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Original paper

Stability problem for Pexiderized Cauchy-Jensen type functional equations of fuzzy number-valued mappings

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Abstract

We investigate the stability problems of the n-dimensional Cauchy-Jensen type and the n-dimensional Pexiderized Cauchy-Jensen type fuzzy number-valued functional equations in Banach spaces by using the metric defined on a fuzzy number space. Under some suitable conditions, some properties of the solutions for these equations such as existence and uniqueness are discussed. Our results can be regarded as important extensions of stability results corresponding to single-valued functional equations and set-valued functional equations, respectively.

Keywords: Fuzzy number-valued mapping, stability, Pexiderized Cauchy-Jensen equation, fuzzy analysis.

1 Introduction

In set-valued analysis, a functional inclusion is called stable if any function which satisfies this inclusion approximately is near to a true solution of the functional inclusion (see [7, 8, 15, 21, 37]). Nikodem and Popa [28] considered the general solution of set-valued functions satisfying linear inclusion relation, which can be regarded as a generalization of the additive single-valued functional equation. By means of the inclusion relation, Park et al. [23, 30] investigated the stability of some set-valued functional equations. Jang et al. [19] and Chu et al. [13] studied the stability of an n-dimensional additive set-valued functional equation and an n-dimensional cubic set-valued functional equation, respectively.

Some interesting results concerning the Cauchy-Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x),$$

and the general linear functional equation with real constants a, b, A, B,

$$f(ax+by)=Af(x)+Bf(y), \tag{1}$$

have been obtained in [6, 17, 20, 23, 24, 29, 32, 39]. In particular, the equation (1) includes the linear equation for $A=a$ and $B=b$. If $t\in(0,1)$, $a=A=t$ and $b=B=1-t$, then (1) has the form

$$f(tx+(1-t)y)=tf(x)+(1-t)f(y),$$

and its solution is called an affine function (see, among others, [29, 33]). Clearly, the equation (1) also includes the Cauchy equation for $a=b=A=B=1$ and the Jensen equation for $a=b=A=B=\frac{1}{2}$.

The functional equation

$$f(x+y)=g(x)+h(y),$$

with three unknown functions f, g, h treated by Pexider [31], which is known as the Pexiderized Cauchy equation. In the case $f=g=h$, this equation reduces to Cauchy equation. The Pexiderized Cauchy equation has a simple

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interpretation: The value of f at $x+y$ is the sum of two values, of which one depends only on x, and the other only on y (separation of variables).

In 1991, Nikodem [27] investigated the stability of the Pexider Cauchy equation. This paper seems to be the first one concerning the stability problem of the Pexider Cauchy equation. Solutions and the stability of various Pexiderized Cauchy and Pexiderized Cauchy-Jensen functional equations in several variables have been investigated by several authors (see, among others, [12, 14, 16, 22, 25, 26]).

The stability of the Cauchy-Jensen and the general linear type fuzzy number-valued functional equations in Banach spaces by using the metric defined on a fuzzy number space investigated by Wu and Jin [39], which are generalizations of the main results obtained in [17, 23]. We refer the reader to [1, 5, 9, 10, 11, 35, 36, 38] for some recent works and discussions on this topic.

The purpose of this paper is to extend the single-valued and set-valued functional equations to fuzzy number-valued functional equations and investigate the stability problems of the n-dimensional Cauchy-Jensen type fuzzy number-valued functional equation

$$f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \sum_{j=2}^{n} f\left(\frac{\sum_{i=1,i\neq j}^{n} x_i - (n-1)x_j}{n}\right) = f(x_1). \tag{2}$$

and the n-dimensional Pexiderized Cauchy-Jensen type fuzzy number-valued functional equation

$$f_1\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) + \sum_{j=2}^{n} f_j\left(\frac{\sum_{i=1,i\neq j}^{n} x_i - (n-1)x_j}{n}\right) = f_{n+1}(x_1), \tag{3}$$

in Banach spaces via the metric defined on a fuzzy number space. Under some suitable conditions, some properties of the solutions for these equations such as existence and uniqueness are discussed. Notice that the supremum metric, as a generalization of the Hausdorff metric, is applied to characterize the fuzzy number-valued functional inequality. It has been shown that the stability of Cauchy, Jensen, Cauchy-Jensen, Pexiderized Cauchy, Pexiderized Jensen, Pexiderized affine, Pexiderized linear and Pexiderized general linear type fuzzy number-valued functional equations can be obtained as special cases from our results. Therefore, the results obtained in the present paper improve and extend the corresponding results in previous works [12, 17, 23, 27, 30, 39].

