Arithmetic operations and ranking of hesitant fuzzy numbers by extension principle

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Abstract

A hesitant fuzzy number (HFN) is important as a generalization of the fuzzy number for hesitant fuzzy analysis and takes some applications that were discussed in recent literature. In this paper, we develop the hesitant fuzzy arithmetic, which is based on the extension principle for hesitant fuzzy sets. Employing this principle, standard arithmetic operations on fuzzy numbers are extended to HFNs and we show that the outcome of these operations on two HFNs are an HFN. Also we use the extension principle in HFSs for the ranking of HFNs, which may be an interesting topic. In this paper, we show that the HFNs can be ordered in a natural way. To introduce a meaningful ordering of HFNs, we use a new lattice operation on HFNs based upon extension principle and defining the Hamming distance on them. Finally, the applications of them are explained on optimization and decision-making problems.

Keywords: Hesitant fuzzy number, extension principle on hesitant fuzzy sets, arithmetic operations on hesitant fuzzy number, ordering of hesitant fuzzy numbers.

1 Introduction

Generally, a hesitant fuzzy number (HFN) is a special case of a hesitant fuzzy set (HFS) of the real line. HFNs could play a fundamental role in arithmetic operations in a hesitant fuzzy environment, analogous to the role played by the fuzzy numbers in fuzzy mathematics. Using these numbers we can do the calculations more easily [34]. So, arithmetic operations and ranking of HFNs are important topics.

HFNs are a generalization of fuzzy numbers. Arithmetic operations, ordering and comparison of fuzzy numbers play a key role in decision-making and optimization problems. The first concepts of fuzzy arithmetic on fuzzy numbers were introduced by Dubois and Prade [12]. Over the years, fuzzy arithmetic improved and played an important role in many applications, such as fuzzy control, decision-making, approximate reasoning, optimization and statistics with imprecise probabilities [21, 23]. Different methods have been given for the ranking of fuzzy numbers. Some of these strategies can only be used in certain environments, and some others can be used only when the membership functions have certain properties such as normal, triangular, trapezoidal and so on. Jain [22] proposed a method using the concept of maximizing set to order the fuzzy numbers. A standard form to extend the natural ordering of real numbers to fuzzy numbers was suggested by Bass and Kwakernaak [3]. Dubios and Prade [12] used maximizing sets to order fuzzy numbers. Yager [47, 48] proposed four indices, which is employed for the purpose of ordering fuzzy quantities in $[0, 1]$. Cheng [9] used the distance between the centroid point and the original point $(0, 0)$ to order the fuzzy numbers. Chu and Tsao [10] used the area between the centroid point and the original point as an index of ranking fuzzy numbers. Mahmodi Nejad and Mashinchi [29] proposed a novel method, based on the areas on the left and the right sides of fuzzy numbers for ranking fuzzy numbers. In the following, many researchers have developed methods to compare and

to rank fuzzy numbers [2, 11, 37, 50].

Intuitionistic fuzzy number (IFN) is a generalization of fuzzy numbers. Mitchell [30] interpreted IFNs as an ensemble of fuzzy numbers and introduced a ranking method for them. Grzegrorzewski [19] defined IFNs of a particular type and introduced a ranking method by using the expected interval of an IFN. Li [25] introduced the concept of a triangular IFN (TIFN) as a special case of the IFN and developed a new methodology for ranking TIFNs.

HFSs are another extension of the fuzzy sets. Many studies have focused on HFSs. In many real-world problems, hesitant situations are very frequent, and HFSs make easy the handling of uncertainty caused by hesitation. By reviewing the specialized literature, we can observe the applicability and quick growth of them [1, 7, 8, 14, 15, 16, 17, 18, 25, 26, 28, 31, 32, 35, 36, 40, 41, 42, 43, 45, 49, 51]. Arithmetic operations, ranking and comparing of HFSs could play a needed role in the decision-making, optimization, expert system problems, and so on, in hesitant fuzzy environments [27, 34, 38]. So far, some kinds of operations and hesitant fuzzy ranking methods have been proposed while we have a discrete reference set of HFSs. Xia and Xu [44] developed a method to deal with multi-attribute decision-making with anonymity based on a series of the hesitant fuzzy aggregation operators. Zhu et al. [52] proposed an approach to address multi-attribute decision-making problems using the weighted hesitant fuzzy geometric Bonferroni mean and the weighted hesitant fuzzy Choquet geometric Bonferroni mean operators. Gu et al. [20] investigated the evaluation model for a risk investment with hesitant fuzzy information and employed the hesitant fuzzy weighted averaging operator to aggregate the hesitant fuzzy information corresponding to each alternative. Farhadinia [13] presented a series of score functions for HFSs providing a variety of new methods for ranking HFSs. Ranjbar and Effati [33] proposed a general definition of order for hesitant fuzzy elements whose members are fuzzy numbers. Also in [34] they proposed a hesitant fuzzy relationship for comparing two HFNs, as an HFS with some additional properties, based on expected intervals by (α, k)-cut approach on them, and finally they give a new binary operation on HFNs. However this method has a high computational volume by increasing the number of elements of HFNs and operators and has not directly used the extension principle on HFSs. In this paper, by the extension principle for HFSs in [1], we propose standard arithmetic operations on HFNs and introduce a partial ordering to compare two HFNs. Also, we propose a meaningful ordering of HFNs based on the Hamming distance.

The rest of this paper has been organized as follows: In Section 2 , we review some concepts related to the paper. In Section 3 we present a method for developing an HFN arithmetic based on the extension principle for HFSs. In Section 4, we propose a new partial and meaningful ordering of HFNs. In Section 5, to illustrate the applicability of the proposed technique, two examples are used. Discussion and comparative analyses of new arithmetic operation are presented in Section 6. Finally, some conclusions are given in Section 7.

2   Preliminaries

In the following, we review some necessary definitions and basic concepts, which are used in this paper.

Definition 2.1. [24] *Let S be a set and $\Re$ be a relation on S. We call $\Re$ a partial ordering on S if it satisfies the following properties:*

(i) *For all $x \in S$, it follows that $(x, x) \in \Re$;*

(ii) *If $x, y \in S$ and both $(x, y) \in \Re$ and $(y, x) \in \Re$, then $x = y$;*

(iii) *If $x, y, z \in S$ and both $(x, y) \in \Re$ and $(y, z) \in \Re$, then $(x, z) \in \Re$.*

We refer to property (i) as reflexivity, to property (ii) as anti-symmetry, and to property (iii) as transitivity.

A noteworthy feature of a partial ordering (as opposed to a total ordering, to be discussed below) is that not every two elements of the set S need be comparable. For example, let us say that $(A, B) \in \Re$ if $A \subset B$. Now if $A = \{x \in \mathbb{R} : 2 < x < 5\}$ and $B = \{x \in \mathbb{R} : 3 < x < 7\}$, then yet neither $(A, B) \in \Re$ nor $(B, A) \in \Re$.
A partial order $\Re$ on a set S is a total or linear order if for all $x, y \in S$, we have $(x, y) \in \Re$ or $(y, x) \in \Re$. The expression of partially ordered set indicates that the elements of the set can be ordered but that it may not be possible to specify an ordering for every pair of elements.
A partially ordered set in which every pair of elements has both a least upper bound (LUB) and a greatest lower bound (GLB) is called a lattice. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

In this paper, we want to define arithmetic operations on HFNs and a method to compare two HFNs. For this purpose, in the following, we give some relevant concepts and notation.

Definition 2.2. [53] *If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}}(x)$ is called the grade of membership or membership function of x in $\tilde{A}$.*

Remark 2.3. *When $X = \{x_1, \ldots, x_n\}$ is a finite set, a fuzzy set on X is expressed as*

$$\tilde{A} = \sum_{i=1}^{n} \frac{\mu_{\tilde{A}}(x_i)}{x_i},$$

where $\mu_{\tilde{A}}(x_i)$ is the grade of membership of x_i in $\tilde{A}$. Also, when $X = [a, b]$ is not finite, we write

$$\tilde{A} = \int_X \frac{\mu_{\tilde{A}}(x)}{x},$$

where $\mu_{\tilde{A}}(x)$ is the membership function of x in $\tilde{A}$ for all $x \in [a, b]$.

Among the several types of fuzzy sets, there are sets on the real numbers under certain conditions, and they are viewed as fuzzy numbers. In order to computational efficiency in arithmetic fuzzy, triangular and trapezoidal fuzzy numbers are often used (see [53]).

In the following, we define the concept of the HFSs as an extension of the fuzzy sets, which was introduced in [42].

