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Original paper

Distributivity laws for quasi-linear means

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Abstract

Aggregation operations play a fundamental role in a large number of disciplines, from mathematics and natural sciences to economics and social sciences. This paper is focused on the problem of distributivity for some special classes of aggregation operations, and quasi-linear means. Characterization of distributivity pairs for uninorms, semi-uninorms and associative a-CAOA vs quasi-linear means is given.

Keywords: Uninorms, semi-uninorms, T-uninorms, S-uninorms, bi-uninorms, semi-t-operators.

1 Introduction

There are many real life situations where the process of combining and merging a certain number of data into a single representative value is required. Mathematical functions carrying out this process are known as aggregation operations (aggregation functions, connectives, merging functions, etc.,). Although the idea of aggregation is very old (think of the arithmetic mean), detailed study of aggregation function in general, their formalization and classification is quite more recent. Therefore, due to high level of applicability, aggregation operations have been intensively studied from 1980's to nowadays. Some purely theoretical researches of this subject can lead to new possibilities of applications as seen in e.g. [3, 5, 12].

The problem of distributivity has roots in [2] and, recently, investigations of this problem are directed towards finding solutions for different classes of aggregation operations such as t-norms, t-conorms, uninorms, nullnorms, semi-uninorms, semi-nullnorms, semi-t-operators, uni-nullnorms, 2-uninorms, quasi-arithmetic means, etc (see [4, 8, 9, 10, 13, 18, 20, 21, 22, 23, 25, 26, 28, 30]). Also, researchers are investigating the problem of distributivity on the restricted domain, i.e., the conditional distributivity. This particular approach produces a larger variety of solutions (see [15, 27, 28]). The significance of the considered contemporary topic (see [13, 24, 27, 28, 30]) follows not only from the theoretical point of view, but also from its applicability in the utility theory for modelling some specific problems [12, 14]. Also, an interesting application of these laws on two Borel-Cantelli lemmas and independence of events for decomposable measures is given in [6].

This paper extends the research from [4, 26] where a characterization of all pairs (M_f, F) where M_f is a quasi-linear mean, and F is a t-norm, t-conorm, Mayor's aggregation operator, nullnorm, semi-nullnorm or semi-t-operator, satisfying distributivity laws on the whole domain is given. Therefore, the main concern of this paper are distributivity equations on the whole domain where one of the unknown functions is a quasi-liner mean and another one is a uninorm, semi-uninorm from the classes $N_e^{min} \cup N_e^{max}$, T-uninorm, S-uninorm and bi-uninorm.

This paper is organized as follows. The Section 2 contains preliminary notions concerning the aggregation operations in general, then the preliminary notions on quasi-linear means, uninorms, semi-uninorms, semi-t-operators, T-uninorms, S-uninorms, bi-uninorms and the distributivity equations. Distributivity of the above mentioned classes of aggregation operations over quasi-linear mean is considered in the third section. Section 4 contains results of distributivity equations in the opposite direction. Some concluding remarks are given in Section 5.

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2 Preliminaries

A short overview of notions that are essential for this research is given throughout this section (see [1, 2, 8, 11, 12, 16, 17, 19, 29]). The basic definition of an aggregation operation is the starting point of this research.

Definition 2.1. [12] *A binary aggregation operation Ag on $[0,1]$ is a function $Ag:[0,1]^2\to[0,1]$ that is nondecreasing in each variable and for which the following holds*

$$Ag(0,0)=0 \quad \text{and} \quad Ag(1,1)=1.$$

The previous definition is in its binary form and it can be easily extended to n-ary aggregation operation on arbitrary nonempty real interval $\mathbb{I}$ (bounded or not). This definition presents the bare bone structure of aggregation operations and, depending on the situation that is being investigated, many other properties can be required. Therefore, depending on the framework in which the aggregation is performed, the additional properties can be commutativity, associativity, idempotence, continuity, bisymmetry, strict increasing monotonicity, etc.

2.1 Quasi-linear means

The focus of this research is on means, perhaps the best known family of aggregation functions with a long history, especially on quasi-linear means (for more details see Chapter 4 from [12]). First, a mean is noting other than an idempotent aggregation function. A well-studied class of means is the class of quasi-arithmetic means introduced as n-ary functions in [1] by Aczél.

