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# Distributivity laws for quasi-linear means

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#### Abstract

Aggregation operations play a fundamental role in a large number of disciplines, from mathematics and natural sciences to economics and social sciences. This paper is focused on the problem of distributivity for some special classes of aggregation operations, and quasi-linear means. Characterization of distributivity pairs for uninorms, semi-uninorms and associative a-CAOA vs quasi-linear means is given.

Keywords: Uninorms, semi-uninorms, T-uninorms, S-uninorms, bi-uninorms, semi-t-operators.

# 1 Introduction

There are many real life situations where the process of combining and merging a certain number of data into a single representative value is required. Mathematical functions carrying out this process are known as aggregation operations (aggregation functions, connectives, merging functions, etc.,). Although the idea of aggregation is very old (think of the arithmetic mean), detailed study of aggregation function in general, their formalization and classification is quite more recent. Therefore, due to high level of applicability, aggregation operations have been intensively studied from 1980's to nowadays. Some purely theoretical researches of this subject can lead to new possibilities of applications as seen in e.g. [3, 5, 12].

The problem of distributivity has roots in [2] and, recently, investigations of this problem are directed towards finding solutions for different classes of aggregation operations such as t-norms, t-conorms, uninorms, nullnorms, semi-uninorms, semi-uninorms, semi-t-operators, uni-nullnorms, 2-uninorms, quasi-arithmetic means, etc (see [4, 8, 9, 10, 13, 18, 20, 21, 22, 23, 25, 26, 28, 30]). Also, researchers are investigating the problem of distributivity on the restricted domain, i.e., the conditional distributivity. This particular approach produces a larger variety of solutions (see [15, 27, 28]). The significance of the considered contemporary topic (see [13, 24, 27, 28, 30]) follows not only from the theoretical point of view, but also from its applicability in the utility theory for modelling some specific problems [12, 14]. Also, an interesting application of these laws on two Borel-Cantelli lemmas and independence of events for decomposable measures is given in [6].

This paper extends the research from [4, 26] where a characterization of all pairs  $(M_f, F)$  where  $M_f$  is a quasilinear mean, and F is a t-norm, t-conorm, Mayor's aggregation operator, nullnorm, semi-nullnorm or semi-t-operator, satisfying distributivity laws on the whole domain is given. Therefore, the main concern of this paper are distributivity equations on the whole domain where one of the unknown functions is a quasi-liner mean and another one is a uninorm, semi-uninorm from the classes  $N_e^{min} \cup N_e^{max}$ , T-uninorm, S-uninorm and bi-uninorm.

This paper is organized as follows. The Section 2 contains preliminary notions concerning the aggregation operations in general, then the preliminary notions on quasi-linear means, uninorms, semi-uninorms, semi-t-operators, T-uninorms, S-uninorms, bi-uninorms and the distributivity equations. Distributivity of the above mentioned classes of aggregation operations over quasi-linear mean is considered in the third section. Section 4 contains results of distributivity equations in the opposite direction. Some concluding remarks are given in Section 5.

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# 2 Preliminaries

A short overview of notions that are essential for this research is given throughout this section (see [1, 2, 8, 11, 12, 16, 17, 19, 29]). The basic definition of an aggregation operation is the starting point of this research.

**Definition 2.1.** [12] A binary aggregation operation Ag on [0,1] is a function  $Ag: [0,1]^2 \rightarrow [0,1]$  that is nondecreasing in each variable and for which the following holds

$$Ag(0,0) = 0$$
 and  $Ag(1,1) = 1$ 

The previous definition is in its binary form and it can be easily extended to n-ary aggregation operation on arbitrary nonempty real interval  $\mathbb{I}$  (bounded or not). This definition presents the bare bone structure of aggregation operations and, depending on the situation that is being investigated, many other properties can be required. Therefore, depending on the framework in which the aggregation is performed, the additional properties can be commutativity, associativity, idempotence, continuity, bisymmetry, strict increasing monotonicity, etc.

#### 2.1 Quasi-linear means

The focus of this research is on means, perhaps the best known family of aggregation functions with a long history, especially on quasi-linear means (for more details see Chapter 4 from [12]). First, a mean is noting other than an idempotent aggregation function. A well-studied class of means is the class of quasi-arithmetic means introduced as n-ary functions in [1] by Aczél.

**Definition 2.2.** [12] Let  $f : [0,1] \to \mathbb{R}$  be a continuous and strictly monotone function. The n-ary quasi-arithmetic mean generated by f is a function  $M_f : [0,1]^n \to [0,1]$  of the form

$$M_f(x_1 \cdots x_n) = f^{-1} \left( \frac{1}{n} \sum_{i=1}^n f(x_i) \right).$$
 (1)

Of course, the class of quasi-arithmetic means comprises most of the algebraic means.