Definition 2.4. [42] *Let X be a collection of objects denoted generically by x; then an HFS $\tilde{\tilde{H}}$ in X is a set of ordered pairs as follows:*

$$\tilde{\tilde{H}} = \{(x, h_{\tilde{\tilde{H}}}(x)) \mid x \in X\}.$$

The same as Remark 2.3 for fuzzy sets, when X is a finite set $\{x_1, \ldots, x_n\}$, an HFS $\tilde{\tilde{H}}$ on X is expressed as

$$\tilde{\tilde{H}} = \sum_{i=1}^{n} \frac{h_{\tilde{\tilde{H}}}(x_i)}{x_i},$$

where $h_{\tilde{\tilde{H}}}(x_i) = \{h^1_{\tilde{\tilde{H}}}(x_i), \ldots, h^{l(x_i)}_{\tilde{\tilde{H}}}(x_i)\}$ with $l(x_i) = |h_{\tilde{\tilde{H}}}(x_i)|$ is the possible membership degrees of the element $x_i \in X$ to the set $\tilde{\tilde{H}}$ for $i = 1, \ldots, n$. Also, when X is infinite an HFS $\tilde{\tilde{H}}$ is written as

$$\tilde{\tilde{H}} = \int_X \frac{h_{\tilde{\tilde{H}}}(x)}{x},$$

where $h_{\tilde{\tilde{H}}}(x) = \{\mu_{\tilde{H}^1}(x), \ldots, \mu_{\tilde{H}^{l(x)}}(x)\}$ with $l(x) = |h_{\tilde{\tilde{H}}}(x)|$ is the possible membership degrees of the element $x \in X$ to the set $\tilde{\tilde{H}}$ given by the membership functions $\mu_{\tilde{H}^j}(x)$ for $j = 1, \ldots, l(x)$.

Remark 2.5. *An HFS $\tilde{\tilde{U}}$ on X is uniformly typical if there is a number p such that $l(x) \leq p$ for each $x \in X$. In [1], a uniformly typical HFS is abbreviated by UHFS. The characteristic of the each uniformly typical $h_{\tilde{\tilde{U}}}$ is well defined due to finiteness as follows:*

$$Char(h_{\tilde{\tilde{U}}}) = \max\{l(x) : x \in X\}.$$

To be easily understood, we express the UHFS $\tilde{\tilde{U}}$ while X is infinite and for all $x \in X$, we have $l(x) = p$, by

$$\tilde{\tilde{U}} = \{\tilde{U}^j\}_{j=1}^{p}.$$

Definition 2.6. *Let $\tilde{\tilde{U}} = \{\tilde{U}^i\}_{i=1}^{p}$ and $\tilde{\tilde{V}} = \{\tilde{V}^j\}_{j=1}^{q}$ be two $UHFSs$ on X. We say that $\tilde{\tilde{U}}$ is a hesitant fuzzy subset of $\tilde{\tilde{V}}$, if there is a one-to-one function as $\varphi : \{1, \ldots, p\} \longrightarrow \{1, \ldots, q\}$ such that $\mu_{\tilde{U}^i}(x) \leq \mu_{\tilde{V}^{\varphi(i)}}(x)$, for all $i \in \{1, \ldots, p\}$ and for all $x \in X$, then we show it as $\tilde{\tilde{U}} \subseteq \tilde{\tilde{V}}$.*

Definition 2.7. *Let $\tilde{\tilde{U}} = \{\tilde{U}^i\}_{i=1}^p$ and $\tilde{\tilde{V}} = \{\tilde{V}^j\}_{j=1}^q$ be two UHFSs on X. We say that $\tilde{\tilde{U}}$ is a strong hesitant fuzzy subset of $\tilde{\tilde{V}}$, if there is a one-to-one function as $\varphi : \{1, \ldots, p\} \longrightarrow \{1, \ldots, q\}$ such that $\mu_{\tilde{U}^i}(x) = \mu_{\tilde{V}^{\varphi(i)}}(x)$, for all $i \in \{1, \ldots, p\}$ and for all $x \in X$, then we show it as $\tilde{\tilde{U}} \sqsubseteq \tilde{\tilde{V}}$.*

Remark 2.8. *It should be noted that if $\tilde{\tilde{U}} \sqsubseteq \tilde{\tilde{V}}$ then $\tilde{\tilde{U}} \subseteq \tilde{\tilde{V}}$, but the opposite is not true. Also $\tilde{\tilde{U}} \subseteq \tilde{\tilde{V}}$ implies that $Char(h_{\tilde{\tilde{U}}}) \le Char(h_{\tilde{\tilde{V}}})$.*

In [1], it was defined a new principle that the mapping $f : X \longrightarrow Y$ can be extended to $\tilde{\tilde{f}} : UHFS(X) \longrightarrow UHFS(Y)$ through the following definition.

Definition 2.9. [1] *For each $h_{\tilde{\tilde{U}}} \in UHFS(X)$ with characteristic p, we decompose $h_{\tilde{\tilde{U}}}(x) = \{h_{\tilde{\tilde{U}}}^{\sigma(1)}(x), \ldots, h_{\tilde{\tilde{U}}}^{\sigma(l(x))}(x)\}$ where $h_{\tilde{\tilde{U}}}^{\sigma(1)}(x) < \cdots < h_{\tilde{\tilde{U}}}^{\sigma(l(x))}(x)$ and $l(x) \le p$ for all $x \in X$ and there must be $x \in X$ with $l(x) = p$. Then the new extension principle allows us to define a UHFS $h_{\tilde{\tilde{V}}}(y)$ in Y by $\tilde{\tilde{V}} = \{(y, h_{\tilde{\tilde{V}}}(y)) \mid y \in Y\}$, where*

$$h_{\tilde{\tilde{V}}}(y) = \bigcup_{j=1,\ldots,p} \{ \sup_{x \in f^{-1}(y)} \{h_{\tilde{\tilde{U}}}^{\sigma(j)}(x) : f(x) = y, x \in X\}\}.$$

Clearly $\tilde{\tilde{V}}$ is another $UHFS$ with characteristic p. By notational convenience, we also denote the standard decomposition of the typical HFE $h_{\tilde{\tilde{V}}}(y)$ as

$$h_{\tilde{\tilde{V}}}(y) = \{h_{\tilde{\tilde{V}}}^1(y), \ldots, h_{\tilde{\tilde{V}}}^p(y)\},$$

where, $h_{\tilde{\tilde{V}}}^j(y) = \sup_{x \in f^{-1}(y)}\{h_{\tilde{\tilde{U}}}^{\sigma(j)}(x) : f(x) = y, x \in X\}$ for each $j = 1, \ldots, p$. It should be noted that $\{\sup \emptyset\} = 0$ when there is no $x \in X$ such that $y = f(x)$.

In [34], it was presented the definition of HFNs, while X is infinite as a special type of HFSs, as follows.

Definition 2.10. [34] *Let $\tilde{\tilde{M}}$ be a UHFS with $Char(h_{\tilde{\tilde{M}}}) = p$ as $\tilde{\tilde{M}} = \{\tilde{M}^j\}_{j=1}^p$. Then we call it an HFN, if*

(1) $\tilde{M}^j \in \mathbb{F}, \quad$ *for all $j = 1, \ldots, p$,*

(2) $\bigcap_{j=1}^p \tilde{M}_1^j \ne \emptyset$,

where $\mathbb{F}$ is space of fuzzy numbers and $\tilde{M}_1^j$ is a 1-cut for the jth ($j = 1, \ldots, p$) element of the UHFS $\tilde{\tilde{M}}$.

The space of HFNs is denoted by $\mathbb{HF}$. In [34], the difference between HFSs and HFNs can be seen.

Definition 2.11. *Let $\tilde{\tilde{M}} = \{\tilde{M}^j\}_{j=1}^p$ be an HFN, with characteristic $l(x) = p$ for all $x \in X$. Then the extension principle allows us to define the HFN $\tilde{\tilde{N}}$ in Y induced by the HFN $\tilde{\tilde{M}}$ in X through $f : X \longrightarrow Y$ as $\tilde{\tilde{N}} = \{\tilde{N}^j\}_{j=1}^p$, where*

$$\tilde{N}^j = \{(y, \mu_{\tilde{N}^j}(y)) \mid y = f(x) \ \ x \in X\},$$

and

$$\mu_{\tilde{N}^j}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \{\mu_{\tilde{M}^{\sigma(j)}}(x)\} & \text{if } f^{-1}(y) \ne \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases} \tag{1}$$

in which $\tilde{M}^{\sigma(j)}$ is the jth smallest membership function in $\tilde{\tilde{M}}$ ordered by a one ranking function in fuzzy number.

In this paper, we use the Yager index from [48] for ordering fuzzy numbers.

Example 2.12. *Let $X = \mathbb{R}$ be a continuous reference set, and take the HFN $\tilde{\tilde{M}} = \{\tilde{M}^1, \tilde{M}^2\}$, where $\tilde{M}^1 = (0, 2, 4)$ and $\tilde{M}^2 = (1, 2, 3)$ are two triangular fuzzy numbers (see Figure 1). Let the function $f : X \to Y$ be given by $y = 2x + 1$. Then the extension principle on HFNs allows us to define the HFN $\tilde{\tilde{N}}$ in Y induced by the HFN $\tilde{\tilde{M}}$ in X through $f : X \longrightarrow Y$ as follows:*

$$\tilde{\tilde{N}} = \{\tilde{N}^1, \tilde{N}^2\},$$

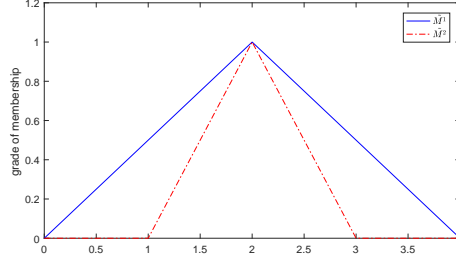*in which $\tilde{N}^1 = (1, 5, 9)$ and $\tilde{N}^2 = (3, 5, 7)$ (see Figure 2).*
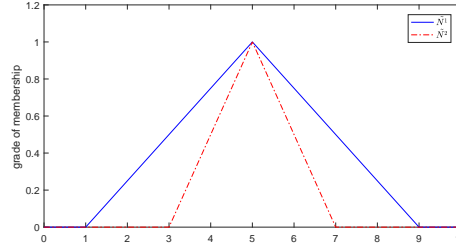
Figure 1: HFN M in Example 2.12.