2 Preliminaries

In this section, we consider some basic concepts and results (see [3, 39]) which will be used in the sequel. From now on, assume that X, Y are Banach spaces and B is a subspace of Y.

Suppose a function $u : X \to [0,1]$ satisfies the following conditions:

1. $[u]^\alpha = \{x \in X : u(x) \geq \alpha\}$ is a non-empty compact convex subset of X, for all $\alpha \in (0,1]$,
2. $[u]^0 = \overline{\{x : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

Then u is called a fuzzy number on X. The set of all fuzzy numbers on X is denoted by X_F.

For $u, v \in X_F$ and $\lambda \in \mathbb{R}$, by Zadeh extension principle, we obtain the following properties about addition $u+v$ and scalar multiplication $\lambda \cdot u$ (see [3, 39]):

(i) $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$,

(ii) $[\lambda \cdot u]^\alpha = \lambda [u]^\alpha$.

Note that $\tilde{0}$ is the zero element in X_F; i.e., $\tilde{0} + u = u + \tilde{0} = u$, for all $u \in X_F$.

Defining the mapping $D : X_F \times X_F \to [0,\infty)$ by

$$D(u,v) = \sup_{\alpha \in [0,1]} d_H([u]^\alpha, [v^\alpha]),$$

where d_H is Hausdorff metric. Then D satisfies the following properties:

(P1) $D(\lambda \cdot u, \lambda \cdot v) = |\lambda| D(u,v)$,

(P2) $D(u+w, v+w) = D(u,v)$,

(P3) $D(u+v, w+e) \leq D(u,w) + D(v,e)$

for all $\lambda \in \mathbb{R}$ and $u, v, w, e \in X_F$, and (X_F, D) is a complete metric space.

Let $u, v \in X_F$. If there exists $w \in X_F$ such that $u = v + w$, then w is called the Hukuhara difference (H-difference) of u and v, and it is denoted by $u - v$ (see [18, 34]).

Notation 2.1. *Let $\lambda \in \mathbb{R}$ and $u, v \in \mathbb{R}_F$. If we denote $\|u\|_F = D(u, \tilde{0})$, then $\|\cdot\|_F$ has the properties of a usual norm on $\mathbb{R}_F$; i.e., $\|u\|_F = 0$ iff $u = \tilde{0}$, $\|\lambda \cdot u\|_F = |\lambda|\|u\|_F$ and $\|u+v\|_F \leq \|u\|_F + \|v\|_F$. Moreover, note that $(\mathbb{R}_F, +, \cdot)$ is not a linear space over $\mathbb{R}$ and consequently $(\mathbb{R}_F, \|\cdot\|_F)$ cannot be a normed space* (see [2] and [4, Theorem 1 and Remark 2]).

3 Stability of the Cauchy-Jensen type fuzzy number-valued functional equation in n variables

In this section, the stability of the equation (2) is established, in which f indicates a fuzzy number-valued mapping. Several special cases are also considered.

Theorem 3.1. *Let $f : B \to X_F$ be a fuzzy number-valued mapping such that*

$$D\left(f\left(\frac{\sum_{i=1}^n x_i}{n}\right) + \sum_{j=2}^n f\left(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}\right), f(x_1)\right) < \varepsilon, \tag{4}$$

for some $\varepsilon > 0$ and for all $x_1, \ldots, x_n \in B$. Then there exists a unique additive mapping $\mathcal{G} : B \to X_F$ such that

$$D\left(f(x), \mathcal{G}(x)\right) \leq \frac{\varepsilon}{n-1},$$

for all $x \in B$.