The next result gives an axiomatization of n-ary quasi-arithmetic means which is due to Aczél.

**Theorem 2.3.** [12] A function  $F : [0,1]^n \to \mathbb{R}$  is commutative, continuous, strictly increasing, idempotent and bisymmetric if and only if there is a continuous and strictly monotonic function  $f : [0,1] \to \mathbb{R}$  such that  $F = M_f$ , i.e., F is a quasi-arithmetic mean generated by f.

Many authors have attempted to generalize the concept of quasi-arithmetic means by relaxing some conditions that characterize the quasi-arithmetic means such as commutativity or strictly increasing monotonicity. The concept considered here is of weighted quasi-arithmetic means, also called quasi-linear means, obtained simply by dropping the commutativity property from the previous theorem.

**Theorem 2.4.** [12] A function  $F : [0,1]^n \to \mathbb{R}$  is continuous, strictly increasing, idempotent and bisymmetric if and only if there is a continuous and strictly monotonic function  $f : [0,1] \to \mathbb{R}$  and real numbers  $w_1, \dots, w_n > 0$  satisfying  $\sum_{i=1}^{n} w_i = 1$  such that

$$F(x_1, \cdots, x_n) = f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right).$$
 (2)

In papers [4, 26] the aggregation operation F from Theorem 2.4 is called quasi-arithmetic mean. In this paper we use terminology from [12], where the quasi-arithmetic mean is given by (1), and function F given by (2) is quasi-linear mean or weighted quasi-arithmetic mean.

Since the function f from (1) and (2) is continuous and strictly monotonic, the range of f is an interval [m, n] where  $m, n \in \mathbb{R}$ . Thus, the working assumption through this paper is that  $f : [0, 1] \to [m, n]$ , and that f is a monotone bijection. Also, this paper deals with binary aggregation operations, therefore the assume is that n = 2 in the equation (2).

This following sub-sections provide overviews of aggregation operations that will be considered as a part of a distributive pair together with the quasi-linear mean.

#### 2.2 Uninorms and semi-uninorms

**Definition 2.5.** [29] A uninorm  $U : [0,1]^2 \to [0,1]$  is an aggregation operation that is commutative, associative, and for which there exists a neutral element  $e \in [0,1]$ , i.e., U(x,e) = x for all  $x \in [0,1]$ .

Recall that when e = 1 uninorm U becomes a t-norm denoted by T, and when e = 0, U is a t-conorm denoted by S. For  $e \in (0, 1)$  the following result holds.

**Theorem 2.6.** [11] Let U be a uninorm with a neutral element  $e \in (0, 1)$ . Then there exists a t-norm  $T_U$ , a t-conorm  $S_U$ , and increasing operator  $C : [0, e) \times (e, 1] \cup (e, 1] \times [0, e) \rightarrow [0, 1]$  that fulfils min  $\leq C \leq \max$ , such that U is given by

$$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e+(1-e)S_U\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ C(x,y) & \text{otherwise.} \end{cases}$$
(3)

A t-norm  $T_U$  and a t-conorm  $S_U$  from the previous theorem are called the underlying t-norm and underlying t-conorm of U. If U(0,1) = 0, the uninorm U is conjunctive and, if U(0,1) = 1 the uninorm U is a disjunctive one.

Semi-uninorms are a form of relaxed uninorms, that are obtained by omitting commutativity and associativity from Definition 2.5. This specific class of operators was introduced in [7]. The family of all semi-uninorms with neutral element e is denoted by  $N_e$ . Specially, the class  $N_1$  consists of t-seminorms or semicopulas, and  $N_0$  t-semiconorms.

Notations  $N_e^{max}$  and  $N_e^{min}$  (see [7]) are used for families of all semi-uninorms with the neutral element  $e \in (0, 1)$  satisfying the following additional conditions

$$\forall x \in (e, 1] \ U(0, x) = U(x, 0) = x \text{ and } \forall x \in [0, e) \ U(1, x) = U(x, 1) = x,$$

respectively.

#### 2.3 Associative a-CAOA: T-uninorms, S-uninorms, bi-uninorms and nullnorms

Another generalization of uninorms necessary for the presented research are associative, commutative aggregation operations with annihilator a, shortly associative a-CAOA, that were studied in [17]. Further on, for any binary operation  $A: [0,1]^2 \to [0,1]$ , and any element  $c \in [0,1]$ , let the section  $A_c: [0,1] \to [0,1]$  given by

$$A_c(x) = A(c, x),$$

be denoted by  $A_c$ . As it will be seen from the following, the continuity (discontinuity) of sections  $A_0$  and  $A_1$  plays a crucial role in classification of associative a-CAOA operations.