Figure 2: HFN N in Example 2.12.

Remark 2.13. *If $\tilde{\tilde{M}}_1, \ldots, \tilde{\tilde{M}}_n$ are HFNs in $X_1, \ldots, X_n$ with the corresponding elements $\{\tilde{M}_1^{\sigma(j)}\}_{j=1}^p, \ldots, \{\tilde{M}_n^{\sigma(j)}\}_{j=1}^p$, respectively. The extension principle of HFNs allows us to extend the crisp function $y = f(x_1, x_2, \ldots, x_n)$ to act on n HFNs of X, such that $\tilde{\tilde{N}} = f(\tilde{\tilde{M}}_1, \ldots, \tilde{\tilde{M}}_n)$. Here the HFN $\tilde{\tilde{N}}$ is defined by $\tilde{\tilde{N}} = \{\tilde{N}^j\}_{j=1}^p$, where*

$$\tilde{N}^j = \left\{ (y, \mu_{\tilde{N}^j}(y)) : y = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n \right\},$$

and

$$\mu_{\tilde{N}^j}(y) = \sup_{y = f(x_1, \ldots, x_n)} \min \left\{ \tilde{M}_1^{\sigma(j)}, \ldots, \tilde{M}_n^{\sigma(j)} \right\}, \tag{2}$$

for $j = 1, \ldots, p$.

Distance measure is a fundamental and important issue of theory of sets. For HFSs, the axiomatic definition of distance measure was addressed by [46] as follows.

Definition 2.14. [46] *Let $\tilde{\tilde{U}}$ and $\tilde{\tilde{V}}$ be two HFSs on $X = \{x_1, x_2, \ldots, x_n\}$. Then the distance measure between $\tilde{\tilde{U}}$ and $\tilde{\tilde{V}}$ is defined as $d(\tilde{\tilde{U}}, \tilde{\tilde{V}})$ satisfying the following properties:*

(1) $0 \le d(\tilde{\tilde{U}}, \tilde{\tilde{V}}) \le 1$;

(2) $d(\tilde{\tilde{U}}, \tilde{\tilde{V}}) = 0$ *if and only if* $\tilde{\tilde{U}} = \tilde{\tilde{V}}$;

(3) $d(\tilde{\tilde{U}}, \tilde{\tilde{V}}) = d(\tilde{\tilde{V}}, \tilde{\tilde{U}})$.

In most cases, $l_{\tilde{\tilde{U}}}(x) \ne l_{\tilde{\tilde{V}}}(x)$. For convenience, let $l(x) = \max\{l_{\tilde{\tilde{U}}}(x), l_{\tilde{\tilde{V}}}(x)\}$ for each x in X. To operate correctly, we should extend the shorter one until both of them have the same length. Drawing on the well-known Hamming distance, a hesitant normalized Hamming distance is defined as follows:

$$d(\tilde{\tilde{U}}, \tilde{\tilde{V}}) = \frac{1}{n} \sum_{i=1}^n \Big[\frac{1}{l(x_i)} \sum_{j=1}^{l(x_i)} |h_{\tilde{\tilde{U}}}^{\sigma(j)}(x_i) - h_{\tilde{\tilde{V}}}^{\sigma(j)}(x_i)| \Big]. \tag{3}$$

Also, for any $x \in [a, b]$, a continuous hesitant normalized Hamming distance for $\tilde{\tilde{M}} = \{\tilde{M}_1^{\sigma(j)}\}_{j=1}^p$ and $\tilde{\tilde{N}} = \{\tilde{N}_1^{\sigma(j)}\}_{j=1}^p$ is defined as follows:

$$d(\tilde{\tilde{\mathrm{M}}}, \tilde{\tilde{\mathrm{N}}}) = \frac{1}{b-a} \sum_{j=1}^{p} \left(\frac{1}{p} \int_{a_j}^{b_j} |\tilde{M}^{\sigma(j)} - \tilde{N}^{\sigma(j)}| \right), \tag{4}$$

where p is characteristic of HFNs $\tilde{\tilde{\mathrm{M}}}$ and $\tilde{\tilde{\mathrm{N}}}$, $[a_j, b_j]$ is union of the support for both the fuzzy numbers $\tilde{M}^j$ and $\tilde{N}^j$ for $j = 1, \ldots, p$, and $a = \min_j \{a_j\}$ and $b = \max_j \{b_j\}$.

3 New arithmetic operations on hesitant fuzzy numbers

The arithmetic of HFNs can be taken as a generalization of the fuzzy arithmetic. The modelling based on the hesitant fuzzy arithmetic is expected to express the situation more realistically. In this section, we present a new method for developing the HFN arithmetic based on the extension principle for HFSs. Recently, in [34], a new binary operation $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ on HFNs is given as follows:

$$\tilde{\tilde{\mathrm{M}}} \tilde{*} \tilde{\tilde{\mathrm{N}}} = \left\{ \tilde{M}^i \tilde{*} \tilde{N}^j \right\}_{i=1,2,\ldots,n, \ j=1,2,\ldots,m}. \tag{5}$$

where $\tilde{\tilde{\mathrm{M}}} = \{\tilde{M}^1, \ldots, \tilde{M}^n\}$ and $\tilde{\tilde{\mathrm{N}}} = \{\tilde{N}^1, \ldots, \tilde{N}^m\} \in \mathbb{HF}$. Then it was shown that the outcome of the operation $\tilde{*}$ on two HFNs is an HFN (see [34, Theorem 3.2]). Section 6 shows that with increasing operators and characteristic of each HFN, this definition has a high computational volume. To solve this problem, in this paper, we introduce a new definition of arithmetic operations on the HFNs as follows.

Definition 3.1. *let $*$ denote any of the four basic arithmetic operations and let $\tilde{\tilde{M}} = \{\tilde{M}_1^{\sigma(j)}\}_{j=1}^{p}$ and $\tilde{\tilde{N}} = \{\tilde{N}_1^{\sigma(j)}\}_{j=1}^{p}$ denote HFNs. Then, we define an HFS on $\mathbb{R}$, $\tilde{\tilde{M}} \underset{\sim}{*} \tilde{\tilde{N}}$, by the result of Remark 2.13 as follows:*

$$\tilde{\tilde{M}} \underset{\sim}{*} \tilde{\tilde{N}}(z) = \bigcup_{j=1,\ldots,p} \left\{ \sup_{z=x*y} \left\{ \min \left\{ \mu_{\tilde{M}^{\sigma(j)}}(x), \mu_{\tilde{N}^{\sigma(j)}}(y) \right\} \right\} \right\}, \tag{6}$$

for all $z \in \mathbb{R}$.

For convenience , we can denote (6) as $\tilde{\tilde{M}} \underset{\sim}{*} \tilde{\tilde{N}}(z) = \left\{ \tilde{M}^{\sigma(j)} \tilde{*} \tilde{N}^{\sigma(j)}(z) \right\}_{j=1}^{p}$.

Remark 3.2. *In some cases, the cardinality of two HFNs is not equal; thus we should extend the shorter one until both have the same length when we perform arithmetic operations on them. To extend the shorter one, the best way is to add the same value several times in it. In fact, we can extend the shorter one by adding any value in it. The selection of this value mainly depends on the decision-makers risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value [46]. In this paper, in such situations, we propose to add the maximum value.*

More specifically, for basic arithmetic operations, we have

$$\tilde{\tilde{\mathrm{M}}} \underset{\sim}{\oplus} \tilde{\tilde{\mathrm{N}}}(z) = \bigcup_{j=1,\ldots,p} \left\{ \sup_{z=x+y} \left\{ \min \left\{ \mu_{\tilde{M}^{\sigma(j)}}(x), \mu_{\tilde{N}^{\sigma(j)}}(y) \right\} \right\} \right\}, \tag{7}$$

$$\tilde{\tilde{\mathrm{M}}} \underset{\sim}{\ominus} \tilde{\tilde{\mathrm{N}}}(z) = \bigcup_{j=1,\ldots,p} \left\{ \sup_{z=x-y} \left\{ \min \left\{ \mu_{\tilde{M}^{\sigma(j)}}(x), \mu_{\tilde{N}^{\sigma(j)}}(y) \right\} \right\} \right\}, \tag{8}$$

$$\tilde{\tilde{\mathrm{M}}} \underset{\sim}{\otimes} \tilde{\tilde{\mathrm{N}}}(z) = \bigcup_{j=1,\ldots,p} \left\{ \sup_{z=x \times y} \left\{ \min \left\{ \mu_{\tilde{M}^{\sigma(j)}}(x), \mu_{\tilde{N}^{\sigma(j)}}(y) \right\} \right\} \right\}, \tag{9}$$

$$\tilde{\tilde{\mathrm{M}}} \underset{\sim}{\oslash} \tilde{\tilde{\mathrm{N}}}(z) = \bigcup_{j=1,\ldots,p} \left\{ \sup_{z=x/y} \left\{ \min \left\{ \mu_{\tilde{M}^{\sigma(j)}}(x), \mu_{\tilde{N}^{\sigma(j)}}(y) \right\} \right\} \right\}. \tag{10}$$

Although $\tilde{\tilde{\mathrm{M}}} \underset{\sim}{*} \tilde{\tilde{\mathrm{N}}}$ is an HFS, in the following, we show that it is an HFN, which is a subject of the following lemma and theorem.

