 18, Number 5, (2021), pp. 37-51
Original paper

Approximating credibilistic constraints by robust counterparts of uncertain linear inequality

N. Q. Liu[1], Y. J. Chen[2] and Y. K. Liu[3]

[1] *College of Management and Economics, Tianjin University, Tianjin 300072, China*
[2,3] *College of Mathematics and Information Science, Hebei University, Baoding 071002, Hebei, China*

liunq@hbu.edu.cn, yanjuchen@hbu.edu.cn, yliu@hbu.edu.cn

Abstract

This paper studies a class of credibilistic optimization (CO) problems, in which a convex objective is minimized subject to ambiguous credibilistic constraints. The considered CO problem is usually computational intractable. Our purpose in this paper is to discuss the robust counterpart approximations of ambiguous credibilistic constraints. Under mild assumptions, the closed property about the feasible region of credibilistic constraint is discussed. Using the obtained results, this paper deals with the robust counterpart approximations of credibilistic constraints under two types of ambiguity sets of possibility distributions. The first type is exponential function-based ambiguity set of possibility distribution, while the second type of ambiguity set is a particular case of the first one, and it is based on range and expectation information of fuzzy variables. The developed approximation techniques are capable to utilize the knowledge of ambiguity sets of possibility distributions when building distribution uncertainty-immunized solutions. As a result, the obtained safe approximations of ambiguous credibilistic constraints are computationally tractable convex/linear constraints. To apply the proposed approximation approach, a portfolio optimization problem is addressed, in which the investor is to find a portfolio to maximize the value-at-risk of his total return under the support and expectation information of uncertain returns. We use two types of robust counterpart approximations to credibilistic constraints. The computational results support our arguments.

Keywords: Credibilistic optimization, credibilistic constraint, ambiguity set, safe approximation.

1 Introduction

Let us consider the following credibilistic optimization (CO) problem

$$\min_{x\in D} f_0(x) \text{ s.t. } \mathrm{Cr}\{\gamma \mid f_i(x,\boldsymbol{\xi}(\gamma)) \leq 0\} \geq 1-\alpha_i, i=1,\ldots,m, \tag{1}$$

where $D \subset \Re^n$, $\alpha_i \in (0,1)$, $\boldsymbol{\xi}$ is a fuzzy vector with possibility distribution μ supported on a subset Ξ of $\Re^L$, and $\mathrm{Cr}\{A\}$ represents credibility of a fuzzy event A (see Liu and Liu [21]). Credibilistic constraints of the form in problem (1) have various applications in practical decision-making problems. In this respect, the interested reader may refer to the literature [1, 2, 3, 10, 11, 14, 18, 17] and the references therein for a more comprehensive picture of CO.

There are several difficulties with numerical processing of credibilistic constraints. The first potential difficulty is that the feasible region of credibilistic constraints may happen to be nonconvex, which makes credibilistic optimization (1) highly problematic. It is only known that when functions $f_i(x,\boldsymbol{\xi})$ are affine in x and $\boldsymbol{\xi}$, and the components of fuzzy vector $\boldsymbol{\xi}$ follow mutually independent triangular, trapezoidal and normal possibility distributions, the feasible regions are convex. In generic situations, we have not such convex conclusions. Second, when credibilistic constraints cannot be turned into their equivalent deterministic convex constraints, it is usually hard to estimate exactly the credibility in the left hand of credibilistic constraints. Even in the nice case that functions $f_i(x,\boldsymbol{\xi})$ are affine in x and $\boldsymbol{\xi}$, we cannot

Corresponding Author: Y. K. Liu
Received: March 2020; Revised: December 2020; Accepted: April 2021.

transform the credibilistic constraints into their equivalent deterministic constraints provided that the components of fuzzy vector $\boldsymbol{\xi}$ are not mutually independent [20]. Third, in reality we often have only partial information on the possibility distribution μ of fuzzy parameter $\boldsymbol{\xi}$, i.e., we only know that μ belongs to a given ambiguity set of possibility distributions. When this is the case, it makes the above two issues even more serious.

To tackle the first two difficulties mentioned above, the existing literature usually employed the technique of fuzzy simulation to check whether or not the credibilistic constraint is satisfied at a given point x. Using fuzzy simulation, a continuous fuzzy vector can be approximated by a sequence of discrete fuzzy vectors and the approximation technique can be ensured by convergence theorems [15]. We refer the interested reader to the recent literature [12, 13, 19, 23, 24] about using fuzzy simulation to solve various decision-making problems. Fuzzy simulation, however, requires the discrete samples with large cardinality, which becomes an actual obstacle when tolerances α_i are small enough. This paper will overcome the above two difficulties from a new viewpoint.

The third difficulty, to the best of our knowledge, has not been addressed in the literature. In present paper, we will resolve the third issue by developing the safe approximation methods for ambiguous credibilistic constraints

$$(\forall \mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\{\gamma \mid f_i(x, \boldsymbol{\xi}(\gamma)) \leq 0\} \geq 1 - \alpha_i, i = 1, \ldots, m, \tag{2}$$

where $\mathcal{M}_\mu$ is a given ambiguity set of possibility distributions.

The above ambiguity set associated with possibility distributions is different from the concept of ambiguity set that has already been in the area of robust optimization. On one hand, we assume to have partial knowledge of possibility distributions about uncertain data $\boldsymbol{\xi}$ on its support Ξ, while robust optimization belongs to the type of uncertain-but-bounded data models, where one only knows about the possible values of data $\boldsymbol{\xi}$ in a given perturbation set, and the distribution information on the perturbation set is unavailable [4]. On the other hand, more often than not there are no reasons to specify model data $\boldsymbol{\xi}$ a stochastic nature. Under this consideration, when building ambiguous credibilistic constraints (2), we work with the ambiguity set of possibility distributions. The underlying assumption in our approach is that data $\boldsymbol{\xi}$ in our model has subjective uncertainty or possibilistic nature. In practice, the experience of related field experts can be used to evaluate the distribution information of parameter $\boldsymbol{\xi}$. In contrast, stochastic nature belongs to objective uncertainty, it is meaningful only in the case that a decision maker can repeat an action many times under the same conditions; when applied to a unique action, stochastic method becomes more problematic.

Our goal in this paper is to develop new techniques that are capable to utilize the information about the ambiguity set $\mathcal{M}_\mu$ of possibility distributions when building distribution uncertainty-immunized solutions. This goal will be realized by a specific transformation of ambiguous credibilistic constraint to the language of uncertain-but-bounded perturbations and the associated robust counterparts. More precisely, we choose a perturbation set $\mathcal{Z}$ such that the x component of robust feasible solution to the following uncertain inequality

$$f_i(x, \boldsymbol{\xi}(\gamma)) \leq 0, \boldsymbol{\xi}(\gamma) \in \mathcal{Z}, \tag{3}$$

is also feasible for ambiguous credibilitic constraints (2). We will employ the techniques of robust optimization (RO) to deal with the robust counterpart of uncertain inequality (3). For details, the interested reader may refer to the literature [4, 6, 8, 9] and the references therein for the developments of RO.

The rest of the paper is organized as follows. Section 2 discusses the closed property about the feasible set of credibilistic constraints. Section 3 introduces the concept of exponential function-based ambiguity set of possibility distributions. Based on this type of ambiguity sets, Section 4 deals with the robust counterpart approximations of ambiguous credibilistic constraints. Furthermore, Section 5 develops the techniques of robust counterpart approximations based on range and expectation information. In Section 6, we provide an application about portfolio optimization. Finally, Section 7 summaries the major results of this paper.