**Definition 2.7.** [17] Let  $A : [0,1]^2 \rightarrow [0,1]$  be an associative a-CAOA.

- A is called a S-uninorm if  $A_0$  is continuous and  $A_1$  is not, and there exists  $e \in (0,1)$  such that e is idempotent,  $A_e$  is continuous and  $A_e(1) = 1$ .
- A is called a T-uninorm if  $A_1$  is continuous and  $A_0$  is not, and there exists  $e \in (0,1)$  such that e is idempotent,  $A_e$  is continuous and  $A_e(0) = 0$ .
- A is called a bi-uninorm if  $A_0$  and  $A_1$  are not continuous, and there exist idempotent elements  $e_0, e_1 \in (0, 1)$  such that  $A_{e_0}$  and  $A_{e_1}$  are continuous and  $A_{e_0}(0) = 0$  and  $A_{e_1}(1) = 1$ .
- A is called a nullnorm (a t-operator) if  $A_0$  and  $A_1$  are continuous.

**Remark 2.8.** Distributivity between semi-nullnorm (or nullnorm) and quasi-linear mean was investigated in full in [26], therefore that type of associative a-CAOA will not be considered further in this paper.

The following overview of results from [17] shows that the form of associative a-CAOA is closely related to uninorms, t-norms and t-conorms, and is essential for investigation that follows. Let  $A : [0, 1]^2 \rightarrow [0, 1]$  be a binary operation.

• A is a S-uninorm if and only if There exists  $a \in [0, 1)$ , a t-conorm S' and a conjunctive uninorm U' with neutral element  $e' \in (0, 1)$  such that A is given by

$$A(x,y) = \begin{cases} aS'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)U'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \end{cases}$$
(4)

• A is a T-uninorm if and only if there exists  $a \in (0, 1]$ , a t-norm T' and a disjunctive uninorm U' with neutral element  $e' \in (0, 1)$  such that A is given by

$$A(x,y) = \begin{cases} aU'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)T'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \end{cases}$$
(5)

• A is a bi-uninorm if and only if there exists  $a \in (0, 1)$ , a disjunctive uninorm  $U'_0$  and a conjunctive uninorm  $U'_1$  with neutral elements  $e'_0, e'_1 \in (0, 1)$ , respectively, such that A is given by

$$A(x,y) = \begin{cases} aU_0'\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ a + (1-a)U_1'\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \end{cases}$$
(6)



Figure 1. Associative a-CAOA and semi-t-operators.

#### 2.4 Semi-t-operators

Now, commutativity is omitted from definition of t-operators, i.e., from nullnorms. Distributivity of this type of operators with respect to quasi-linear mean was also studied in [26], however some corrections are needed and they are given in the fourth section of this paper.

**Definition 2.9.** [8] A semi-t-operator  $F : [0,1]^2 \rightarrow [0,1]$  is an associative aggregation operation such that the functions  $F_0, F_1, F^0, F^1$ , where  $F_0(x) = F(0,x), F_1(x) = F(1,x), F^0(x) = F(x,0), F^1(x) = F(x,1)$ , are continuous.

The family of all semi-t-operators such that F(0,1) = a and F(1,0) = b is denoted with  $\mathcal{F}_{a,b}$ . The following result holds.

**Theorem 2.10.** [19] Let  $F : [0,1]^2 \to [0,1]$ , F(0,1) = a, F(1,0) = b. The operation  $F \in \mathcal{F}_{a,b}$  if and only if there exists an associative t-seminorm  $T_F$  and an associative t-semiconorm  $S_F$  such that

$$F(x,y) = \begin{cases} aS_F\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ b + (1-b)T_F\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ x & \text{otherwise,} \end{cases}$$
(7)

for  $a \leq b$  and

$$F(x,y) = \begin{cases} bS_F\left(\frac{x}{b}, \frac{y}{b}\right) & if(x,y) \in [0,b]^2, \\ a + (1-a)T_F\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & if(x,y) \in [a,1]^2, \\ a & if x \le a \le y, \\ b & if y \le b \le x, \\ y & otherwise, \end{cases}$$
(8)

for  $b \leq a$ .

#### 2.5 Distributivity equations

Finally, let us recall the functional equations that are called left and right distributivity laws ([2], p. 318).

**Definition 2.11.** Let F and G be some binary aggregation operations. F is distributive over G, if the following two laws hold:

(LD) F is a left distributive over G, i.e.,

$$F(x, G(y, z)) = G(F(x, y), F(x, z)), \text{ for all } x, y, z \in [0, 1],$$

and

(RD) F is a right distributive over G, i.e.,

$$F(G(y, z), x) = G(F(y, x), F(z, x)), \text{ for all } x, y, z \in [0, 1].$$

Of course, for commutative F (LD) and (RD) coincide. Since for this research results for (RD) are analogous to the results for (LD), further on, the focus will be only on (LD) case for non-commutative F.