Lemma 3.3. *Let $\tilde{\tilde{V}}$ be an HFN. If $\tilde{\tilde{U}} \sqsubseteq \tilde{\tilde{V}}$, then $\tilde{\tilde{U}}$ is an HFN.*

Proof. Let $\tilde{\tilde{U}} = \{\tilde{U}^i\}_{i=1}^p$ and $\tilde{\tilde{V}} = \{\tilde{V}^j\}_{j=1}^q$, due to the $\tilde{\tilde{U}} \sqsubseteq \tilde{\tilde{V}}$, we know by Definition 2.7 that there is a one-to-one function as $\varphi : \{1,\ldots,p\} \longrightarrow \{1,\ldots,q\}$ such that $\tilde{U}^i = \tilde{V}^{\varphi(i)}$ for all $i \in \{1,\ldots,p\}$. Since $\tilde{\tilde{V}}$ is an HFN, we have $\tilde{V}^{\varphi(i)} \in \mathbb{F}$ for each $i \in \{1,\ldots,p\}$. Hence $\tilde{U}^i \in \mathbb{F}$. On the other hand, it can be written

$$\bigcap_{j=1}^q \tilde{V}_1^j \subseteq \bigcap_{i=1}^p \tilde{V}_1^{\varphi(i)} = \bigcap_{i=1}^p \tilde{U}_1^i. \tag{11}$$

Now, since $\tilde{\tilde{V}} \in \mathbb{HF}$, we have $\bigcap_{j=1}^q \tilde{V}_1^j \neq \emptyset$. Hence, it follows from (11) that $\bigcap_{i=1}^p \tilde{U}_1^i \neq \emptyset$. Hence, due to Definition 2.10, it shows $\tilde{\tilde{U}} \in \mathbb{HF}$. $\qquad\square$

Notation 3.4. *It should be noted that in Lemma 3.3, the condition $\tilde{\tilde{A}} \sqsubseteq \tilde{\tilde{B}}$ cannot be replaced by the weaker condition $\tilde{\tilde{A}} \subseteq \tilde{\tilde{B}}$.*

Theorem 3.5. *Let $* \in \{\oplus, \ominus, \otimes, \oslash\}$, and let $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$ be two HFNs with equal characteristics. Then, the HFS $\tilde{\tilde{M}} \underset{\approx}{*} \tilde{\tilde{N}}$ in Definition 3.1 is an HFN.*

Proof. Let $\tilde{\tilde{M}} = \{\tilde{M}^i\}_{i=1}^p$ and $\tilde{\tilde{N}} = \{\tilde{N}^i\}_{i=1}^p$. Also, let $\tilde{\tilde{U}} = \tilde{\tilde{M}} \underset{\approx}{*} \tilde{\tilde{N}} = \{\tilde{U}^i\}_{i=1,\ldots,p}$, where $\tilde{U}^i = \tilde{M}^{\sigma(i)} \tilde{*} \tilde{N}^{\sigma(i)}$, and let $\tilde{\tilde{V}} = \tilde{\tilde{M}}\tilde{*}\tilde{\tilde{N}} = \{\tilde{V}^{(i,j)}\}_{i,j=1,\ldots,p}$, where $\tilde{V}^{(i,j)} = \tilde{M}^{\sigma(i)} \tilde{*} \tilde{N}^{\sigma(j)}$.

Now, consider function $\varphi : \{1,\ldots,p\} \longrightarrow \{(i,j)\}_{i,j=1,\ldots,p}$ that is defined by $\varphi(i) = (i,i)$. It is clear that φ is a one-to-one function and for each $i \in \{1,\ldots,p\}$ we have

$$\tilde{U}^i = \tilde{M}^{\sigma(i)} \tilde{*} \tilde{N}^{\sigma(i)} = \tilde{V}^{(i,i)} = \tilde{V}^{\varphi(i)}.$$

So due to Definition 2.7, it can be concluded that $\tilde{\tilde{U}} \sqsubseteq \tilde{\tilde{V}}$. On the other hand, according to [34, Theorem 3.2], we know $\tilde{\tilde{V}} = \tilde{\tilde{M}}\tilde{*}\tilde{\tilde{N}} \in \mathbb{HF}$. Hence, from Lemma 3.3 we have $\tilde{\tilde{U}} = \tilde{\tilde{M}} \underset{\approx}{*} \tilde{\tilde{N}} \in \mathbb{HF}$, which completes the proof. $\qquad\square$

The following example shows the new arithmetic operations on HFNs.

Example 3.6. *Let $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$ be two HFNs with the continuous reference sets as follows:*

$$\tilde{\tilde{M}} = \left\{ \left(x, \{\mu_{\tilde{M}^1}(x), \mu_{\tilde{M}^2}(x)\}\right)|x \in \mathbb{R} \right\}, \quad \tilde{\tilde{N}} = \left\{ \left(x, \{\mu_{\tilde{N}^1}(x), \mu_{\tilde{N}^2}(x)\}\right)|x \in \mathbb{R} \right\},$$

where $\tilde{M}^1 = (3,5,6,8), \tilde{M}^2 = (4,6,8), \tilde{N}^1 = (2,3,4)$ and $\tilde{N}^2 = (1,3,6)$ (see Figure 3).
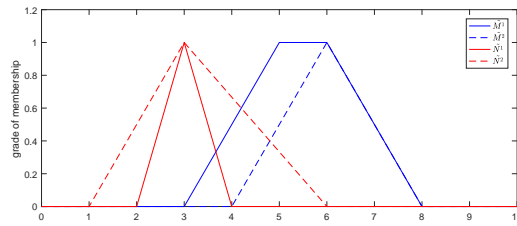

Figure 3: HFNs M and N in Example 3.6.

Using (7) for the operator $\underset{\approx}{\oplus}$, we have $\tilde{\tilde{A}} = \tilde{\tilde{M}} \underset{\approx}{\oplus} \tilde{\tilde{N}}(z) = \left\{ \left(x, \{\mu_{\tilde{A}^1}(x), \mu_{\tilde{A}^2}(x)\}\right)|x \in \mathbb{R} \right\}$, where

$$\mu_{\tilde{A}^1}(x) = \begin{cases} 0 & for\ x \in (-\infty, 5], \\ \frac{x-5}{3} & for\ x \in [5,8], \\ 1 & for\ x \in [8,9], \\ \frac{12-x}{3} & for\ x \in [9,12], \\ 0 & for\ x \in [12,\infty), \end{cases} \qquad \mu_{\tilde{A}^2}(x) = \begin{cases} 0 & for\ x \in (-\infty, 5], \\ \frac{x-5}{4} & for\ x \in [5,9], \\ \frac{14-x}{5} & for\ x \in [9,14], \\ 0 & for\ x \in [14,\infty). \end{cases}$$

Using (8) for the operator $\ominus_{\approx}$, we have $\tilde{\tilde{B}} = \tilde{\tilde{M}} \ominus_{\approx} \tilde{\tilde{N}}(z) = \left\{ \left(x, \{\mu_{\tilde{B}^1}(x), \mu_{\tilde{B}^2}(x)\}\right) | x \in \mathbb{R} \right\}$, where

$$\mu_{\tilde{B}^1}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, -1], \\ \frac{x+1}{3} & \text{for } x \in [-1, 2], \\ 1 & \text{for } x \in [2, 3], \\ \frac{6-x}{3} & \text{for } x \in [3, 6], \\ 0 & \text{for } x \in [6, \infty), \end{cases} \qquad \mu_{\tilde{B}^2}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, -2], \\ \frac{x+2}{5} & \text{for } x \in [-2, 3], \\ \frac{7-x}{4} & \text{for } x \in [3, 7], \\ 0 & \text{for } x \in [7, \infty). \end{cases}$$

Using (9) for the operator $\otimes_{\approx}$, we have $\tilde{\tilde{C}} = \tilde{\tilde{M}} \otimes_{\approx} \tilde{\tilde{N}}(z) = \left\{ \left(x, \{\mu_{\tilde{C}^1}(x), \mu_{\tilde{C}^2}(x)\}\right) | x \in \mathbb{R} \right\}$, where

$$\mu_{\tilde{C}^1}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 6], \\ \frac{\sqrt{8x+1}-7}{4} & \text{for } x \in [6, 15], \\ 1 & \text{for } x \in [15, 18], \\ 4 - \sqrt{\frac{x}{2}} & \text{for } x \in [18, 32], \\ 0 & \text{for } x \in [32, \infty), \end{cases} \qquad \mu_{\tilde{C}^2}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 4], \\ \frac{\sqrt{4x+9}-5}{4} & \text{for } x \in [4, 18], \\ 3 - \sqrt{\frac{x+6}{6}} & \text{for } x \in [18, 48], \\ 0 & \text{for } x \in [48, \infty). \end{cases}$$