Definition 2.2. [12] *Let $f:[0,1]\to\mathbb{R}$ be a continuous and strictly monotone function. The n-ary quasi-arithmetic mean generated by f is a function $M_f:[0,1]^n\to[0,1]$ of the form*

$$M_f(x_1\cdots x_n)=f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f(x_i)\right). \tag{1}$$

Of course, the class of quasi-arithmetic means comprises most of the algebraic means.

The next result gives an axiomatization of n-ary quasi-arithmetic means which is due to Aczél.

Theorem 2.3. [12] *A function $F:[0,1]^n\to\mathbb{R}$ is commutative, continuous, strictly increasing, idempotent and bisymmetric if and only if there is a continuous and strictly monotonic function $f:[0,1]\to\mathbb{R}$ such that $F=M_f$, i.e., F is a quasi-arithmetic mean generated by f.*

Many authors have attempted to generalize the concept of quasi-arithmetic means by relaxing some conditions that characterize the quasi-arithmetic means such as commutativity or strictly increasing monotonicity. The concept considered here is of weighted quasi-arithmetic means, also called quasi-linear means, obtained simply by dropping the commutativity property from the previous theorem.

Theorem 2.4. [12] *A function $F:[0,1]^n\to\mathbb{R}$ is continuous, strictly increasing, idempotent and bisymmetric if and only if there is a continuous and strictly monotonic function $f:[0,1]\to\mathbb{R}$ and real numbers $w_1,\cdots,w_n>0$ satisfying $\sum_{i=1}^n w_i=1$ such that*

$$F(x_1,\cdots,x_n)=f^{-1}\left(\sum_{i=1}^{n}w_if(x_i)\right). \tag{2}$$

In papers [4, 26] the aggregation operation F from Theorem 2.4 is called quasi-arithmetic mean. In this paper we use terminology from [12], where the quasi-arithmetic mean is given by (1), and function F given by (2) is quasi-linear mean or weighted quasi-arithmetic mean.

Since the function f from (1) and (2) is continuous and strictly monotonic, the range of f is an interval $[m,n]$ where $m,n\in\mathbb{R}$. Thus, the working assumption through this paper is that $f:[0,1]\to[m,n]$, and that f is a monotone bijection. Also, this paper deals with binary aggregation operations, therefore the assume is that $n=2$ in the equation (2).

This following sub-sections provide overviews of aggregation operations that will be considered as a part of a distributive pair together with the quasi-linear mean.

2.2 Uninorms and semi-uninorms

Definition 2.5. [29] *A uninorm $U : [0,1]^2 \to [0,1]$ is an aggregation operation that is commutative, associative, and for which there exists a neutral element $e \in [0,1]$, i.e., $U(x,e) = x$ for all $x \in [0,1]$.*

Recall that when $e = 1$ uninorm U becomes a t-norm denoted by T, and when $e = 0$, U is a t-conorm denoted by S. For $e \in (0,1)$ the following result holds.

Theorem 2.6. [11] *Let U be a uninorm with a neutral element $e \in (0,1)$. Then there exists a t-norm T_U, a t-conorm S_U, and increasing operator $C : [0,e) \times (e,1] \cup (e,1] \times [0,e) \to [0,1]$ that fulfils $\min \leq C \leq \max$, such that U is given by*

$$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ C(x,y) & \text{otherwise.} \end{cases} \tag{3}$$

A t-norm T_U and a t-conorm S_U from the previous theorem are called the underlying t-norm and underlying t-conorm of U. If $U(0,1) = 0$, the uninorm U is conjunctive and, if $U(0,1) = 1$ the uninorm U is a disjunctive one.

Semi-uninorms are a form of relaxed uninorms, that are obtained by omitting commutativity and associativity from Definition 2.5. This specific class of operators was introduced in [7]. The family of all semi-uninorms with neutral element e is denoted by N_e. Specially, the class N_1 consists of t-seminorms or semicopulas, and N_0 t-semiconorms.