# 3 Distributivity of some aggregation operations over quasi-linear means

In this section we present results concerning distributivity of previously described classes of aggregation operations over quasi-linear means  $M_f$ , where  $f : [0,1] \to [m,n]$ . Further on, the assumption is that generator f is an increasing bijection, i.e., that f(0) = m and f(1) = n.

#### 3.1 Distributivity of uninorms and semi-uninorms over quasi-linear means

**Proposition 3.1.** Let U be a disjunctive uninorm with a neutral element  $e \in (0,1)$ , and  $M_f$  be a quasi-linear mean. Then U is not distributive over  $M_f$ .

*Proof.* Let us suppose that a disjunctive uninorm U with a neutral element  $e \in (0, 1)$  is distributive over quasi-linear mean  $M_f$ . Similarly to the proof of Theorem 7 from [4] we obtain that

$$U(x,y) = f^{-1}(A(x)f(y) + B(x)),$$
(9)

or

$$U(x,y) = f^{-1}(C(x)),$$
(10)

where A, B and C are three functions on [0, 1].

Equation (10) is not possible solution for the uninorm U, since, according to (10), the uninorm U does not depend on y. This implies that for all  $x \in [0,1]$  x = U(x,e) = U(x,1) = 1 which is impossible. Therefore, the only possible solution is (9).

It follows from (9) and the structure of U that

$$1 = U(0,1) = f^{-1}(A(0)f(1) + B(0))$$
 and  $0 = U(0,0) = f^{-1}(A(0)f(0) + B(0))$ .

Hence, the following is obtained

$$n = A(0)n + B(0)$$
 and  $m = A(0)m + B(0)$ , i.e.,  $n - m = A(0)(n - m)$ .

Since  $n \neq m$ , it follows that A(0) = 1 and B(0) = 0. Again, from (9) and the structure of U follows

$$0 = U(0, e) = f^{-1}(A(0)f(e) + B(0))$$
, i.e.,  $f(0) = m = f(e)$ .

Since f is a bijection, 0 = e is obtained and this is in contradiction with the assumption  $e \in (0, 1)$ . Therefore, disjunctive uninorm U with a neutral element  $e \in (0, 1)$  is not distributive over quasi-linear mean  $M_f$ .

As a consequence of the previous theorem and results from [4] the following result can be obtained.

**Theorem 3.2.** Let U be a disjunctive uninorm with a neutral element  $e \in [0, 1)$ , and  $M_f$  be a quasi-linear mean. U is distributive over  $M_f$  if and only if U is a t-conorm given by

$$U(x,y) = f^{-1}\left(\frac{n-f(x)}{n-m}f(y) + n\frac{f(x)-m}{n-m}\right).$$
(11)

That is, distributivity in this case reduces disjunctive uninorms to t-conorms of a specific form. For distributivity of conjunctive uninorm over quasi-linear mean results are similar.

**Proposition 3.3.** Let U be a conjunctive uninorm with a neutral element  $e \in (0,1)$ , and  $M_f$  be a quasi-linear mean. Then U is not distributive over  $M_f$ .

**Theorem 3.4.** Let U be a conjunctive uninorm with a neutral element  $e \in (0,1]$ , and  $M_f$  be a quasi-linear mean. U is distributive over  $M_f$  if and only if U is a t-norm given by

$$U(x,y) = f^{-1}\left(\frac{f(x) - m}{n - m}f(y) + m\frac{n - f(x)}{n - m}\right).$$
(12)

Now conjunctive uninorms are reduced to t-norms of a specific form.

Using similar arguments as in Proposition 3.1 and Proposition 3.3 the following result can be proved.

**Theorem 3.5.** Let U be a semi-uninorm from the classes  $N_e^{\min} \cup N_e^{\max}$  with a neutral element  $e \in (0,1)$  and let  $M_f$  be a quasi-linear mean. Then, U is not left (right) distributive over  $M_f$ .

### 3.2 Distributivity of associative a-CAOA over quasi-linear means

Now, the focus is on distributivity of aggregation operations with an annihilator over quasi-linear mean.