Proof. Setting $x_1 = nx$ and $x_i = 0$ $(2 \leq i \leq n)$ in (4), we obtain

$$D\left(f(x), \frac{1}{n}f(nx)\right) < \frac{\varepsilon}{n},$$

for all $x \in B$. Then, by induction on m, we have

$$D\left(f(x), \frac{1}{n^m}f(n^m x)\right) < \sum_{j=1}^m (\frac{1}{n})^j \varepsilon. \tag{5}$$

Let $f_0(x) = f(x)$ and $f_m(x) = \frac{1}{n^m}f(n^m x)$, where $m \in \mathbb{N}$ and $x \in B$. Then, by (5) and the properties (P2) and (P3) of D, we obtain

$$\begin{aligned} D\left(f_m(x), f_{m-1}(x)\right) &= D\left(\frac{1}{n^m}f(n^m x) + f(x), f(x) + \frac{1}{n^{m-1}}f(n^{m-1}x)\right) \\ &\leq D\left(\frac{1}{n^m}f(n^m x), f(x)\right) + D\left(f(x), \frac{1}{n^{m-1}}f(n^{m-1}x)\right) \\ &< \sum_{j=1}^m (\frac{1}{n})^j \varepsilon + \sum_{j=1}^{m-1} (\frac{1}{n})^j \varepsilon, \end{aligned}$$

for all $x \in B$. Thus the sequence $\{f_m(x)\}$ is Cauchy in X_F. Since (X_F, D) is complete, we can define $\mathcal{G}(x) = \lim_{m\to\infty} f_m(x)$ for any $x \in B$.

The next step is to show that $\mathcal{G}$ is additive. By (4), we can conclude that

$$\begin{aligned} &D\left(f_m\left(\frac{\sum_{i=1}^n x_i}{n}\right) + \sum_{j=2}^n f_m\left(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}\right), f_m(x_1)\right) \\ &= D\left(\frac{1}{n^m}f\left(\frac{\sum_{i=1}^n n^m x_i}{n}\right) + \frac{1}{n^m}\sum_{j=2}^n f\left(\frac{\sum_{i=1, i\neq j}^n n^m x_i - n^m(n-1)x_j}{n}\right), \frac{1}{n^m}f(n^m x_1)\right) < \frac{\varepsilon}{n^m}, \end{aligned}$$

for all $x_1, \ldots, x_n \in B$. Hence,

$$\lim_{m\to\infty} D\left(f_m\left(\frac{\sum_{i=1}^n x_i}{n}\right) + \sum_{j=2}^n f_m\left(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}\right), f_m(x_1)\right) = 0.$$

By the continuity of the metric D, we obtain

$$D\left(\mathcal{G}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)+\sum_{j=2}^{n}\mathcal{G}\left(\frac{\sum_{i=1,i\neq j}^{n} x_i-(n-1)x_j}{n}\right),\mathcal{G}(x_1)\right)=0,$$

meaning

$$\mathcal{G}\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)+\sum_{j=2}^{n}\mathcal{G}\left(\frac{\sum_{i=1,i\neq j}^{n} x_i-(n-1)x_j}{n}\right)=\mathcal{G}(x_1), \tag{6}$$

for all $x_1,\ldots,x_n \in B$. Letting $x_i=0\ \ (1\le i\le n)$ in (6), we obtain $\mathcal{G}(0)=\tilde{0}$. Setting $x_1=x$ and $x_i=0\ \ (2\le i\le n)$ in (6), we get

$$n\mathcal{G}\left(\frac{x}{n}\right)=\mathcal{G}(x), \tag{7}$$

for all $x\in B$. Setting $x_i=0\ \ (3\le i\le n)$ in (6) and using (7), we have

$$\frac{n-1}{n}\mathcal{G}\left(x_1+x_2\right)+\frac{1}{n}\mathcal{G}\left(x_1-(n-1)x_2\right)=\mathcal{G}(x_1), \tag{8}$$

for all $x_1,x_2\in B$. Putting $x_1=x_1+(n-1)x_2$ in (8), one finds

$$\frac{n-1}{n}\mathcal{G}\left(x_1+nx_2\right)+\frac{1}{n}\mathcal{G}\left(x_1\right)=\mathcal{G}\left(x_1+(n-1)x_2\right), \tag{9}$$