Using (10) for the operator $\oslash_{\approx}$, we have $\tilde{\tilde{D}} = \tilde{\tilde{M}} \oslash_{\approx} \tilde{N}(z) = \left\{ \left(x, \{\mu_{\tilde{D}^1}(x), \mu_{\tilde{D}^2}(x)\}\right) | x \in \mathbb{R} \right\}$, where

$$\mu_{\tilde{D}^1}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, \frac{3}{4}], \\ \frac{4x-3}{x+2} & \text{for } x \in [\frac{3}{4}, \frac{5}{3}], \\ 1 & \text{for } x \in [\frac{5}{3}, 2], \\ \frac{8-2x}{x+2} & \text{for } x \in [2, 4], \\ 0 & \text{for } x \in [4, \infty), \end{cases} \qquad \mu_{\tilde{D}^2}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, \frac{2}{3}], \\ \frac{6x-4}{3x+2} & \text{for } x \in [\frac{2}{3}, 2], \\ \frac{8-x}{2x+2} & \text{for } x \in [2, 8], \\ 0 & \text{for } x \in [8, \infty). \end{cases}$$

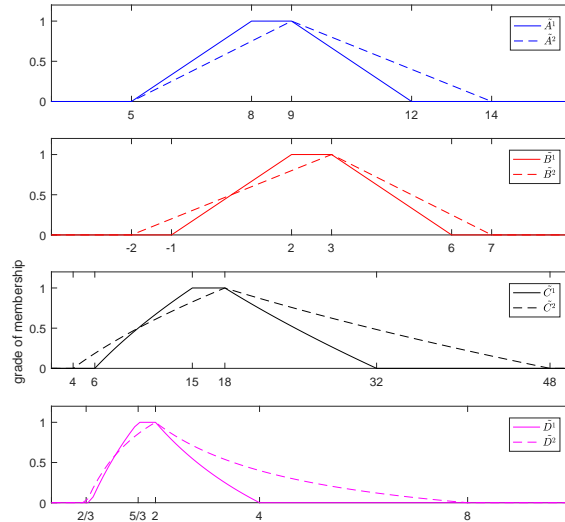*The four arithmetic operations performed in this example are illustrated in Figure 4.*

Figure 4: Illustration of arithmetic operations on HFNs in Example 3.6.

Remark 3.7. *For more convenience in practical calculations, the HFNs $\tilde{\tilde{M}} \otimes_{\approx} \tilde{\tilde{N}}(z)$ and $\tilde{\tilde{M}} \oslash_{\approx} \tilde{\tilde{N}}(z)$ whose elements are not necessarily triangular and trapezoidal fuzzy numbers, can be approximated by a suitable triangular and trapezoidal fuzzy numbers using the concept of left and right divergence (see [4, 5]). The HFN so obtained the approximation*

of the given HFN, is called. For example, a possible approximation of $\tilde{\tilde{C}} = \tilde{\tilde{M}} \underset{\approx}{\otimes} \tilde{\tilde{N}}(z)$ in Example 3.6, is the HFN
$\tilde{\tilde{C}} = \{(6, 15, 18, 32), (4, 18, 48)\}$.

4 Partial and meaningful ordering of hesitant fuzzy numbers

To introduce a meaningful ordering of HFNs, we first extend the lattice operations on fuzzy numbers to HFNs. Let $\mathbb{F}$ denote the set of all fuzzy numbers. In [24], it was shown that the pair $(\mathbb{F}, \lesssim)$ is a distributive lattice, where $\lesssim$ is a partial ordering defined as:

$$\tilde{A} \lesssim \tilde{B} \iff LUB(\tilde{A}, \tilde{B}) = \tilde{B}, \tag{12}$$

where

$$LUB(\tilde{A}, \tilde{B})(c) = sup_{c=\max\{a,b\}} \min\{\mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)\}.$$

The linear ordering of real numbers does not extend to fuzzy numbers, but in [24], it was shown that fuzzy numbers with the pair $(\mathbb{F}, \lesssim)$ can be ordered partially in a natural way. To introduce a meaningful ordering of HFNs, we first extend the lattice operation LUB on fuzzy numbers, to the corresponding operations on HFNs. For this purpose, for any two HFNs $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$, by using Remark 3.2 we define

$$LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \left\{ LUB(\tilde{M}^{\sigma(j)}, \tilde{N}^{\sigma(j)}) \right\}_{j=1}^{p}. \tag{13}$$

The next theorem shows that $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}})$ is an HFN.

Theorem 4.1. *Let $\tilde{\tilde{M}}, \tilde{\tilde{N}} \in \mathbb{HF}$. Then $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) \in \mathbb{HF}$.*

Proof. Let $\Box : \mathrm{R}^2 \to \mathrm{R}$ and let $x \Box y = \max\{x, y\}$. Based on Definition 3.1, $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \tilde{\tilde{M}} \underset{\approx}{\Box} \tilde{\tilde{N}}$, since $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}})$ is a new binary operation in (6) and max is a continuous operator. Hence it follows from Theorem 3.5 that $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}})$ is an HFN. $\square$

Recall that $\mathbb{HF}$ denotes the set of all HFNs. Let $\lessapprox$ be a relation defined as follows:

$$\tilde{\tilde{M}} \lessapprox \tilde{\tilde{N}} \iff LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \tilde{\tilde{N}}. \tag{14}$$

In the following, we show the pair $(\mathbb{HF}, \lessapprox)$ is a partial ordering. This is a subject of the following lemma and theorem.

Lemma 4.2. *Let $\tilde{\tilde{M}} = \{\tilde{M}^i\}_{i=1}^{p}$ and $\tilde{\tilde{N}} = \{\tilde{N}^i\}_{i=1}^{p}$ be two HFNs. Then $\tilde{\tilde{M}} \lessapprox \tilde{\tilde{N}} \iff \tilde{M}^{\sigma(i)} \lesssim \tilde{N}^{\sigma(i)}$, for all $i = 1, \ldots, p$.*

Proof. We first show that for each $i = 1, \ldots, p$

$$\tilde{\tilde{M}} \lessapprox \tilde{\tilde{N}} \Rightarrow \tilde{M}^{\sigma(i)} \lesssim \tilde{N}^{\sigma(i)}.$$

Let $\tilde{\tilde{M}} \lessapprox \tilde{\tilde{N}}$. From (14) we have $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \tilde{\tilde{N}}$, which implies that

$$\left\{ LUB(\tilde{M}^{\sigma(i)}, \tilde{N}^{\sigma(i)}) \right\}_{i=1}^{p} = \left\{ \tilde{N}^{\sigma(i)} \right\}_{i=1}^{p}.$$

Therefore $LUB(\tilde{M}^{\sigma(i)}, \tilde{N}^{\sigma(i)}) = \tilde{N}^{\sigma(i)}$, for each $i = 1, \ldots, p$. Hence, from (12), for $i = 1, \ldots, p$, we have $\tilde{M}^{\sigma(i)} \lesssim \tilde{N}^{\sigma(i)}$. The proof of the reverse relation is analogous. $\square$

Theorem 4.3. *The relation $\lessapprox$ introduced in (14) on $\mathbb{HF}$ is a partial ordering.*

Proof. We should show that the relation satisfies in three properties reflexivity, anti-symmetry and transitivity. The reflexivity property is clear, namely, for each $\tilde{\tilde{M}} \in \mathbb{HF}$, we have $LUB(\tilde{\tilde{M}}, \tilde{\tilde{M}}) = \tilde{\tilde{M}}$, that is, $\tilde{\tilde{M}} \lessapprox \tilde{\tilde{M}}$.
For anti-symmetry property, let both $\tilde{\tilde{M}} \lessapprox \tilde{\tilde{N}}$ and $\tilde{\tilde{N}} \lessapprox \tilde{\tilde{M}}$ be true. Therefore

$$LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \tilde{\tilde{N}}, \tag{15}$$

$$LUB(\tilde{\tilde{N}}, \tilde{\tilde{M}}) = \tilde{\tilde{M}}. \tag{16}$$

Due to [24, Theorem 4.3], we have

$$LUB(\tilde{M}, \tilde{N}) = LUB(\tilde{N}, \tilde{M}), \tag{17}$$

hence, from (13) and (17), it can be concluded

$$LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = LUB(\tilde{\tilde{N}}, \tilde{\tilde{M}}). \tag{18}$$

Therefore, from (15), (16), and (18), it is clear that $\tilde{\tilde{M}} = \tilde{\tilde{N}}$. Finally, we prove the transitivity property. For each $\tilde{\tilde{M}}, \tilde{\tilde{N}}, \tilde{\tilde{U}} \in \mathbb{HF}$, let both $\tilde{\tilde{M}} \lesssim \tilde{\tilde{N}}$ and $\tilde{\tilde{N}} \lesssim \tilde{\tilde{U}}$ be true. Then from Lemma 4.2, for each $i = 1, \dots, p$, we get

$$\tilde{M}^{\sigma(i)} \lesssim \tilde{N}^{\sigma(i)}, \tag{19}$$

$$\tilde{N}^{\sigma(i)} \lesssim \tilde{U}^{\sigma(i)}. \tag{20}$$

Since the relation $\lesssim$ is a partial ordering (see [24]), hence for each $i = 1, \dots, p$, we have

$$\tilde{M}^{\sigma(i)} \lesssim \tilde{U}^{\sigma(i)}. \tag{21}$$

Thus, due to Lemma 4.2, we have $\tilde{\tilde{M}} \lesssim \tilde{\tilde{U}}$, which completes the proof. $\square$

The set $(\mathbb{HF}, \lesssim)$ is not linearly ordered; thus some HFNs are not directly comparable. In the following, we propose a new method to compare two HFNs based upon the definition of the Hamming distance. We compare two HFNs $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$ by the following steps:

1- At first, we determine their $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}})$.