Notations N_e^{max} and N_e^{min} (see [7]) are used for families of all semi-uninorms with the neutral element $e \in (0,1)$ satisfying the following additional conditions

$$\forall x \in (e,1]\ U(0,x) = U(x,0) = x \ \text{ and } \ \forall x \in [0,e)\ U(1,x) = U(x,1) = x,$$

respectively.

2.3 Associative a-CAOA: T-uninorms, S-uninorms, bi-uninorms and nullnorms

Another generalization of uninorms necessary for the presented research are associative, commutative aggregation operations with annihilator a, shortly associative a-CAOA, that were studied in [17]. Further on, for any binary operation $A : [0,1]^2 \to [0,1]$, and any element $c \in [0,1]$, let the section $A_c : [0,1] \to [0,1]$ given by

$$A_c(x) = A(c,x),$$

be denoted by A_c. As it will be seen from the following, the continuity (discontinuity) of sections A_0 and A_1 plays a crucial role in classification of associative a-CAOA operations.

Definition 2.7. [17] *Let $A : [0,1]^2 \to [0,1]$ be an associative a-CAOA.*

- *A is called a S-uninorm if A_0 is continuous and A_1 is not, and there exists $e \in (0,1)$ such that e is idempotent, A_e is continuous and $A_e(1) = 1$.*
- *A is called a T-uninorm if A_1 is continuous and A_0 is not, and there exists $e \in (0,1)$ such that e is idempotent, A_e is continuous and $A_e(0) = 0$.*
- *A is called a bi-uninorm if A_0 and A_1 are not continuous, and there exist idempotent elements $e_0, e_1 \in (0,1)$ such that A_{e_0} and A_{e_1} are continuous and $A_{e_0}(0) = 0$ and $A_{e_1}(1) = 1$.*
- *A is called a nullnorm (a t-operator) if A_0 and A_1 are continuous.*

Remark 2.8. *Distributivity between semi-nullnorm (or nullnorm) and quasi-linear mean was investigated in full in* [26]*, therefore that type of associative a-CAOA will not be considered further in this paper.*

The following overview of results from [17] shows that the form of associative a-CAOA is closely related to uninorms, t-norms and t-conorms, and is essential for investigation that follows. Let $A : [0,1]^2 \to [0,1]$ be a binary operation.

- A is a S-uninorm if and only if There exists $a \in [0,1)$, a t-conorm S' and a conjunctive uninorm U' with neutral element $e' \in (0,1)$ such that A is given by

$$A(x,y)=\begin{cases} aS'\left(\frac{x}{a},\frac{y}{a}\right) & \text{if } (x,y)\in[0,a]^2,\\ a+(1-a)U'\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a]\end{cases} \tag{4}$$

- A is a T-uninorm if and only if there exists $a\in(0,1]$, a t-norm T' and a disjunctive uninorm U' with neutral element $e'\in(0,1)$ such that A is given by

$$A(x,y)=\begin{cases} aU'\left(\frac{x}{a},\frac{y}{a}\right) & \text{if } (x,y)\in[0,a]^2,\\ a+(1-a)T'\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a]\end{cases} \tag{5}$$

- A is a bi-uninorm if and only if there exists $a\in(0,1)$, a disjunctive uninorm U_0' and a conjunctive uninorm U_1' with neutral elements $e_0',e_1'\in(0,1)$, respectively, such that A is given by

$$A(x,y)=\begin{cases} aU_0'\left(\frac{x}{a},\frac{y}{a}\right) & \text{if } (x,y)\in[0,a]^2,\\ a+(1-a)U_1'\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right) & \text{if } (x,y)\in[a,1]^2,\\ a & \text{if } (x,y)\in[0,a]\times[a,1]\cup[a,1]\times[0,a]\end{cases} \tag{6}$$

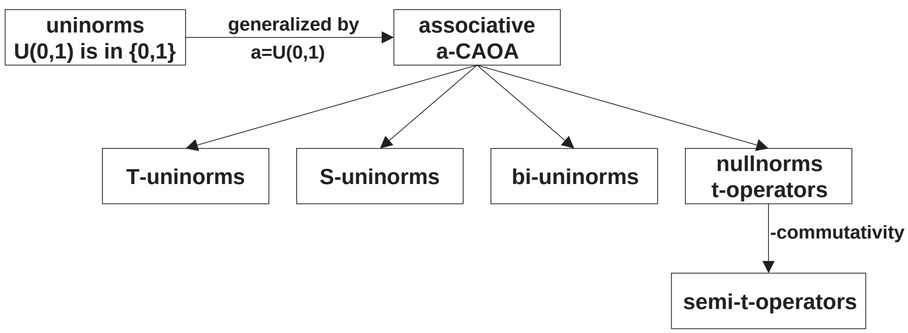

Figure 1. Associative a-CAOA and semi-t-operators.