for all $x_1,x_2\in B$. Replacing x_1 by 0 and x_2 by x into (9) and using (7), one gets

$$\mathcal{G}\left((n-1)x\right)=(n-1)\mathcal{G}(x), \tag{10}$$

for all $x\in B$. Replacing x_1 by 0 and x_2 by x into (8) and using (10), we obtain $\mathcal{G}(-x)=-\mathcal{G}(x)$ for all $x\in B$; that is, $\mathcal{G}$ is odd. Setting $x_2=x_2-x_1$ in (8), we get

$$\frac{n-1}{n}\mathcal{G}\left(x_2\right)+\frac{1}{n}\mathcal{G}\left(nx_1-(n-1)x_2\right)=\mathcal{G}\left(x_1\right), \tag{11}$$

for all $x_1,x_2\in B$. Replacing x_1 by $\frac{x_1}{n}$ and x_2 by $-\frac{x_2}{n-1}$ into (11) and using (7), (10) and the oddness of $\mathcal{G}$, we have

$$\mathcal{G}(x_1+x_2)=\mathcal{G}(x_1)+\mathcal{G}(x_2),$$

for all $x_1,x_2\in B$; that is, $\mathcal{G}$ is additive.

By letting $m\to\infty$ in (5), we immediately achieve $D\left(f(x),\mathcal{G}(x)\right)\le\dfrac{\varepsilon}{n-1}$ for all $x\in B$.

Finally, we prove the uniqueness of $\mathcal{G}$. Assume that there exist additive mappings $\mathcal{G}_1,\mathcal{G}_2:B\to X_F$ satisfying the last inequality. Then,

$$\begin{aligned} D\left(\mathcal{G}_1(x),\mathcal{G}_2(x)\right) &= \frac{1}{m}D\left(m\mathcal{G}_1(x),m\mathcal{G}_2(x)\right)\\ &\le \frac{1}{m}\left(D\left(\mathcal{G}_1(mx),f(mx)\right)+D\left(f(mx),n\mathcal{G}_2(mx)\right)\right)\\ &< \frac{2\varepsilon}{m(n-1)}, \end{aligned}$$

for all $x\in B$ and for any $m\in\mathbb{N}$. Since the right-hand side of the last inequality tends to zero as $m\to\infty$, we conclude that $\mathcal{G}_1(x)=\mathcal{G}_2(x)$ for all $x\in B$, as desired. □

From Theorem 3.1, we easily obtain the following.

Corollary 3.2. ([39, Theorem 1]) *If a fuzzy number-valued mapping $f:B\to X_F$ satisfes the inequality (4) with $n=2$, then there exists a unique additive mapping $\mathcal{G}:B\to X_F$ such that*

$$D\left(f(x),\mathcal{G}(x)\right)\le\varepsilon, \tag{12}$$

for all $x\in B$.

In the next result, we establish the stability of the Cauchy type fuzzy number-valued functional equation.

Corollary 3.3. *Let $f : B \to X_F$ be a fuzzy number-valued mapping such that*

$$D\left(f(x)+f(y), f(x+y)\right) < \varepsilon, \tag{13}$$

for some $\varepsilon > 0$ and for all $x, y \in B$. Then there exists a unique additive mapping $\mathcal{G} : B \to X_F$ such that (12) *holds for all $x \in B$.*

Proof. Substituting $x = \frac{x+y}{2}$ and $y = \frac{x-y}{2}$ in (13) gives

$$D\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right), f(x)\right) < \varepsilon,$$

for all $x, y \in B$. Then f satisfies the inequality (4) with $n = 2$. Then, by Theorem 3.1, we find a unique additive mapping $\mathcal{G} : B \to X_F$ such that (12) holds for all $x \in B$. □

Now, we obtain the stability of the Jensen type fuzzy number-valued functional equation.

Corollary 3.4. *If a fuzzy number-valued mapping $f : B \to X_F$ satisfes the inequality*

$$D\left(2f(\frac{x+y}{2}), f(x)+f(y)\right) < \varepsilon, \tag{14}$$

for some $\varepsilon > 0$ and for all $x, y \in B$, then there exists a unique additive mapping $\mathcal{G} : B \to X_F$ such that

$$D\left(f(x)-f(0), \mathcal{G}(x)\right) \leq 2\varepsilon,$$

for all $x \in B$.