2- Then, we calculate the Hamming distances $d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{M}})$ and $d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{N}})$ by (4),

3- Finally, we have

$$\tilde{\tilde{M}} \leqq \tilde{\tilde{N}} \quad \Leftrightarrow \quad d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{M}}) \geq d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{N}}) \tag{22}$$

Remark 4.4. *It should be noted that if $\tilde{\tilde{M}} \lesssim \tilde{\tilde{N}}$, then $\tilde{\tilde{M}} \leqq \tilde{\tilde{N}}$, but the opposite is not true.*

Example 4.5. *Let $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$ be two HFNs with the continuous reference sets in Example 3.6. We compare $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$. At first, we obtain $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}})$ as follows:*

$$LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \{ LUB(\tilde{M}^{\sigma(1)}, \tilde{N}^{\sigma(1)}), LUB(\tilde{M}^{\sigma(2)}, \tilde{N}^{\sigma(2)}) \},$$

where

$$LUB(\tilde{M}^{\sigma(1)}, \tilde{N}^{\sigma(1)}) = \tilde{M}^{\sigma(1)}, \quad LUB(\tilde{M}^{\sigma(2)}, \tilde{N}^{\sigma(2)}) = \tilde{M}^{\sigma(2)}.$$

Therefore we have $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \tilde{\tilde{M}}$. Hence, due to (14), we have $\tilde{\tilde{N}} \lesssim \tilde{\tilde{M}}$ and by Remark 4.4, we get $\tilde{\tilde{M}} \leqq \tilde{\tilde{N}}$.

Example 4.6. *Let $\tilde{\tilde{M}} = \{ \tilde{M}^{\sigma(1)}, \tilde{M}^{\sigma(2)} \}$ and $\tilde{\tilde{N}} = \{ \tilde{N}^{\sigma(1)}, \tilde{N}^{\sigma(2)} \}$ be two HFNs with the continuous reference sets, where $\tilde{M}^{\sigma(1)} = (0, 1, 3), \tilde{M}^{\sigma(2)} = (-2, 1, 2, 5), \tilde{N}^{\sigma(1)} = (0, 2, 3)$ and $\tilde{N}^{\sigma(2)} = (1, 2, 3)$ are fuzzy numbers (see Figure 5). We want to compare $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$. At first, we obtain $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}) = \{ LUB(\tilde{M}^{\sigma(1)}, \tilde{N}^{\sigma(1)}), LUB(\tilde{M}^{\sigma(2)}, \tilde{N}^{\sigma(2)}) \}$, in which*

$$LUB(\tilde{M}^{\sigma(1)}, \tilde{N}^{\sigma(1)}) = \tilde{N}^{\sigma(1)}, \quad LUB(\tilde{M}^{\sigma(2)}, \tilde{N}^{\sigma(2)}) \neq \tilde{N}^{\sigma(2)},$$

where

$$\mu_{LUB(\tilde{M}^{\sigma(2)}, \tilde{N}^{\sigma(2)})}(x) = \begin{cases} 0 & x < 1 \\ x - 1 & 1 \leq x < 2 \\ \dfrac{5 - x}{3} & 2 \leq x < 5 \\ 0 & x \geq 5 \end{cases}.$$

Figure 6 show $LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}})$. In this example, according to (14), we cannot compare $\tilde{\tilde{M}}$ and $\tilde{\tilde{N}}$; in other words, $\tilde{\tilde{N}} \lesssim \tilde{\tilde{M}}$.
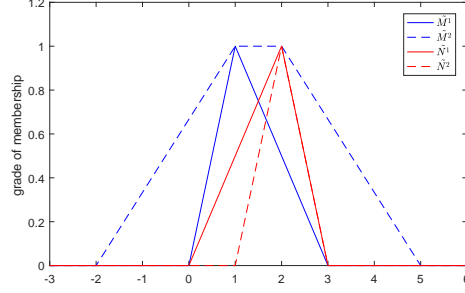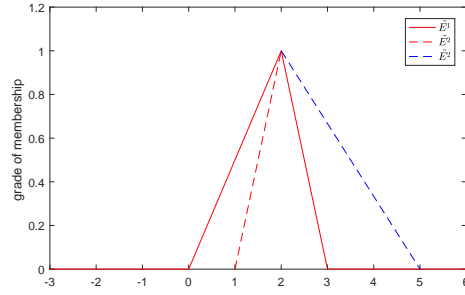
Figure 5: HFNs M and N in Example 4.6.

Figure 6: The lattice operation LUB on HFNs in Example 4.6.

Now, due to (22) we obtain $d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{M}})$ and $d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{N}})$, respectively, as follows:

$$d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{M}}) = \frac{1}{5-(-2)}\Big(\frac{1}{2}\sum_{j=1}^{2}\int_{-2}^{5}|LUB(\tilde{M}^{\sigma(j)}, \tilde{N}^{\sigma(j)}) - \tilde{M}^{\sigma(j)}|\Big)$$

$$= \frac{1}{14}\Big(\int_{0}^{3}|LUB(\tilde{M}^{\sigma(1)}, \tilde{N}^{\sigma(1)}) - \tilde{M}^{\sigma(1)}| + \int_{-2}^{5}|LUB(\tilde{M}^{\sigma(2)}, \tilde{N}^{\sigma(2)}) - \tilde{M}^{\sigma(2)}|\Big) = 0.1964$$

, and

$$d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{N}}) = \frac{1}{5-0}\Big(\frac{1}{2}\sum_{j=1}^{2}\int_{0}^{5}|LUB(\tilde{M}^{\sigma(j)}, \tilde{N}^{\sigma(j)}) - \tilde{N}^{\sigma(j)}|\Big)$$

$$= \frac{1}{10}\Big(\int_{0}^{3}|LUB(\tilde{M}^{\sigma(1)}, \tilde{N}^{\sigma(1)}) - \tilde{N}^{\sigma(1)}| + \int_{1}^{5}|LUB(\tilde{M}^{\sigma(2)}, \tilde{N}^{\sigma(2)}) - \tilde{N}^{\sigma(2)}|\Big) = 0.1889.$$

Due to $d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{M}}) \geq d(LUB(\tilde{\tilde{M}}, \tilde{\tilde{N}}), \tilde{\tilde{N}})$, we have $\tilde{\tilde{M}} \leqq \tilde{\tilde{N}}$.

5   Applications of new arithmetic operations on HFNs

Proper arithmetic operations and ordering method on HFNs can be important in a hesitant fuzzy environment, and by using these operations, we can do the calculations more easily. Also, those can be a very useful tool in several applications of hesitant fuzzy sets such as decision-making, optimization and so on. For example, consider a hesitant fuzzy linear programming (HFLP) problem as follows:

$$Max \ \ \tilde{\tilde{c}}^T x$$
$$s.t.$$
$$\tilde{\tilde{a}}_i x \lessapprox \tilde{\tilde{b}}_i, \quad i = 1, \ldots, m \tag{23}$$
$$x \geq 0,$$

where, $\tilde{\tilde{c}} = (\tilde{\tilde{c}}_1, \ldots, \tilde{\tilde{c}}_n)$, $\tilde{\tilde{a}}_i = (\tilde{\tilde{a}}_{i1}, \ldots, \tilde{\tilde{a}}_{in})$, and $x = (x_1, \ldots, x_n)^T$. Also, $\tilde{\tilde{c}}_j, \tilde{\tilde{b}}_i$, and $\tilde{\tilde{a}}_{ij}$ $(j = 1, 2, \ldots, n, \ i = 1, 2, \ldots, m)$ denote the hesitant fuzzy parameters, where the possibility distribution of hesitant fuzzy parameters is assumed to be characterized by HFNs. We needed a arithmetic operations to determine optimal value in such problems.

In [34], to solve this problem, a new approach with a binary operation on HFNs was given, but we show that this method has a high computational volume by increasing the number of elements of HFNs. In the following, an example to illustrate the superiority of the new approach in optimization problems is provided.