**Theorem 3.6.** Let A be a S-uninorm and let  $M_f$  be a quasi-linear mean. Then, A is not distributive over  $M_f$ .

*Proof.* Let suppose that a S-uninorm A with an annihilator  $a \in (0, 1)$  is distributive over quasi-linear mean  $M_f$ . The case when a = 0 is solved in Proposition 3.3, since in that case S-uninorm reduces to conjunctive uninorm with neutral element  $e \in (0, 1)$ . Again, similarly to the proof of Theorem 7 from [4] the following can be obtained

$$A(x,y) = f^{-1}(A_1(x)f(y) + B(x)),$$
(13)

or

$$A(x,y) = f^{-1}(C(x)),$$
(14)

where  $A_1, B$  and C are three functions on [0, 1]. As in Proposition 3.1, (14) is not possible solution for S-uninorm A, and only (13) remains as an option.

It follows from (13) and the structure of A for  $x \in (0, a)$  that

$$x = A(x,0) = f^{-1}(A_1(x)f(0) + B(x))$$
 and  $a = A(x,a) = f^{-1}(A_1(x)f(a) + B(x))$ 

Hence,

$$f(x) = A_1(x)m + B(x)$$
 and  $f(a) = A_1(x)f(a) + B(x)$ ,

i.e.,

$$f(a) - f(x) = A_1(x)(f(a) - m).$$

In order to determine  $A_1(x)$ , the following two cases have to be considered:

- f(a) = m. This implies a = 0, however it contradicts the starting assumption  $a \in (0, 1)$ .
- $f(a) \neq m$ . Then  $A_1(x) = \frac{f(a) f(x)}{f(a) m}$  and  $B(x) = f(a) \frac{f(x) m}{f(a) m}$ . Again, from the structure of A for  $x \in (0, a)$ , it follows that A(x, 1) = a. From (13) follows

$$f(a) = \frac{f(a) - f(x)}{f(a) - m}n + f(a)\frac{f(x) - m}{f(a) - m}$$

which implies f(x) = f(a) or f(a) = n = f(1). This results contradict the assumption of the strict monotonicity of f, because  $x \in (0, a)$  and  $a \in (0, 1)$ .

Therefore, S-uninorm A is not distributive over quasi-linear mean  $M_f$ .

Similar claims can be shown for T-uninorms and bi-uninorms. That is, the following two theorems hold.

**Theorem 3.7.** Let A be a T-uninorm and  $M_f$  be a quasi-linear mean. Then A is not distributive over  $M_f$ .

**Theorem 3.8.** Let A be a bi-uninorm and  $M_f$  be a quasi-linear mean. Then A is not distributive over  $M_f$ .

The results from this section show that the distributivity of disjunctive (conjunctive) uninorms over quasi-linear means is being reduced into the distributivity of t-conorms (t-norms) over quasi-linear means. Also, for a given quasi-linear mean  $M_f$ , there does not exist a uninorm or a semi-uninorm from the class  $N_e^{min} \cup N_e^{max}$  with the neutral element  $e \in (0, 1)$ , nor a T-uninorm, a S-uninorm or a bi-uninorm, that is distributive with respect to  $M_f$ .

## 4 Distributivity of quasi-linear means over some aggregation operations

Results in this section concern the revers situation, i.e., distributivity of a quasi-linear mean  $M_f$  over some classes of aggregation operations.

#### 4.1 Distributivity of quasi-linear means over uninorms and semi-uninorms

**Theorem 4.1.** Let  $M_f$  be a quasi-linear mean, and let U be a disjunctive uninorm with a neutral element  $e \in (0, 1)$ . Then,  $M_f$  is not left (right) distributive over U.

Proof. Let suppose that (LD) holds. From the representation of  $M_f$ , it follows  $M_f(x, y) = f^{-1}(w_1f(x) + w_2f(y))$ , for all  $(x, y) \in [0, 1]^2$  where  $w_1 + w_2 = 1$ . Letting x = y = z in equation (LD), the following is obtained  $M_f(x, U(x, x)) = U(M_f(x, x), M_f(x, x))$ . Now, from structure of  $M_f$  there follows

$$w_1f(x) + w_2f(U(x,x)) = f(U(x,x))$$
, that is,

$$w_1 f(x) = (1 - w_2) f(U(x, x)) = w_1 f(U(x, x)).$$

Since f is a bijection, U(x, x) = x for all  $x \in [0, 1]$ , i.e., U is an idempotent uninorm. Since U is an idempotent uninorm, there holds (see [16])  $U|_{[0,e]^2} = \min, U|_{[e,1]^2} = \max$ . On the remaining part of the unit square  $[0,1]^2$ , that is for  $x < e < y, U(x,y) \in \{x,y\}$ .

Now, by letting  $x = 1, y \in (0, e), z = e$  in (LD), there holds

$$M_f(1,y) = M_f(1, U(y,e)) = U(M_f(1,y), M_f(1,e)).$$
(15)

Since y < e, it follows  $M_f(1, y) < M_f(1, e)$ . Also,  $M_f(1, e) > M_f(e, e) = e$ .