Proof. Let $g(x) := f(x) - f(0)$ for all $x \in B$. Then $g(0) = \tilde{0}$ and (14) yields that

$$D\left(2g(\frac{x+y}{2}), g(x)+g(y)\right) < \varepsilon, \tag{15}$$

for all $x, y \in B$. Replacing x by $x + y$ and y by 0 in (15), we get

$$D\left(2g(\frac{x+y}{2}), g(x+y)\right) < \varepsilon, \tag{16}$$

for all $x, y \in B$. It follows from (15) and (16) that

$$\begin{aligned} D\left(g(x+y), g(x)+g(y)\right) &= D\left(g(x+y)+2g(\frac{x+y}{2}), 2g(\frac{x+y}{2})+g(x)+g(y)\right) \\ &\leq D\left(g(x+y), 2g(\frac{x+y}{2})\right) + D\left(2g(\frac{x+y}{2}), g(x)+g(y)\right) \\ &< 2\varepsilon, \end{aligned}$$

for all $x, y \in B$. Using the same method as in the proof of Corollary 3.3, we find a unique additive mapping $\mathcal{G} : X \to Y$ such that

$$D\left(g(x), \mathcal{G}(x)\right) \leq 2\varepsilon,$$

holds for all $x \in B$. This completes the proof of this result. □

4 Stability of the Pexiderized Cauchy-Jensen type fuzzy number-valued functional equation in n variables

In this section, the stability of the equation (3) is established, in which f indicates a fuzzy number-valued mapping. Some special cases involving the stability of Pexiderized Cauchy, Pexiderized Jensen, Pexiderized affine, Pexiderized linear and Pexiderized general linear type fuzzy number-valued functional equations are also presented.

Theorem 4.1. *Let $f_1, \dots, f_{n+1} : B \to X_F$ be fuzzy number-valued mappings with $f_1(0) = \cdots = f_n(0) = \tilde{0}$ such that*

$$D\left(f_1\left(\frac{\sum_{i=1}^n x_i}{n}\right) + \sum_{j=2}^{n} f_j\left(\frac{\sum_{i=1, i\neq j}^n x_i - (n-1)x_j}{n}\right), f_{n+1}(x_1)\right) < \varepsilon, \tag{17}$$

for some $\varepsilon > 0$ and for all $x_1, \dots, x_n \in B$. Then there exists a unique additive mapping $\mathcal{G} : B \to X_F$ such that

$$D\left(f_k(x), \mathcal{G}(x)\right) \leq \frac{2n}{n-1}\varepsilon, \quad (k = 1, 2, ..., n), \tag{18}$$

$$D\left(f_{n+1}(x), \mathcal{G}(x)\right) \leq \frac{n+1}{n-1}\varepsilon, \tag{19}$$

for all $x \in B$.

Proof. Letting $x_1 = x$ and $x_i = \frac{(n-2)x+ny}{2n-2}$ $(2 \leq i \leq n)$ in (17), we get

$$D\left(f_1\left(\frac{x+y}{2}\right) + \sum_{j=2}^{n} f_j\left(\frac{x-y}{2n-2}\right), f_{n+1}(x)\right) < \varepsilon, \tag{20}$$

for all $x, y \in B$. Setting $x = \frac{x+y}{2}$ and $y = \frac{x-y}{2}$ in (20), we have

$$D\left(f_1\left(\frac{x}{2}\right) + \sum_{j=2}^{n} f_j\left(\frac{y}{2n-2}\right), f_{n+1}(\frac{x+y}{2})\right) < \varepsilon, \tag{21}$$

for all $x, y \in B$. Letting $x_j = -x$ $(j = 2, ..., n)$ separately, $x_1 = x$ and $x_i = 0$ $(i = 2, ..., n$ and $i \neq j)$ in (17), we get

$$D\left(f_j\left(x\right), f_{n+1}\left(x\right)\right) < \varepsilon, \tag{22}$$