2.4 Semi-t-operators

Now, commutativity is omitted from definition of t-operators, i.e., from nullnorms. Distributivity of this type of operators with respect to quasi-linear mean was also studied in [26], however some corrections are needed and they are given in the fourth section of this paper.

Definition 2.9. [8] *A semi-t-operator* $F:[0,1]^2\to[0,1]$ *is an associative aggregation operation such that the functions* F_0,F_1,F^0,F^1*, where* $F_0(x)=F(0,x)$*,* $F_1(x)=F(1,x)$*,* $F^0(x)=F(x,0)$*,* $F^1(x)=F(x,1)$*, are continuous.*

The family of all semi-t-operators such that $F(0,1)=a$ and $F(1,0)=b$ is denoted with $\mathcal{F}_{a,b}$. The following result holds.

Theorem 2.10. [19] *Let* $F:[0,1]^2\to[0,1]$*,* $F(0,1)=a$*,* $F(1,0)=b$*. The operation* $F\in\mathcal{F}_{a,b}$ *if and only if there exists an associative t-seminorm* T_F *and an associative t-semiconorm* S_F *such that*

$$F(x,y)=\begin{cases} aS_F\left(\frac{x}{a},\frac{y}{a}\right) & \textit{if } (x,y)\in[0,a]^2,\\ b+(1-b)T_F\left(\frac{x-b}{1-b},\frac{y-b}{1-b}\right) & \textit{if } (x,y)\in[b,1]^2,\\ a & \textit{if } x\le a\le y,\\ b & \textit{if } y\le b\le x,\\ x & \textit{otherwise,}\end{cases} \tag{7}$$

for $a\le b$ *and*

$$F(x,y)=\begin{cases} bS_F\left(\frac{x}{b},\frac{y}{b}\right) & \textit{if } (x,y)\in[0,b]^2,\\ a+(1-a)T_F\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right) & \textit{if } (x,y)\in[a,1]^2,\\ a & \textit{if } x\le a\le y,\\ b & \textit{if } y\le b\le x,\\ y & \textit{otherwise,}\end{cases} \tag{8}$$

for $b\le a$.

2.5 Distributivity equations

Finally, let us recall the functional equations that are called left and right distributivity laws ([2], p. 318).

Definition 2.11. *Let F and G be some binary aggregation operations. F is distributive over G, if the following two laws hold:*

(LD) *F is a left distributive over G ,i.e.,*

$$F(x, G(y, z)) = G(F(x, y), F(x, z)), \quad \textit{for all}\, x, y, z \in [0, 1],$$

and

(RD) *F is a right distributive over G ,i.e.,*

$$F(G(y, z), x) = G(F(y, x), F(z, x)), \quad \textit{for all}\, x, y, z \in [0, 1].$$

Of course, for commutative F (LD) and (RD) coincide. Since for this research results for (RD) are analogous to the results for (LD), further on, the focus will be only on (LD) case for non-commutative F.

3 Distributivity of some aggregation operations over quasi-linear means

In this section we present results concerning distributivity of previously described classes of aggregation operations over quasi-linear means M_f, where $f : [0, 1] \to [m, n]$. Further on, the assumption is that generator f is an increasing bijection, i.e., that $f(0) = m$ and $f(1) = n$.

3.1 Distributivity of uninorms and semi-uninorms over quasi-linear means

Proposition 3.1. *Let U be a disjunctive uninorm with a neutral element $e \in (0, 1)$, and M_f be a quasi-linear mean. Then U is not distributive over M_f.*

Proof. Let us suppose that a disjunctive uninorm U with a neutral element $e \in (0, 1)$ is distributive over quasi-linear mean M_f. Similarly to the proof of Theorem 7 from [4] we obtain that

$$U(x, y) = f^{-1}(A(x)f(y) + B(x)), \tag{9}$$

or

$$U(x, y) = f^{-1}(C(x)), \tag{10}$$

where A, B and C are three functions on $[0, 1]$.