2 The closed feasible set of credibilistic constraint

Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ be a fuzzy vector with a joint possibility distribution μ defined on a closed support $\Xi \subset \Re^L$. For any $\boldsymbol{u} = (u_0, u_1, \ldots, u_L)^T \in \Re^{L+1}$, and $\alpha \in (0, 1)$, we next discuss the closedness of feasible set about the following credibilistic constraint:

$$\mathrm{Cr}\left\{\gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) \leq 0\right\} \geq 1 - \alpha, \tag{4}$$

which, by the self-duality of credibility measure (see Liu and Liu [21]), is equivalent to

$$\mathrm{Cr}\left\{\gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0\right\} \leq \alpha. \tag{5}$$

For this purpose, we introduce the following notations of fuzzy events:

$$A(\boldsymbol{u}) = \left\{ \gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0 \right\}, \ B(\boldsymbol{u}) = \left\{ \gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) = 0 \right\}, \text{ and } C(\boldsymbol{u}) = \left\{ \gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) \le 0 \right\}.$$

Using the above notations, we obtain the following properties about upper and lower limits of fuzzy events:

Lemma 2.1. *Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_L)^T$ be a fuzzy vector and $\{\boldsymbol{u}_n\}$ some sequence converging to $\boldsymbol{u}$. Then one has*
(i) $\liminf_{n\to\infty} A(\boldsymbol{u}_n) \supset A(\boldsymbol{u})$;
(ii) $\limsup_{n\to\infty} A(\boldsymbol{u}_n) \subset A(\boldsymbol{u}) \cup B(\boldsymbol{u})$.

Proof. Let $\gamma \in A(\boldsymbol{u})$. Then, one has $u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0$. Hence, there is an integer $N(\gamma)$ such that $u_{n,0} + \sum_{l=1}^{L} u_{n,l} \xi_l(\gamma) > 0$ whenever $n > N(\gamma)$, i.e., $\gamma \in A(\boldsymbol{u}_n)$ whenever $n > N(\gamma)$. It follows that $\gamma \in \liminf_{n\to\infty} A(\boldsymbol{u}_n)$. Assertion (i) is valid.

We next prove assertion (ii). In fact, if $\gamma \in \limsup_{n\to\infty} A(\boldsymbol{u}_n)$, then there exists some subsequence $\{n_k\}$ of $\{n\}$ such that $u_{n_k,0} + \sum_{l=1}^{L} u_{n_k,l} \xi_l(\gamma) > 0$, which implies $u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) \ge 0$. If $u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0$, then $\gamma \in A(\boldsymbol{u})$; otherwise $\gamma \in B(\boldsymbol{u})$. The proof of lemma is complete. □

On the basis of Lemma 2.1, the following lemma deals with the upper semicontinuity of credibility function:

Lemma 2.2. *Assume that $\boldsymbol{\xi} = (\xi_1, \dots, \xi_L)^T$ is a fuzzy vector with $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ at $\boldsymbol{u}$. If $\mathrm{Cr}(A(\boldsymbol{u})) \le 0.5$, then the function $c(\boldsymbol{u}) = \mathrm{Cr}(C(\boldsymbol{u}))$ is upper semicontinuous at $\boldsymbol{u}$.*

Proof. Let $\{\boldsymbol{u}_n\}$ be some sequence converging to $\boldsymbol{u}$. It follows from Lemma 2.1 and the monotonicity of credibility measure that

$$\mathrm{Cr}\left(\liminf_{n\to\infty} A(\boldsymbol{u}_n)\right) \ge \mathrm{Cr}(A(\boldsymbol{u})), \text{ and } \mathrm{Cr}\left(\limsup_{n\to\infty} A(\boldsymbol{u}_n)\right) \le \mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})).$$

By premise of proposition, $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$, it follows that

$$\mathrm{Cr}\left(\liminf_{n\to\infty} A(\boldsymbol{u}_n)\right) = \mathrm{Cr}\left(\limsup_{n\to\infty} A(\boldsymbol{u}_n)\right) = \mathrm{Cr}(A(\boldsymbol{u})).$$

If $\mathrm{Cr}(A(\boldsymbol{u})) \le 0.5$, then it follows from the semicontinuity property of credibility measure that

$$\mathrm{Cr}(A(\boldsymbol{u})) = \mathrm{Cr}\left(\liminf_{n\to\infty} A(\boldsymbol{u}_n)\right) \le \liminf_{n\to\infty} \mathrm{Cr}(A(\boldsymbol{u}_n)).$$

By the self-duality of credibility measure, one has

$$c(\boldsymbol{u}) = \mathrm{Cr}(C(\boldsymbol{u})) = 1 - \mathrm{Cr}(A(\boldsymbol{u})) \ge \limsup_{n\to\infty} \mathrm{Cr}(C(\boldsymbol{u}_n)) = \limsup_{n\to\infty} c(\boldsymbol{u}_n),$$

which implies that $c(\boldsymbol{u})$ is upper semicontinuous at $\boldsymbol{u}$. The proof of lemma is complete. □

As a consequence of Lemma 2.2, we obtain the following main result about the closedness of feasible set about credibilistic constraint (4):

Theorem 2.3. *Assume that $\boldsymbol{\xi} = (\xi_1, \dots, \xi_L)^T$ is a fuzzy vector with $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ for every $\boldsymbol{u}$. If $\alpha \le 0.5$, then the following feasible set*

$$D_{\mu,\alpha} = \left\{ \boldsymbol{u} \in \Re^{L+1} \mid \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu} \left\{ \gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) \le 0 \right\} \ge 1 - \alpha \right\},$$

of credibilistic constraint (4) is a closed subset of $\Re^{L+1}$.

Proof. We first represent the feasible set $D_{\mu,\alpha}$ as the following equivalent form

$$D_{\mu,\alpha} = \left\{ \boldsymbol{u} \in \Re^{L+1} \mid c(\boldsymbol{u}) \ge 1 - \alpha \right\}.$$

Since $\alpha \le 0.5$, one has

$$\mathrm{Cr}(A(\boldsymbol{u})) = 1 - \mathrm{Cr}(C(\boldsymbol{u})) = 1 - c(\boldsymbol{u}) \le 0.5,$$

for every $\boldsymbol{u}$. It follows from Lemma 2.2 that $c(\boldsymbol{u})$ is semicontinuous with respect to $\boldsymbol{u}$. By the property of semicontinuous functions [22], the level set of the type

$$\left\{ \boldsymbol{u} \in \Re^{L+1} \mid c(\boldsymbol{u}) \ge 1 - \alpha \right\},$$

is a closed subset of $\Re^{L+1}$. The proof of theorem is complete. □

3 Exponential function-based ambiguity set of possibility distributions

Before we present a new ambiguity set to characterize possibility distributions of fuzzy variables, we obtain the following observation:

Theorem 3.1. *Let ξ be a fuzzy variable with finite expected value. Then for any $u \in \Re$, the expected value* $\mathrm{E}[\exp\{u\xi\}]$ *of fuzzy variable* $\exp\{u\xi\}$ *can be represented as*

$$\mathrm{E}[\exp\{u\xi\}] = \int_{\Re} \exp\{ur\}\mathrm{d}(\mathrm{Cr}\{\xi \le r\}).$$

Proof. For the sake of presentation, we assume that ξ has a finite support $[a, b]$. In virtue of the definition about the expected value of fuzzy variable [21], one has

$$\mathrm{E}[\exp\{u\xi\}] = \int_0^{\infty} \mathrm{Cr}\left\{\exp\{u\xi\} \ge r\right\} \mathrm{d}r.$$

In particular, when $u = 0$, we have the following computational result

$$\mathrm{E}[\exp\{0\}] = 1 = \int_a^b \mathrm{d}(\mathrm{Cr}\{\xi \le r\}).$$

Next, we assume $u > 0$. In this case, one has

$$\mathrm{E}[\exp\{u\xi\}] = \exp\{ua\} + \int_{\exp\{ua\}}^{\exp\{ub\}} \mathrm{Cr}\left\{\exp\{u\xi\} \ge r\right\} \mathrm{d}r.$$

Integrating by parts, we can obtain the following desired result

$$\int_{\exp\{ua\}}^{\exp\{ub\}} \mathrm{Cr}\left\{\exp\{u\xi\} \ge r\right\} \mathrm{d}r = -\exp\{ua\} + \int_{[a,b]} \exp\{ur\}\mathrm{d}(\mathrm{Cr}\{\xi \le r\}).$$

The case $u < 0$ can be proved similarly. The proof of proposition is complete. □

On the basis of Theorem 3.1, we can characterize the possibility distribution of fuzzy vector $\boldsymbol{\xi}$ by an exponential function-based ambiguity set, which is formally defined as follows:

Definition 3.2. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector with finite expected value. All we know about the joint possibility distribution μ of $\boldsymbol{\xi}$ is that it belongs to the following exponential function-based ambiguity set of possibility distributions*

$$\mathcal{M}_\mu = \left\{ \mu \mid \mathrm{E}_{\mu_l}[\exp\{u_l\xi_l\}] \le \exp\left\{ \max[m_l^+ u_l, m_l^- u_l] + \frac{1}{2}s_l^2 u_l^2 \right\} \text{ for any } u_l \in \Re \text{ and } l \le L \right\}, \tag{6}$$

where m_l^+, m_l^- and s_l are known parameters with $m_l^- \le m_l^+$ and $s_l \ge 0$, and $\mu_l, l \le L$, are the marginal possibility distributions of fuzzy variables ξ_l (see Liu [16]*).*

In the construction of exponential function-based ambiguity set $\mathcal{M}_\mu$, the parameters m_l^+, m_l^- and s_l are known and reflect the information about the first-order moment and second-order moment of fuzzy variables ξ_l, $l \le L$. Specifically, the true value of expectation $\mathrm{E}_{\mu_l}[\xi_l]$ belongs to the interval $[m_l^-, m_l^+]$, while the true value of variance of fuzzy variables ξ_l is not more than s_l^2, $l \le L$. Hence, the exponential function will bring us convenience from the viewpoint of computation when we deal with the safe approximation of credibilistic constraint.