for all $x \in B$, where $j \in \{2, ..., n\}$. Setting $x = 2x$ and $y = 0$ in (21), we have

$$D\left(f_1\left(x\right), f_{n+1}(x)\right) < \varepsilon, \tag{23}$$

for all $x \in B$. It follows from (22), (23) and the property (P3) of D that

$$D\left(\sum_{j=1, j\neq k}^{n} f_j\left(x\right), (n-1)f_{n+1}(x)\right) < (n-1)\varepsilon, \tag{24}$$

for all $x \in B$, where $k \in \{1, ..., n\}$. Putting $x = 2x$ and $y = (2n-2)x$ in (21), we obtain

$$D\left(\sum_{j=1}^{n} f_j\left(x\right), f_{n+1}(nx)\right) < \varepsilon, \tag{25}$$

for all $x \in B$. Using the inequalities (22) to (25) and the properties (P1) to (P3) of D, we conclude that

$$\begin{aligned}
&D\left(f_k(nx), nf_k(x)\right) \\
&\quad = D\left(f_k\left(nx\right) + f_{n+1}(nx), f_{n+1}(nx) + nf_k(x)\right) \\
&\quad \leq D\left(f_k\left(nx\right), f_{n+1}(nx)\right) + D\left(f_{n+1}(nx), nf_k(x)\right) \\
&\quad < \varepsilon + D\left(f_{n+1}(nx) + \sum_{j=1, j\neq k}^{n} f_j\left(x\right), \sum_{j=1, j\neq k}^{n} f_j\left(x\right) + f_k(x) + (n-1)f_k(x)\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon + D\left(f_{n+1}(nx), \sum_{j=1}^{n} f_j\left(x\right)\right) + D\left(\sum_{j=1,j\neq k}^{n} f_j\left(x\right), (n-1)f_k(x)\right) \\
&< \varepsilon + \varepsilon + D\left(\sum_{j=1,j\neq k}^{n} f_j\left(x\right), (n-1)f_k(x)\right) \\
&= 2\varepsilon + D\left(\sum_{j=1,j\neq k}^{n} f_j\left(x\right) + (n-1)f_{n+1}(x), (n-1)f_{n+1}(x) + (n-1)f_k(x)\right) \\
&\le 2\varepsilon + D\left(\sum_{j=1,j\neq k}^{n} f_j\left(x\right), (n-1)f_{n+1}(x)\right) + D\left((n-1)f_{n+1}(x), (n-1)f_k(x)\right) \\
&< 2\varepsilon + (n-1)\varepsilon + D\left(\sum_{j=1,j\neq k}^{n} f_j\left(x\right), (n-1)f_{n+1}(x)\right) \\
&< 2\varepsilon + (n-1)\varepsilon + (n-1)\varepsilon = 2n\varepsilon,
\end{aligned}
$$

for all $x \in B$, where $k \in \{1, ..., n\}$. Rewriting the above, we obtain

$$D\left(f_k(x), \frac{1}{n} f_k(nx)\right) \le 2\varepsilon,$$

for all $x \in B$, where $k \in \{1, ..., n\}$. Then, by induction on m, it follows that

$$D\left(f_k(x), \frac{1}{n^m} f_k(n^m x)\right) \le \sum_{j=0}^{m-1} \frac{2}{n^j}\varepsilon,$$

for all $x \in B$, where $k \in \{1, ..., n\}$. Using the same method as in the proof of Theorem 3.1, we find a unique additive mapping $\mathcal{G} : X \to Y$ such that (18) holds for $k \in \{1, ..., n\}$.