Equation (10) is not possible solution for the uninorm U, since, according to (10), the uninorm U does not depend on y. This implies that for all $x \in [0, 1]$ $x = U(x, e) = U(x, 1) = 1$ which is impossible. Therefore, the only possible solution is (9).

It follows from (9) and the structure of U that

$$1 = U(0, 1) = f^{-1}(A(0)f(1) + B(0)) \;\text{ and }\; 0 = U(0, 0) = f^{-1}(A(0)f(0) + B(0).$$

Hence, the following is obtained

$$n = A(0)n + B(0) \;\text{ and }\; m = A(0)m + B(0)\text{,i.e., } n - m = A(0)(n - m).$$

Since $n \neq m$, it follows that $A(0) = 1$ and $B(0) = 0$. Again, from (9) and the structure of U follows

$$0 = U(0, e) = f^{-1}(A(0)f(e) + B(0)) \text{ ,i.e., } f(0) = m = f(e).$$

Since f is a bijection, $0 = e$ is obtained and this is in contradiction with the assumption $e \in (0, 1)$. Therefore, disjunctive uninorm U with a neutral element $e \in (0, 1)$ is not distributive over quasi-linear mean M_f. □

As a consequence of the previous theorem and results from [4] the following result can be obtained.

Theorem 3.2. *Let U be a disjunctive uninorm with a neutral element $e \in [0,1)$, and M_f be a quasi-linear mean. U is distributive over M_f if and only if U is a t-conorm given by*

$$U(x,y) = f^{-1}\left(\frac{n-f(x)}{n-m}f(y) + n\frac{f(x)-m}{n-m}\right). \tag{11}$$

That is, distributivity in this case reduces disjunctive uninorms to t-conorms of a specific form.

For distributivity of conjunctive uninorm over quasi-linear mean results are similar.

Proposition 3.3. *Let U be a conjunctive uninorm with a neutral element $e \in (0,1)$, and M_f be a quasi-linear mean. Then U is not distributive over M_f.*

Theorem 3.4. *Let U be a conjunctive uninorm with a neutral element $e \in (0,1]$, and M_f be a quasi-linear mean. U is distributive over M_f if and only if U is a t-norm given by*

$$U(x,y) = f^{-1}\left(\frac{f(x)-m}{n-m}f(y) + m\frac{n-f(x)}{n-m}\right). \tag{12}$$

Now conjunctive uninorms are reduced to t-norms of a specific form.

Using similar arguments as in Proposition 3.1 and Proposition 3.3 the following result can be proved.

Theorem 3.5. *Let U be a semi-uninorm from the classes $N_e^{min} \cup N_e^{max}$ with a neutral element $e \in (0,1)$ and let M_f be a quasi-linear mean. Then, U is not left (right) distributive over M_f.*

3.2 Distributivity of associative a-CAOA over quasi-linear means

Now, the focus is on distributivity of aggregation operations with an annihilator over quasi-linear mean.

Theorem 3.6. *Let A be a S-uninorm and let M_f be a quasi-linear mean. Then, A is not distributive over M_f.*

Proof. Let suppose that a S-uninorm A with an annihilator $a \in (0,1)$ is distributive over quasi-linear mean M_f. The case when $a = 0$ is solved in Proposition 3.3, since in that case S-uninorm reduces to conjunctive uninorm with neutral element $e \in (0,1)$. Again, similarly to the proof of Theorem 7 from [4] the following can be obtained

$$A(x,y) = f^{-1}(A_1(x)f(y) + B(x)), \tag{13}$$

or

$$A(x,y) = f^{-1}(C(x)), \tag{14}$$

where A_1, B and C are three functions on $[0,1]$. As in Proposition 3.1, (14) is not possible solution for S-uninorm A, and only (13) remains as an option.