Example 5.1. (see [34]) *Consider the following HFLP problem with hesitant fuzzy parameters:*

$$\max \quad \tilde{\tilde{z}} = \tilde{\tilde{12}}x_1 + \tilde{\tilde{8}}x_2$$
$$s.t.$$
$$\tilde{\tilde{2}}x_1 + \tilde{\tilde{3}}x_2 \leq \tilde{\tilde{100}}$$
$$\tilde{\tilde{4}}x_1 + \tilde{\tilde{2}}x_2 \leq \tilde{\tilde{80}}$$
$$x_1, x_2 \geq 0,$$

where

$\tilde{\tilde{c}}_1 = \tilde{\tilde{12}} = \{(10, 12, 13), (11, 12, 15)\}, \tilde{\tilde{c}}_2 = \tilde{\tilde{8}} = \{(7, 8, 9), (5, 8, 12), (6, 8, 9, 11)\},$
$\tilde{\tilde{a}}_{11} = \tilde{\tilde{2}} = \{(1.5, 2, 2.5), (1, 2, 3)\}, \tilde{\tilde{a}}_{12} = \tilde{\tilde{3}} = \{(2, 3, 4), (0, 3, 5)(1, 3, 4, 4)\},$
$\tilde{\tilde{b}}_1 = \tilde{\tilde{100}} = \{(90, 100, 120), (80, 100, 115), (85, 95, 105, 115)\}, \tilde{\tilde{a}}_{21} = \tilde{\tilde{4}} = \{(3, 4, 5), (3, 3.5, 4.5, 5), (2, 4, 6)\},$
$\tilde{\tilde{a}}_{22} = \tilde{\tilde{2}} = \{(1, 2, 3.5), (1.5, 2, 2.5), (0, 2, 4)\}, \tilde{\tilde{b}}_2 = \tilde{\tilde{80}} = \{(70, 80, 100), (60, 80, 110), (50, 70, 90, 120)\}.$

After solving this problem using the proposed approach in [34], the acceptable hesitant fuzzy optimal solutions are, respectively, obtained as follows for $k = 2$ and $k = 3$:

$$x_1^*(0.6, 2) = 1.0665, \quad x_2^*(0.6, 2) = 31.4963 \tag{24}$$

$$x_1^*(0.9, 3) = 4.8617, \quad x_2^*(0.9, 3) = 25.1630. \tag{25}$$

Now, by substituting the optimum solutions (24) and (25) in the objective function and using the binary operation on HFNs in [34], we obtain an HFN with characteristic $p = 6$ for each optimum value as follows:

$\tilde{\tilde{z}} = \tilde{\tilde{12}}x_1 \tilde{\oplus} \tilde{\tilde{8}}x_2 = \{(10, 12, 13)x_1^* \tilde{\oplus} (7, 8, 9)x_2^*, (10, 12, 13)x_1^* \tilde{\oplus} (5, 8, 12)x_2^*, (10, 12, 13)x_1^* \tilde{\oplus} (6, 8, 9, 11)x_2^*, (11, 12, 15)x_1^*$
$\tilde{\oplus} (7, 8, 9)x_2^*, (11, 12, 15)x_1^* \tilde{\oplus} (5, 8, 12)x_2^*, (11, 12, 15)x_1^* \tilde{\oplus} (6, 8, 9, 11)x_2^*\}.$

Now, for $k = 2$ and $k = 3$ we have, respectively.

$\tilde{\tilde{z}}^*(0.6, 2) = \{(231.1390, 264.7683, 297.3311), (168.1464, 264.7683, 391.8199), (199.6427, 264.7683, 296.2646, 360.3237),$
$(232.2055, 264.7683, 299.4641), (169.2129, 264.7683, 393.9529), (200.7092, 264.7683, 296.2646, 362.4567)\}$

$\tilde{\tilde{z}}^*(0.9, 3) = \{(224.7579, 259.6443, 289.6690), (174.4318, 259.6443, 365.1581), (199.5949, 259.6443, 284.8037, 339.9951),$
$(229.6196, 259.6443, 299.3923), (179.2935, 259.6443, 374.8814), (204.4565, 259.6443, 284.8073, 349.7184)\}$

Figures 7 and 8 show the HFNs $\tilde{\tilde{z}}^*(0.6, 2)$ and $\tilde{\tilde{z}}^*(0.9, 3)$ using the binary operation on HFNs in [34], respectively.
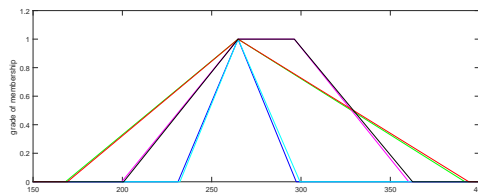

Figure 7: Optimum value for $k = 2$ as a HFN in Example 5.1 by the binary operation on HFNs in [34].
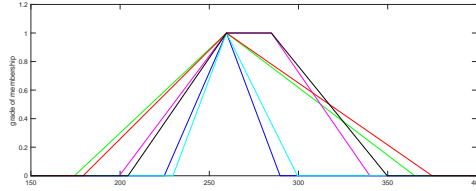
Figure 8: Optimum value for $k = 3$ as a HFN in Example 5.1 by the binary operation on HFNs in [34].

Now, use the new arithmetic operations defined in this paper, for this example. Since the cardinality of two HFNs $\tilde{\tilde{12}}$ and $\tilde{\tilde{8}}$ are not equal, at first, due to Remark 3.2, we should extend the shorter one until both have the same length. Therefore

$$\tilde{\tilde{c}}_1 = \tilde{\tilde{12}} = \{(10, 12, 13), (11, 12, 15), (11, 12, 15)\}, \quad \tilde{\tilde{c}}_2 = \tilde{\tilde{8}} = \{(7, 8, 9), (5, 8, 12), (6, 8, 9, 11)\}.$$

Now, we use the optimum solutions (24) and (25) and substitute them in the objective function. Using the new arithmetic operations, then we obtain an HFN with characteristic $p = 3$ for each optimum value as follows:

$$\tilde{\tilde{z}} = \tilde{\tilde{12}}x_1 \underset{\tilde{\tilde{z}}}{\oplus} \tilde{\tilde{8}}x_2 = \big\{(10, 12, 13)x_1^* \tilde{\oplus}(7, 8, 9)x_2^*, (11, 12, 15)x_1^* \tilde{\oplus}(5, 8, 12)x_2^*, (11, 12, 15)x_1^* \tilde{\oplus}(6, 8, 9, 11)x_2^*\big\}.$$

Now, for $k = 2$ and $k = 3$ we get, respectively.

$$\tilde{\tilde{z}}^*(0.6, 2) = \{(231.1390, 264.7683, 297.3311), (169.2129, 264.7683, 393.9529), (200.7092, 264.7683, 296.2646, 362.4567)\}$$

$$\tilde{\tilde{z}}^*(0.9, 3) = \{(224.7579, 259.6443, 289.6690), (179.2935, 259.6443, 374.8814), (204.4565, 259.6443, 284.8073, 349.7184)\}$$

Figures 9 and 10 show the HFNs $\tilde{\tilde{z}}^*(0.6, 2)$ and $\tilde{\tilde{z}}^*(0.9, 3)$ by using the new arithmetic operations, respectively. In
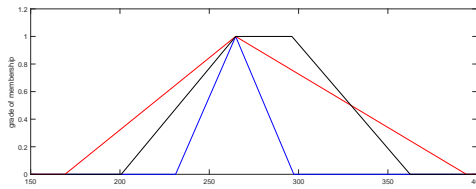

Figure 9: Optimum value for $k = 2$ as a HFN in Example 5.1 by using the new arithmetic operations.
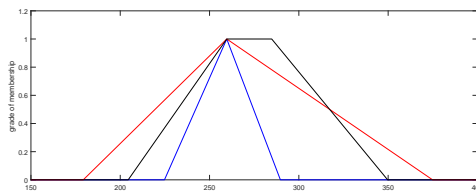

Figure 10: Optimum value for $k = 3$ as a HFN in Example 5.1 by using the new arithmetic operations.

this problem, the obtained optimal solutions by new arithmetic operations are a strong subset of the obtained optimal solutions in [34].

The following example is another application of the proposed arithmetic operations and ordering method on HFNs in a hesitant fuzzy decision-making (HFDM) problems.