It follows from (23), (24), (25) and the properties (P2) and (P3) of D that

$$
\begin{aligned}
&D\left(f_{n+1}(nx), nf_{n+1}(x)\right) \\
&\quad = D\left(f_{n+1}(nx) + \sum_{j=1}^{n} f_j\left(x\right), \sum_{j=1}^{n} f_j\left(x\right) + nf_{n+1}(x)\right) \\
&\quad \le D\left(f_{n+1}(nx), f_1\left(x\right) + \sum_{j=2}^{n} f_j\left(x\right)\right) + D\left(f_1(x) + \sum_{j=2}^{n} f_j\left(x\right), nf_{n+1}(x)\right) \\
&\quad \le D\left(\sum_{j=1}^{n} f_j\left(x\right), f_{n+1}(nx)\right) + D\left(f_1(x), f_{n+1}(x)\right) + D\left(\sum_{j=2}^{n} f_j\left(x\right), (n-1)f_{n+1}(x)\right) \\
&\quad < \varepsilon + \varepsilon + (n-1)\varepsilon = (n+1)\varepsilon,
\end{aligned}
$$

for all $x \in B$. We deduce from the last inequality that

$$D\left(f_{n+1}(x), \frac{1}{n} f_{n+1}(nx)\right) \le \frac{n+1}{n}\varepsilon,$$

for all $x \in B$. Then, by induction on m, we get

$$D\left(f_{n+1}(x), \frac{1}{n^m} f_{n+1}(n^m x)\right) \le \sum_{j=1}^{m} \frac{n+1}{n^j}\varepsilon,$$

for all $x \in B$. Using the same method as in the proof of Theorem 3.1, we find a unique additive mapping $\mathcal{G} : X \to Y$ such that (19) holds. □

Now, we investigate the stability problem for the Pexiderized general linear type fuzzy number-valued functional equation in n variables.

Corollary 4.2. *Let $f_1, \dots, f_{n+1} : B \to X_F$ be fuzzy number-valued mappings with $f_1(0) = \dots = f_n(0) = \tilde{0}$ such that*

$$D\left(\sum_{i=1}^{n} A_i f_i(x_i), f_{n+1}(\sum_{i=1}^{n} a_i x_i)\right) < \varepsilon, \tag{26}$$

for all $x_1, ..., x_n \in B$, where $\varepsilon, a_i, A_i > 0$. Then there exists a unique additive mapping $\mathcal{G} : B \to X_F$ such that

$$D(f_k(x), \mathcal{G}(x)) \leq \frac{2n}{A_k(n-1)}\varepsilon, \quad (k = 1, 2, ..., n), \tag{27}$$

$$D(f_{n+1}(x), \mathcal{G}(x)) \leq \frac{n+1}{n-1}\varepsilon, \tag{28}$$

for all $x \in B$.

Proof. Setting $x_i = \frac{x_i}{a_i}$ for $i = 1, ..., n$ in (26), we get

$$D\left(\sum_{i=1}^{n} A_i f_i(\frac{x_i}{a_i}), f_{n+1}(\sum_{i=1}^{n} x_i)\right) < \varepsilon, \tag{29}$$

for all $x_1, ..., x_n \in B$. For $i = 1, ..., n$, define

$$G_i(x_i) := A_i f_i\left(\frac{x_i}{a_i}\right), \tag{30}$$

for all $x_i \in B$. Then, by (29),

$$D\left(\sum_{i=1}^{n} G_i(x_i), f_{n+1}(\sum_{i=1}^{n} x_i)\right) < \varepsilon, \tag{31}$$

for all $x_1, ..., x_n \in B$. Letting $x_1 = \frac{\sum_{i=1}^{n} x_i}{n}$ and $x_i = \frac{\sum_{j=1, j\neq i}^{n} x_j - (n-1)x_i}{n}$ for $i = 2, ..., n$ in (31), we get

$$D\left(G_1(\frac{\sum_{i=1}^{n} x_i}{n}) + \sum_{i=2}^{n} G_i(\frac{\sum_{j=1, j\neq i}^{n} x_j - (n-1)x_i}{n}), f_{n+1}(x_1)\right) < \varepsilon,$$

for all $x_1, ..., x_n \in B$. Using the same method as in the proofs of Theorems 3.1 and 4.1, the limits

$$\lim_{m\to\infty} \frac{1}{n^m} G_1(n^m x) = \dots = \lim_{m\to\infty} \frac{1}{n^m} G_n(n^m x) = \lim_{m\to\infty} \frac{1}{n^m} f_{n+1}(n^m x),$$

exist for all $x \in B$ and are unique additive mappings satisfying

$$D\left(G_k(x), \lim_{m\to\infty} \frac{1}{n^m} G_k(n^m x)\right) \leq \frac{2n}{n-1}\varepsilon, \tag{32}$$

$$D\left(f_{n+1}(x), \lim_{m\to\infty} \frac{1}{n^m} f_{n+1}(n^m x)\right) \leq \frac{n+1}{n-1}\varepsilon, \tag{33}$$

for all $x \in B$, where $k \in \{1, ..., n\}$. Now, (27) and (28) follow from (30), (32), (33) and the property (P1) of D. □

Corollary 4.2 with $n = 2$ implies the following.