It follows from (13) and the structure of A for $x \in (0,a)$ that

$$x = A(x,0) = f^{-1}(A_1(x)f(0) + B(x)) \text{ and } a = A(x,a) = f^{-1}(A_1(x)f(a) + B(x).$$

Hence,

$$f(x) = A_1(x)m + B(x) \text{ and } f(a) = A_1(x)f(a) + B(x),$$

i.e.,

$$f(a) - f(x) = A_1(x)(f(a) - m).$$

In order to determine $A_1(x)$, the following two cases have to be considered:

- $f(a) = m$. This implies $a = 0$, however it contradicts the starting assumption $a \in (0,1)$.
- $f(a) \neq m$. Then $A_1(x) = \frac{f(a)-f(x)}{f(a)-m}$ and $B(x) = f(a)\frac{f(x)-m}{f(a)-m}$. Again, from the structure of A for $x \in (0,a)$, it follows that $A(x,1) = a$. From (13) follows

$$f(a) = \frac{f(a)-f(x)}{f(a)-m}n + f(a)\frac{f(x)-m}{f(a)-m},$$

which implies $f(x) = f(a)$ or $f(a) = n = f(1)$. This results contradict the assumption of the strict monotonicity of f, because $x \in (0,a)$ and $a \in (0,1)$.

Therefore, S-uninorm A is not distributive over quasi-linear mean M_f. □

Similar claims can be shown for T-uninorms and bi-uninorms. That is, the following two theorems hold.

Theorem 3.7. *Let A be a T-uninorm and M_f be a quasi-linear mean. Then A is not distributive over M_f.*

Theorem 3.8. *Let A be a bi-uninorm and M_f be a quasi-linear mean. Then A is not distributive over M_f.*

The results from this section show that the distributivity of disjunctive (conjunctive) uninorms over quasi-linear means is being reduced into the distributivity of t-conorms (t-norms) over quasi-linear means. Also, for a given quasi-linear mean M_f, there does not exist a uninorm or a semi-uninorm from the class $N_e^{min} \cup N_e^{max}$ with the neutral element $e \in (0,1)$, nor a T-uninorm, a S-uninorm or a bi-uninorm, that is distributive with respect to M_f.

4 Distributivity of quasi-linear means over some aggregation operations

Results in this section concern the revers situation, i.e., distributivity of a quasi-linear mean M_f over some classes of aggregation operations.

4.1 Distributivity of quasi-linear means over uninorms and semi-uninorms

Theorem 4.1. *Let M_f be a quasi-linear mean, and let U be a disjunctive uninorm with a neutral element $e \in (0,1)$. Then, M_f is not left (right) distributive over U.*

Proof. Let suppose that (LD) holds. From the representation of M_f, it follows $M_f(x,y) = f^{-1}(w_1 f(x) + w_2 f(y))$, for all $(x,y) \in [0,1]^2$ where $w_1 + w_2 = 1$. Letting $x = y = z$ in equation (LD), the following is obtained $M_f(x, U(x,x)) = U(M_f(x,x), M_f(x,x))$. Now, from structure of M_f there follows

$$w_1 f(x) + w_2 f(U(x,x)) = f(U(x,x)), \text{ that is,}$$

$$w_1 f(x) = (1 - w_2) f(U(x,x)) = w_1 f(U(x,x)).$$

Since f is a bijection, $U(x,x) = x$ for all $x \in [0,1]$, i.e., U is an idempotent uninorm. Since U is an idempotent uninorm, there holds (see [16]) $U|_{[0,e]^2} = \min$, $U|_{[e,1]^2} = \max$. On the remaining part of the unit square $[0,1]^2$, that is for $x < e < y$, $U(x,y) \in \{x,y\}$.

Now, by letting $x = 1, y \in (0,e), z = e$ in (LD), there holds

$$M_f(1,y) = M_f(1, U(y,e)) = U(M_f(1,y), M_f(1,e)). \tag{15}$$

Since $y < e$, it follows $M_f(1,y) < M_f(1,e)$. Also, $M_f(1,e) > M_f(e,e) = e$.