Example 5.2. *Suppose that at a university, the post of a professor is vacant, and two candidates A_1 and A_2 remain. A committee has convened to decide which applicant is the best qualified for the job according to two attributes educational capabilities (x_1) and research capabilities (x_2). The committee has three members with different evaluations in a hesitant fuzzy environment and they have identified the following hesitant fuzzy decision matrix that the possibility distribution*

Table 1: Hesitant fuzzy decision matrix by three members in Example 5.2.

	x_1	x_2
A_1	$\tilde{\tilde{a_{11}}} = \{(0.6, 0.8, 1),$ $(0.7, 0.8, 0.9),$ $(0.5, 0.7, 0.9, 1)\}$	$\tilde{\tilde{a_{12}}} = \{(0.4, 0.5, 0.6, 0.7),$ $(0.4, 0.6, 0.8),$ $(0.3, 0.5, 0.6, 0.7)\}$
A_2	$\tilde{\tilde{a_{21}}} = \{(0.4, 0.5, 0.7),$ $(0.3, 0.4, 0.6, 0.8),$ $(0.4, 0.5, 0.8)\}$	$\tilde{\tilde{a_{22}}} = \{(0.6, 0.7, 0.9, 1),$ $(0.7, 0.8, 1),$ $(0.5, 0.8, 0.9)\}$

of its elements is assumed to be characterized by HFNs (see Table 1). The hesitant fuzzy weights of attributes x_1 and x_2 are, respectively, given as follows:

$$\tilde{\tilde{w_1}} = \big\{(0.3, 0.6, 0.7), (0.4, 0.6, 0.7), (0.3, 0.5, 0.7, .0.8)\big\}, \quad \tilde{\tilde{w_2}} = \big\{(0.3, 0.4, 0.5, 0.6), (0.3, 0.4, 0.7), (0.3, 0.35, 0.45, 0.6)\big\}.$$

Based on the extension of Bonissone's approach in the fuzzy decision-making problems [6] with the help of the approximated algebraic operations for HFNs in Section 3, we can quickly compute the performance of alternative A_i $(i = 1, \ldots, n)$ with respect to attributes, x_j $(j = 1, \ldots, m)$ using

$$\tilde{\tilde{U_i}} = \sum_{j=1}^{m} \tilde{\tilde{w_j}} \otimes_{\tilde{\approx}} \tilde{\tilde{a_{ij}}}. \tag{26}$$

The hesitant fuzzy utility of A_1 is computed using Eq.(26) as

$$\tilde{\tilde{U_1}} = \tilde{\tilde{w_1}} \otimes_{\tilde{\approx}} \tilde{\tilde{a_{11}}} \oplus_{\tilde{\approx}} \tilde{\tilde{w_2}} \otimes_{\tilde{\approx}} \tilde{\tilde{a_{12}}} = \{(0.24, 0.59, 0.81, 1.11), (0.4, 0.68, 0.78, 1.05), (0.3, 0.64, 0.8, 1.36)\}.$$

Similarly, we can obtain

$$\tilde{\tilde{U_2}} = \tilde{\tilde{w_1}} \otimes_{\tilde{\approx}} \tilde{\tilde{a_{21}}} \oplus_{\tilde{\approx}} \tilde{\tilde{w_2}} \otimes_{\tilde{\approx}} \tilde{\tilde{a_{22}}} = \{(0.24, 0.52, 0.72, 1.1), (0.34, 0.58, 0.75, 1.09), (0.33, 0.57, 0.67, 1.34)\}.$$

Due to Eq.(22), we can show $d(LUB(\tilde{\tilde{U_1}}, \tilde{\tilde{U_2}}), \tilde{\tilde{U_1}}) \leq d(LUB(\tilde{\tilde{U_1}}, \tilde{\tilde{U_2}}), \tilde{\tilde{U_2}})$, therefore $\tilde{\tilde{U_2}} \leqq \tilde{\tilde{U_1}}$.

To show the correctness of the obtained results, we can use the fuzzy group decision-making by taking the average value of evaluations three members of the committee (see Table 2). By Bonissone's approach in the fuzzy decision

Table 2: Fuzzy decision matrix by taking the average value of the three members in Example 5.2.

	x_1	x_2
A_1	$\tilde{0.8} = (0.6, 0.77, 0.83, 0.97)$	$\tilde{0.6} = (0.37, 0.53, 0.6, 0.73)$
A_2	$\tilde{0.5} = (0.37, 0.47, 0.53, 0.77)$	$\tilde{0.8} = (0.6, 0.77, 0.83, 97)$
w	$\tilde{w_1} = (0.33, 0.57, 0.363, 0.7)$	$\tilde{w_2} = (0.3, 0.38, 0.45, 0.63)$

making problems [6], the fuzzy utility of A_1 and A_2 is, respectively, computed as follows:

$$\tilde{U_1} = (0.31, 0.64, 0.79, 1.17), \quad \tilde{U_2} = (0.30, 0.56, 0.71, 1.17).$$

We find that the ordering for A_1 and A_2 remains the same, but the results of group decision-making problem in the hesitant fuzzy environment are more desirable than the results obtained in fuzzy environment, because we can see more information in the hesitant fuzzy approach in the results.

6 Discussion and comparative analyses

While modelling certain problems in the decision-making, optimization, physical sciences and engineering, it is often observed that the parameters of the problem are not known precisely but rather lie in an interval. In the past, such situations have been handled by the application of interval arithmetic which allows mathematical computations to be performed at intervals and obtains meaningful estimates of desired quantities also in terms of intervals. Arithmetic of

fuzzy numbers can be taken as a generalization of the interval arithmetic where rather than considering intervals at one level only, several levels in $[0, 1]$ are considered. HFNs are the major generalization of fuzzy numbers to describe situations in which they permit the membership degree of an element of a set to be represented in several possible values between 0 and 1.

In the different applications for HFNs, hesitant fuzzy arithmetic operations can be perform in order to solve mathematical equations that use HFNs. Thus, using a suitable arithmetic for these numbers is very important. According to Definition 2.10 an HFN is a special type of HFSs. In many studies, the members from elements of the HFSs got some values in the interval $[0,1]$ with a discrete and finite reference set of X, but in this paper for an HFN we use a and infinite reference set of X on the real numbers.

In this paper, we use the extension principle in HFSs for the new arithmetic operation, which has the following advantages:

(1) Reducing the volume of calculations.

(2) Using the knowledge of experts in all problem-solving processes.

(3) More effective than the fuzzy approach in group decision-making problems.

For example, for (1), consider an HFLP problem similar to Example 5.1 with n variables and HFNs with characteristic p_i for $i = 1, 2, \ldots, n$, based on the binary operation $(\tilde{\tilde{\oplus}})$ in [34]. We have an HFN as an optimal solution with $p_1 \times p_2 \times \cdots \times p_n$ elements, but by applying the new arithmetic operations $(\underset{\approx}{\oplus})$ in this paper, we have an HFN as an optimal solution with $p = \max\{p_1, p_2, \ldots, p_n\}$ elements. Therefore, the new approach significantly reduces the computational volume to determine the optimal solution (see Table 3).

Table 3: Complexity comparison of the new arithmetic operation with binary operation in [34] in an HFLP problem.

Methods	Binary operation in [34]	New arithmetic operation
Characteristic of $\tilde{\tilde{z}}$	$p_1 \times p_2 \times \ldots \times p_n$	p
Computational volume	$p_1 \times p_2 \times \ldots \times p_n$	$(n-1)p$

Table 4 shows a complexity comparison for the binary operation in [34] with the new arithmetic operations on the HFDM problem similar to Example 5.2, based on the extension of Bonissone's approach in the fuzzy decision-making problems [6], with m attributes, n alternatives and HFNs with characteristic p. In fact, the information obtained by the new arithmetic operation is the strong hesitant fuzzy subset of the information obtained by the binary arithmetic operation.

Table 4: Complexity comparison of the new arithmetic operation with binary operation in [34] in an HFDM problem.

Methods	Binary operation in [34]	New arithmetic operation
Characteristic of $\tilde{\tilde{U}}_i$	p^{2m}	p
Computational volume	np^{2m}	$n(2m-1)p$

Tables 3 and 4 show that the new arithmetic operation for HFNs can reduce the computational volume in large-scale problems. This means that if the characteristic of the input HFNs is large or the numbers of operators are large, the use of new arithmetic operation is recommended for hesitant fuzzy arithmetic. In addition, to the calculations with HFNs allow the incorporation of uncertainty on parameters and use the evaluations of various experts.

Also for (2) and (3) in Example 5.2, by these arithmetic operations, we use the knowledge of experts in all problem-solving processes, and it is more effective than the fuzzy approach by aggregating the opinions of experts. Generally, the arithmetic calculations on HFNs can be done by two different approaches: (a) The extension principle approach (b) The interval arithmetic approach. In this paper, we use the extension principle approach for this purpose. In later

studies, the interval arithmetic approach on the (α, k)-cuts of given HFNs can be investigated. Accordingly, developing computational methods for the implementation of hesitant fuzzy arithmetic operations contributes to the more effective use of HFNs in different applications.

There are many different methods for ordering or comparing the fuzzy numbers. In the same way, given the importance of this issue for HFNs, in this paper, we introduce a meaningful ordering of HFNs by the extension principle which may be required in the final selection of decision options after performing arithmetic operations on HFNs. Due to the existence of various methods in the order of fuzzy numbers, the study and research on the order for HFNs can be an interesting topic in the future.

7   Conclusions

In this paper, a new arithmetic operations and ordering method on HFNs based on the extension principle were introduced. The selection of proper arithmetic operations and ordering method for HFNs have many applications in different fields such as optimization problems, decision-making problems, expert systems, and so on. In an application, we showed that the new approach reduces the computational volume to obtain optimal solutions in the special case of the HFLP problems. However, this issue can be important in hesitant fuzzy set environments while we use HFNs.

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Arithmetic operations and ranking of hesitant fuzzy numbers by extension principle

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عملیات حسابی و رتبه‌بندی اعداد فازی مردد بر اساس اصل توسیع

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