- If  $M_f(1, y) = e$ , then from (15) follows  $e = M_f(1, e)$ . This is in contradiction with  $M_f(1, e) > e$ .
- If  $M_f(1, y) > e$ , then from (15) and since  $U|_{[e,1]^2} = \max$ , follows

$$M_f(1,y) = \max(M_f(1,y), M_f(1,e)) = M_f(1,e),$$

which implies y = e, because  $M_f$  is strictly increasing. This is in contradiction with  $y \in (0, e)$ .

Therefore, for all  $y \in (0, e)$ , there should be  $M_f(1, y) < e$ . Now, since  $M_f$  is continuous, it follows

$$M_f(1, e) = \lim_{y \to e^-} M_f(1, y) \le e.$$

Again, this is in contradiction with  $M_f(1, e) > e$ .

Therefore, the quasi-linear mean  $M_f$  is not left distributive over disjunctive uninorm U with a neutral element  $e \in (0, 1)$ .

Similar result can be obtained for a conjunctive uninorm U.

**Theorem 4.2.** Let  $M_f$  be a quasi-linear mean, and let U be a conjunctive uninorm with a neutral element  $e \in (0, 1)$ . Then,  $M_f$  is not left (right) distributive over U.

Therefore, according to [4], the only conclusion is that quasi-linear mean  $M_f$  is left (right) distributive over a conjunctive uninorm with a neutral element e = 1, (t-norm) and over a disjunctive uninorm with a neutral element e = 0 (t-conorm). That is, the following theorem hold.

**Theorem 4.3.** [4] Let  $M_f$  be a quasi-linear mean, and let F be a t-norm or a t-conorm.  $M_f$  is left (right) distributive over F if and only if  $F = \min$  or  $F = \max$ .

Using similar arguments as in Theorem 4.1 and Theorem 4.2 the following result can be proven.

**Theorem 4.4.** Let U be a semi-uninorm from the classes  $N_e^{\min} \cup N_e^{\max}$  with a neutral element  $e \in (0,1)$  and  $M_f$  be a quasi-linear mean. Then  $M_f$  is not left (right) distributive over U.

#### 4.2 Distributivity of quasi-linear means over associative a-CAOA

Now, we consider the left (right) distributivity of quasi-linear means with respect to the classes of aggregation operations with an annihilator.

**Theorem 4.5.** Let A be a T-uninorm, and let  $M_f$  be a quasi-linear mean. Then,  $M_f$  is not left (right) distributive over A.

*Proof.* Let suppose that a T-uninorm A has an annihilator  $a \in (0, 1)$ . The case a = 1 is solved by Theorem 4.1, since in that case T-uninorm reduces to a disjunctive uninorm with neutral element  $e \in (0, 1)$ .

Let suppose that  $M_f$  is left distributive over a T-uninorm A. Similarly to the proof of Theorem 4.1 idempotency of A is obtained.

Setting x = 1, y = 1, z = 0 in (LD), the following is obtained

$$M_f(1,a) = M_f(1,A(1,0)) = A(M_f(1,1),M_f(1,0)) = A(1,M_f(1,0)).$$

Now, from the structure of T-uninorm A follow:

- if  $M_f(1,0) \ge a$ , then  $M_f(1,a) = M_f(1,0)$  which implies a = 0,
- if  $M_f(1,0) < a$ , then  $M_f(1,a) = a = M_f(a,a)$  which implies a = 1.

Both cases are in contradiction with the assumption that  $a \in (0, 1)$ . Therefore,  $M_f$  is not left distributive over a T-uninorm A.

Similarly, we can obtain results for S-uninorms and bi-uninorms.

**Theorem 4.6.** Let A be a S-uninorm and  $M_f$  be a quasi-linear mean. Then  $M_f$  is not left (right) distributive over A.

**Theorem 4.7.** Let A be a bi-uninorm and  $M_f$  be a quasi-linear mean. Then  $M_f$  is not left (right) distributive over A.

From the previously presented results can be deduced that for a given uninorm and a semi-uninorm U from the classes  $N_e^{min} \cup N_e^{max}$  with a neutral element  $e \in (0, 1)$ , does not exist a quasi-linear mean  $M_f$  that is left (right) distributive with respect to U. Also, the same conclusion holds for T-uninorms, S-uninorms and bi-uninorms.

#### 4.3 Distributivity of quasi-linear means over semi-t-operators

The focus of this section are semi-t-operators. Wang and Qin in [26] proved the following for a quasi-linear mean  $M_f$ and a semi-t-operator  $F \in \mathcal{F}_{a,b}$ :

- if a < b, then  $M_f$  is left (right) distributive over F if and only if  $F = \max$  or  $F = \min$  or F(x, y) = x for all  $x, y \in [0, 1]$ ;
- if a > b, then  $M_f$  is left (right) distributive over F if and only if  $F = \max$  or  $F = \min$  or F(x, y) = y for all  $x, y \in [0, 1]$ .