In order to apply the ambiguity set $\mathcal{M}_\mu$ of possibility distributions, it is required to specify the values of parameters m_l^-, m_l^+ and s_l. In practical applications, the knowledge of related field experts can be used to evaluate the expectation range $[m_l^-, m_l^+]$ and variance information s_l. In the following, we provide such an example, which will be useful in the rest of paper.

Example 3.3. *Let ξ_l be fuzzy variables with possibility distributions μ_l. Suppose all we know about μ_l is that they are supported on the interval $[-1, 1]$ and the expected values of fuzzy variables ξ_l belong to the interval $[m_l^-, m_l^+]$, where $-1 \le m_l^- \le m_l^+ \le 1$.*

In this example, given $u_l \in \Re$, one has

$$\mathrm{E}[\exp\{u_l\xi_l\}] \leq \cosh(u_l) + m_l \sinh(u_l),$$

where m_l is the true expected value of fuzzy variable ξ_l.

The above inequality becomes an equality when fuzzy variables ξ_l have the following possibility distribution:

$$\mathrm{Pos}\{\xi_l = -1\} = 1 - m_l, \mathrm{Pos}\{\xi_l = 1\} = 1, i = 1, \ldots, L.$$

Furthermore, by calculation, one has

$$\ln(\mathrm{E}[\exp\{u_l\xi_l\}]) \leq \max[m_l^+ u_l, m_l^- u_l] + \frac{1}{2}\mathrm{S}_l^2(m_l^-, m_l^+)u_l^2,$$

where

$$\mathrm{S}_l(m_l^-, m_l^+) = \min\left\{s \mid \ln[\cosh(u_l) + m_l \sinh(u_l)] \leq \max[m_l^+ u_l, m_l^- u_l] + \frac{1}{2}s^2 u_l^2\right\},$$

for any $m_l \in [m_l^-, m_l^+]$ and $u_l \in \Re$. Hence, the possibility distributions μ_l of fuzzy variables ξ_l satisfy the conditions in Definition 3.2 with the parameters m_l^-, m_l^+ and $s_l = \mathrm{S}_l(m_l^-, m_l^+)$.

In next section, we will develop safe approximation methods that are capable to utilize the knowledge about the ambiguity set $\mathcal{M}_\mu$ of possibility distributions when building uncertainty-immunized solutions. We achieve this goal via a specific transformation of ambiguous credibilistic constraint to the language of uncertain-but-bounded perturbations and the associated robust counterparts.

4 Robust counterpart approximations

Let us denote Ψ as the following function

$$\Psi(u_1, \ldots, u_L) = \sum_{l=1}^{L}\left[\max[m_l^+ u_l, m_l^- u_l] + \frac{1}{2}s_l^2 u_l^2\right],$$

for any $(u_1, \ldots, u_L)^T \in \Re^L$. Then we obtain the following observation:

Proposition 4.1. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector with finite expected value, and the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{M}_\mu$. If $\alpha \in (0,1)$, then the following set*

$$X_\alpha^0 = \left\{\boldsymbol{u} \in \Re^{L+1} \mid \exists k > 0,\ s.t.\ ku_0 + \Psi(k(u_1, \ldots, u_L)) \leq \ln(\alpha)\right\},$$

is contained in the feasible set $\mathcal{D}_\alpha$ of ambiguous credibilistic constraint

$$\forall(\mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid u_0 + \sum_{l=1}^{L} u_l\xi_l(\gamma) \leq 0\right\} \geq 1 - \alpha.$$

That is, $X_\alpha^0 \subset \mathcal{D}_\alpha$ with $\mathcal{D}_\alpha = \cap_{\mu\in\mathcal{M}_\mu} D_{\mu,\alpha}$.

Proof. Since $u_0 + \sum_{l=1}^{L} u_l\xi_l > 0$ is equivalent to $\exp\left\{k\left[u_0 + \sum_{l=1}^{L} u_l\xi_l\right]\right\} > 1$, for all $k > 0$. As a consequence, for any $k > 0$, one has

$$\mathrm{Cr}(A(\boldsymbol{u})) \leq \int_{\Re^L} \exp\left\{k\left[u_0 + \sum_{l=1}^{L} u_l r_l\right]\right\} \mathrm{d}\left(\prod_{l=1}^{L} \mathrm{Cr}\{\xi_l \leq r_l\}\right).$$

By the property of L–S multiple integral [7] and Theorem 3.1, for all $k > 0$, one has

$$\mathrm{Cr}(A(\boldsymbol{u})) \leq \exp\{ku_0 + \Psi(k(u_1, \ldots, u_L))\}.$$

Thus, if $\boldsymbol{u} \in X_\alpha^0$, then $\mathrm{Cr}(A(\boldsymbol{u})) \leq \alpha$. By the self-duality of credibility measure, one has $\mathrm{Cr}(C(\boldsymbol{u})) \geq 1 - \alpha$ for all $\mu \in \mathcal{M}_\mu$. The proof of proposition is complete. □

To find an explicit description about the safe approximations of ambiguous credibilistic constraints, we need the following propositions 4.2 and 4.3. In credibilistic constraint (4), the parameter α represents a prespecified small tolerance that the inequality $u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) \leq 0$ is violated, which is equivalent to require that this inequality holds true with a high credibility $1-\alpha$. Under this consideration, we next find the safe approximations of ambiguous credibilistic constraints only in the case of prespecified small tolerance $\alpha \in (0, 0.5]$.

Proposition 4.2. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector with finite expected value, and $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ for every $\boldsymbol{u}$. If the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{M}_\mu$ and $\alpha \in (0, 0.5]$, then the following explicit convex constraint*

$$u_0 + \sum_{l=1}^{L} \max[m_l^+ u_l, m_l^- u_l] + \sqrt{2\ln(1/\alpha)} \sqrt{\sum_{l=1}^{L} s_l^2 u_l^2} \leq 0, \tag{7}$$

in variable $\boldsymbol{u}$ is a safe approximation of ambiguous credibilistic constraint

$$\forall(\mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu} \left\{ \gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) \leq 0 \right\} \geq 1 - \alpha.$$

Proof. By Theorem 2.3 and the premise of proposition, the feasible set $\mathcal{D}_\alpha$ of ambiguous credibilistic constraint

$$\forall(\mu \in \mathcal{M}_\mu) : \mathrm{Cr}\{\gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0\} \leq \alpha,$$

is closed. It follows from Proposition 4.1 that $\mathcal{D}_\alpha \supset X_\alpha^0$. Thus, one has $\mathcal{D}_\alpha \supset X_\alpha = \mathrm{cl}(X_\alpha^0)$, where $\mathrm{cl}(X_\alpha^0)$ represents the closure of set X_α^0.