- If $M_f(1,y) = e$, then from (15) follows $e = M_f(1,e)$. This is in contradiction with $M_f(1,e) > e$.
- If $M_f(1,y) > e$, then from (15) and since $U|_{[e,1]^2} = \max$, follows

  $$M_f(1,y) = \max(M_f(1,y), M_f(1,e)) = M_f(1,e),$$

  which implies $y = e$, because M_f is strictly increasing. This is in contradiction with $y \in (0,e)$.

Therefore, for all $y \in (0,e)$, there should be $M_f(1,y) < e$. Now, since M_f is continuous, it follows

$$M_f(1,e) = \lim_{y \to e^-} M_f(1,y) \leq e.$$

Again, this is in contradiction with $M_f(1,e) > e$.

Therefore, the quasi-linear mean M_f is not left distributive over disjunctive uninorm U with a neutral element $e \in (0,1)$. □

Similar result can be obtained for a conjunctive uninorm U.

Theorem 4.2. *Let M_f be a quasi-linear mean, and let U be a conjunctive uninorm with a neutral element $e \in (0,1)$. Then, M_f is not left (right) distributive over U.*

Therefore, according to [4], the only conclusion is that quasi-linear mean M_f is left (right) distributive over a conjunctive uninorm with a neutral element $e = 1$, (t-norm) and over a disjunctive uninorm with a neutral element $e = 0$ (t-conorm). That is, the following theorem hold.

Theorem 4.3. [4] *Let M_f be a quasi-linear mean, and let F be a t-norm or a t-conorm. M_f is left (right) distributive over F if and only if $F = \min$ or $F = \max$.*

Using similar arguments as in Theorem 4.1 and Theorem 4.2 the following result can be proven.

Theorem 4.4. *Let U be a semi-uninorm from the classes $N_e^{min} \cup N_e^{max}$ with a neutral element $e \in (0,1)$ and M_f be a quasi-linear mean. Then M_f is not left (right) distributive over U.*

4.2 Distributivity of quasi-linear means over associative a-CAOA

Now, we consider the left (right) distributivity of quasi-linear means with respect to the classes of aggregation operations with an annihilator.

Theorem 4.5. *Let A be a T-uninorm, and let M_f be a quasi-linear mean. Then, M_f is not left (right) distributive over A.*

Proof. Let suppose that a T-uninorm A has an annihilator $a \in (0,1)$. The case $a = 1$ is solved by Theorem 4.1, since in that case T-uninorm reduces to a disjunctive uninorm with neutral element $e \in (0,1)$.

Let suppose that M_f is left distributive over a T-uninorm A. Similarly to the proof of Theorem 4.1 idempotency of A is obtained.

Setting $x = 1, y = 1, z = 0$ in (LD), the following is obtained

$$M_f(1, a) = M_f(1, A(1, 0)) = A(M_f(1, 1), M_f(1, 0)) = A(1, M_f(1, 0)).$$

Now, from the structure of T-uninorm A follow:

- if $M_f(1, 0) \geq a$, then $M_f(1, a) = M_f(1, 0)$ which implies $a = 0$,
- if $M_f(1, 0) < a$, then $M_f(1, a) = a = M_f(a, a)$ which implies $a = 1$.

Both cases are in contradiction with the assumption that $a \in (0,1)$.

Therefore, M_f is not left distributive over a T-uninorm A. □

Similarly, we can obtain results for S-uninorms and bi-uninorms.

Theorem 4.6. *Let A be a S-uninorm and M_f be a quasi-linear mean. Then M_f is not left (right) distributive over A.*

Theorem 4.7. *Let A be a bi-uninorm and M_f be a quasi-linear mean. Then M_f is not left (right) distributive over A.*

From the previously presented results can be deduced that for a given uninorm and a semi-uninorm U from the classes $N_e^{min} \cup N_e^{max}$ with a neutral element $e \in (0,1)$, does not exist a quasi-linear mean M_f that is left (right) distributive with respect to U. Also, the same conclusion holds for T-uninorms, S-uninorms and bi-uninorms.

4.3 Distributivity of quasi-linear means over semi-t-operators

The focus of this section are semi-t-operators. Wang and Qin in [26] proved the following for a quasi-linear mean M_f and a semi-t-operator $F \in \mathcal{F}_{a,b}$:

- if $a < b$, then M_f is left (right) distributive over F if and only if $F = \max$ or $F = \min$ or $F(x, y) = x$ for all $x, y \in [0,1]$;
- if $a > b$, then M_f is left (right) distributive over F if and only if $F = \max$ or $F = \min$ or $F(x, y) = y$ for all $x, y \in [0,1]$.