However, the next results improve ones from [26] by showing that a can not be 1, and b can not be 0, that is that  $F \neq \max$  and  $F \neq \min$ .

**Theorem 4.8.** Let  $M_f$  be a quasi-linear mean, and let  $F \in \mathcal{F}_{a,b}$  be a semi-t-operator with a < b. Then,  $M_f$  is left (right) distributive over F if and only if F(x, y) = x for all  $x, y \in [0, 1]$ .

*Proof.* ( $\Rightarrow$ ) Here, only the case when  $M_f$  is left distributive over F is considered since the right distributivity proof is analogous. Analogously to the proof of Theorem 4.1 there can be obtained that F(x, x) = x for all  $x \in [0, 1]$ , i.e., that F is an idempotent semi-t-operator. Then, by letting x = 0, y = 0, z = 1 in (LD), the following is obtained

$$M_f(0,a) = M_f(0,F(0,1)) = F(M_f(0,0),M_f(0,1)) = F(0,M_f(0,1)).$$

- If  $M_f(0,1) < a$ , then it follows from the structure of F that  $M_f(0,a) = F(0, M_f(0,1)) = M_f(0,1)$ , which implies a = 1. This contradicts the assumption  $a < b \le 1$ .
- If  $M_f(0,1) \ge a$ , that it follows from the structure of F that  $M_f(0,a) = F(0, M_f(0,1)) = a = M_f(a,a)$ , which implies a = 0.

Now, letting x = 1, y = 1, z = 0 in (LD), the following is obtained

$$M_f(1,b) = M_f(1,F(1,0)) = F(1,M_f(1,0)).$$

- If  $M_f(1,0) \leq b$ , then it follows from the structure of F that  $M_f(1,b) = F(1,M_f(1,0)) = b = M_f(b,b)$  which implies b = 1. Now, Theorem 2.10 implies that F(x,y) = x for all  $x, y \in [0,1]$ .
- If  $M_f(1,0) > b$ , that it follows from the structure of F that  $M_f(1,b) = F(1,M_f(1,0)) = M_f(1,0)$ , which implies b = 0. This is contradiction to the assumption that  $0 \le a < b$ .

The converse statement is obvious.

The analogous result for semi-t-operator  $F \in \mathcal{F}_{a,b}$  with a > b can be obtain in the same manner.

**Theorem 4.9.** Let  $M_f$  be a quasi-linear mean, and let  $F \in \mathcal{F}_{a,b}$  be a semi-t-operator with a > b. Then  $M_f$  is left (right) distributive over F if and only if F(x,y) = y for all  $x, y \in [0,1]$ .

It should be emphasized that Theorem 4.8 and Theorem 4.9 show that structure of semi-t-operator F is the same as in the case of left (right) distributivity of semi-t-operator with respect to quasi-linear mean  $M_f$  (see Theorem 3.7 and Corollary 3.8 from [26]).

# 5 Conclusion

This paper deals with distributivity equations on the whole domain for quasi-linear means on one side, and uninorms, semi-uninorms from the classes  $N_e^{\min} \cup N_e^{\max}$  with a neutral element  $e \in (0, 1)$ , T-uninorms, S-uninorms and biuninorms on the other. The results in Section 3 and Section 4 confirm the fact that distributivity law is strong condition because distributivity equations between disjunctive (conjunctive) uninorms and quasi-linear means degrades into the distributivity between t-conorms (t-norms) and quasi-linear means. Also, as seen, in the other cases distributivity laws do not hold. Additionally, Theorem 4.8 and Theorem 4.9 improve corresponding ones from [26], i.e., semi-toperator F can be neither max nor min operator, it is being reduced to F(x, y) = x or F(x, y) = y for all  $x, y \in$ [0, 1]. In the forthcoming work analogous study to the one given in this paper will be done for the other classes of aggregation operations such as uni-nullnorms and 2-uninorms. Also, since pairs of aggregation operators that are satisfying distributivity law play an important role in utility theory (see [12, 14]), further investigations will go to this direction as well.

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## References

- [1] J. Aczél, On mean values, Bulletin of the American Mathematical Society, 54 (1948), 392-400.
- [2] J. Aczél, Lectures on functional equations and their applications, Academic Press, New York, 1966.