Furthermore, to find an explicit description of the set X_α, it is easy to show that $\boldsymbol{u} \in X_\alpha$ if and only if $\boldsymbol{u}$ satisfies the following explicit convex constraint

$$u_0 + \sum_{l=1}^{L} \max[m_l^+ u_l, m_l^- u_l] + \sqrt{2\ln(1/\alpha)} \sqrt{\sum_{l=1}^{L} s_l^2 u_l^2} \leq 0,$$

which completes the proof of proposition. □

The following proposition develops an important credibility inequality:

Proposition 4.3. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector with finite expected value, and $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ for every $\boldsymbol{u}$. If the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{M}_\mu$, then for every deterministic vector $(u_1, \ldots, u_L)^T \in \Re^L$ and constant Θ with $\exp\{-\Theta^2/2\} \leq 0.5$ one has*

$$\forall(\mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu} \left\{ \sum_{l=1}^{L} u_l \xi_l > \sum_{l=1}^{L} \max[m_l^+ u_l, m_l^- u_l] + \Theta \sqrt{\sum_{l=1}^{L} s_l^2 u_l^2} \right\} \leq \exp\{-\Theta^2/2\}. \tag{8}$$

Proof. If we denote

$$u_0 = -\sum_{l=1}^{L} \max[m_l^+ u_l, m_l^- u_l] - \Theta \sqrt{\sum_{l=1}^{L} s_l^2 u_l^2},$$

and $\alpha = \exp\{-\Theta^2/2\}$, then inequality (7) holds true.

Combining the premise of proposition and Proposition 4.2, one has $\mathrm{Cr}\left\{\gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0\right\} \leq \alpha$. According to the definitions of u_0 and α, the proof of proposition is complete. □

Consider the following uncertain linear inequality

$$a^T x \leq b, [a; b] = [a^n; b^n] + P\boldsymbol{\xi}, \tag{9}$$

where ^{n} represents the nominal data and P is perturbation matrix with P_l its lth column. Assume that the perturbation vector $\boldsymbol{\xi}$ is fuzzy with only partial information on possibility distribution μ, i.e., we know only that μ belongs to a given ambiguity set $\mathcal{M}_\mu$ of possibility distributions. When this is the case, it makes sense to build the following ambiguous credibilistic constraint

$$\forall(\mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid ([a^n]^T x - b^n) + \sum_{l=1}^{L} \xi_l(\gamma) P_l^T [x; -1] \leq 0\right\} \geq 1 - \alpha, \tag{10}$$

which is equivalent to

$$\forall(\mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid ([a^n]^T x - b^n) + \sum_{l=1}^{L} \xi_l(\gamma) P_l^T [x; -1] > 0\right\} \leq \alpha. \tag{11}$$

Furthermore, assume that a perturbation set $\mathcal{Z}$ has the following conic quadratic representation

$$\mathcal{Z} = \left\{\boldsymbol{\xi}(\gamma) \in \Re^L \mid \exists \boldsymbol{w} \in \Re^L, \text{ s.t. } m_l^- \leq \xi_l(\gamma) - w_l \leq m_l^+, \sqrt{\sum_{l=1}^{L} w_l^2/s_l^2} \leq \sqrt{2\ln(1/\alpha)}\right\}. \tag{12}$$

We next summarize our main result about robust counterpart approximation:

Theorem 4.4. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector with finite expected value, and $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ for every $\boldsymbol{u}$. If the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{M}_\mu$ and $\alpha \in (0, 0.5]$, then the following explicit convex inequality*

$$([a^0]^T x - b^0) + \sum_{l=1}^{L} \max\{m_l^- P_l^T[x;-1], m_l^+ P_l^T[x;-1]\} + \sqrt{2\ln(1/\alpha)}\sqrt{\sum_{l=1}^{L} s_l^2 (P_l^T[x;-1])^2} \leq 0, \tag{13}$$

in variable x is a computationally tractable robust counterpart approximation to ambiguous credibilistic constraint (10).

Proof. We first show that inequality (13) is the robust counterpart of uncertain linear inequality (9) corresponding to perturbation set (12).

In fact, if x is robust feasible for uncertain linear inequality (9) corresponding to the perturbation set (12), then we have

$$([a^n]^T x - b^n) + \sum_{l=1}^{L} \xi_l(\gamma) P_l^T [x; -1] \leq 0 \quad \forall \boldsymbol{\xi}(\gamma) \in \mathcal{Z}.$$

If we denote $u_0 = [a^n]^T x - b^n$ and $u_l = P_l^T[x; -1]$, then we obtain $u_0 + \sup_{\boldsymbol{\xi}(\gamma)\in\mathcal{Z}} \boldsymbol{\xi}^T [u_1; \ldots; u_L] \leq 0$. When $\mathcal{Z}$ has a conic quadratic representation (12), it is easy to show that

$$\sup_{\boldsymbol{\xi}(\gamma)\in\mathcal{Z}} \boldsymbol{\xi}^T [u_1; \ldots; u_L] = \sum_{l=1}^{L} \max\{m_l^- u_l, m_l^+ u_l\} + \sqrt{2\ln(1/\alpha)}\sqrt{\sum_{l=1}^{L} s_l^2 u_l^2},$$

which verifies that x is feasible for the robust counterpart in question.

Furthermore, for every solution to the following convex inequality

$$u_0 + \sum_{l=1}^{L} \max\{m_l^- u_l, m_l^+ u_l\} + \sqrt{2\ln(1/\alpha)}\sqrt{\sum_{l=1}^{L} s_l^2 u_l^2} \leq 0,$$

by Proposition 4.2, we have $\mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid u_0 + \sum_{l=1}^{L} \xi_l(\gamma) u_l > 0\right\} \leq \alpha$, for all μ in ambiguity set $\mathcal{M}_\mu$ of possibility distributions, which completes the proof of theorem. □

In the following, we assume that fuzzy variables ξ_l have bounded supports:

$$\mathrm{Cr}\left\{b_l^- \leq \xi_l \leq b_l^+\right\} = 1, l = 1, \ldots, L.$$

In this case, Theorem 4.4 admits the following refinement:

Theorem 4.5. *Assume that* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ *is a fuzzy vector with finite expected value, and* $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ *for every* $\boldsymbol{u}$ *and* $\mathrm{Cr}\left\{b_l^- \le \xi_l \le b_l^+\right\} = 1$ *for* $l = 1, \ldots, L$. *If the joint possibility distribution* μ *of* $\boldsymbol{\xi}$ *belongs to ambiguity set* $\mathcal{M}_\mu$ *of possibility distributions and* $\alpha \in (0, 0.5]$, *then the following explicit system of convex constraints*

$$\begin{array}{ll} (i) & y_0 + z_0 = [a^n]^T x - b^n, \\ (ii) & y_l + z_l = P_l^T[x; -1], l = 1, \ldots, L, \\ (iii) & y_0 + \sum\limits_{l=1}^{L} \max\{b_l^- y_l, b_l^+ y_l\} \le 0, \\ (iv) & z_0 + \sum\limits_{l=1}^{L} \max\{m_l^- z_l, m_l^+ z_l\} + \sqrt{2\ln(1/\alpha)}\sqrt{\sum\limits_{l=1}^{L} s_l^2 z_l^2} \le 0, \end{array} \tag{14}$$

in variables x, y, z *is a computationally tractable robust counterpart approximation to ambiguous credibilistic constraint (10).*

Proof. First, according to the theory of RO, the system of convex constraints (14) is equivalent to the fact that x is robust feasible for uncertain linear inequality $a^T x \le b, [a; b] = [a^n; b^n] + P\boldsymbol{\xi}(\gamma)$ corresponding to perturbation set

$$\mathcal{Z} = \left\{ \boldsymbol{\xi}(\gamma) \in \Re^L \mid \exists \boldsymbol{w} \in \Re^L, \text{ s.t. } m_l^- \le \xi_l(\gamma) - w_l \le m_l^+, b_l^- \le \xi_l(\gamma) \le b_l^+, \sqrt{\sum_{l=1}^{L} w_l^2/s_l^2} \le \sqrt{2\ln(1/\alpha)} \right\}.$$

Second, assume that μ belongs to ambiguity set $\mathcal{M}_\mu$ of possibility distributions, we will demonstrate that whenever x can be extended to a feasible solution (x, y, z) of system (14), x is feasible for credibilistic constraint (10).

In fact, let $u_0 = [a^n]^T x - b^n$, $u_l = P_l^T[x; -1], l = 1, \ldots, L$. By $(i), (ii)$ in (14), one has

$$u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) = y_0 + \sum_{l=1}^{L} y_l \xi_l(\gamma) + z_0 + \sum_{l=1}^{L} z_l \xi_l(\gamma).$$

Since $\mathrm{Cr}\left\{b_l^- \le \xi_l \le b_l^+\right\} = 1$ for $l = 1, \ldots, L$, by (iii) in (14), one has $\mathrm{Cr}\left\{\gamma \mid y_0 + \sum_{l=1}^{L} y_l \xi_l(\gamma) > 0\right\} = 0$.