However, the next results improve ones from [26] by showing that a can not be 1, and b can not be 0, that is that $F \neq \max$ and $F \neq \min$.

Theorem 4.8. *Let M_f be a quasi-linear mean, and let $F \in \mathcal{F}_{a,b}$ be a semi-t-operator with $a < b$. Then, M_f is left (right) distributive over F if and only if $F(x, y) = x$ for all $x, y \in [0,1]$.*

Proof. ($\Rightarrow$) Here, only the case when M_f is left distributive over F is considered since the right distributivity proof is analogous. Analogously to the proof of Theorem 4.1 there can be obtained that $F(x,x) = x$ for all $x \in [0,1]$, i.e., that F is an idempotent semi-t-operator. Then, by letting $x = 0, y = 0, z = 1$ in (LD), the following is obtained

$$M_f(0,a) = M_f(0,F(0,1)) = F(M_f(0,0), M_f(0,1)) = F(0, M_f(0,1)).$$

- If $M_f(0,1) < a$, then it follows from the structure of F that $M_f(0,a) = F(0, M_f(0,1)) = M_f(0,1)$, which implies $a = 1$. This contradicts the assumption $a < b \leq 1$.
- If $M_f(0,1) \geq a$, that it follows from the structure of F that $M_f(0,a) = F(0, M_f(0,1)) = a = M_f(a,a)$, which implies $a = 0$.

Now, letting $x = 1, y = 1, z = 0$ in (LD), the following is obtained

$$M_f(1,b) = M_f(1,F(1,0)) = F(1, M_f(1,0)).$$

- If $M_f(1,0) \leq b$, then it follows from the structure of F that $M_f(1,b) = F(1, M_f(1,0)) = b = M_f(b,b)$ which implies $b = 1$. Now, Theorem 2.10 implies that $F(x,y) = x$ for all $x, y \in [0,1]$.
- If $M_f(1,0) > b$, that it follows from the structure of F that $M_f(1,b) = F(1, M_f(1,0)) = M_f(1,0)$, which implies $b = 0$. This is contradiction to the assumption that $0 \leq a < b$.

The converse statement is obvious. □

The analogous result for semi-t-operator $F \in \mathcal{F}_{a,b}$ with $a > b$ can be obtain in the same manner.

Theorem 4.9. *Let M_f be a quasi-linear mean, and let $F \in \mathcal{F}_{a,b}$ be a semi-t-operator with $a > b$. Then M_f is left (right) distributive over F if and only if $F(x,y) = y$ for all $x, y \in [0,1]$.*

It should be emphasized that Theorem 4.8 and Theorem 4.9 show that structure of semi-t-operator F is the same as in the case of left (right) distributivity of semi-t-operator with respect to quasi-linear mean M_f (see Theorem 3.7 and Corollary 3.8 from [26]).

5 Conclusion

This paper deals with distributivity equations on the whole domain for quasi-linear means on one side, and uninorms, semi-uninorms from the classes $N_e^{min} \cup N_e^{max}$ with a neutral element $e \in (0,1)$, T-uninorms, S-uninorms and bi-uninorms on the other. The results in Section 3 and Section 4 confirm the fact that distributivity law is strong condition because distributivity equations between disjunctive (conjunctive) uninorms and quasi-linear means degrades into the distributivity between t-conorms (t-norms) and quasi-linear means. Also, as seen, in the other cases distributivity laws do not hold. Additionally, Theorem 4.8 and Theorem 4.9 improve corresponding ones from [26], i.e., semi-t-operator F can be neither max nor min operator, it is being reduced to $F(x,y) = x$ or $F(x,y) = y$ for all $x, y \in [0,1]$. In the forthcoming work analogous study to the one given in this paper will be done for the other classes of aggregation operations such as uni-nullnorms and 2-uninorms. Also, since pairs of aggregation operators that are satisfying distributivity law play an important role in utility theory (see [12, 14]), further investigations will go to this direction as well.

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