Corollary 4.3. (Compare with [39, Theorem 2]). *If fuzzy number-valued mappings $f, g, h : B \to X_F$ with $f(0) = g(0) = \tilde{0}$ satisfy the inequality*

$$D\left(Af(x) + Bg(y), h(ax + by)\right) < \varepsilon, \tag{34}$$

for all $x, y \in B$, where $\varepsilon, a, b, A, B > 0$, then there exists a unique additive mapping $\mathcal{G} : B \to X_F$ such that

$$D\left(f(x), \mathcal{G}(x)\right) \leq \frac{4}{A}\varepsilon, \tag{35}$$

$$D\left(g(x),\mathcal{G}(x)\right)\leq \frac{4}{B}\varepsilon, \tag{36}$$

$$D\left(h(x),\mathcal{G}(x)\right)\leq 3\varepsilon, \tag{37}$$

for all $x\in B$.

Notation 4.4. *Let $f,g,h:B\to X_F$ be fuzzy number-valued mappings with $f(0)=g(0)=\tilde{0}$. Then*

(i) If f,g,h satisfy the inequality (34) with $A=B=a=b=1$, then there exists a unique additive mapping $\mathcal{G}:B\to X_F$ such that (35) to (37) hold for $A=B=1$.

(ii) If f,g,h satisfy the inequality (34) with $A=B=a=b=\frac{1}{2}$, then there exists a unique additive mapping $\mathcal{G}:B\to X_F$ such that (35) to (37) hold for $A=B=\frac{1}{2}$.

(iii) If f,g,h satisfy the inequality (34) with $A=a=t\in(0,1)$ and $B=b=1-t$, then there exists a unique additive mapping $\mathcal{G}:B\to X_F$ such that (35) to (37) hold for $A=t$ and $B=1-t$.

(iv) If f,g,h satisfy the inequality (34) with $A=a$ and $B=b$, then there exists a unique additive mapping $\mathcal{G}:B\to X_F$ such that (35) to (37) hold for $A=a$ and $B=b$.

5 Conclusions

In this paper, we proved the stability of several types of additive fuzzy number-valued functional equations, including Cauchy, Jensen, Cauchy-Jensen, Pexiderized Cauchy, Pexiderized Jensen, Pexiderized affine, Pexiderized linear and Pexiderized general linear type fuzzy number-valued functional equations. Our results generalized certain important results obtained by other authors for these equations when they are a single-valued or a set-valued one. Obviously, this paper provided us a novel idea to discuss the stability of functional equations from a more unified perspective. Certainly, further work will focus on the stability of other types of (functional, difference, differential, integral) equations by using this idea.

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Stability problem for Pexiderized Cauchy-Jensen type functional equations of fuzzy number-valued mappings

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مسأله پایداری برای معادلات تابعی کوشی-ینسن پکسیدر شده نگاشت‌های عددی-مقدار فازی

چکیده. ما مسائل پایداری معادلات تابعی عددی-مقدار فازی کوشی-ینسن n-بعدی و کوشی-ینسن پکسیدر شده n-بعدی در فضاهای باناخ با استفاده متریک تعریف شده روی یک فضای اعدد فازی را بررسی می‌کنیم. تحت شرایط مناسب، برخی ویژگی‌ها از قبیل وجود و یکتایی حل‌های این معادلات مورد بحث قرار می‌گیرند. نتایج ما را می‌توان به‌عنوان توسیع‌های مهمی به‌ترتیب از نتایج پایداری مربوط به معادلات تابعی تک-مقدار و معادلات تابعی مجموعه-مقدار درنظر گرفت.