Furthermore, by (iv) in (14) and Proposition 4.2, one has $\mathrm{Cr}\left\{\gamma \mid z_0 + \sum_{l=1}^{L} z_l \xi_l(\gamma) > 0\right\} \le \alpha$.

Since

$$\left\{\gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0\right\} \subset \left\{\gamma \mid y_0 + \sum_{l=1}^{L} y_l \xi_l(\gamma) > 0\right\} \cup \left\{\gamma \mid z_0 + \sum_{l=1}^{L} z_l \xi_l(\gamma) > 0\right\},$$

by the subadditivity of credibility measure [16], we have $\mathrm{Cr}\left\{\gamma \mid u_0 + \sum_{l=1}^{L} u_l \xi_l(\gamma) > 0\right\} \le \alpha$, which completes the proof of theorem. □

The following theorem shows that ambiguous credibilistic constraint (10) can be approximated by a system of linear constraints.

Theorem 4.6. *Assume that* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ *is a fuzzy vector with finite expected value, and* $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ *for every* $\boldsymbol{u}$ *and* $\mathrm{Cr}\left\{b_l^- \le \xi_l \le b_l^+\right\} = 1$ *for* $l = 1, \ldots, L$. *If the joint possibility distribution* μ *of* $\boldsymbol{\xi}$ *belongs to ambiguity set* $\mathcal{M}_\mu$ *and* $\alpha \in (0, 0.5]$, *then the following explicit system of linear constraints*

$$\begin{array}{ll} (i) & y_0 + z_0 = [a^n]^T x - b^n, \\ (ii) & y_l + z_l = P_l^T[x; -1], l = 1, \ldots, L, \\ (iii) & y_0 + \sum\limits_{l=1}^{L} \max\{b_l^- y_l, b_l^+ y_l\} \le 0, \\ (iv) & z_0 + \sum\limits_{l=1}^{L} \max\{m_l^- z_l, m_l^+ z_l\} + \sqrt{2L\ln(1/\alpha)} \max\limits_{l} |s_l z_l| \le 0, \end{array} \tag{15}$$

in variables x, y, z *is a computationally tractable robust counterpart approximation to ambiguous credibilistic constraint (10).*

Proof. First, according to the theory of RO, the system of convex constraints (15) is equivalent to the fact that x is robust feasible for uncertain linear inequality $a^T x \leq b, [a; b] = [a^n; b^n] + P\boldsymbol{\xi}(\gamma)$, corresponding to perturbation set

$$\mathcal{Z} = \left\{ \boldsymbol{\xi}(\gamma) \in \Re^L \mid \exists \boldsymbol{w} \in \Re^L, \text{ s.t. } m_l^- \leq \xi_l(\gamma) - w_l \leq m_l^+, b_l^- \leq \xi_l(\gamma) \leq b_l^+, \sum_{l=1}^L |\frac{w_l}{s_l}| \leq \sqrt{2L\ln(1/\alpha)} \right\}.$$

Second, assume that μ belongs to ambiguity set $\mathcal{M}_\mu$ of possibility distributions, and (x, y, z) is feasible for (15). Then it follows from the following inequality $\sqrt{\sum_{l=1}^L s_l^2 z_l^2} \leq \sqrt{L} \max_l |s_l z_l|$, that condition (iv) in (15) implies

$$z_0 + \sum_{l=1}^L \max\{m_l^- z_l, m_l^+ z_l\} + \sqrt{2\ln(1/\alpha)} \sqrt{\sum_{l=1}^L s_l^2 z_l^2} \leq 0.$$

As a consequence, (x, y, z) is feasible for system (14), and its component x is feasible for ambiguous credibilistic constraint (10). The proof of theorem is complete. □

5 Approximations based on range and expectation information

In this section, we consider the case of credibilistic constraint (10) where all we know about fuzzy variables ξ_l is that

$$\mathrm{E}_{\mu_l}[\xi_l] = 0 \quad \& \quad |\xi_l| \leq 1, l = 1, \ldots, L. \tag{16}$$

In this case, it is required in the following definition of ambiguity set $\mathcal{P}_\mu$ of possibility distributions:

Definition 5.1. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector. All we know about the joint possibility distribution μ of $\boldsymbol{\xi}$ is that it belongs to the following expectation-based ambiguity set $\mathcal{P}_\mu$ of possibility distributions*

$$\mathcal{P}_\mu = \{\mu \mid \mathrm{E}_{\mu_l}[\xi_l] = 0 \quad \& \quad |\xi_l| \leq 1 \text{ for } l \leq L\ \}, \tag{17}$$

where $\mu_l, l \leq L$, are the marginal possibility distributions of fuzzy variables ξ_l. That is, the ambiguity set $\mathcal{P}_\mu$ is composed of all possibility distributions satisfying (16).

In a more general case, if fuzzy variables ξ_l take values in given finite intervals centered at the expected values of ξ_l, then we can reduce this general case to (16) by maps $\zeta_l = a_l\xi_l + b_l$ with appropriate deterministic constants a_l, b_l.

As a corollary of Proposition 4.3, we arrive:

Proposition 5.2. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector and $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ for every $\boldsymbol{u}$. If the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{P}_\mu$, then for every deterministic vector $(u_1, \ldots, u_L)^T \in \Re^L$ and constant Θ with $\exp\{-\Theta^2/2\} \leq 0.5$ one has*

$$\forall(\mu \in \mathcal{P}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu} \left\{ \sum_{l=1}^L u_l \xi_l > \Theta \sqrt{\sum_{l=1}^L u_l^2} \right\} \leq \exp\{-\Theta^2/2\}. \tag{18}$$

Proof. By the premise of proposition, we know that the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{P}_\mu$. When this is the case, it follows from Example 3.3 that the joint possibility distribution μ of $\boldsymbol{\xi}$ also belongs to ambiguity set $\mathcal{M}_\mu$ with parameters $m_l^- = m_l^+ = 0, s_l = 1, l = 1, \ldots, L$. As a particular case of Proposition 4.3, the assertion of proposition is valid. □

In the following, we will develop safe approximation techniques that are capable to utilize the knowledge about the ambiguity set $\mathcal{P}_\mu$ of possibility distributions when building uncertainty-immunized solutions. This goal can be achieved via a specific transformation of ambiguous credibilistic constraint to the language of uncertain-but-bounded perturbations and the associated robust counterparts.

Theorem 5.3. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector with $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ for every $\boldsymbol{u}$. If the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{P}_\mu$ and the constant Θ satisfies $\exp\{-\Theta^2/2\} \leq 0.5$, then the following explicit convex inequality*

$$\Theta \sqrt{\sum_{l=1}^L (P_l^T [x; -1])^2} \leq b^n - [a^n]^T x, \tag{19}$$

is a computationally tractable safe approximation of the following ambiguous credibilistic constraint:

$$\forall(\mu \in \mathcal{P}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] > b^n - [a^n]^T x\right\} \leq \exp\{-\Theta^2/2\}. \tag{20}$$

Particularly, when $\Theta \geq \sqrt{2\ln(1/\alpha)}$, *(19) is a robust counterpart approximation to ambiguous credibilistic constraint*

$$\forall(\mu \in \mathcal{P}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] \leq b^n - [a^n]^T x\right\} \geq 1-\alpha. \tag{21}$$

Proof. Let x be a solution of the convex inequality $\Theta\sqrt{\sum_{l=1}^{L}(P_l^T[x;-1])^2} \leq b^n - [a^n]^T x$. Then we have

$$\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] > b^n - [a^n]^T x\right\} \subset \left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] > \Theta\sqrt{\sum_{l=1}^{L}(P_l^T[x;-1])^2}\right\}.$$

When the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{P}_\mu$, by Proposition 5.2, we have

$$\mathrm{Cr}\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] > \Theta\sqrt{\sum_{l=1}^{L}(P_l^T[x;-1])^2}\right\} \leq \exp\{-\Theta^2/2\},$$

which implies x is a feasible solution to ambiguous credibilistic constraint (20).