- [3] G. Beliakov, A. Pradera, T. Calvo, Aggregation functions: A guide for practitioners, Springer, Berlin, Heidelberg, 2007.
- [4] T. Calvo, On some solutions of the distributivity equations, Fuzzy Sets and Systems, 104 (1999), 85-96.
- [5] T. Calvo, G. Mayor, R. Mesiar, Aggregation operators: New trends and applications, Springer, Berlin, Heidelberg, 2002.
- B. Cavallo, L. D'Apuzzo, M. Squillante, Independence and convergence in non-additive settings, Fuzzy Optimization and Decision Making, 8 (2009), 29-43.
- [7] J. Drewniak, P. Drygaś, E. Rak, Distributivity between uninorms and nullnorms, Fuzzy Sets and Systems, 159 (2008), 1646-1657.
- [8] P. Drygaś, Distributivity between semi-t-operators and semi-nullnorms, Fuzzy Sets and Systems, 264 (2015), 100-109.
- P. Drygaś, F. Qin, E. Rak, Left and right distributivity equations for semi-t-operators and uninorms, Fuzzy Sets and Systems, 325 (2017), 21-34.
- [10] P. Drygaś, E. Rak, Distributivity equation in the class of 2-uninorms, Fuzzy Sets and Systems, 291 (2016), 82-97.
- [11] J. C. Fodor, R. R. Yager, A. Rybalov, Structure of uninorms, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 5 (1997), 411-427.
- [12] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, Aggregations functions, Cambridge University Press, New York, 2009.
- [13] D. Jočić, P. Drygaś, I. Štajner-Papuga, Left and right distributivity equations in the class of semi-t-operators, Fuzzy Sets and Systems, 372 (2019), 62-80.
- [14] D. Jočić, I. Štajner-Papuga, Some implications of the restricted distributivity of aggregation operators with absorbing elements for utility theory, Fuzzy Sets and Systems, 291 (2016), 54-65.
- [15] D. Jočić, I. Stajner-Papuga, Distributivity and conditional distributivity for T-uninorms, Information Sciences, 424 (2018), 91-103.
- [16] M. Mas, S. Massanet, D. Ruiz-Aguilera, J. Torrens, A survey on the existing classes of uninorms, Journal of Intelligent and Fuzzy Systems, 29 (2015), 1021-1037.
- [17] M. Mas, R. Mesiar, M. Monserat, J. Torrens, Aggregation operations with annihilator, International Journal of General Systems, 34 (2005), 1-22.
- [18] F. Qin, Y. M. Min, Distributivity for semi-nullnorms over semi-t-operators, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 24(5) (2016), 685-702.
- [19] W. Sander, Associative aggregation operators, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators, Physica-Verlag, Heidelberg, (2002), 124-158.
- [20] Y. Su, H. W. Liu, J. V. Riera, D. Ruiz-Aguilera, J. Torrens, The distributivity equation for uninorms revisited, Fuzzy Sets and Systems, 334 (2018), 1-23.
- [21] Y. Su, H. W. Liu, D. Ruiz-Aguilera, J. Vicente Riera, J. Torrens, On the distributivity property for uninorms, Fuzzy Sets and Systems, 287 (2016), 184-202.
- [22] Y. Su, W. Zong, H. W. Liu, On distributivity equations for uninorms over semi-t-operators, Fuzzy Sets and Systems, 299 (2016), 41-65.
- [23] Y. Su, W. Zong, H. W. Liu, P. Xue, The distributivity equations for semi-t-operators over uninorms, Fuzzy Sets and Systems, 287 (2016), 172-183.
- [24] X. R. Sun, H. W. Liu, The left (right) distributivity of semi-t-operators over 2-uninorms, Iranian Journal of Fuzzy Systems, 17(3) (2020), 103-116.

- [25] J. M. Wang, H. W. Liu, On the distributivity equation for uni-nullnorms, Kybernetika, 55(1) (2019), 24-43.
- [26] J. M. Wang, F. Qin, On the characterization of distributivity equations about quasi-arithmetic means, Aequationes Mathematicae, 90 (2016), 501-515.
- [27] G. Wang, F. Qin, W. H. Li, Conditional distributivity for uni-nullnorms with continuous and archimedean underlying t-norms and t-conorms, Journal of Intelligent and Fuzzy Systems, 36 (2019), 2773-2785.
- [28] G. Wang, F. Qin, W. H. Li, Distributivity and conditional distributivity for uni-nullnorms, Fuzzy Sets and Systems, 372 (2019), 34-49.
- [29] R. R. Yager, A. Rybalov, Uninorm aggregation operators, Fuzzy Sets and Systems, 80 (1996), 111-120.
- [30] F. X. Zhang, E. Rak, J. Bazan, On the distributivity of continuous triangular norms and triangular conorms with respect to 2-uninorms, Fuzzy Sets and Systems, (2019), Doi: 10.1016/j.fss.2019.09.002