Note that convex inequality (19) is nothing but the robust counterpart of the following uncertain linear inequality $a^T x \leq b, [a;b] = [a^n;b^n] + P\boldsymbol{\xi}(\gamma)$, corresponding to perturbation set $\mathcal{Z} = \{\boldsymbol{\xi}(\gamma) \in \Re^L \mid \|\boldsymbol{\xi}(\gamma)\|_2 \leq \Theta\}$.

As a consequence, when $\Theta \geq \sqrt{2\ln(1/\alpha)}$, inequality (19) is a robust counterpart approximation to ambiguous credibilistic constraint (21). □

The following result is a refinement of Theorem 5.3.

Theorem 5.4. *Assume that* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ *is a fuzzy vector and* $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ *for every* $\boldsymbol{u}$. *If the joint possibility distribution* μ *of* $\boldsymbol{\xi}$ *belongs to ambiguity set* $\mathcal{P}_\mu$ *and the constant* Θ *satisfies* $\exp\{-\Theta^2/2\} \leq 0.5$, *then the following system of convex constraints*

$$\begin{array}{ll} (i) & y_l + z_l = -P_l^T[x;-1], l = 1, \ldots, L, \\ (ii) & \sum_{l=1}^{L} |y_l| + \Theta\sqrt{\sum_{l=1}^{L} z_l^2} \leq b^n - [a^n]^T x, \end{array} \tag{22}$$

in variables x, y, z *is a computationally tractable robust counterpart approximation to ambiguous credibilistic constraint*

$$\forall(\mu \in \mathcal{P}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] \leq b^n - [a^n]^T x\right\} \geq 1 - \exp\{-\Theta^2/2\}. \tag{23}$$

Proof. First, it is easy to show the system of convex constraints (22) is equivalent to the robust counterpart of uncertain linear inequality $a^T x \leq b, [a;b] = [a^n;b^n] + P\boldsymbol{\xi}(\gamma)$, corresponding to perturbation set $\mathcal{Z} = \{\boldsymbol{\xi}(\gamma) \in \Re^L \mid \|\boldsymbol{\xi}(\gamma)\|_\infty \leq 1, \|\boldsymbol{\xi}(\gamma)\|_2 \leq \Theta\}$.

Now assume that μ belongs to ambiguity set $\mathcal{P}_\mu$ of possibility distributions, and (x, y, z) is feasible for (22), then by (i) in (22), the following inequality $\sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] > b^n - [a^n]^T x$, implies

$$-\sum_{l=1}^{L} y_l \xi_l(\gamma) - \sum_{l=1}^{L} z_l \xi_l(\gamma) > b^n - [a^n]^T x.$$

In virtue of $\|\boldsymbol{\xi}(\gamma)\|_\infty \leq 1$, one has $\sum_{l=1}^{L} |y_l| - \sum_{l=1}^{L} z_l \xi_l(\gamma) > b^n - [a^n]^T x$.

According to (ii) in (22), one has

$$-\sum_{l=1}^{L} z_l \xi_l(\gamma) > b^n - [a^n]^T x - \sum_{l=1}^{L} |y_l| \geq \Theta\sqrt{\sum_{l=1}^{L} z_l^2}.$$

As a consequence, for any possibility distribution μ belonging to ambiguity set $\mathcal{P}_\mu$, one has

$$\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] > b^n - [a^n]^T x\right\} \subset \left\{\gamma \mid -\sum_{l=1}^{L} z_l \xi_l(\gamma) > \Theta \sqrt{\sum_{l=1}^{L} z_l^2}\right\}.$$

It follows from Proposition 5.2 that

$$\mathrm{Cr}\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] > b^n - [a^n]^T x\right\} \leq \mathrm{Cr}\left\{\gamma \mid -\sum_{l=1}^{L} z_l \xi_l(\gamma) > \Theta \sqrt{\sum_{l=1}^{L} z_l^2}\right\} \leq \exp\{-\Theta^2/2\},$$

which completes the proof of theorem. □

The following theorem shows that ambiguous credibilistic constraint can be approximated by a system of linear constraints.

Theorem 5.5. *Assume that $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_L)^T$ is a fuzzy vector and $\mathrm{Cr}(A(\boldsymbol{u}) \cup B(\boldsymbol{u})) = \mathrm{Cr}(A(\boldsymbol{u}))$ for every $\boldsymbol{u}$. If the joint possibility distribution μ of $\boldsymbol{\xi}$ belongs to ambiguity set $\mathcal{P}_\mu$ and the constant θ satisfies $\exp\{-\frac{\theta^2}{2L}\} \leq 0.5$, then the following system of linear constraints*

$$\begin{array}{ll} (i) & y_l + z_l = -P_l^T[x;-1], l = 1, \ldots, L, \\ (ii) & \sum_{l=1}^{L} |y_l| + \theta \max_l |z_l| \leq b^n - [a^n]^T x, \end{array} \tag{24}$$

in variables x, y, z is a computationally tractable robust counterpart approximation to ambiguous credibilistic constraint

$$\forall(\mu \in \mathcal{P}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\left\{\gamma \mid \sum_{l=1}^{L} \xi_l(\gamma) P_l^T[x;-1] \leq b^n - [a^n]^T x\right\} \geq 1 - \exp\left\{-\frac{\theta^2}{2L}\right\}. \tag{25}$$

Proof. First, it is easy to show the system of linear constraints (24) is equivalent to the robust counterpart of uncertain linear inequality $a^T x \leq b, [a;b] = [a^n;b^n] + P\boldsymbol{\xi}(\gamma)$, corresponding to perturbation set $\mathcal{Z} = \{\boldsymbol{\xi}(\gamma) \in \Re^L \mid \|\boldsymbol{\xi}(\gamma)\|_\infty \leq 1, \|\boldsymbol{\xi}(\gamma)\|_1 \leq \theta\}$.

Now assume that possibility distribution μ belongs to ambiguity set $\mathcal{P}_\mu$, and (x, y, z) is feasible for (24). Since the following inequality $\sqrt{\sum_{l=1}^{L} z_l^2} \leq \sqrt{L} \max_l |z_l|$, holds true, condition (ii) in (24) implies

$$\sum_{l=1}^{L} |y_l| + \frac{\theta}{\sqrt{L}} \sqrt{\sum_{l=1}^{L} z_l^2} \leq b^n - [a^n]^T x. \tag{26}$$

Combining (26) and (i) in (24), we deduce that (x, y, z) is feasible for (22) with $\Theta = \frac{\theta}{\sqrt{L}}$. It follows from Theorem 5.4 that assertion (25) is valid. □

Before ending this section, we compare our approach with the stochastic method in the existing literature [5, 25] via the following remark:

Remark 5.6. *The uncertainty types of model data in two optimization methods are different. More often there are no reasons to specify model data $\boldsymbol{\xi}$ a stochastic nature. Stochastic method is meaningful only in the case that one can executes many similar actions in parallel under the same conditions. When applied to a unique action, stochastic method may become more problematic. Under this consideration, in the construction of ambiguity set, we assume to have partial knowledge of possibility distributions about uncertain data $\boldsymbol{\xi}$ on its support Ξ. The underlying assumption in our approach is that data $\boldsymbol{\xi}$ in our model has subjective uncertainty or possibilistic nature.*

As a consequence, the theoretic foundation of our approach is different from that of stochastic method. The stochastic method is based on classic probability theory, in which probability Pr *is an additive measure of uncertain event; while our approach is based on credibility theory* [21, 16]*, in which credibility* Cr *is a nonadditive measure of uncertain event. In virtue of credibility measure, our approach has some advantage over stochastic one when dealing with joint chance constraint. For example, a general joint credibilistic constraint can be represented as*

$$(\forall \mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\{\gamma \mid f_i(x, \boldsymbol{\xi}(\gamma)) \leq 0, i = 1, \ldots, m\} \geq 1 - \alpha, \tag{27}$$

where $\mathcal{M}_\mu$ is a prespecified ambiguity set of possibility distributions. In practical decision making problem, the tolerance level α is usually required to be not more than 0.5. In this case, the above joint credibilistic constraint is equivalent to the following m separate credibilistic constraints:

$$(\forall \mu \in \mathcal{M}_\mu) : \mathrm{Cr}_{\boldsymbol{\xi}\sim\mu}\{\gamma \mid f_i(x, \boldsymbol{\xi}(\gamma)) \leq 0\} \geq 1-\alpha, i = 1, \ldots, m. \tag{28}$$

Therefore, the main results obtained in this paper about approximating single credibilistic constraint can be extended to the case of joint credibilistic constraint. In contrast, compared with separate probabilistic constraints, stochastic approach is more complex to approximate joint probabilistic constraint [4, 25].

6 An application in portfolio optimization

An investor wants to invest in J risk assets and a risk-free asset with his/her unit capital. The maximum number of assets in a portfolio is assumed to be m. The investment proportion in the asset i is x_i. Assume that the risk assets' returns are mutually independent. The return of risk asset i at the end of the investment period is denoted as r_i. The return of the risk-free asset is $r_{J+1} = 1.0169$. For the ith asset, the lower and upper bounds of investment proportion x_i are l_i and u_i, respectively. Moreover, we assume that short sales are not permitted, i.e., all available wealth should be invested. The goal of an investor is to find a portfolio that maximizes the value-at-risk of his/her total return under prescribed risk level.

In a real financial problem, there exist many factors that influence the open price, the highest price, the lowest price, and the close price of each day in the future. Usually, the possibility distribution of risk assets' returns cannot be exactly evaluated by the available historical data. In this case, one can refer to stock experts' experiences, and specify uncertain returns at the end of the investment period as fuzzy variables with partial knowledge about their possibility distributions $r_i = r_i^n + \eta_i\sigma_i$, where r_i^n is the nominal return, σ_i is the maximal deviation from the nominal return, $i = 1, \ldots, J$. Here η_i is the uncertain perturbation such that $\mathrm{E}[\eta_i] = 0$ and its realization $\eta_i(\gamma)$ is in the support $[-1, 1]$, $i = 1, \ldots, J$. In addition, we assume that η_i, $i = 1, \ldots, J$ are mutually independent fuzzy variables. In what follows, we denote the collection of all possibility distributions that satisfy the above distribution information as the ambiguity set $\mathcal{P}$ of possibility distributions.

As a consequence, the above portfolio selection problem is built as the following fuzzy optimization model under ambiguous credibilistic chance constraint:

$$\max \ t \tag{29}$$

$$\text{s.t.} \ \mathrm{Cr}_{\eta\sim\mu}\left\{\sum_{i=1}^{J+1} r_i x_i \geq t\right\} \geq 1-\alpha, \forall \mu \in \mathcal{P}, \tag{30}$$

$$\sum_{i=1}^{J+1} n_i \leq m, \ n_i = 0 \text{ or } 1, i = 1, 2, \ldots, J+1, \tag{31}$$

$$l_i n_i \leq x_i \leq u_i n_i, i = 1, 2, \ldots, J+1, \tag{32}$$

$$\sum_{i=1}^{J+1} x_i = 1, x_i \geq 0, j = 1, 2, \ldots, J+1. \tag{33}$$

In model (29)-(33), the possibility distribution μ of fuzzy vector $\eta = (\eta_1, \eta_2, \cdots, \eta_J)^T$ belongs to the following ambiguity set of possibility distributions

$$\mathcal{P} = \{\mu = (\mu_1, \cdots, \mu_J^T) \mid \mathrm{E}_{\mu_i}[\eta_i] = 0, \eta_i \sim \mu_i, \eta_i \in [-1, 1], i = 1, \ldots, J\}.$$

Furthermore, if we take

$$\mathcal{Z} = \mathcal{Z}_1 \triangleq \{\eta(\gamma) \in \Re^L \mid \|\eta(\gamma)\|_\infty \leq 1, \|\eta(\gamma)\|_2 \leq \Theta\},$$

as the perturbation set of $\eta = (\eta_1, \eta_2, \cdots, \eta_J)^T$, and set the constant Θ satisfies $\alpha = \exp\{-\Theta^2/2\} \leq 0.5$, then according to Theorem 5.4, ambiguous credibilistic constraint (30) has the following robust counterpart approximation

$$\begin{array}{ll} (i) & y_i + z_i = -P_i^T[x; -1], i = 1, \ldots, J, \\ (ii) & \sum\limits_{i=1}^{J} |y_i| + \Theta\sqrt{\sum\limits_{i=1}^{J} z_i^2} \leq b^n - [a^n]^T x, \end{array} \tag{34}$$

where $x = (x_1, x_2, \cdots, x_J, x_{J+1}, -t)^T$, $a^n = -(r_1^n, r_2^n, \cdots, r_J^n, r_{J+1}, -1)^T$, $b^n = 0$, $P_i = -\sigma_i \varepsilon_i^T, i = 1, \cdots, J$, and $\varepsilon_i \in \Re^{J+2}$ is a unit vector with 1 as the ith component.

From the analysis above, we can obtain a computationally tractable robust counterpart approximation to model (29)-(33), in which we set the values of model parameters as $m = 6$, $l_i = 0.05, u_i = 0.2$, $r_i^n = 1.045 + 0.25\frac{200-i}{199}$, $\sigma_i = 0.045 + 0.55\frac{200-i}{199}, i = 1, 2, \ldots, 49$. For various values of risk level α, we use the CPLEX software to solve the above models in our personal computer (Think-PC with Intel(R) Core(TM) i5-5200U CPU 2.20 GHz and RAM 8.00 GB). The computational results are reported in Table 1.

Table 1: The robust optimal solutions and VaR values under $\mathcal{Z}_1$ and risk level α

α	Θ	t	x_{44}	x_{45}	x_{46}	x_{47}	x_{48}	x_{49}	x_{50}
0.05	2.45	0.8194	0	0	0.2	0.2	0.2	0.2	0.2
0.1	2.15	0.83328	0	0.15723	0.1584	0.15988	0.16145	0.16305	0.2
0.15	1.95	0.86675	0	0.15733	0.15843	0.15987	0.16141	0.16296	0.2
0.2	1.79	0.89494	0.1633	0.16476	0.16576	0.16719	0.16872	0.17027	0
0.25	1.67	0.91792	0.16341	0.16483	0.16577	0.16716	0.16866	0.17017	0
0.3	1.55	0.94091	0.16354	0.16493	0.16578	0.16713	0.16858	0.17005	0

On the other hand, if we take

$$\mathcal{Z} = \mathcal{Z}_2 \triangleq \{\eta(\gamma) \in \Re^L \mid \|\eta(\gamma)\|_\infty \leq 1, \|\eta(\gamma)\|_1 \leq \theta\},$$

as the perturbation set of $\eta = (\eta_1, \eta_2, \cdots, \eta_J)^T$, and set the constant θ satisfies $\alpha = \exp\{-\frac{\theta^2}{2J}\} \leq 0.5$, then it follows from Theorem 5.5 that ambiguous credibilistic constraint (30) has the following robust counterpart approximation

$$\begin{array}{ll} (i) & y_i + z_i = -P_i^T[x; -1], i = 1, \ldots, J, \\ (ii) & \sum_{i=1}^{J} |y_i| + \theta \max_i |z_i| \leq b^n - [a^n]^T x, \end{array} \tag{35}$$

where $x = (x_1, x_2, \cdots, x_J, x_{J+1}, -t)^T$, $a^n = -(r_1^n, r_2^n, \cdots, r_J^n, r_{J+1}, -1)^T$, $b^n = 0$, $P_i = -\sigma_i \varepsilon_i^T, i = 1, \cdots, J$, and $\varepsilon_i \in \Re^{J+2}$ is a unit vector with 1 as the ith component.

As a consequence, we derive a computationally tractable robust counterpart approximation to model (29)-(33), in which we set the values of model parameters as $\alpha = 0.05$, $m = 6$, $l_i = 0.05$, $r_i^n = 1.045 + 0.25\frac{200-i}{199}$, $\sigma_i = 0.045 + 0.55\frac{200-i}{199}, i = 1, 2, \ldots, 99$. For different values of u_i, we use the CPLEX software to solve the above models. The computational results are reported in Table 2, which implies that the upper bound u_i of investment proportion affects the robust optimal investment strategy.

From the computational results in our numerical experiments, we conclude that the obtained robust optimal investment strategies are able to immunize the distribution uncertainty, and may provide suitable references for investors to make his/her investment decision.

Table 2: The robust optimal solutions and VaR values under $\mathcal{Z}_2$ and $\alpha = 0.05$

u_i	t	x_{96}	x_{97}	x_{98}	x_{99}	x_{100}
0.2	0.87976	0.2	0.2	0.2	0.2	0.2
0.25	0.8888	0	0.25	0.25	0.25	0.25
0.3	0.8977	0	0.1	0.3	0.3	0.3

7 Conclusions

This paper studied a class of CO problems subject to ambiguous credibilistic constraints, where the distribution information about uncertain model parameters was only partially known and described by ambiguity sets of possibility distributions. The major new results of this paper included the following several aspects.

First, the closed property about the feasible set of credibilistic constraints has been established. This property is useful to find the explicit description of safe approximation of credibilistic constraints. As a result, an important credibility inequality about credibilistic constraint has been derived.

Second, an ambiguity set of possibility distributions based on exponential function was introduced. Using the knowledge of ambiguity set, this paper dealt with the robust counterpart approximation of ambiguous credibilistic constraints and obtained the computationally tractable convex/linear constraints.

Third, the second type ambiguity set was a particular case of the first one, which is based on the range and expectation information of fuzzy variables. According to the information provided by ambiguity set, this paper transformed ambiguous credibilistic constraints into their associated computationally tractable robust counterparts. The refinements about the safe approximations of ambiguous credibilistic constraints were also addressed for two types of ambiguity sets of possibility distributions.

Finally, we applied the proposed approximation approach to a portfolio optimization problem, in which the possibility distributions about the returns of risk assets are uncertain and assumed to belong to an ambiguity set of possibility distributions. In our portfolio optimization problem, the goal of investor was to find a portfolio to maximize the value-at-risk of his/her total return under the support and expectation information of uncertain returns. We used two types of robust counterpart approximations to credibilistic constraints. The obtained two approximating optimization models are computationally tractable using the conventional optimization software. The computational results supported our arguments.

Acknowledgement

This work was supported by National Natural Science Foundation of China (Nos.61773150, 61374184).

References

[1] X. Bai, J. Gao, Y. Liu, *Prepositioning emergency supplies under uncertainty: A parametric optimization method*, Engineering Optimization, **50** (2018), 1114-1133.

[2] X. Bai, X. Li, R. Jia, Y. Liu, *A distributionally robust credibilistic optimization method for the economic-environmental-energy-social sustainability problem*, Information Sciences, **501** (2019), 1-18.

[3] X. Bai, F. Zhang, Y. Liu, *Modeling fuzzy data envelopment analysis under robust input and output data*, RAIRO-Operations Research, **52** (2018), 619-643.

[4] A. Ben-Tal, L. Ghaoui, A. Nemirovski, *Robust optimization*, Princeton Univ Press, Princeton, 2009.

[5] A. Ben-Tal, A. Nemirovski, *Selected topics in robust convex optimization*, Mathematical Programming, **112** (2008), 125-158.

[6] D. Bertsimas, D. Brown, C. Caramanis, *Theory and applications of robust optimization*, SIAM Review, **53** (2011), 464-501.

[7] M. Carter, B. Brunt, *The lebesgue-stieltjes integral*, Springer, New York, 2000.

[8] V. Gabrel, C. Murat, A. Thiele, *Recent advances in robust optimization: An overview*, European Journal of Operational Research, **235** (2014), 471-483.

[9] B. Gorissen, I. Yanikoglu, D. Den Hertog, *A practical guide to robust optimization*, Omega-International Journal of Management Science, **53** (2015), 124-137.

[10] Z. Guo, Y. Liu, Y. Liu, *Coordinating a three level supply chain under generalized parametric interval-valued distribution of uncertain demand*, Journal of Ambient Intelligence and Humanized Computing, **8** (2017), 677-694.

[11] X. Li, *Credibilistic programming*, Springer, New York, 2013.

[12] X. Li, *A numerical-integration-based simulation algorithm for expected values of strictly monotone functions of ordinary fuzzy variables*, IEEE Transactions on Fuzzy Systems, **23** (2015), 964-972.

[13] X. Li, J. Wu, H. Ma, X. Li, R. Kang, *A random fuzzy accelerated degradation model and statistical analysis*, IEEE Transactions on Fuzzy Systems, **26** (2018), 1638-1650.

[14] B. Liu, *Theory and practice of uncertain programming*, Physica-Verlag, Herdelberg, 2009.

[15] Y. Liu, *Convergent results about the use of fuzzy simulation in fuzzy optimization problems*, IEEE Transactions on Fuzzy Systems, **14** (2006), 295-304.

[16] Y. Liu, *Credibility measure theory: A modern methodology of handling subjective uncertainty*, Science Press, Beijing, 2018.

[17] Y. Liu, X. Bai, K. Yang, *Parametric credibilistic optimization*, Science Press, Beijing, 2017.

[18] N. Liu, Y. Chen, Y. Liu, *Optimizing portfolio selection problems under credibilistic CVaR criterion*, Journal of Intelligent Fuzzy Systems, **34** (2018), 335-347.

[19] Y. Liu, Y. Chen, G. Yang, *Developing multiobjective equilibrium optimization method for sustainable uncertain supply chain planning problems*, IEEE Transactions on Fuzzy Systems, **27**(5) (2019), 1037-1051.

[20] Y. Liu, J. Gao, *The independence of fuzzy variables with applications to fuzzy random optimization*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **15** (2007), 1-20.

[21] B. Liu, Y. Liu, *Expected value of fuzzy variable and fuzzy expected value models*, IEEE Transactions on Fuzzy Systems, **10** (2002), 445-450.

[22] R. Rockafellar, R. Wets, *Variational analysis*, Springer-Verlag Berlin Heidelberg, 1998.

[23] G. Yang, Y. Liu, *Optimizing an equilibrium supply chain network design problem by an improved hybrid biogeography based optimization algorithm*, Applied Soft Computing, **58** (2017), 657-668.

[24] H. Zhai, J. Zhang, *Equilibrium reliability measure for structural design under twofold uncertainty*, Information Sciences, **477** (2019), 466-489.

[25] S. Zymler, D. Kuhn, B. Rustem, *Distributionally robust joint chance constraints with second-order moment information*, Mathematical Programming, **137** (2013), 167-198.

Iranian Journal of Fuzzy Systems
Volume 18, Number 5 (2021), pp. 203
University of Sistan and Baluchestan

Approximating credibilistic constraints by robust counterparts of uncertain linear inequality

N. Q. Liu, Y. J. Chen and Y. K. Liu

تقریب محدودیت‌های اعتبار توسط معادل‌های قوی نامساوی خطی نامشخص

چکیده. این مقاله یک کلاس از مسائل بهینه‌سازی اعتبار (CO) را مطالعه می‌کند، که در آن یک هدف محدب با محدودیت‌های اعتبار مبهم به حداقل می‌رسد. مسئله CO در نظر گرفته شده معمولاً سخت قابل حل است. هدف ما در این مقاله بحث در مورد تقریب‌های قوی معادل محدودیت‌های اعتبار مبهم است. تحت فرضیات معتدل، خاصیت بسته، مربوط به ناحیه امکان‌پذیر از محدودیت اعتبار بحث شده‌است. با استفاده از نتایج بدست آمده، این مقاله به تقریب‌های قوی معادل محدودیت‌های اعتبار تحت دو نوع مجموعه‌ی مبهم از توزیع احتمال می‌پردازد. نوع اول مجموعه مبهم، مبتنی بر تابع نمایی توزیع امکان است، در حالی که نوع دوم مجموعه‌ی مبهم یک مورد جزئی از نوع اول است و بر اساس اطلاعات دامنه و انتظار متغیرهای فازی است. تکنیک‌های تقریب توسعه یافته هنگام ایجاد راه حل‌های ایمن‌سازی شده با عدم اطمینان توزیع قادر به استفاده از دانش مجموعه‌های مبهم هستند. در نتیجه، تقریب‌های ایمن بدست آمده از محدودیت‌های اعتبار مبهم، از نظر محاسباتی، محدودیت‌های محدب/خطی هستند. برای استفاده از رویکرد تقریب پیشنهادی، به یک مسئله بهینه‌سازی اوراق بهادار پرداخته می شود، که درآن سرمایه‌گذار قرار است یک نمونه از اوراق برای به حداکثر رساندن ارزش در معرض خطر بازده کل خود تحت اطلاعات پشتیبانی و انتظارات بازده نامشخص را بیابد. ما از دو نوع تقریب معادل قوی برای محدودیت‌های اعتبار استفاده می‌کنیم. نتایج محاسباتی، استدلال‌های ما را پشتیبانی می‌کند.